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A two-dimensional model for three-dimensional convective patterns in wide containers

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Abstract. — We derive a model of convection appropriate for containers with large width to height ratios. It couples a real rapidly varying convective variable to a slowly varying « drift » velocity field given by its stream-function. The derivation assumes stress-free boundary conditions at the top and the bottom. We discuss its extension to the more realistic no-slip case and the analogies/differences with amplitude equations.

1. Introduction.

Convection experiments performed in containers with lateral dimensions large compared to their height show that structures which develop freely above the threshold are not made up of a single set of perfect straight rolls but are rather disordered [1]. The global pattern is a « texture » [2] with « domains » of regular rolls — sometimes straight but most often curved — separated by « grain-boundaries » with other defects such as dislocations or disclinations also present.

A detailed understanding of the nature and dynamics of such patterns has not yet been obtained. It seems that some progress could be expected from the study of models at a complexity level intermediate between the primitive Boussinesq equations [3] which contain too much « microscopic » information and amplitude equations [4-6] which, at least to lowest order, turn out to be insufficient for discussing the long term evolution of textured convective structures.

Such « semi-microscopic » models have already been presented and used for theoretical analysis [7, 8] or numerical simulations [9, 10]. They have led to some interesting results about the possible equilibrium configurations [2, 10] but they have not provided a reliable account of the dynamics of textures. In particular these models neglected the slowly varying « drift flows » induced by curvature which have been shown to play a crucial role especially at low Prandtl numbers [11]. This paper is devoted to the derivation of a more realistic model which includes these drift terms and thus is expected to contribute to a better understanding of weak turbulence at large aspect ratios.

The method of derivation is basically a projection technique which eliminates the dependence on the vertical coordinate z — thought to be irrelevant — leaving a dependence on time t and the two remaining horizontal coordinates x, y only. The model will consist of a set of two partial differential equations coupled by their nonlinear terms. The first equation will govern a real quantity associated with the convection cells: the vertical velocity of the convective flow at a given point. The second equation will account for the evolution of the stream-function of the large scale drift flow alluded to above.

Here we consider mainly the case of « stress-free » boundary conditions for the velocity since this leads to simpler calculations than the realistic case of...
« no-slip » conditions even though it may have some undesirable peculiarities. This simplicity derives from the special form — pure trigonometric lines — of the vertical profile of the solution of the linearized Boussinesq equations. The detailed calculation parallels that of Swift and Hohenberg [12] but instead of working in Fourier space we stay in physical space. This implies approximations which make use of the remark made by C. Normand [13] that, when applied to a variation at a wave-vector \( q \sim q_c \), the horizontal Laplacian \( \Delta_h = \partial^2_{xx} + \partial^2_{yy} \) is equal to \( -q_c^2 \) to leading order \( (Ra - Ra_c)^{1/2} \) with \( q_c \) the critical wave-vector, \( Ra = \) the Rayleigh number and \( Ra_c = \) the convection threshold.

The procedure used is reminiscent of the derivation of amplitude equations [6]. Actual similarities and differences will be discussed in § 4 after a brief examination of the problems posed by the extension to the realistic « no-slip » case (§ 3).

2. Explicit derivation.

As usual the starting point is the well-known set of Boussinesq equations [3]. Here we shall work mainly with the equation which governs the vertical component \( w \) of the velocity \( V = (u, v, w) \). After elimination of the pressure field and the horizontal flow, it reads:

\[
\partial_t \Delta w + NLW_w = Pr(\Delta^2 w + \Delta \theta) \tag{1a}
\]

where \( \Delta = \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz} \) is the 3-dimensional Laplacian and

\[
NLW_w = \Delta \theta (V \cdot \nabla w) - \partial_z \left[ \Delta \theta (V \cdot \nabla u) + \partial_z (V \cdot \nabla v) \right]. \tag{1a'}
\]

The temperature fluctuation \( \theta \) can further be eliminated from the r.h.s. of (1a) using:

\[
\partial_t \theta + V \cdot \nabla \theta = \Delta \theta + Ra \cdot w. \tag{2}
\]

