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A reformulation of Schrödinger and Dirac equations in terms of observable local densities and electromagnetic fields: a step towards a new interpretation of quantum mechanics?

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Résumé. On montre que l'équation de Schrödinger, l'équation de Dirac à 3 dimensions et l'équation de Dirac à 4 dimensions peuvent être transformées de manière à éliminer tous les effets de phase et de jauge. Des densités locales réelles sont définies et l'on montre que les tenseurs obéissent à des « équations probabilistes » réelles qui sont équivalentes aux équations de la mécanique quantique. Ces nouvelles équations, qui sont construites explicitement, ne font intervenir que les densités locales et le champ électromagnétique extérieur. De nouvelles conditions aux limites sont aussi établies. L'expérience d'Aharonov-Bohm est réinterprétée dans le cadre de ce nouveau formalisme.

Abstract. It is shown that the Schrödinger equation, the 3-dimensional Dirac equation and the 4-dimensional Dirac equation can be transformed so as to eliminate all phase and gauge effects. Real local densities are defined and it is shown that these tensors obey real « probabilistic equations » which are equivalent to the equations of quantum mechanics. These new equations, which are explicitly constructed, involve only local densities and the external electromagnetic field. New boundary conditions are also derived. The Aharonov-Bohm experiment is reinterpreted in the framework of this new formalism.

« I think one can make a safe guess that uncertainty relations in their present form will not survive in the physics of the future. »

P.A.M. Dirac, 1962
(Scientific American, 1963)

1. Introduction.

Quantum mechanics is extremely successful but many physicists [1], even young ones, do not consider its orthodox interpretation as completely satisfactory. The difficulties are purely philosophical but real. Quantum mechanics is an efficient recipe. Is that enough?

Actually, some physicists think that quantum mechanics should also present a picture of reality. However, concerning this question, physicists have different opinions. In this introduction the author will present his views and his motivations. If the reader does not agree with them and is just interested in facts, he should pass directly to the following sections.

The author thinks that some difficulties may be connected with the fact that a wave function is not an observable.

Of course, physicists working now on particle theory do not explicitly use wave functions but more powerful tools like functional integrals or the S matrix. However, when they calculate a physical quantity, they generally use, in the course of the calculation process, quantities which are gauge and phase dependent.

The situation would be clearer if quantum mechanics could be directly formulated in terms of probabilities which are observables with a physical meaning.

Let us dream. These probabilities should obey partial derivative equations just as the probability distributions of classical statistical mechanics obey a Fokker-Planck equation. These « probabilistic equations » cannot be trivial because the vacuum is not
a simple system; it is certainly as complex as a heat bath.

Of course dissipation is not present since a particle may have a constant motion in the vacuum but polarization effects should be important (think of the complexity of the vacuum-vacuum diagrams in field theory!). These probabilistic equations should describe the evolution of real probabilities or observables. They should also present a more realistic view of nature than ordinary quantum mechanics. Therefore, we may expect that it would be easier to interpret them.

In statistical mechanics, we describe the motion of a particle interacting with an external potential and a heat bath by writing a Langevin equation and, from this Langevin equation, we deduce the corresponding Fokker-Planck equation; in this case, the Langevin equation provides a physical interpretation of the Fokker-Planck equation. We may expect a similar situation in quantum mechanics [2].

At the present time, this is only a dream. However, we may ask the question « Is it possible to replace a Schrödinger equation or a Dirac equation by an equivalent probabilistic equation describing how local densities (local mean values) vary with the local electromagnetic field? ». Another formulation of the same question is « Can we transform quantum mechanics so as to eliminate all phase and gauge effects? ».

Our answer is « Yes » and the aim of the present article is to prove the validity of this affirmation. The reader, however, might be tempted to say « No ». In fact, when we try to interpret the famous « Aharonov-Bohm experiment » [3] (see Fig. 1) we discover that the interference pattern of a particle interfering with itself can be modified without applying any electromagnetic field to the particle. In the experiment, the particle beams define a toroidal space and to modify the interference pattern we have only to change the

![Fig. 1. — Propagation of a wave in a toroidal domain. The circuit C is represented by the dashed line. \( \Phi \) is the magnetic flux through the hole.](image-url)
paradox also occurs in semiconductors and in that case, the situation is quite clear. The anomaly which is observed in strongly repulsive potential is only a mark of the instability of the vacuum (or ground state) with respect to the production of pairs of particles of opposite signs. Thus, all these arguments brought us to believe that Dirac equation may be more realistic that Schrödinger equation.

Accordingly, we derive here the «probabilistic equations» which can be associated with the usual four-dimensional Dirac equation (3 dimensions of space, 1 dimension of time). The system however is not very simple. Consequently, we begin with a simpler problem, the derivation of the probabilistic equations » which can be associated with the three-dimensional Dirac equation (2 dimensions of space, 1 dimension of time).

In section 3, we show how the simple three-dimensional Dirac equation can be transformed into a probabilistic equation (and conversely). In section 4, a similar treatment is applied to the four-dimensional Dirac equation. However, this case is rather complicated because the number of parameters which are involved in the calculation is much larger. Consequently, if the reader is only interested in methods and results, he may skip all the details of the calculation and pass directly to subsection 4.6 where the results are listed.

We also show that the probabilistic equations are not always sufficient to determine the system. It is sometimes necessary to add special boundary conditions, especially when the particle propagates in a domain which is not simply connected (toroidal).

Section 5 is more physical and is devoted to a discussion of the Aharonov-Bohm experiment. Finally, in the conclusion, we suggest that the preceding results may contribute to a better understanding of quantum mechanics.

The Einstein convention of summation over repeated indices is used everywhere in this article.

2. A transformation of the Schrödinger equation.

2.1 LOCAL DENSITIES AND PROBABILISTIC EQUATIONS.

Using proper units, we may write the one-particle Schrödinger equation in the following form (j=x, y, z):

\[ i \frac{\partial \phi}{\partial t} = - \frac{1}{2M} \left( \partial_j - iA_j \right) \left( \partial_j - iA_j \right) \phi + V \phi \]  

(1)

where \( V \) is the potential, and \( A_j \) a component of the vector potential \( A \). The quantities \( \phi, V, A_j \) are functions of \( x, y, z, \) and \( t \). We shall also use the complex conjugate of equation (1):

\[ -i \partial \phi^* = - \frac{1}{2M} \left( \partial_j + iA_j \right) \left( \partial_j + iA_j \right) \phi^* + V \phi^*. \]  

(2)

The density \( P \) and the current density \( J \) at a given point are defined by

\[ P = \phi^* \phi \]

\[ J_j = -\frac{i}{2M} \left[ \phi^*(\partial_j - iA_j) \phi - \phi(\partial_j + iA_j) \phi^* \right] \]

\[ = -\frac{i}{2M} \left( \phi^* \partial_j \phi - \phi \partial_j \phi^* \right) - \frac{1}{M} A_j \phi^* \phi. \]  

(3)

It will also be convenient to introduce the velocity \( v \) by setting

\[ J_j = P v_j. \]  

(4)

We note that the wave function can be written in the form

\[ \phi = P^{1/2} e^{i\alpha}, \]  

(5)

where \( \alpha \) is the phase of the wave function. Now equation (3) gives

\[ J_j = \frac{P}{M} (\partial_j \alpha - A_j), \]  

(6)

and therefore according to equation (6)

\[ v_j = \frac{1}{M} (\partial_j \alpha - A_j). \]  

(7)

From this equation, we deduce immediately the condition

\[ \partial_j p_j - \partial_k v_j = -\frac{1}{M} (\partial_j A_k - \partial_k A_j) = -\frac{1}{M} \epsilon_{jkm} B_m \]  

(8)

where \( B \) is the magnetic induction (\( \epsilon_{jkm} \) is completely antisymmetric and \( \epsilon_{xxx} = 1 \). Thus, equation (8) is a condition which has to be fulfilled [8] independently of the propagation equations which we shall now consider.

Let us calculate \( \partial_t P \):

\[ \partial_t P = \phi^* \partial_t \phi + \phi \partial_t \phi^*, \]

thus by using equations (1) and (2), we find

\[ \partial_t P = \frac{i}{2M} (\phi^*(\partial_j - iA_j) \partial_j - iA_j) \phi - \phi(\partial_j + iA_j) \partial_j \phi^* \]

\[ + \frac{i}{M} A_j \phi^* \partial_j \phi - \phi \partial_j \phi^* \phi \]

\[ = \frac{1}{M} \partial_j \left( \frac{i}{2} (\phi^* \partial_j \phi - \phi \partial_j \phi^*) + A_j \phi^* \phi \right), \]
and therefore, using definition (3), we obtain the conservation equation
\[ \partial_t P = - \partial_j J_j, \] (9)
which is the first probabilistic equation. We have now to calculate \( \partial_i J_j \):
\[
 M \partial_i J_j = - \frac{i}{2} \left[ (\partial_i \phi^*) (\partial_j \phi) + \phi^* \partial_j \partial_i \phi 
 - (\partial_i \phi) (\partial_j \phi^*) - \phi \partial_j \partial_i \phi^* \right] 
 - (\partial_i A_j) \phi^* \phi - A_j (\phi^* \phi \partial_i \phi^*). \] (10)

In this equality, we replace \( \partial_t \phi \) and \( \partial_t \phi^* \) by the values obtained from equations (1) and (2). A straightforward (but not very short) calculation shows that \( \partial_i J_j \) can be written in the following way
\[
 M \partial_i J_j = - \partial_i T_{ji} + (- \partial_j V - \partial_i A_j) P + (\partial_j A_i - \partial_i A_j) J_i 
 + (\partial_j A_i - \partial_i A_j) J_i. \]

We note that equation (6) involves only the electric field \( E_j \) and the magnetic induction \( B_j \)
\[
 E_j = - \partial_j V - \partial_i A_j 
 B_j = \frac{1}{2} \varepsilon_{jm} (\partial_j A_i - \partial_i A_j). \] (13)
Thus the propagation equations are
\[
 \partial_t P = - \partial_i J_i 
 M \partial_i J_j = - \partial_i T_{ji} + PE_j + \varepsilon_{jm} J_i B_m, \] (14)
where \( T_{ji} \) is given by equation (12).

It is convenient to write these equations by expressing \( J_j \) in terms of \( P \) and \( v_j \) (see Eq. (4)). The propagation equations obtained in this way and equation (8) form the set of probabilistic equations which we wanted to obtain, i.e.,
\[
 \partial_t v_k - \partial_k v_j = - \frac{1}{M} \varepsilon_{km} B_m 
 \partial_t P + v_i \partial_i P = - P \partial_i v_i 
 M(\partial_i v_j + v_i \partial_j v_j) = \partial_t W + E_j - \varepsilon_{jm} B_i v_m \] (15)
where
\[
 W = \frac{1}{4 M} \left[ \frac{\partial_i P}{P} - \frac{(\partial_i P)^2}{2 P^2} \right]. \] (16)

We see that these equations are hydrodynamic equations where \( W \) plays the role of a quantum potential or pressure. Incidentally we remark that if \( B = 0 \), we have an irrotational flow. We also note that equation (8) can be considered as an initial condition. In fact, the propagation equations are such that if equation (8) is satisfied for \( t = t_0 \), it remains satisfied for \( t > t_0 \) as can be easily verified. We also note that equations (15) give the electric field \( E \) and the magnetic induction \( B \) in terms of the local densities; the magnetic induction is given by the first equation (15) and the electric field by the third equation (15) after replacement, in this equation, of \( \varepsilon_{jm} B_i \) by the value extracted from the first equation (15).

Thus a knowledge of \( P, v_j \) at all times determines completely the values, at all times, of the electromagnetic fields applied to the particle. The fact that these quantities obey a closed set of equations which determine their time dependence, is in itself very remarkable (very different situations occur in statistical mechanics). We note that the time dependence of \( P \) is given by the continuity equation (second equation (15)) which is linear with respect to \( P \), in agreement with probability theory. On the contrary, the third equation (15) is not linear in \( P \); this fact which may look strange cannot be directly interpreted, but, in some way, this equation looks like a « self-consistent equation ».

Let us now show that the preceding set of equations is equivalent to the Schrödinger equation. This means that starting from these equations, we can reconstruct the Schrödinger equation. For this purpose, we shall proceed as follows. The electromagnetic fields which appear in equation (15) obey Maxwell equations but are defined independently of any gauge. In order to fix the gauge, we have to choose a potential \( V \) and a vector potential \( A \) which have to obey equation (13) but which otherwise are arbitrary. Now we can define a wave function \( \phi \) by setting
\[
 \phi = P^{1/2} \phi^*, \] (17)
where the phase \( \alpha \) at a point \( A \) is given by a path integral going from the origin \( O \) to the point \( A \):
\[
 \alpha = \int_0^A dr_j (M v_j + A_j). \] (18)
It will be assumed that the domain in which the particle moves is simply connected. In this case, the integral depends on the positions of \( A \) and \( O \) but as a consequence of equation (8), it does not depend on the shape of the path. Thus, the phase \( \alpha \) of \( \phi \) is well defined.

Let us now calculate \( i \partial_t \phi \):
\[
 i \partial_t \phi = \frac{i}{2} \phi^* \frac{\partial_i P}{P} - \phi \int_0^A dr_j (M \partial_i v_j + \partial_i A_j). \] (19)
Let us now use equations (15); we obtain

\[ i \partial_t \phi = -\frac{i}{2} \phi \frac{\delta(P v_t)}{P} \]

\[ - \phi \int_0^\Lambda \mathrm{d}r f(-M v_t \partial_r v_j + \partial_r W + E_0 - \omega_i \omega_j B_i v_m + \partial_r A_j). \]  

(20)

The integral can be transformed by using equations (8) and (13). We find

\[ i \partial_t \phi = -\frac{i}{2} \phi \delta(P v_t) \]

\[ - \phi \int_0^\Lambda \mathrm{d}r f(-M v_t \partial_r v_i + \partial_r W - \partial_r V). \]

We can now integrate and we obtain

\[ i \partial_t \phi = -\frac{i}{2} \phi \delta(P v_t) \]

\[ + \phi \left( \frac{M}{2} v_t v_i - W - V + V_0 \right) \]

where \( V_0 \) is just a constant (which can be absorbed in \( V \)). More explicitly, we have (see Eq. (16))

\[ i \partial_t \phi = -\frac{i}{2} \phi \delta(P v_t) \]

\[ + \phi \left[ \frac{M}{2} v_t v_i - \frac{1}{4} M^2 \partial_t \partial_i v_i + \frac{1}{8} M^2 \partial_t^2 (\partial_i v_i)^2 + V - V_0 \right]. \]

(21)

Let us now calculate \((\partial_t - iA_t)^2 \phi\). We have

\[ (\partial_t - iA_t) \phi = \frac{e^{i \alpha}}{2 P^{1/2}} \partial_t P \partial_t + i e^{i \alpha} (\partial_t \alpha - A_t) P^{1/2}. \]

On the other hand, equation (18) gives

\[ \partial_t \alpha = M v_t + A_t \]

and therefore using this equation and equation (17), we find

\[ (\partial_t - iA_t) \phi = \phi \frac{P}{2 \partial_i P} \partial_i P + iM \phi v_t. \]

(22)

Now we have

\[ (\partial_t - iA_t)^2 \phi = \left( \frac{1}{2} \partial_t P \partial_t + iM v_t \right) (\partial_t - iA_t) \phi + \phi \partial_t \left( \frac{1}{2} \partial_t P \partial_t + iM v_t \right) \]

\[ = \phi \left( \frac{1}{2} \partial_t P \partial_t + iM v_t \right)^2 + \phi \partial_t \left( \frac{1}{2} \partial_t P \partial_t + iM v_t \right). \]

(23)

Now, we must compare \( i \partial_t \phi \) which is give by equation (21) and \(-\frac{1}{2 M} (\partial_t - iA_t)^2 \phi \) which is given by equation (23). A trivial calculation shows that the function \( \phi \) which is defined by equations (17) and (18) obeys the equation

\[ i \partial_t \phi = -\frac{1}{2} \partial_t (\partial_t - iA_t)^2 \phi + (V - V_0) \phi. \]

(24)

Thus, we see that the set of equations (15) is completely equivalent to the Schrödinger equation (1), since the constant \( V_0 \) can always be absorbed in \( V \).
However this equivalence is only formal because the condition \( P \neq 0 \) has been implicitly assumed. If \( P = 0 \) at some point, singularities may occur; the effect of such singularities will be studied in section 2.3.

2.2 Multiply connected domains. Additional boundary conditions. — To determine a solution of the probabilistic equations, we must add boundary conditions. Thus, we have to assume that \( P \) and \( J \) vanish at the border of the domain (and also \( \partial / \partial n \) since \( \phi = 0 \) on the border).

These conditions are not sufficient if the domain is not simply connected. Let us consider, in a two-dimensional space, the propagation of a wave of wave vector \( k \) inside a rectangular strip with cyclic boundary conditions. The corresponding values of \( P \) and \( J \) are constants and the probabilistic equations do not give the value of \( k \). We need an additional boundary condition.

Thus, let us consider a multiply connected domain (a torus) and, in this space, a circuit \( C \) which cannot be reduced to a point by continuous deformation (see Fig. 1). Let \( \alpha \) be the phase of \( \phi \) at a given point

\[ \phi = P^{1/2} e^{i\alpha} \]

Quantum mechanics tell us that the variation of \( \alpha \) along \( C \) must be a multiple of \( 2\pi \)

\[ \int_C d\gamma = 2\pi n \]  \hspace{1cm} (25)

On the other hand, equation (7) gives

\[ v_j = \frac{1}{M} (\partial / \partial n - A_j) \]

and equation (25) can be written in the form

\[ \int_C d\gamma (M v_j + A_j) = 2\pi n \]  \hspace{1cm} (26)

The circuit \( C \) can be considered as the edge of a two-sided surface \( S \) and we have (Stokes theorem):

\[ \int_C d\gamma A_j = \frac{1}{2} \int_S d\gamma (\partial_j A_k - \partial_k A_j) = \int_S d\sigma_j B_j \]

where \( d\sigma_j \) defines an element of the surface \( S \). Consequently, equation (26) gives

\[ \int_C d\gamma M v_j = -\int_S d\sigma_j B_j + 2\pi n \]  \hspace{1cm} (27)

Thus we obtain a global boundary condition involving only local densities and the flux of the magnetic induction through \( S \). Of course it is necessary to write as many conditions as there are independent irreducible nontrivial circuits in the domain (their number is the genus of the domain). These boundary conditions remind us of the Bohr quantization conditions but they are as exact as the Schrödinger equation. With the probabilistic equations and the boundary conditions on the border of the domain, they determine the solutions \( P(x, y, z, t) \) and \( J_j(x, y, z, t) \) corresponding to the physical situation.

2.3 Singular solutions. — If \( P \) vanishes at a point, \( v_j \) may be singular. Consider for instance, the function

\[ \varphi(x, y, 0) = e^{im} J_m(kr) e^{-ikx} \]

which is a solution of the equation

\[ \partial_t \varphi = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \varphi \]

in the circle \( r < R \), for values of \( k \) given by \( J_m(kR) = 0 \).

In the vicinity of the origin

\[ \varphi(x, y, 0) \approx \frac{1}{2^m m!} (x + iy)^m \]

\[ P = \frac{1}{(2^m m!)^2} (x^2 + y^2)^m \]

\[ V_x = -\frac{y}{x^2 + y^2}, \quad v_x = \frac{x}{x^2 + y^2} \]

The velocity is singular at the origin and the phase of the wave function is also singular at this point.

In this case, we need an additional boundary condition. If a solution of the probabilistic equation is singular at the origin, we quantize it by assuming that it is defined in the domain obtained by removing the origin from the square box. This domain is not simply connected, consequently we draw a circuit around the origin and we write the corresponding boundary condition given by equation (27). In the present case \( n = m \).

In general, in three dimensional space, if \( \varphi \) is complex, \( P \) vanishes on a line which can be considered as a kind of vortex line. The solutions of the probabilistic equations are quantized by writing boundary conditions corresponding to circuits drawn around these vortex lines.

If, on the contrary, \( \varphi \) is real, \( J_j \) vanishes everywhere and therefore \( v_j \) also vanishes everywhere. In three dimensional space, \( P \) may vanish on surfaces and the gradient of \( P^{1/2} \) has to be continuous on the surface.

3. A transformation of the three-dimensional Dirac equation.

3.1 The equation; algebraic considerations. — Let us consider a quantum particle with spin 1/2 in a three-dimensional relativistic space-time. The coordinates of a point are \( x, y \) and \( t \). The metric tensor \( g_{\mu\nu} \) has only diagonal elements

\[ g_{tt} = -g_{xx} = -g_{yy} = 1 \]  \hspace{1cm} (28)
We assume that a classical electromagnetic field $F_{ij}$ is interacting with the particle. We have

$$F_{ij} = \partial_j A_i - \partial_i A_j ,$$

(29)

where $A_j$ is a component of the vector potential. The components $F_{xt}$ and $F_{yt}$ define the electric field; the third component $F_{xy}$ defines the magnetic induction (and can be considered as a pseudo-scalar).

The motion of a particle of spin $\frac{1}{2}$ and mass $M$ can be described by a three-dimensional Dirac equation. Using proper units ($\hbar = 1$, e = 1, c = 1) we write this equation in the usual way [7, 9].

$$\gamma^l (\partial_l - iA_l) \phi + iM\phi = 0$$

(30)

where

$$\gamma^l \gamma^l + \gamma^l \gamma_l = 2 g^{ll} \mathbb{1} \quad (j = x, y, t)$$

and

$$\gamma^l \phi = \beta \gamma^l \phi .$$

(32)

(here the particle has a negative charge).

Here $\phi$ has two complex components. Accordingly the symbols $\gamma^l$ and $\beta$ represent $2 \times 2$ matrices and $\mathbb{1}$ is the unit $2 \times 2$ matrix. These matrices can be expressed in terms of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$. For instance, we may set

$$\gamma^x = i\sigma_1, \quad \gamma^y = i\sigma_2, \quad \gamma^z = \sigma_3 .$$

(33)

The algebra of the matrices $\gamma^l$ is determined by the relation

$$\gamma^l \gamma^x \gamma^y = g^{kl} \mathbb{1} - i\delta^{kl} \gamma_1 ,$$

(34)

where $\delta^{kl}$ is completely anti-symmetric and $\epsilon^{xyz} = 1$.

At a given point $(t, x, y)$, the density $P$ and the current density $J^l$ are defined by

$$P = \langle \bar{\phi}\phi \rangle$$

$$J^l = \langle \bar{\phi}\gamma^l \phi \rangle ,$$

(35)

where $P$ and $J^l$ are functions of $(t, x, y)$.

At a given point, $\phi$ depends on 4 real parameters but the phase is not observable. Thus, as we shall see, the preceding quantities cannot be independent. We shall use the identity

$$2 \delta_{ab} \delta_{ba} = \delta_{ab} \delta_{ba} + \sigma_{ab} \sigma_{ba}$$

(36)

where $\sigma_{ab}$ is a matrix element of $\sigma_x$.