This leads to:

\[
-Pr^{-1} \Delta \partial^2_{tt} w + (1 + Pr^{-1}) \Delta^2 \partial_z w + NLW_b = \Delta^3 w - Ra \cdot \Delta w. \tag{1b}
\]

and

\[
NLW_b = Pr^{-1}(\Delta - \partial_z) NLW_w - \Delta \theta (V \cdot \nabla \theta). \tag{1b'}
\]

The fluctuations \( u, v, \theta \) cannot be eliminated from \( NLW_b \), and have to be computed. The determination of the horizontal flow \( V_h = (u, v) \) turns out to be more transparent if one separates its rotational component from its irrotational one. Thus we set:

\[
V_h = V_h \varphi + \nabla \times (\psi \hat{z}) \tag{3}
\]

(where \( \hat{z} \) is the unit vector in the \( z \)-direction) or:

\[
u = \partial_x \varphi + \partial_y \psi \tag{3a}
\]

\[
v = \partial_z \varphi - \partial_x \psi. \tag{3b}
\]

The rotational component \( \psi \) is linked to the vertical vorticity \( \zeta = \partial_y u - \partial_x v \) by \( \zeta = -\Delta \psi \) so that the equation for \( \psi \) reads:

\[
(\partial_t - Pr \Delta) \Delta \psi = \partial_x (V \cdot \nabla \psi) - \partial_z (V \cdot \nabla u) \tag{4}
\]

while the continuity equation:

\[
V \cdot V = \partial_x u + \partial_y v + \partial_z w = 0 \tag{5}
\]

leads immediately to the equation for \( \varphi \):

\[
\Delta \varphi = -\partial_z w. \tag{6}
\]

Equations (1, 2, 4, 6) are totally general and stand as the starting point for any theoretical analysis of the Rayleigh-Bénard instability. In this section we assume perfectly conducting « stress-free » surfaces at the top \( (z = 1) \) and at the bottom \( (z = 0) \). The boundary conditions are \( \theta = 0 \) and \( w = \partial^2_{zz} w = 0 \) at \( z = 0 \) and \( z = 1 \). In this case it is well known that the eigen-modes of the linearized problem are pure trigonometric lines [14]. What we wish to do is basically a Galerkin expansion of the solution using this set of basis functions. Thus we search for \( w(x, y, z, t) \) in the form:

\[
w = \sum_a \omega_a(x, y, t) \sin (n \pi z). \tag{7}
\]

The critical mode corresponds to \( n = 1 \) and the critical conditions are given by the minimum of the marginal curve:

\[
Ra^* (q) = \frac{(q^2 + \pi^2)^3}{q^2} \tag{8}
\]

at \( q_c = \pi \sqrt{2} \) and \( Ra_c = 27 \pi^4/4 \).

Returning to (1b), we shall first solve the complete linearized problem for the critical mode \( \omega_1 \) and next turn to the nonlinear contributions.

2.1 LINEARIZED PROBLEM. — Let us define

\[
e = \frac{Ra - Ra_c}{Ra_c}. \tag{9}
\]

The linear part of the equation which governs the horizontal dependence of \( w \) is obtained by assuming that \( \zeta = \partial^2_{xx} + \partial^2_{yy} + q_c^2 \) is at least of order \( e^{1/2} \) and \( \partial_t \) at least of order \( e \). Using \( \partial^2_{zz} = -\pi^2 = -2 q_c^2 \) one can replace \( \Delta \) by \( \zeta - 3 q_c^2 \) in (1b) which is further expanded in powers of \( e \). The lowest orders cancel out due to the actual value of \( Ra \) and \( q_c \) and one gets:

\[
9 q_c^4 (1 + Pr^{-1}) \partial_z \omega_1 = [27 q_c^6 e - 9 q_c^2 \zeta^2] \omega_1 \tag{10}
\]

(the coefficients in (7) relate to the relaxation time and coherence length as will be seen in (22)).
At the linear stage the order of magnitude of $\psi_1$ is not fixed and unstable modes close to $q_c$ grow indefinitely. However nonlinearities enforce the presence of harmonics which in turn limit the growth of the critical modes. In order to evaluate these harmonics we have first to determine the other unknowns which enter the nonlinear terms. Let us consider first the temperature fluctuation $\theta$ as governed by (2). It can be expanded as:

$$\theta = \sum_n \Theta_n (x, y, t) \sin(n\pi z).$$

At lowest order we can neglect the l.h.s. of (2), replace $\Delta$ by $-3 q_c^2$ and $Ra$ by $Ra'$, which leads to

$$\Theta_1 = 9 q_c^2 \psi_1.$$  

(8)

From the continuity equation (5) we see that the expansion for $V_h$ has to be of the form:

$$(u, v) = \sum_n (\psi_{n1}(x, y, t), \psi_{n1}(x, y, t)) \cos(n\pi z)$$

in which the term $n = 0$ (i.e, independent of $z$) should not be omitted as was shown by Siggia and Zippelius [11].