To derive this formula, we have only to consider the group of quaternions and its representation by Pauli matrices:

$$E = \mathbb{1}, \quad \Sigma_1 = -i\sigma_1, \quad \Sigma_2 = -i\sigma_2, \quad \Sigma_3 = -i\sigma_3$$

$$\bar{E} = -\mathbb{1}, \quad \overline{\Sigma}_1 = i\sigma_1, \quad \overline{\Sigma}_2 = i\sigma_2, \quad \overline{\Sigma}_3 = i\sigma_3 ,$$

(37)

(where $\mathbb{1}$ is the $2 \times 2$ unit matrix). Then a general theorem [10] gives

$$\Sigma_{a,ab}(\Sigma_a)^{-1} = 4 \delta_{ab} \delta_{ba}$$

which leads to equation (29). Using the representation (22) of the matrices $\gamma^l$, we see that we may also write:

$$2 \delta_{ab} \delta_{ba} = \delta_{ab} \delta_{ba} + \gamma^l_{ab} \gamma^l_{ba} .$$

(37)

Let us multiply the preceding equation by the product $\bar{\phi}_a \phi_x \phi_b \phi_y$ of components of $\phi$. By summing over all indices, we obtain

$$P^2 = \bar{J}^l J^l .$$

(38)

The current density has 3 components $J^l$ which are real. This vector field defines the state of the system since the phase of $\phi$ at each point is not well defined. In fact, we can make the gauge transformation

$$\phi(t, x, y) \rightarrow \phi(t, x, y) e^{i\alpha(t, x, y)} ,$$

$$A^l(t, x, y) \rightarrow A^l(t, x, y) + \partial^l \alpha(t, x, y) .$$

The equations (30) and (31) are invariant; the quantities $P(t, x, y), J^l(t, x, y)$ and also the tensor $F_{ij}(t, x, y)$ remain unchanged. We want now to eliminate the phase and gauge dependent quantities. Starting from the Dirac equation, we shall derive « probabilistic equations ». In these equations, the current density will replace the wave function and the electromagnetic field will replace the vector potential.

3.2 The Probabilistic Equations. — An equation of conservation is easily obtained. Let us multiply equation (30) by $\phi$ to the left and equation (31) by $\bar{\phi}$ to the right. By adding the resulting equalities, we find

$$\partial_j (\bar{\phi}) \gamma^l \phi = 0$$

and therefore

$$\partial_j J^l = 0 .$$

(39)

We have now to derive dynamical equations. These equations will express the value of the electromagnetic field applied to the particle at a given point, in terms of the components of the current density and of their derivatives at the same point.

We note that by applying various terms to the lefthand side of equation (30) and to the righthand side of equation (31), it is possible to obtain several equations. From these equations we shall extract the value of the vector potential. Let us multiply equation (30) by $\bar{\phi}_a \phi_x$ to the left and equation (31) by $\gamma^x \phi$ to the right. We obtain

$$\langle \bar{\phi}_a \gamma^x \gamma^l \phi \rangle - iA^l \langle \bar{\phi}_a \gamma^x \phi \rangle + iM \langle \bar{\phi}_a \gamma^x \phi \rangle = 0$$

$$- (\partial^l \bar{\phi}) \gamma^x \phi - iA^l \langle \bar{\phi}_a \gamma^x \phi \rangle + iM \langle \bar{\phi}_a \gamma^x \phi \rangle = 0 .$$

(40)
Let us add these equalities. Using equation (34), we find
\[
A_k(\phi, \phi) = \frac{i}{2} (\phi \partial_k \phi) + \frac{i}{2} (\partial_k \phi) \phi - \frac{1}{2} \varepsilon_{kh} \partial^l \phi \partial^l j^k
+ M(\phi, \gamma_k \phi). \tag{41}
\]
Thus, in addition to equation (39), we obtain three real equations. These equations seem independent from one another. We may assume that they are equivalent to the initial Dirac equation. Now, we may set
\[
C_k = -\frac{i}{2(\phi \phi)} [(\phi \partial_k \phi) - (\partial_k \phi)] \tag{42}
\]
and we find
\[
A_k = C_k - \frac{1}{2} \varepsilon_{kh} \partial^l j^l + M \frac{J_k}{\phi}. \tag{43}
\]
We shall now calculate the electromagnetic field
\[
F_{jk} = \partial_k A_j - \partial_j A_k. \tag{44}
\]
We may set
\[
\psi_{jk} = \partial_j C_k - \partial_k C_j \tag{45}
\]
This equality is the « probabilistic equation » which we want, but the calculation is not finished. By looking at equation (46) we see that \(\psi_{jk}\) is gauge invariant and does not depend on the phase of \(\phi\). We have now to show that this quantity can be expressed in terms of components \(J^j\) of the current density and derivatives of these components. Equations (42) and (45) give
\[
\psi_{jk} = -\frac{i}{(\phi \phi)} [(\phi \partial_k \phi) - (\partial_k \phi)]
- \frac{i}{(\phi \phi)^2} [(\phi \partial_k \phi) (\partial_j \phi) - (\phi \partial_j \phi) (\partial_k \phi)]. \tag{47}
\]
Now it can be directly verified that \(\psi_{jk}\) remains invariant under the transformation
\[
\phi(t, x, y) \rightarrow \phi(t, x, y) e^{i\alpha(t, x, y)}. \tag{48}
\]
Let us transform the first term on the righthand side. By using the identity (37), we find that
\[
(\phi \phi) [(\phi \partial_k \phi) - (\partial_k \phi)] = 1 \frac{1}{2} [(\phi \partial_k (\phi \phi)) - (\phi \partial_k \phi) (\phi \phi)]
+ \frac{1}{2} [(\phi \partial_k (\phi \phi)) (\partial_j \phi) - (\phi \partial_k \phi) (\partial_j \phi)]. \tag{49}
\]
Let us bring this result in equation (47) (first part of the righthand side). We obtain
\[
\psi_{jk} = -\frac{i}{4(\phi \phi)^2} \{ - \partial_j \phi [(\phi \partial_k \phi) - (\phi \partial_k \phi)]
- (\phi \partial^l \phi) [(\phi \partial_l \gamma_j) - (\partial_l \phi)]
+ (\phi \partial^l \phi) [(\phi \partial_l \gamma_j) - (\partial_l \phi)]
- \partial_j \phi [(\phi \partial_k \phi) - (\partial_k \phi)] \} \tag{50}
\]
Using the definitions (35), we may also write
\[
\psi_{jk} = -\frac{i}{4} \frac{p}{2} \{ - \partial_j p [(\phi \partial_k \phi) - (\phi \partial_k \phi)]
+ \partial_k p [(\phi \partial_k \phi) - (\partial_k \phi)]
+ \partial_j [(\phi \partial_k \phi) - (\partial_k \phi)]
- \partial_j [(\phi \partial_k \phi) - (\partial_k \phi)] \} \tag{51}
\]
The equation
\[ P \partial_j P = J^1 \partial_j J^1 , \]
which is a direct consequence of equation (38) can be used to eliminate \( \partial_j P \) from equation (51). In this way, we obtain
\[ \psi_{jk} = \frac{1}{4p^3} (\partial_j J_m I_{k}^m - \partial_k J_m I_{j}^m) \]  
(52)
where
\[ I_{k}^m = - i \overline{\phi} \phi [ (\phi \gamma^m \partial_k \phi) - (\partial_k \phi \phi^m \phi) ] + i \overline{\phi} \phi^m \phi [ (\overline{\phi} \partial_k \phi) - (\partial_k \phi \phi) ] , \]
or after a trivial reordering of the right-hand side
\[ I_{k}^m = - i [ (\overline{\phi} \phi) (\phi \gamma^m \partial_k \phi) + (\partial_k \phi \phi^m \phi) ] + i [ (\overline{\phi} \phi^m \phi) (\partial_k \phi) + (\overline{\phi} \phi) (\phi \gamma^m \phi) ] . \]
Both brackets on the right-hand side are now phase independent. More explicitly, we may write
\[ I_{k}^m = - i [ (\overline{\phi} \phi) \phi \phi^m \phi + \partial_k (\overline{\phi} \phi) \phi \phi^m \phi ] \gamma_{k}^m \]
and therefore
\[ I_{k}^m = - i [ (\overline{\phi} \phi) \phi \phi^m \phi ] \gamma_{k}^m \]
(53)
Again, we shall use equation (37). Multiplying this equation by \( \phi \phi^m \phi \) and summing over \( b \) and \( \alpha' \), we obtain
\[ 2 \overline{\phi} \phi \phi^m \phi = \delta_{ab} (\overline{\phi} \phi) + \gamma_{ab} (\phi \gamma_i \phi) = \delta_{ab} P + \gamma_{ab} J_1 . \]
(54)
Let us use this identity to transform all terms of the form \( \overline{\phi} \phi \phi^m \phi \) which appear in equation (53). We obtain
\[ I_{k}^m = - \frac{i}{4} \delta_{ab} P + \gamma_{ab} J_1 (\partial_k \phi \phi \phi^m \phi) + \frac{i}{4} (\partial_k \phi \phi \phi^m \phi) \gamma_{k}^m \]
or in a more compact way
\[ I_{k}^m = - \frac{i}{4} \text{Tr} \{ (P + \gamma^1 J_1 ) \gamma^m (\partial_k \phi \phi \phi^m \phi) \}
+ \frac{i}{4} \text{Tr} \{ (P + \gamma^1 J_1 ) (\partial_k \phi \phi \phi^m \phi) \gamma^m \}
\]
and therefore
\[ I_{k}^m = - \frac{i}{4} \text{Tr} \{ (P + \gamma^1 J_1 ) [ \gamma^m, \gamma^m ] \partial_k J_1 \} \gamma_{k}^m \]
(55)
Equations (33) and (34) tell us that
\[ [ \gamma^m, \gamma^m ] = - 2 i \epsilon^{mpl} \gamma_p \]
(56)
\[ \text{Tr} \gamma^p = 0 \quad \text{Tr} (\gamma^m \gamma_p) = 2 \delta_{mp} . \]
Thus equation (55) gives
\[ I_{k}^m = - \epsilon^{ml} J_1 \partial_k J_1 . \]
The preceding equality and equation (52) give
\[ \psi_{jk} = -\frac{1}{2P^3} \epsilon^{lmn} J_l \partial_j J_m \partial_k J_n. \]  
(58)

Let us bring this expression in equation (46).

We obtain the result
\[ F_{jk} = -\frac{1}{2} \epsilon^{lmn} J_l \partial_j J_m \partial_k J_n \]
\[ \quad - \partial_j \left( \frac{1}{2P} \epsilon_{klm} \partial^l J^m \right) + \partial_k \left( \frac{1}{2P} \epsilon_{jlm} \partial^j J^m \right) \]
\[ \quad + M \partial_j \left( \frac{1}{P} J^k \right) - M \partial_k \left( \frac{1}{P} J^j \right). \]
(59)

This equation and the relations (38) and (39)
\[ P^2 = J^j J_j \quad \partial J^j = 0 \]
define the « probabilistic equations » associated with the three-dimensional Dirac equation.