To lowest order from (4) we get $\Psi_1 = 0$ and from (6) $-q_c^2 \Phi_1 = -\pi \psi_1$. Thus:

$$\left(\psi_{21}, \psi_{12}\right) = \frac{\sqrt{2}}{q_c} (\partial_x, \partial_y) \psi_1.$$  

(9)

2.2 Formally quadratic nonlinearities. — We are now in position to evaluate nonlinear contributions to (1) which are formally quadratic in $\psi_1$, then cubic, etc... In this section we are interested in the second vertical harmonics ($\psi_2, \Theta_2$) sin (2 $\pi z$), the corresponding horizontal flow ($\psi_2, \psi_2$) cos (2 $\pi z$) and the drift component independent of $z$ ($\psi_0, \Theta_0$).

2.2.1 Vertical velocity and temperature. — These second harmonics are expected to follow the critical mode in an adiabatic way. Except for this simplification the equations for $\psi_2$ and $\Theta_2$ read:

$$(\Delta_n - 4 \pi^2) \psi_2 + \Delta_n \Theta_2 = \frac{3}{\pi} Pr^{-1} \Delta_n \psi_1.$$  

(10a)

$$(\Delta_n - 4 \pi^2) \Theta_2 + Ra \psi_2 = \frac{9}{2} \pi \psi_0.$$  

(10b)

with the definition:

$$\mathcal{M} = (\nabla_h \psi_1)^2 + q_c^2 \psi_1^2,$$

from which it is readily seen that, for a single set of straight rolls, $\mathcal{M}$ is exactly the squared modulus of the associated complex amplitude [4].

On the r.h.s. of (10) we have pair-wise combinations of wave-vectors the length of which varies from 0 to 2 $q_c$. An accurate solution for $\psi_2$ and $\Theta_2$ requires an exact inversion of (10). Such an inversion is possible in Fourier space but leads to a complicated convolution for the coupling at the next order [21, 4a, 6]. Here we shall content ourselves with an approximate solution which takes advantage of the fact that for such wave-vectors the 3-dimensional Laplacian $\Delta$ reads $-q^2 - 4 \pi^2$ with $q^2 < (2 q_c)^2 = 2 \pi^2 \ll 4 \pi^2$ so that we can replace $\Delta$ by $-4 \pi^2$ everywhere.

Using this simplification we obtain:

$$\psi_2 = \frac{3}{128 \pi^3} (8 Pr^{-1} + 3) \Delta_n \psi_1.$$  

(11)

and

$$\Theta_2 = \frac{9}{8 \pi} \left\{ \frac{9}{256 \pi^2} (8 Pr^{-1} + 3) \Delta_n - 1 \right\} \psi_0.$$  

(12)

2.2.2 Second harmonic horizontal flow component. — The irrotational part of the horizontal velocity field at cos (2 $\pi z$) is given by:

$$\Delta_n \Phi_2 = \frac{3}{64 \pi^2} (8 Pr^{-1} + 3) \Delta_n \psi_1.$$  

(13)

which after << simplification >> of $\Delta_n$ gives:

$$\psi_{21} = \frac{3}{64 \pi^2} (8 Pr^{-1} + 3) \left(\partial_x, \partial_y\right) \psi_0.$$  

(14)