Another form of the same equation can be obtained by setting
\[ J^j = P v^j. \]

In this way, we obtain
\[ v^j v_j = 1 \]
\[ \partial_j (P v^j) = 0 \]
(60)

\[ F_{jk} = -\frac{1}{2} \epsilon^{lmn} v_l \partial_j p_m \partial_k v_n \]
\[ \quad - \frac{1}{2} \partial_j (\epsilon_{klm} \partial^l v^m) + \frac{1}{2} \partial_k (\epsilon_{jlm} \partial^j v^m) \]
\[ \quad - \frac{1}{2} \partial_j \left( \epsilon_{klm} \frac{\partial^l P}{P} v^m \right) + \frac{1}{2} \partial_k \left( \epsilon_{jlm} \frac{\partial^j P}{P} v^m \right) \]
\[ \quad + M (\partial_j p_k - \partial_k p_j). \]
(61)

3.3 REMARKS CONCERNING THE « PROBABILISTIC EQUATIONS ». — We note that the local densities determine completely the electromagnetic field. This is the best we can do since the forces which produce the motion of the particle are completely determined by this motion. We also note that at each point the wave function depends on three « physical » parameters and that equation (59) gives three relations at each point. Thus equation (59) should be completely equivalent to the Dirac equation.

This equivalence is explicitly proved in appendix A. The proof is valid in regions where \( P \neq 0 \). When \( P = 0 \) singularities occur, in general, the components \( v_j \) become infinite on 2-dimensional surfaces which are boundaries of regions of the 3-dimensional space. In this case, as we have a simple pole on the surface, the matching condition is trivial.

These probabilistic equations have great merits; they are real and all the variables which appear in them are « physical » quantities. These physical quantities are the current density and the electromagnetic field which are gauge independent.

Unfortunately, to reach this interesting result, we had to pay a high price : the simple linearity of the Dirac equation has been lost. Thus, from a practical point of view. Equation (59) seems useless. We may only hope that it will lead to a more fundamental interpretation of quantum mechanics.

3.4 MULTIPLY CONNECTED DOMAINS. ADDITIONAL BOUNDARY CONDITIONS. — When the domain in which the particle propagates is multiply connected, additional boundary conditions have to be introduced as was explained in section 2.2. We consider a circuit \( C \) which cannot be reduced to a point by continuous deformation (see Fig. 1) and we want to express the fact that the variation along \( C \) of the global phase \( \alpha \) of \( \phi \) is a multiple of \( 2 \pi \).
The phase $\alpha$ is defined by writing the components $\phi_1$ and $\phi_2$ of $\phi$ as follows

$$\phi_1 = a e^{i(\theta + \phi)}$$
$$\phi_2 = b e^{i(\theta - \phi)}$$

where $a$ and $b$ are real positive.

The boundary condition is

$$\int_C \dot{r} \cdot (\partial_{\alpha} \alpha) = 2 \pi n. \quad (62)$$

We shall express this condition in terms of local densities by using equation (43)

$$A_k = C_k - \frac{1}{2} \rho_k \partial^j J^j + M_k^j \frac{J_k}{P}$$

where (see Eq. (35))

$$C_k = - \frac{i}{2 (\phi \phi)} (\dot{\phi} \partial_x \phi - \partial_x \phi \dot{\phi}).$$

In fact we shall apply equation (52) along $C$.

We note that $C_k = \partial_k \alpha + \text{terms independent of } \alpha$ and consequently that $(C_k - \partial_k \alpha)$ is a uniform function of $x, y, t$.

Accordingly, we have (see Eq. (45))

$$\int_C \dot{r}^k (C_k - \partial_k \alpha) = \frac{1}{2} \int_S \dot{r}^j \wedge \dot{r}^k (\partial_j C_k - \partial_k C_j)$$
$$= \frac{1}{2} \int_S \dot{r}^j \wedge \dot{r}^k \psi_{jk}. \quad (63)$$

Now using equation (63), let us calculate the circulation of $A_k$ along $C$.

With the help of equations (62) and (43), we find

$$\int_C \dot{r}^k A_k - 2 \pi n = \int_C \dot{r}^k \left( - \frac{1}{2} \rho_k \partial^j J^j + M_k^j \frac{J_k}{P} \right) +$$
$$+ \frac{1}{2} \int_S \dot{r}^j \wedge \dot{r}^k \psi_{jk}.$$

Finally, we obtain the boundary condition

$$\int_C \dot{r}^k \left( - \frac{1}{2} \rho_k \partial^j J^j + M_k^j \frac{J_k}{P} \right) +$$
$$+ \frac{1}{2} \int_S \dot{r}^j \wedge \dot{r}^k \psi_{jk} =$$
$$= \frac{1}{2} \int_S \dot{r}^j \wedge \dot{r}^k F_{jk} - 2 \pi n. \quad (64)$$

This equation is written in a relativistic form. However, it is convenient to assume that along $C$ we have $\dot{r}_t = 0$ and that $S$ is perpendicular to the $t$ axis. In this case

$$\frac{1}{2} \int_S \dot{r}^j \wedge \dot{r}^k F_{jk} = \int_S d\sigma^j B_j = \Phi$$

where $\Phi$ is the flux of the magnetic induction through the circuit $C$.

3.5 CLASSICAL LIMIT. — For reasons of simplicity, we wrote $\partial^j$ instead of $\partial^j$ in Dirac equation (30). If we had kept the factor $\hbar$ there, it would also appear in equation (61) in front of the terms which contain two derivatives. Thus, we see that the classical limit ($\hbar = 0$) of equation (61) is

$$F_{jk} = M(\partial_j v_k - \partial_k v_j). \quad (65)$$

This equation has also been obtained many years ago (from Klein Gordon equation) by Takabayasi [11].

The preceding equality gives:

$$M(v^k \partial_j v_k - v^j \partial_k v_j) = F_{jk} v^k.$$ 

On the other hand equation (60) gives

$$v^k v_k = 1$$

which leads to

$$v^k \partial_j v^k = 0.$$ 

Thus by combining the preceding equations we find

$$M v^k \partial_j v^j = - F_{jk} v_k. \quad (66)$$

The operation $v^k \partial_k$ is a differentiation along the trajectory. Introducing

$$(dx)^2 : dx^j dx^j$$

we may also write equation (66) in the form

$$M \frac{d}{ds} v^j = - F_{jk} v_k \quad (67)$$

which give the classical relativistic motion of the electron (Lorentz equation).

4. A transformation of the four-dimensional Dirac equation.

4.1 THE DIRAC EQUATION AND THE ALGEBRA OF THE $\gamma^j$. — We consider now the motion of a quantum particle with spin $1/2$ in a four-dimensional relativistic space-time. The coordinates of a point are $x, y, z,$ and $t$. The metric tensor $g_{ij}$ has only diagonal elements

$$g_{tt} = - g_{xx} = - g_{yy} = - g_{zz} = 1.$$ 

We assume that a classical electromagnetic field $F_{jk}$ is interacting with the particle. We have

$$F_{jk} = \partial_j A_k - \partial_k A_j, \quad (68)$$

where $A_j$ is a component of the vector potential.
The motion of a particle of spin $1/2$ and mass $M$ is described by the usual Dirac equation which will be written \[8\]

\[\gamma^l (\partial_j - i A_j) \phi + i M \phi = 0\]

where

\[\gamma^l \gamma^l + \gamma^l \gamma^l = 2 \gamma^4 \mathds{1}\]

\[(\gamma^4)^* = -\gamma^4 \quad (\gamma^4)^* = -\gamma^4 \quad (\gamma^4)^* = -\gamma^4 \quad (\gamma^4)^* = -\gamma^4 \quad \bar{\phi} = \phi^* \gamma^4 .\]

The field $\phi$ has now four components and the symbols $\gamma^l$ represent $4 \times 4$ matrices; $\mathds{1}$ is now the $4 \times 4$ unit matrix. It will be useful to define the spin tensor $\sigma^{ij}$ and the matrix $\gamma_5$ by setting

\[\sigma^{ij} = \frac{i}{2} (\gamma^i \gamma^j - \gamma^j \gamma^i) ,\]

and accordingly we have

\[\gamma^j \gamma^k = g^{jk} \mathds{1} - i \sigma^{jk} .\]

We also define $\gamma_5$ which has the properties

\[\gamma_5 = -i \gamma^2 \gamma^3 \gamma^4 \quad (\gamma_5)^* = \gamma_5 \quad \gamma_5 \gamma^j + \gamma^j \gamma_5 = 0 .\]

We shall also use the notation

\[\overline{e}^{ijm} = i \gamma_5 e^{ijm} \quad (73)\]

where $e^{ijm}$ is completely antisymmetric and $e^{xyz} = 1$.

Let us now derive a useful identity (similar to Eq. (26)). We note that the space $E$ of the $4 \times 4$ matrices which are of the form $\gamma^j u$ where $u$ is an hermitian $4 \times 4$ matrix can be spanned by using the following matrices

\[
\begin{array}{c|c}
\Gamma^A & \mathds{1} \\
\gamma^l & \gamma^j \\
\sigma^{jk} & \frac{i}{2} (\gamma^j \gamma^k - \gamma^k \gamma^j) \\
\gamma^5 & \gamma^j \\
i \gamma^3 & \\
\end{array}
\]

(74)

Two matrices $\Gamma^A$ and $\Gamma^B$ belonging to this set of 16 matrices obey the relation

\[\text{Tr} (\Gamma^A \Gamma_B) = 4 \delta_{AB} \varepsilon(A) .\]

(75)

where

\[\varepsilon(\mathds{1}) = \varepsilon(\gamma^j) = \varepsilon(\sigma^{jk}) = - \varepsilon(\gamma^5) = - \varepsilon(\gamma^j) = 1 .\]

(76)

Thus any $4 \times 4$ matrix $x$ belonging to $E$ can be expanded in the form

\[x = X_A \Gamma^A \quad (X_A \text{ real})\]

where according to equation (78)

\[X_A = \frac{\varepsilon(A)}{4} \text{Tr} (x \Gamma_A) .\]

Thus for any matrix $X$ belonging to $E$

\[x = \frac{\varepsilon(A)}{4} \Gamma_A \text{Tr} (x \Gamma_A) .\]

(77)

We shall now use these equalities to obtain the identity which defines the « Fierz transformations » [12]. The matrix $x$ has matrix elements $X_{ab}$ and a particular $X$ will be defined by setting

\[X_{ab} = \delta_{a \alpha} \delta_{b \beta} \]

where $a_0$ and $b_0$ have fixed values.

For this $X$, we have

\[(x \Gamma_A)_{ab} = \delta_{a \alpha} \Gamma_{A,b \beta} \text{Tr} (x \Gamma_A) = \Gamma_{A,b \beta} \text{ and therefore, equation (77) gives}

\[\delta_{a \alpha} \delta_{b \beta} = \frac{\varepsilon(A)}{4} \Gamma_{A,b \beta} \Gamma_{A,a \alpha} .\]

(78)

In a more standard form, we may write

\[4 \delta_{a \alpha} \delta_{b \beta} = \varepsilon(A) \Gamma^A_{ab} \Gamma_{A,b \beta} .\]

(78)

and this basic identity will be used several times in the following.

4.2 LOCAL DENSITIES. — At each point $(x, y, z, t)$ the 16 matrices $\Gamma^A$ generate 16 local densities of the form $\phi \Gamma^A \phi$, namely

\[P = (\bar{\phi} \phi) \quad J^l = (\bar{\phi} \gamma^l \phi) \quad S^{jk} = (\bar{\phi} \sigma^{jk} \phi) \quad K^l = (\bar{\phi} \gamma_5 \gamma^l \phi) \quad Q = (\bar{\phi} \gamma_5 \phi) .\]

We shall also use the notation

\[\overline{S}^{jk} = (\bar{\phi} \sigma^{jk} \phi) = (\bar{\phi} \gamma_5 \sigma^{jk} \phi) = \frac{i}{2} \varepsilon^{ijm} S_{im} .\]

(80)

It is not difficult to show and it is important to note that all the quantities which are defined by equations (79) are automatically real, as we want.