The rotational part is evaluated from (4). As is readily seen by a naive replacement:

$$\nabla \cdot \nabla u |_{2} = \frac{1}{r^2} \partial_r \psi_0.$$  

so that there seems to be no rotational contribution. However using the continuity equation one can write:

$$\nabla \cdot \nabla u = \partial_r (u^2) + \partial_r (uw) + \partial_r (uw)$$

which gives:

$$\nabla \cdot \nabla u |_{2} = \frac{1}{q_c^2} \left( \partial_r (\partial_x \psi_0)^2 + \partial_r (\partial_y \psi_0 \psi_1) + q_c^2 \partial_x \psi_0 \psi_1 \right)$$

so that the equation for $\psi_2$ reads:

$$[\partial_t - Pr(\Delta_n - 4 \pi^2)] \Delta_n \psi_2 = \mathcal{N}.$$  

(15)

with:

$$\mathcal{N} = \frac{1}{q_c^2} \left\{ \partial_{xy} [\partial_x (\partial_x \psi_0)^2 - (\partial_x \psi_1)^2] + (\partial_{xx} - \partial_{yy}) (\partial_x \psi_0 \partial_y \psi_0) \right\}. $$  

(15')
Without any additional approximation this expression can be rewritten as:

\[ N' = \frac{1}{q_c^2} \left\{ \partial_y \omega_1 \partial_x \Delta \omega_1 - \partial_x \omega_1 \partial_y (\Delta \omega_1) \right\} \]  

(15')

which may be more practical in certain circumstances.

In order to evaluate the actual order of the r.h.s. let us assume

\[ \omega_1 = (W_a \exp(iq_0 x) + W_b \exp(iq_0 x)) \]

which leads to contributions of the form:

\[ \frac{1}{q_c^2} W_a W_b (q_0^2 - q_c^2) (q_{ax} q_{by} - q_{bx} q_{ay}) \exp[i(q_0 + q_c)x]. \]

Assuming \( W_a, W_b \sim \varepsilon^{1/2} \) and \( q_0^2 \sim q_c^2 \sim \varepsilon^{1/2} \) one observes that the r.h.s. is of the order of \( \varepsilon^{3/2} \) since in general \( q_{ax} q_{by} - q_{bx} q_{ay} \sim q_0^2 \). Under these circumstances \( \Psi_2 \) and thus \( (\Omega_{1x}^2, \Omega_{1y}^2) \) are of the order of \( \varepsilon^{3/2} \) while \( \Theta_2 \) and \( \Psi_2 \) and thus \( (\Omega_{2x}^2, \Omega_{2y}^2) \) are of the order of \( \varepsilon^2 \) so that \( (\Omega_{1x}^2, \Omega_{1y}^2) \) can be neglected. A particular case occurs when one deals with a single set of slowly modulated rolls. Then with \( q_{ax} \sim q_c, q_{ay} = 0, q_{by} \sim \varepsilon^{1/4} \), a similar evaluation gives \( N' \) of the order of \( \varepsilon^{7/4} \) and \( (\Omega_{1x}^2, \Omega_{1y}^2) \) of the order of \( \varepsilon^{3/2} \). Meanwhile one can see that \( (\Omega_{2x}^2, \Omega_{2y}^2) \) are of the order of \( \varepsilon^{3/2} \) while \( \Theta_2 \) remains of the order of \( \varepsilon^3 \), so that all the formally quadratic contributions to the velocity field can be neglected.

2.2.3 Drift flow. — Averaging the continuity equation shows that the \( z \)-independent part of the horizontal flow is purely rotational. Now \( \Delta \) reduces itself to \( \Delta_0 \) and one gets:

\[ (\partial_t - \partial_x \Delta_0) \Delta_0 \Psi_0 = N'. \]  

(16)

As before the r.h.s. is of the order of \( \varepsilon^{3/2} \) in general and of the order of \( \varepsilon^{7/4} \) for nearly parallel rolls. The differential operator on the l.h.s. remains of the order of \( q_c^4 \) in the general case so that \( (\Omega_{1w}, \Omega_{1b}) \) could be neglected but for nearly parallel rolls it is now of the order of \( \varepsilon \). This leads to a drift flow of the order of \( \varepsilon^3 \) which must not be neglected as first noticed by Siggia and Zippelius [11].

2.3 Formally Cubic Nonlinearities. — We have now to evaluate the formally cubic terms which have a nontrivial projection onto \( \sin(nz) \) in (1b). These terms present themselves as the product of formally quadratic terms determined in the preceding section by the linearized solution calculated in \( \S \, 2.1 \). The computation is straightforward but inserting the complete solution at 2nd order leads to a very complicated expression for \( NLW_b \) which is not very enlightening. In order to simplify the coupling term we shall make use of the remark made at the end of \( \S \, 2.2.2 \) according to which the 2nd harmonics are negligible in most of the cases of interest. But of course we shall keep the \( z \)-independent part which must not be neglected. Then the nonlinear contributions to equation (1a) read:

\[ NLW_a \rightarrow 3 \Delta_0 (\Omega_0 \partial_x \omega_1 + \Omega_0 \partial_y \omega_1) - \\
- 2 \partial_x \omega_1 \partial_x \Delta_0 - 2 \partial_y \omega_1 \partial_y \Delta_0. \]  

(17)

Similarly, from equation (2) for \( \theta \) we get:

\[ \nabla \cdot \Psi \rightarrow 9 q_c^2 (\Omega_0 \partial_x \omega_1 + \Omega_0 \partial_y \omega_1) + \frac{q_c^2}{2} \omega_1. \]  

(18)

In order to simplify a little more the expression for \( NLW_b \) we shall assume that \( \Delta_0 = 0 \) or \( - q_c \) according to whether it acts on an essentially slowly or rapidly varying quantity. Then eliminating \( \theta \) leads immediately to:

\[ NLW_b = 9 q_c^2 (1 + Pr^{-1}) (\Omega_0 \partial_x \omega_1 + \Omega_0 \partial_y \omega_1) + \\
+ \frac{q_c^2}{2} \omega_1. \]

None of these simplifications are indispensable but using them greatly reduces the complexity of the nonlinear coupling. The price to be paid will be examined later.

Finally, dividing by \( 27 q_c^6 \), dropping index 1 for \( \omega_1 \) and 0 for \( \Psi_0, \Omega_{1w}, \Omega_{1b} \) and gathering terms we get:

\[ \tau_0 \partial_t \omega = \left[ - (\zeta_2^2/4 q_c^2) (\Delta_0 + q_c^2) - \right] \omega - \\
- \partial_x \omega (\partial_x \Psi)^2 + q_c^2 (\partial_x \Psi)^2 - \partial_y (\partial_y \Psi + \partial_x \Psi) \]  

(19a)

with:

\[ \Omega = \partial_y \Psi, \quad \nabla = - \partial_x \Psi \]

and:

\[ \tau_0 = \frac{2}{3} \frac{1}{\pi^2} (1 + Pr^{-1}), \quad \zeta_2^2 = \frac{8}{\pi^2}, \quad g = \frac{1}{6 \pi^2}. \]  

(19c)

2.4 Boundary Conditions. — In this two-dimensional model, « convection » takes place in a domain \( D \) of the horizontal plane \( (x, y) \) and the conditions on \( \Psi \) and \( \Psi_0 \) at the boundary \( \partial D \) have to be specified.

It is quite natural to assume no-slip conditions for the velocity at the boundary. The condition \( \Psi = 0 \) immediately leads to:

\[ \Psi(x, y) |_{\partial D} = 0. \]  

(20a)

The first harmonic horizontal flow being related to the vertical component by (9), \( \tilde{\mathbf{n}} \) and \( \tilde{\mathbf{i}} \) denoting unit vectors normal and tangential to the boundary, we get \( \tilde{\mathbf{i}} \cdot \nabla \Psi |_{\partial D} = 0 \) which is automatically fulfilled as a consequence of (20a) above and:

\[ \tilde{\mathbf{n}} \cdot \nabla \Psi(x, y) |_{\partial D} = 0. \]  

(20b)
(these boundary conditions have already been derived by C. Normand [13]). Notice that (1b) is a 6th order equation with respect to horizontal derivatives while the model is a 4th order equation. From the 3 boundary conditions required for the full problem it is natural to drop the one relating to $\theta$ which is linked to $w$ in the model. In addition to two pairs of periodic solutions at a given $Ra$ the linearized Boussinesq equations have a pair of exponential solutions which allow the boundary condition on $\theta$ to be satisfied. The exponential solutions are absent in the model which yields only the periodic ones.