The 16 local densities defined by equations (79) are not independent. In fact, the wave function $\phi$ has four complex components. Thus, as the local densities do not depend on the global phase of $\phi$, they are functions of 7 independent variables. Thus, we must find 9 independent relations among these quantities.

These relations can be deduced from the identity (78). In this way, we find (see Appendix B) the
following (complete) sets of equations:
\[ J^J J_j = - K^j K_j = P^2 + Q^2 \]  
\[ K^J J_j = 0 \]  

and
\[ S^\mu S_\mu = 2(P^2 - Q^2) \]
\[ J_\mu S^\mu = - QK^\mu \]
\[ K_\mu S^\mu = - QJ^\mu \]
\[ S_\mu S^\mu = 4PQ \]
\[ J_\mu \bar{S}^\mu = PK^\mu \]
\[ K_\mu \bar{S}^\mu = PJ^\mu \]  

It is also possible to find these relations by using special representations of the \( \gamma^J \) and an adequate system of reference. The local densities \( P, Q, J^J, \) and \( K^J \) which obey equations (81) are just sufficient to define the state of the particle at a point \((t, x, y, z)\). In particular, we may set
\[ T^\mu = \frac{1}{P^2 + Q^2}(K^J J^J - J^J K^J) \]
\[ \bar{T}^\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda\mu} T_{\nu\lambda} \]  

Thus, \( T^\mu \) is associated with the plane containing the vectors \( J_\mu \) and \( K_\mu \) and \( \bar{T}^\mu \) with the plane perpendicular to these vectors. Now, equations (82) show that the antisymmetric tensor \( S^\mu \) can be expressed in terms of \( T^\mu \) and \( \bar{T}^\mu \)
\[ S^\mu = QT^\mu + \bar{P} \bar{T}^\mu \]  

and this expression is just a compact form of equations (82).
(This equality can be easily established by taking the axis \( O\bar{t} \) along \( J \) and the axis \( O\bar{z} \) along \( K \), which is possible according to equations (81).)

We also find
\[ T^\mu T_\mu = - \bar{T}^\mu \bar{T}_\mu = - 2 \]
\[ \bar{S}^\mu = - P T^\mu + Q \bar{T}^\mu \]  

(If the result in the first line can be used to derive the second one.)

It has to be shown that the local densities defined by the preceding equalities obey probabilistic equations, which are to be deduced from the Dirac equation.

Accordingly, we shall now derive:

1) Kinematic probabilistic equations which involve only local densities.
2) Dynamical probabilistic equations which involve local densities and the external electromagnetic field.

Incidentally, we may note with M. Gaudin [13] that it would be certainly easier to write these equations in terms of the phase independent elements \( \phi_a \phi_b \) of the density matrix at the point \((t, x, y, z)\). However, these matrix elements are not defined in a unique way, they are not independent, and they are not tensors. Thus, in principle, it is better to introduce local densities.

4.3 The Kinematic Probabilistic Equations. — Let us start by writing the Dirac equation
\[ \gamma^J(\partial_J - iA_J) \phi + iM \phi = 0 \]  
\[ (\partial^\mu \phi + iA_\mu \phi) \gamma^\mu - iM \phi = 0 \]  

It is easy to find out current conservation equations. Let us multiply equation (86) by \( \phi \) to the left and equation (87) by \( \phi \) to the right. By adding the resulting equalities, we obtain
\[ \partial_j (\bar{\phi} \gamma^j \phi) = 0 \]  

and therefore according to equation (79)
\[ \partial_j J^j = 0 \quad \text{(an equation which is similar to (39))} \]  

Let us now multiply equation (86) by \( \bar{\phi} \gamma_3 \) to the left and equation (87) by \( - \gamma^3 \phi \) to the right. By adding the resulting equalities, we obtain
\[ \partial_j (\bar{\phi} \gamma^3 \gamma^j \phi) + 2M(\bar{\phi} \gamma_3 \phi) = 0 \]

and therefore, according to equations (89)
\[ \partial_j K^j + 2MQ = 0 \]  

We can now derive a more complicated set of kinematical equations. Let us multiply (86) by \( - i\bar{\phi} \gamma_k \) to the left and (87) by \( i\gamma_3 \phi \) to the right.

By adding these equalities, we obtain
\[ A_4(\bar{\phi} \phi) = - \frac{i}{2} (\bar{\phi} \partial_4 \phi) + \frac{i}{2} (\partial_4 \bar{\phi} \phi) + \frac{1}{2} \partial^4 (\bar{\phi} \sigma_4 \phi) + + M(\bar{\phi} \gamma_4 \phi) \]  

an equation which is very similar to (41). Another equation of the same kind can be obtained as follows. We multiply (86) by \( - i\bar{\phi} \gamma_3 \gamma_k \) to the left and (87) by \( i\gamma_k \gamma_3 \phi \) to the right. By adding these equalities, we obtain
\[ A_4(\bar{\phi} \gamma_3 \gamma_k \phi) = - \frac{i}{2} (\bar{\phi} \gamma_3 \partial_k \phi) + \frac{i}{2} (\partial_k \bar{\phi} \gamma_3 \phi) + + \frac{1}{2} \partial^4 (\bar{\phi} \gamma_3 \phi \gamma_k \phi) \]  

Equations (90) and (91) are simple because both of them express \( A_4 \) in terms of matrix elements. For this reason, these equations will be used as a starting point to obtain new probabilistic equations.

In particular, a new set of kinematical equations is obtained by eliminating \( A_4 \) from these equations. Let us set
\[ D_k = (\bar{\phi} \gamma_3 \phi) \left[ (\bar{\phi} \partial_k \phi) - (\partial_k \bar{\phi} \phi) \right] - (\bar{\phi} \phi) \left[ (\bar{\phi} \gamma_3 \partial_k \phi) - (\partial_k \bar{\phi} \gamma_3 \phi) \right] \]  

However these matrix elements are not defined in a unique way, they are not independent, and they are not tensors. Thus, in principle, it is better to introduce local densities.
Then, by combining (90) and (91), we obtain the equation
\[ D_k = -\langle {\phi}_i \rangle \, \partial_j (\Phi \sigma^k \Phi) + \langle \Phi \partial_j (\Phi \sigma^k \Phi) \rangle \] (93)
and using definitions (79) and (80), we may also write
\[ D_k = -Q \partial S_{\Phi} + P \partial S_{\Phi}. \] (94)

The vector \( D_k \) defined by (92) is phase independent (gauge invariant); thus to derive a new kinematic probabilistic equation, we have only to express \( D_k \) in terms of local densities (and derivatives of these quantities).

With any \( 4 \times 4 \) matrix \( U \), we can associate the phase independent quantity
\[ H_4(U) = \langle \Phi \partial \Phi \rangle \, (\Phi \partial \Phi) + \langle \Phi \partial \Phi \rangle \, (\Phi \partial \Phi) \]
and we see from equation (92) that \( D_k \) is of that form. In fact, we have
\[ D_k = H_4(\gamma_3). \] (96)

Let us show how \( H_4(U) \) can be expressed in terms of local densities. Equation (95) can be written as follows
\[
H_4(U) = \frac{\varepsilon(A) \varepsilon(B)}{16} \left( \langle \Phi \partial \Phi \rangle \, (\Phi \partial \Phi) \right) \partial_k (\Phi \partial \Phi) \, Tr \left( U \left[ \Gamma^A, \Gamma^B \right] \right).
\] (99)

Thus according to equation (96), we have
\[ D_k = \frac{\varepsilon(A) \varepsilon(B)}{16} \left( \langle \Phi \partial \Phi \rangle \, (\Phi \partial \Phi) \right) \partial_k (\Phi \partial \Phi) \, Tr \left( \gamma_3 \left[ \Gamma^A, \Gamma^B \right] \right) \] (100)

and we see that on the right hand side of this equation all matrix elements are local densities.

Finally equations (93) and (100) lead to the kinematic probabilistic equations
\[ \frac{\varepsilon(A) \varepsilon(B)}{16} \left( \langle \Phi \partial \Phi \rangle \, (\Phi \partial \Phi) \right) \partial_k (\Phi \partial \Phi) \, Tr \left( \gamma_3 \left[ \Gamma^A, \Gamma^B \right] \right) + Q \partial S_{\Phi} - P \partial S_{\Phi} = 0. \] (101)

4.4 THE DYNAMIC PROBABILISTIC EQUATION. — In the preceding section, we have found for \( A_k \) two different expressions (Eqs. (90) and (91)) which give two independent conditions. By eliminating \( A_4 \), we found the kinematic equation (101). We have to take into account another equation and we shall choose equation (90) (see also Eqs. (79)).

\[ A_4(\Phi \Phi) = -\frac{i}{2} (\Phi \partial \Phi) + \frac{i}{2} (\partial_k \Phi \Phi) + \frac{1}{2} \partial S_{\Phi} \] (102)

From this equation, we can deduce the strength of the electromagnetic field \( F_{jk} \)
\[ F_{jk} = \partial_j A_k - \partial_k A_j. \] (103)

The aim of this operation is the elimination of all phase dependent (gauge dependent) terms. Using equations (90), let us write \( F_{jk} \) as the sum of two terms
\[ F_{jk} = \Phi_{jk} + \psi_{jk}. \] (104)

The first term is fairly simple
\[ \Phi_{jk} = \partial \left[ \frac{1}{2 P} \partial S_{\Phi} + \frac{M}{P} J \right] - \partial \left[ \frac{1}{2 P} \partial S_{\Phi} + \frac{M}{P} J \right], \] (105)

since it is directly expressed in terms of local densities.
The second term is more complicated:

\[ \psi_{jk} = \partial_j \left\{ -\frac{i}{2(\phi \phi)} \left[ \left( \partial \phi \partial_\phi \right) - \left( \partial_\phi \partial \phi \right) \right] \right\} - \partial_k \left\{ -\frac{i}{2(\phi \phi)} \left[ \left( \partial \phi \partial_\phi \right) - \left( \partial_\phi \partial \phi \right) \right] \right\}. \] (106)

The problem is to express \( \psi_{jk} \) in terms of local densities. We have (this equality is similar to Eq. (47) but now \( \psi \) has four components instead of two).