For the drift flow, written in terms of $\Psi$ the condition $(U, V) = 0$ reads:

$$\Psi(x, y) |_{\text{in}} = 0 = \mathbf{n} \cdot \nabla \Psi (x, y) |_{\text{in}}.$$  \hspace{4cm} (21)

3. Qualitative extension to no-slip conditions.

Since the work of Rayleigh (1916, ref. [14]), the stress-free condition is known to lead to much simpler calculations than the actual zero velocity condition on the horizontal plates which confine the fluid. Here the projection is made easy by the combination rules of the sines and cosines which are exact solutions of the linearized Boussinesq equations. In principle there is no objection to carrying out the Galerkin expansion using as a basis the exact solutions of the linearized problem with the no-slip conditions at top and bottom. This is quite straightforward at the linear stage and one gets the linear part of (19a) with the corresponding values for $Ra$, $q$, $\xi_0$ and $\zeta_0$ [3b, 6, 24c]. Unfortunately things become much more complicated at the non-linear stage. As every basis function is a linear combination of 3 (complex) trigonometric lines, the number of terms to consider grows rapidly. The form of the interaction coefficient for 3 modes at $q_c$ can be deduced in Fourier space from the classical work of Schlüter, Lortz and Busse [21], which leads to an amplitude equation consistent to order $\varepsilon^{3/2}$ [6] but we are interested rather in the corresponding formula in physical space. We can consider an extension of (19a) in which the coefficient $g$ of the cubic term is replaced by an appropriate function of $Pr$ deduced from [21] for straight rolls. The limitations inherent to this replacement will be the same as those pointed out at the end of the discussion below (§ 4).

The form of the drift flow contribution to the equation for $\Psi$ is dictated by dimensional considerations and thus is not expected to change. The main difference should appear in the equation for the drift flow. According to the argument developed by Siggia and Zippelius [22] or Cross [23] the replacement of the $z$-independent flow by a Poiseuille-like flow fitting the no-slip condition implies that the 3-dimensional Laplacian amounts to a constant multiplicative factor upon averaging over the thickness. The l.h.s. of (19b) will be modified to read simply $-c Pr \Delta_h \Psi$. Since the r.h.s. is still $O(\varepsilon^{3/2})$ in general and $O(\varepsilon^{7/4})$ for slightly bent rolls, the drift flow is no longer $O(\varepsilon)$ but $O(\varepsilon^{3/2})$. However, it must be kept in the model since it introduces a qualitatively new feature. From the point of view of boundary conditions, the equation for the drift flow is now of 2nd order instead of 4th order, which obliges us to drop one of the two conditions (21).

It seems more natural to assume that the lateral walls remain impervious. In terms of $\Psi$ the vanishing of the component normal to the boundary simply reads $\Psi |_{\text{in}} = 0$, which leaves open the possibility that the parallel component be non-zero. A different procedure would be to keep the two conditions and a 4th order l.h.s. retaining $\Delta_h$ in the full Laplacian even though it is small when compared to $\partial_x^2$. This would give:

$$(\partial_t - Pr(\Delta_h + c)) \Delta_h \Psi$$

on the l.h.s. of (19b).

4. Discussion.

In this article we have derived a system of partial differential equations which governs the horizontal dependence of the local intensity of convection coupled to slowly varying drift flows. The derivation has assumed stress-free boundary conditions on the horizontal plates and the modifications involved for a better account of actual boundary conditions have been briefly examined. It now seems timely to discuss the extent to which the model generalizes standard amplitude equations.

First of all, one should notice that the derivation of an amplitude equation also involves a projection onto the eigen-modes of the linearized problem in order to get rid of the irrelevant aspects linked to the vertical dependence of the fluctuations. However in this latter case the horizontal dependence of the modes is written explicitly, e.g. Fourier modes for an infinite pattern, while in the model it is left unspecified. The model can thus be seen rather as an explicit equation for wave-packets, which may eventually make easier the description of local events and the treatment of lateral boundary conditions.