Let us use equation (78). Multiplying this equation by \( \partial_\phi \partial_\phi \partial_\phi \partial_\phi \), we find

\[ 4(\partial \phi \partial_\phi \partial_\phi \partial_\phi) = \varepsilon(A) (\partial \phi \partial_\phi \partial_\phi \partial_\phi), \]

an expression which will be used to transform the first bracket on the right hand side of equation (107). In this way, we obtain

\[ \psi_{jk} = -\frac{i}{4(\phi \phi)^2} \left[ \varepsilon(A) (\partial_j \phi \Gamma_A \phi) (\phi \Gamma_A \partial_k \phi ) - \varepsilon(A) (\partial_k \phi \Gamma_A \phi) (\phi \Gamma_A \partial_j \phi ) - \right. \]

\[ \left. - 4(\partial_j \phi \partial_\phi \partial_\phi \partial_\phi) + 4(\partial_k \phi \partial_\phi \partial_\phi \partial_\phi) \right] \] (108)

an equation which can also be written as follows

\[ \psi_{jk} = -\frac{i}{8(\phi \phi)^2} \left[ \varepsilon(A) \partial_j (\phi \Gamma_A \phi) \left[ (\phi \Gamma_A \partial_k \phi ) - (\partial_k \phi \Gamma_A \phi) \right] - 4 \partial_j (\phi \phi) \left[ (\phi \partial_k \phi ) - (\partial_k \phi \phi) \right] - \right. \]

\[ \left. - \varepsilon(A) \partial_k (\phi \Gamma_A \phi) \left[ (\phi \partial_j \phi ) - (\partial_j \phi \phi) \right] + 4 \partial_k (\phi \phi) \left[ (\phi \partial_j \phi ) - (\partial_j \phi \phi) \right] \right]. \] (109)

Let us again use equation (78). Multiplying equation (78) by \( \partial_\phi \partial_\phi \partial_\phi \partial_\phi \), we find

\[ 4(\phi \phi)^2 = \varepsilon(A) (\phi \Gamma_A \phi) (\phi \Gamma_A \phi), \]

and by differentiation

\[ 4(\phi \phi) \partial_j (\phi \phi) = \varepsilon(A) (\phi \Gamma_A \phi) \partial_j (\phi \Gamma_A \phi). \] (110)

We may now replace \( \partial_j (\phi \phi) \) in equation (109) by the value given by the preceding equation and we do the same for \( \partial_k (\phi \phi) \). Then, it is easy to see that \( \psi_{jk} \) can be written the following form

\[ \psi_{jk} = \frac{\varepsilon(A)}{8 P^3} \left[ \partial_j (\phi \Gamma_A \phi) I_A - \partial_k (\phi \Gamma_A \phi) I_A \right]. \] (111)

where

\[ I_A = -i(\phi \phi) \left[ (\phi \Gamma_A \partial_k \phi ) - (\partial_k \phi \Gamma_A \phi) \right] + i(\phi \Gamma_A \phi) \left[ (\phi \partial_k \phi ) - (\partial_k \phi \phi) \right] \] (112)

and we note that \( I_A \) is phase invariant.

By comparing equations (112) and (95), we see that

\[ I_A = iH_A (\Gamma_A), \]

and using the result (99), we find

\[ I_A = \frac{i\varepsilon(B) \varepsilon(C) P}{16} (\phi \Gamma_B \phi) \partial_k (\phi \Gamma_C \phi) \text{Tr} (\Gamma_A [\Gamma_B, \Gamma_C]). \] (113)

Let us bring this result in equation (111). We find

\[ \psi_{jk} = -\frac{i\varepsilon(A) \varepsilon(B) \varepsilon(C) P}{8} \left[ \phi \Gamma_A \phi \right] \partial_j (\phi \Gamma_B \phi) \partial_k (\phi \Gamma_C \phi) \text{Tr} (\Gamma_A [\Gamma_B, \Gamma_C]). \] (114)
Equations (104), (105), and (114) give the dynamical equation

$$F_{jk} = - \frac{i e(A) e(B) e(C)}{8P^3} \left( \phi \Gamma_A \phi \right) \left( \bar{\phi} \Gamma_B \phi \right) \left( \bar{\phi} \Gamma_C \phi \right) \text{Tr} \left( \Gamma_j \left[ \Gamma_B, \Gamma_C \right] \right) +$$

$$+ \partial_j \left[ \frac{1}{2P} \partial^l S_{lk} + \frac{M}{P} J_k \right] - \partial_k \left[ \frac{1}{2P} \partial^l S_{lj} - \frac{M}{P} J_j \right]. \quad (115)$$

(Note that $i \text{Tr} \left( \Gamma_j \left[ \Gamma_B, \Gamma_C \right] \right)$ is always real.)

4.5 Multiply Connected Domains. Additional Boundaries Conditions. — When the domains in which the particle propagates is multiply connected additional boundary conditions have to be introduced. The method used in the present case and the method described in section 3.4 are very similar. The global phase $\alpha$ is defined by writing the components $\phi_1, \phi_2, \phi_3, \phi_4$ of $\phi$ as follows

$$\phi_1 = a e^{i(\alpha + \beta + \gamma)}$$
$$\phi_2 = b e^{i(\alpha - \beta - \gamma)}$$
$$\phi_3 = c e^{i(\alpha - \beta + \gamma)}$$
$$\phi_4 = d e^{i(\alpha + \beta - \gamma)}$$

where $a, b, c, d$ are real positive. The boundary condition corresponding to a circuit $C$ (see Fig. 1) is

$$\int_C d\mathbf{r} \cdot (\partial_\alpha) = 2\pi n. \quad (116)$$

We shall express this condition in terms of local probabilities by using equation (102)

$$A_k = - \frac{i}{2} \left[ (\bar{\phi} \partial \phi) - (\bar{\phi} \partial \phi) \right] +$$

$$+ \frac{1}{2P} \partial^l S_{lk} + \frac{M}{P} J_k. \quad (117)$$

We calculate the circulation of $A_k$ along the circuit $C$ and by reasoning exactly as in section 3.4, we find

$$\int_C d\mathbf{r} \wedge d\mathbf{r} \wedge d\mathbf{r} = 2\pi \int_C d\mathbf{r} \cdot (\partial \alpha) = 2\pi n.$$

Finally, we obtain

$$\int_C d\mathbf{r} \left( \frac{1}{2P} \partial^l S_{lk} + \frac{M}{P} J_k \right) + \frac{1}{2} \int_S d\mathbf{r} \wedge d\mathbf{r} \wedge d\mathbf{r} =$$

$$= \frac{1}{2} \int_S d\mathbf{r} \wedge d\mathbf{r} \wedge d\mathbf{r} F_{jk} - 2\pi n \quad (118)$$

and we note that these boundary conditions involve only local densities and the electromagnetic field.

4.6 RESULTS. — We summarize here the results which have been found for the four-dimensional Dirac equation. We have introduced a set of complex matrices $\Gamma^A$ (see Eqs. (74)) and defined local densities of the form $(\phi \Gamma^A \phi)$, these local densities are real tensors. With each $A$, we also associate a number $e(A)$ (see Eqs. (75) and (76)).

The local densities $P, J^j, S^j, K^j$, and $Q$ (see Eqs. (79)) obey the following conditions:

$$J^j J_j = - K^j K_j = P^2 + Q^2 \quad (119)$$

and

$$S^j = QT^j + P\bar{T}^j,$$

where

$$T^j = \frac{1}{P^2 + Q^2} (K^j J^j - J^j K^j) \quad (120)$$

(see Eqs. (81), (83), (84)).

From Dirac equation, we deduced:

1) conservation equations (Eqs. (88), (89))

$$\partial_j J^j = 0 \quad (121)$$

$$\partial_j K^j + 2MQ = 0.$$

2) a set of kinematic equations (Eq. (101))

$$\frac{e(A) e(B)}{16} (\bar{\phi} \Gamma_A \phi) \partial_j (\bar{\phi} \Gamma_B \phi) \text{Tr} \left( \gamma_5 [\Gamma^A, \Gamma^B] \right) +$$

$$+ Q \partial^l S_{lk} - P \partial^l S_{lk} = 0 \quad (122)$$

3) a set of dynamical equations (Eq. (115))
The local densities depend on seven independent parameters. By looking at equations (90) and (91) from which the preceding probabilistic equations are deduced we can count the number of independent equations \((2 \times 4 - 1)\) and it is not difficult to realize that the preceding equations should be sufficient to determine the time dependence of the local densities. The same conclusion can be reached by looking at the preceding equations. We see that we have four kinematic equations and six dynamical equations. However the dynamical equations are not independent.

The second group of Maxwell equations

\[
e^{iklm} \partial_l F_{jk} = \partial_l F^{lm} = 0
\]

is automatically satisfied (see Eq. (103)), this gives three independent conditions and we have only three independent dynamical equations. Thus, this discussion shows that equations (120), (121), and (122) can be considered as equivalent to the Dirac equation.

However, if the domain is multiply connected we must add boundary conditions of the form (see Eq. (118)):

\[
\int_{\mathcal{C}} d^k \left( \frac{1}{2} \rho \partial^l s_{lk} + \frac{M}{P} J_k \right) + \frac{1}{2} \int_{\mathcal{B}} d^l \wedge d^k \psi_{jk} = \frac{1}{2} \int_{\mathcal{B}} d^l \wedge d^k F_{jk} - 2 \pi n,
\]

where \(\psi_{jk}\) is (see Eq. (114))

\[
\psi_{jk} = \frac{i\alpha(A) \alpha(B) \alpha(C)}{8 \, P^3} \left( \phi \Gamma_A \phi \right) \left( \phi \Gamma_B \phi \right) \left( \phi \Gamma_C \phi \right) \text{Tr} \left( \Gamma_A [\Gamma_B, \Gamma_C] \right).
\]

5. Probabilistic description of the Aharonov-Bohm experiment.

5.1 INTERSECTING BEAMS. — Before discussing the Aharonov-Bohm experiment in connection with the existence of probabilistic equations, we shall consider a simpler experiment. We consider waves propagating in a plane inside two linear channels of width \(2a\), crossing each other at right angles (see Fig. 1). The arms of the cross will be called \(X^+\), \(X^-\), \(Y^+\), and \(Y^-\). The waves are assumed to be solutions of the time independent Schrödinger equation

\[
E\phi = -\frac{1}{2M} \left( \partial_x^2 \partial_y + \partial_y^2 \partial_x \right) \phi
\]

where \(E = \frac{1}{2M} \left( k^2 + u^2 \right)\) and \(ua = \frac{\pi}{2}\).

We want to study the interference (at the crossing) of two beams propagating along \(Ox\) and \(Oy\). We note that far from the center of the cross, in one channel, the local densities \(P(x, y)\) and \(J(x, y)\) should not \textit{a priori} depend on the global phase of the beam propagating in this channel. However, the interference pattern at the centre of the cross depends on the difference between the phases of the beams; thus this phase difference has a physical meaning; therefore it should appear in \(P(x, y)\) and \(J(x, y)\). How can we expect such an effect? This is the question which will be solved here.

Fig. 2. — Quantum waves propagate along \(Ox\) and \(Oy\) in a cross-shaped wave guide and interfere at the centre of the cross.
We may associate solutions of the Schrödinger equation with the following irreducible representations of the group of the cross.

1) \( X^+ + X^- + Y^+ + Y^- \),
2) \( X^+ + X^- - Y^+ - Y^- \),
3) \( X^+ - X^- \\
    Y^+ - Y^- \).

A representation will be denoted by the index \( j (j = 1, 2, 3) \) and the phase shift \( v_j \) will be associated with it. With the symbols \( X^+ \), \( X^- \), \( Y^+ \), and \( Y^- \), we shall associate functions \( X^+(x, y), X^-(x, y), Y^+(x, y), \) and \( Y^-(x, y) \) which are solutions of the Schrödinger equation in the corresponding arms and vanish elsewhere. By definition, we have (asymptotically):

\[
X_j^+(x, y) = X_j^-(x, y) = \cos (kx - v_j) \cos uy \quad x \gg a \\
Y_j^+(x, y) = Y_j^-(x, y) = \cos (ky - v_j) \cos ux \quad y \gg a.
\]

Thus far from the centre of the cross, we may write the general solution of energy

\[
E = \frac{1}{2M} (k^2 + u^2)
\]

in the form

\[
\phi(x, y) = A[X_1^+(x, y) + X_1^-(x, y) + Y_1^+(x, y) + Y_1^-(x, y)] + \\
+ B[X_2^+(x, y) + X_2^-(x, y) - Y_2^+(x, y) - Y_2^-(x, y)] + C[X_3^+(x, y) - X_3^-(x, y)] + D[Y_3^+(x, y) - Y_3^-(x, y)].
\]

Incidentally, we remark:

1) that the solutions associated with representation (3) propagate only along one axis and are not scattered side-ways, that they vanish at the centre of the cross, and therefore that they cannot interfere with any other solution;

2) that in the absence of any scattering, we would get \( v_1 = v_2 = 0, v_3 = \pi/2 \).

Let us consider now a solution \( \phi(x, y) \). It will be convenient to set

\[
\phi_x^+(x) = \phi(x, 0) \quad x \gg a \\
\phi_x^-(x) = \phi(x, 0) \quad x \ll -a \\
\phi_y^+(y) = \phi(0, y) \quad y \gg a \\
\phi_y^-(y) = \phi(0, y) \quad y \ll -a.
\]

In the same way, we associate with \( P(x, y) \) and \( J(x, y) \), the quantities \( P_x(x), J_x(x) \) and \( P_y(y), J_y(y) \). In particular, we have:

\[
P_x(x) = \phi_x^+(x) \phi_x(x) \\
J_x(x) = -\frac{i}{2} [\phi_x^+(x) \partial_x \phi_x(x) - \phi_x(x) \partial_x \phi_x^+(x)].
\]

In order to understand the situation better, we shall first assume that we have only one incoming beam along Ox (see Fig. 3). Along \( X^+ \), \( Y^+ \), and \( Y^- \), we have only pure outgoing waves. It is not difficult to see that

![Fig. 3. — A wave coming from \(-\infty\) on the x-axis is scattered by the cross.](image-url)
this solution is given by setting

\[ A = U e^{-i\nu_1} \]
\[ B = U e^{-i\nu_2} \]
\[ C = -2U e^{-i\nu_3} \]
\[ D = 0. \]

Then far from the centre of the cross, we have

\[ \phi_+^+(x) = \frac{U}{2} (e^{-2i\nu_1} + e^{-2i\nu_2} - 2 e^{-2i\nu_3}) e^{ikx} \]
\[ \phi_-^-(x) = 2U e^{ikx} + \frac{U}{2} (e^{-2i\nu_1} + e^{-2i\nu_2} + 2 e^{-2i\nu_3}) e^{-ikx} \]
\[ \phi_+^+(y) = \frac{U}{2} (e^{-2i\nu_1} - e^{-2i\nu_3}) e^{iky} \]
\[ \phi_-^-(y) = \frac{U}{2} (e^{-2i\nu_1} - e^{-2i\nu_3}) e^{-iky} \]

The corresponding local densities are

\[ P_+^+(x) = \frac{1}{2} UU^*[3 + \cos(\nu_1 - \nu_2) - 2 \cos(\nu_3 - \nu_1) - 2 \cos(\nu_3 - \nu_2)] \]
\[ P_-^-(x) = \frac{1}{2} UU^*[11 + \cos(\nu_1 - \nu_2) + 2 \cos(\nu_3 - \nu_1) + 2 \cos(\nu_3 - \nu_2)] + 
\quad + 2 UU^* \cos 2kx[\cos 2\nu_1 + \cos 2\nu_2 + \cos 2\nu_3] - 2 UU^* \sin 2kx[\sin 2\nu_1 + \sin 2\nu_2 + \sin 2\nu_3] \]
\[ P_+^+(y) = P_-^-(y) = \frac{1}{2} UU^*[1 - \cos(\nu_1 - \nu_2)] \] (132)

and

\[ J_+^+(x) = \frac{1}{2} kUU^*[3 + \cos(\nu_1 - \nu_2) - 2 \cos(\nu_3 - \nu_1) - 2 \cos(\nu_3 - \nu_2)] \]
\[ J_-^-(x) = \frac{1}{2} kUU^*[5 - \cos(\nu_1 - \nu_2) - 2 \cos(\nu_3 - \nu_1) - 2 \cos(\nu_3 - \nu_2)] \]
\[ J_+^+(y) = J_-^-(y) = \frac{1}{2} kUU^*[1 - \cos(\nu_1 - \nu_2)]. \] (133)

Thus, we see that the incoming wave is scattered in all directions. In particular, it is scattered backwards and, in \( X^- \), the outgoing wave interferes with the incoming wave (look at the expression of \( P^-_x(x) \)). In the other channels, the density is uniform. We could treat in the same way the case where we have one incoming beam along \( Oy \).

Now, it would be possible to treat the case where we have two incoming beams along \( Ox \) and \( Oy \) (see Fig. 4) by applying the superposition principle. The result would be complicated but we guess what happens in this case: the beams are scattered at the crossing; in particular the beam propagating in the arm \( X^- \) towards the centre of the cross is scattered sideways in the arm \( Y^- \) and interferes with the other beam (and conversely). Thus, if the incoming beams are out of phase the phase difference will appear in the interference terms. In particular, this phase difference will appear in the expressions of \( P^+_x(x) \) and \( P^-_y(y) \).

Fig. 4. — Two waves coming from \(-\infty\) on the \( x\)-axis and the \( y\)-axis interfere at the centre of the cross and are scattered as shown on the picture.
The conclusion of this discussion is the following. When two incoming beams propagating in a wave guide cross, producing an interference pattern at the centre of the cross, interference effects can also be felt far from the crossing point. Thus, there exists a correlation between the behaviour of the local densities far from the crossing point and the interference pattern at the centre of the cross. This effect would appear by writing explicitly the expressions of \( P_x(x) \), \( P_x^+(y) \), and \( P_x^-(y) \) corresponding to the present situation. In this case, the solution of the probabilistic equation depends on three real parameters as can be shown in various ways. We may say for instance that these parameters are the amplitudes of the incoming beams and their phase difference which is a physical quantity.

5.2 REMARKS CONCERNING THE AHARONOV-BOHM EXPERIMENT. — The setup of the experiment is shown in figure 5. An incoming beam is split in \( O \) into two beams which interfere in \( I \). In the middle, a coil may produce a magnetic flux through the central hole.

In the preceding section, we discussed the situation in which two beams cross each other and we noted that the probabilistic equations had several solutions. The solution which corresponds to a given physical situation must be determined by supplementary boundary conditions. The same remark applies here. At a given energy, several solutions are possible but only one solution is compatible with a given magnetic flux through the hole. The preceding statement is supported by the following analysis.

We assume that the waves propagate in narrow channels of width \( 2a \); they are assumed to be solutions of the time independent Schrödinger equation

\[
E = -\frac{1}{2M} \left[ (\partial_x - iA_x) (\partial_x - iA_x) + \right. \\
+ \left. (\partial_y - iA_y) (\partial_y - iA_y) \right] \phi + V \quad (134)
\]

where \( E = \frac{1}{2M} (k^2 + u^2) \) and \( u = \pi/2 \).

We can solve this Schrödinger equation in the channels and at the intersection points \( O \) and \( I \). We assume that everywhere in the domain where the particle propagates, the electromagnetic field vanishes. Thus, when considering local solutions of the Schrödinger equation, we may always assume that \( A_x = 0 \), \( A_y = 0 \), and \( V = 0 \) in equation (134).

In the preceding section, we have studied the nature of the solutions of the Schrödinger equation at a crossing like \( I \). We can make a similar analysis of the solutions of the Schrödinger equation at the bifurcation \( O \). For simplicity, we may consider that the bifurcation is completely symmetric and this « triskele » is drawn on figure 6. Its arms will be called \( T^0 \), \( T^+ \), and \( T^- \). We associate solutions of the Schrödinger equations in this triskele with the following irreducible representation of the group of the triskele

1) \( T^0 + T^+ T^- \)

2) \( T^0 + e^{2i\pi/3} T^+ + e^{-2i\pi/3} T^- \)

\[
T^0 + e^{-2i\pi/3} T^+ + e^{2i\pi/3} T^-
\]

A representation will be denoted by the index \( j \) (where \( j = 1, 2 \)) and the phase shift \( \psi_j \) will be associated with it.

With the symbols \( T^0, T^+, T^- \), we associate functions \( T^0(x, y) \), \( T^+(x, y) \), and \( T^-(x, y) \) which are solutions of the Schrödinger equation in the corresponding arms (far from \( O \)) and which vanish elsewhere.

For instance

\[
T_j^0(x, y) = \cos(-kx - \psi_j) \cos u y \quad x \ll a.
\]

Thus, far from the centre of the triskele, we may
write the general solution energy $E$ in the form
\[ \phi(x, y) = A[T_{10}^0 + T_{1}'(x, y) + T_{1}^\pm(x, y)] + B[T_{20}^0(x, y) + e^{i2\pi/3} T_{2}^+(x, y) + e^{-i2\pi/3} T_{2}^- (x, y)] + C[T_{30}^0(x, y) + e^{-i2\pi/3} T_{3}^+(x, y) + e^{i2\pi/3} T_{3}^- (x, y)]. \] (135)

Thus a wave function in the triskele depends on three complex parameters, and as we have seen in the preceding section, a wave function in the cross depends on four complex parameters. Instead of looking at the wave functions we may consider the corresponding local densities which are solutions of the probabilistic equations. In the triskele, the local densities depend on five real parameters, in the cross on seven real parameters.

To determine the total number of independent parameters, we have now to examine how many conditions we have to fulfill.

Firstly, the local densities $P(x, y)$ and $J(x, y)$ at the entrance and at the exit of a channel must be compatible. In a channel, the wave function $\phi(x, y)$ is of the form
\[ \phi(x, y) = (A e^{i k x} + B e^{-i k x}) \cos \psi y , \] (136)
where $x$ is the coordinate along the channel and $y$ the perpendicular coordinate.

Secondly, in the Aharonov-Bohm experiment, the waves propagating in the arms $X^+$ and $Y^+$ are purely outgoing waves. Equation (136) immediately shows that for each channel this gives two real conditions, and therefore, in this way, we find four real conditions.

Let us summarize. The triskele and the cross give twelve parameters ($5 + 7$) and we found ten conditions ($6 + 4$). There remains two independent parameters. Clearly one parameter is an amplitude which is determined by the amplitude of the incoming wave in channel $T^0$. The other parameter corresponds to a phase difference between the channels $(T^+ X^-)$ and $(T^- Y^-)$. Now, we observe that, until now, we took into account only local boundary conditions. This is not sufficient here because the domain of propagation drawn on figure 5 is not simply connected. We have to draw a circuit $C$ around the hole and we must take into account the boundary condition (see Eq. (20))
\[ \int_C (dx v_x + dy v_y) = -\int_S d\sigma \mathbf{B} + 2 \pi n = \Phi + 2 \pi n \]
where $S$ is the surface bounded by $C$, $\mathbf{B}$ the magnetic induction perpendicular to the plane $(x, y)$ and $\Phi$ the flux through the hole. Now everything is determined.

Thus, this rather long discussion leads to a simple conclusion: in an Aharonov-Bohm experiment, all physical effects can be predicted by solving probabilistic equations and by applying boundary conditions involving only local densities and electromagnetic fields. Consequently, all phase and gauge effects can be eliminated.

6. Conclusion.

In this article, we have studied one-particle quantum equations and we have shown that these equations could be transformed so as to eliminate all phase and gauge effects, and all ambiguities related to the representation of the Dirac matrices (1). We believe that this is an interesting result.

The reader may object that nothing has been gained in this process, that it is not always useful to restrict oneself to work only with group invariants and that using reference frames is quite legitimate. However, we think that this point of view which may be sensible in other occasions is not adapted to the present situation:

1) because we do not want to make practical calculations but to get a more basic insight into quantum mechanics,
2) because the group which is eliminated is unphysical,
3) because it seems that difficulties of interpretation in quantum mechanics may be related to the fact that the wave function is not an observable and experience in other fields (for instance critical phenomena) has taught us that it is sometimes very important to work with observable quantities.

We must also emphasize another point. We have shown that in quantum mechanics a limited number of mean values obey a closed set of equations which determine their time dependence. This is a very remarkable result: it was not a priori obvious that closed sets of equations should exist. In fact, very different situations occur in statistical mechanics. Then, in general, mean values obey an infinite set of equations and finite sets are obtained only by truncation.

From a conceptual point of view, the new formalism which is presented here, has the great advantage of being more physical since all equations are real and since it eliminates all quantities which cannot be defined without ambiguity.

We note that the equations which we obtained are local but we do not know how to interpret this fact.

(1) In this respect, our probabilistic equations differ from other recent reformulations [16, 17] of Dirac equation, which « a priori » may look similar.
The problem of locality has been much discussed in quantum mechanics [14, 15]. However, this interesting question is outside the scope of the present article.

We have also to emphasize that the probabilistic equations which have been derived in this article cannot be directly interpreted. These equations cannot be considered as really fundamental and a deeper analysis of their nature and of their meaning is needed.

From another point of view, the transformation which has been made does not allow us to use the superposition principle and practically (and also theoretically), this is a serious drawback. However, the probabilistic equations may be only a consequence, of more primitive and simple physical processes, which have to be discovered. This is what we expect.

The preceding transformation applies only to one-particle quantum equations, but it seems that it should be possible to extend it to systems with several interacting particles. Again one should establish equations describing the evolution of local densities and an analysis of such equations may help to solve the paradoxes of quantum mechanics. In any case, the new formalism does not allow to discuss « the reduction of the wave packet » since all wave functions are eliminated!

It would also be interesting to transform field theory in the same spirit namely by eliminating all phase and gauge dependence at every stage of the theory. However, it is hard to foretell whether this ambitious program can lead to really useful results ! It is not obvious that it may provide a clear reinterpretation of quantum mechanics. The present article only shows that much work remains to be done in this domain.

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Appendix A. — Reconstruction of the 3-dimensional Dirac equation from the corresponding probabilistic equation.

We start from the probabilistic equations given by (59), (60) and (39) and we want to reconstruct the Dirac equation (30).

Firstly, we have to define a wave function. We choose matrices γ' which obey (32). We want to show that it is possible to find a two component function \( \psi = (\psi_1, \psi_2) \) which obeys the relations

\[
\bar{\psi} = P \\
(\bar{\psi})' = J^j
\]

where \( \bar{\psi} \) is related to \( \psi \) in the usual way. Here \( P \) and \( J^j \) are real and obey (60)

\[
P^2 = J^j J_j = (J^x)^2 - (J^y)^2 - (J^z)^2.
\]

Using for \( \gamma' \) the representation given by (33), we may write equations (A.1) in the form

\[
\begin{align*}
\psi_1^+ \psi_1 - \psi_2^+ \psi_2 &= P \\
\psi_1^+ \psi_1 + \psi_2^+ \psi_2 &= J^j \\
i(\psi_1^+ \psi_2 - \psi_2^+ \psi_1) &= J^x \\
\psi_1^+ \psi_2 + \psi_2^+ \psi_1 &= J^y.
\end{align*}
\]

It will be assumed that \( J^j \geq 0 \).

Now, in accordance with equation (A.2), we may set

\[
\begin{align*}
J^j &= \rho^2 \\
(\rho^2 \cos \mu) + (\rho^2 \sin \mu) &= \rho^4 \sin^2 \mu \\
(\rho^2 \sin \mu \sin \nu) + (\rho^2 \sin \mu \cos \nu) &= \rho^4 \sin \nu,
\end{align*}
\]

where \( \rho, \mu \) and \( \nu \) are real.

In this way, we obtain solutions of equation (A.3) of the form

\[
\begin{align*}
\psi_1 &= \rho \cos \frac{\mu}{2} e^{i \beta} e^{i \alpha} \\
\psi_2 &= \rho \sin \frac{\mu}{2} e^{-i \beta} e^{i \alpha}
\end{align*}
\]

where \( \alpha \) is arbitrary.

Thus, using equations (A.1), we may replace \( P \) and \( J^j \) in (59) and (39), by their expressions in terms of \( \psi_1, \psi_2 \) and \( \gamma' \). Now, by using the identity (37) and by proceeding backwards, we get (46). Let us now consider the equation

\[
\partial_t A_k - \partial_k A_j = F_{jk}
\]

where \( F_{jk} \) is given by (45) and (46). Obviously, the quantity \( A_k \) which is defined by (41) in terms of \( \psi \), is a solution of equation (A.6) and we may assume that the vector potential coincide with this solution (we note that the arbitrary phase \( \alpha \) defines the gauge).

Thus, we have only to show that Dirac equation can be deduced from the equations

\[
\partial_t (\bar{\psi} \gamma^j \psi) = 0
\]

\[
A_k (\bar{\psi} \gamma^j \psi) = -\frac{i}{2} (\bar{\psi} \partial_t \psi) + \frac{i}{2} (\psi (\bar{\psi})' - \bar{\psi} (\psi)' - M(\bar{\psi} \gamma_k \psi)
\]

(Eqs. (41) and (34) lead to the second equation).
Equations (A.8) and (70) give:

\[ A_k(\phi) = -\frac{i}{2}(\bar{\phi} \gamma^k \partial \phi) + \frac{i}{2}(\partial \bar{\phi} \gamma^k \gamma_k \phi) + M(\bar{\phi} \gamma_k \phi). \]  \hspace{1cm} (A.9)

We shall now multiply both sides of equation (A.9) by \( \gamma^k \phi \) (which is a two component vector) and we shall transform the right hand side by using the identity (70) which reads:

\[ 2 \delta_{ab'} \delta_{b''} = \delta_{ab'} \delta_{b''} + \gamma^k \gamma_{k,b''}. \]  \hspace{1cm} (A.10)

Firstly, let us deduce, from this equation, a few useful identities. Let us multiply (A.10) by \( \gamma^l \partial \phi_{\mu'} \phi_{\nu'} \); we obtain

\[ 2 \gamma^l \partial \phi_{\mu'} \phi_{\nu'} = \gamma^l \phi(\partial \bar{\phi} \gamma^l \gamma_k \phi) + \gamma^l \phi(\bar{\phi} \gamma^l \partial \phi) \]  \hspace{1cm} (A.11)

Let us multiply (A.10) by \( \partial \phi_{\mu'} \phi_{\nu'} \); we obtain

\[ \phi(\partial \phi_{\mu'} \phi_{\nu'}) = \gamma^l \phi(\partial \bar{\phi} \gamma^l \gamma_k \phi) \]  \hspace{1cm} (A.12)

By subtracting (A.11) and (A.12) and by taking (A.6) into account, we find

\[ 2 \gamma^l \partial \phi_{\mu'} \phi_{\nu'} = \gamma^l \phi[(\bar{\phi} \gamma^l \partial \phi) - (\partial \bar{\phi} \gamma^l \gamma_k \phi)] \]  \hspace{1cm} (A.13)

Now let us multiply (A.10) by \( \phi_{\mu'} \bar{\phi}_{\nu'} \); we obtain

\[ \phi(\partial \phi_{\mu'} \phi_{\nu'}) = \gamma^l \phi(\bar{\phi} \gamma^l \gamma_k \phi) \]  \hspace{1cm} (A.14)

Finally, by using equations (A.13) and (A.14), the product of (A.8) and \( \gamma^k \phi \) can be transformed into

\[ A_k \gamma^k \phi(\phi) = -i \gamma^l \partial \phi(\bar{\phi} \phi) + M \phi(\bar{\phi} \phi). \]  \hspace{1cm} (A.15)

In general \( \bar{\phi} \phi \neq 0 \) and therefore the preceding equation coincide with Dirac equation:

\[ (\partial_j - iA_j) \gamma^j \phi + i M \phi = 0. \]  \hspace{1cm} (A.16)

Thus, we have shown that the 3-dimensional Dirac equation is completely equivalent to the corresponding probabilistic equations.

Appendix B. — Relations between local densities.

We want to establish relations between the local densities \( P, J^l, S^k, K^l \), and \( Q \) defined by equations (79). We start from equations (81):

\[ 4 \delta_{ab'} \delta_{b''} = \sigma(A) \Gamma_{ab'}^l \Gamma_{,ab'}^l. \]  \hspace{1cm} (B.1)

Let us introduce two arbitrary \( 4 \times 4 \) matrices \( U \) and \( V \). Equation (B.1) leads to the following identity

\[ 4(\bar{\phi} U \phi)(\bar{\phi} V \phi) = \epsilon(A)(\bar{\phi} U \Gamma^A \phi)(\bar{\phi} \Gamma_A \phi) \]  \hspace{1cm} (B.2)

and by using (76) and (79), we can write it in the explicit form

\[ 4(\bar{\phi} U \phi)(\bar{\phi} V \phi) = (\bar{\phi} U \Gamma^A \phi) P + (\bar{\phi} U \gamma^l V \phi) J^l + \frac{1}{2}(\bar{\phi} U \sigma^k \phi) S^k - (\bar{\phi} U \gamma_5 \gamma^l V \phi) K^l - (\bar{\phi} U \gamma_5 \phi) Q, \]  \hspace{1cm} (B.3)

(the factor \( \frac{1}{2} \) comes from the fact that \( \sigma^{jk} \) and \( \sigma^{kj} \) correspond to the same \( \Gamma^A \)).

We can now replace \( U \) and \( V \) by well defined matrices and obtain in this way relations between local densities. For these calculations, the following algebraic equations are useful (see Eqs. (70), (71), (72)):

\[ \gamma^l \gamma^k = g^{lk} - i \sigma^{lk} \]

\[ \sigma^{jk} = \frac{i}{2}(\gamma^j g^{kl} - \gamma^k g^{jl}) - \delta^{jk} \gamma_5 \gamma_\mu \]  \hspace{1cm} (B.4)

(with \( \varepsilon^{xyz} = 1 \))

Choosing for \( U \) and \( V \) the following values, we obtain respectively

1) \( U = 1 \quad V = 1 \)
\[ 3 P^2 = J^l J_j + \frac{1}{2} S^R S_R - K^l K_j - Q^2 \]  \hspace{1cm} (B.5)

2) \( U = i \gamma_5 \quad V = i \gamma_5 \)
\[ 3 Q^2 = - P^2 + J^l J_j - \frac{1}{2} S^R S_R - K^l K_j \]  \hspace{1cm} (B.6)

3) \( U = i \gamma_5 \quad V = 1 \)
\[ 4 Q P = \bar{S}^R S_R \]  \hspace{1cm} (B.7)

4) \( U = \gamma^l \quad V = \gamma_1 \)
\[ 3 J^l J_j = 2 P^2 - K^l K_j + 2 Q^2 \]  \hspace{1cm} (B.8)

5) \( U = 1 \quad V = \gamma^l \)
\[ P J^l = \bar{S}^R K_j \]  \hspace{1cm} (B.9)

6) \( U = i \gamma_5 \quad V = \gamma^l \)
\[ Q J^l = - S^R K_j \]  \hspace{1cm} (B.10)

7) \( U = 1 \quad V = \gamma_5 \gamma^l \)
\[ P K^l = \bar{S}^R J_j \]  \hspace{1cm} (B.11)

8) \( U = i \gamma_5 \quad V = \gamma_5 \gamma^l \)
\[ Q K^l = - S^R J_j \]  \hspace{1cm} (B.12)

Thus, by combining equations (B.5), ..., (B.12), we obtain the sets of equations (81) and (82). (The reader may note that for \( P \neq 0 \) or \( Q \neq 0 \) it is easy to show that \( \frac{1}{2} K_j = 0 \), but that this relation remains true if \( P = Q = 0 \).)
References