Secondly, an amplitude equation can be said to be accurate to a given order with respect to some small parameter, usually the distance to the threshold. Moreover slow space-time variations are linked to this small parameter [4] and even if this is not indispensable at a formal level it turns out to be so for any practical application. On the other hand, one cannot say that our model is accurate to some given order due to the approximations which have been made in order to have a sufficiently simple system. These approximations allow us to avoid the computation of the complicated nonlinear coupling which makes the amplitude equations completely intractable except in some ideal cases such as slightly bent rolls, squares or hexagons [6], isolated dislocations [16, 15] or simple grain-boundaries [2, 17]. Moreover there is no explicit separation of scales in the model. Everything is written in the original coordinates. Separation of scales is only implicit, via the assumptions $\mathcal{L} \sim \varepsilon^{1/2}$ and $\partial_t \sim \varepsilon$. The result is a simplified version of the primitive equations, not an approximate solution of these equa-
tions at a given order, as are the amplitude equations. The immediate consequence is that the model may be the starting point for a standard multiple scale analysis. For example, assuming a single set of nearly perfect rolls, one readily gets the same pair of generalized amplitude equations as those derived by Siggia and Zippelius [11]. Indeed, assuming \( \Omega = (A, e^{i \omega x} + cc) \) with \( \delta_x \rightarrow i q_x + \delta_x, \delta_y \rightarrow i \delta_y \) and \( \delta_t \rightarrow \delta_T \) one obtains:

\[
\tau_0 \delta_T A = \varepsilon A + \xi_0^2 \left( \delta_x + \frac{1}{2 i q_x} \delta_{TT} \right)^2 A - 4 q_x^2 A |A|^2 - i q_x \tau_0 U A \quad (22a)
\]

\[
(\partial_T - Pr \Delta_T) \Delta_T \Psi = -2 \partial_i \left[ A^* \left( \delta_x + \frac{1}{2 i q_x} \delta_{TT} \right) A + cc \right]. \quad (22b)
\]

In this form it is apparent that \( \xi_0 \) is the natural coherence length and that \( \tau_0 \) is the natural relaxation time [3b, 4]. The coupling constant has the correct value because terms neglected in the course of the derivation are effectively of higher order in this particular case.

In the same way one can derive the amplitude equations for two systems of rolls at right angles. One then finds a set identical to that of Brown and Stewartson [18] except for the numerical value of the interaction coefficient which is \( Pr \)-independent with the simplified nonlinear coupling. In order to introduce more realism into this and other similar situations it would be necessary to use the refined solution at the first nonlinear step (§ 2.2).

Once the contribution of the large scale flow is dropped, the amplitude equations at order \( \varepsilon^{3/2} \) have a variational structure [6]. On the contrary, even with the simplified nonlinear coupling and \textit{a fortiori} with the complete one, our model cannot be derived from a Lyapunov functional. In this respect, the model already includes a qualitatively new feature, present in the primitive equations but absent from the amplitude equation at lowest order. Of course only rather trivial nonpotential corrections are included: those linked to the interactions between 3 modes close to \( q_c \).

It does not include those of higher order, variational or not, linked to interactions between 4 modes or more. The latter may be of the same order of magnitude as the former quantitatively speaking but taking the former into account may already have interesting consequences. In particular the statement that the temporal behaviour of the textures is relaxational close to the threshold is a perturbative result. As such it may break down at a finite (but possibly very small [19]) distance from the threshold. Thus nonvariational corrections included automatically in the model can give some useful hints to help better understand the actual behaviour even if the description is still incomplete.

Another limitation of the amplitude equations is implied by the formal link between the slow space-time scales and the distance to the threshold. For example amplitude equations at an order higher than \( \varepsilon^{3/2} \) are required to discuss reliably the wave-length selection by boundary effects [8, 25], or to show the difference between the selection criterion for concentric rolls [24a, 23], the condition of marginal stability against zig-zags [24b, 23] and the condition for a dislocation to stay at rest [16, 15, 26]. The derivation of amplitude equations at orders higher than \( \varepsilon^{3/2} \) starting with the Boussinesq equations is very painful and it may be preferable to develop such expansions from a simplified model. The derivation will gain in transparency even if the final results cannot be quantitatively correct due to the neglect of some important contributions from the very beginning [27].

In order to derive this model we have proceeded in much the same way as somebody who performs a numerical simulation of the Boussinesq equations with stress-free boundaries. Indeed the vertical dependence is generally treated through a Galerkin projection technique onto the same basis as that used here. Of course numerically one uses several vertical modes while here we restrict ourselves to the lowest ones. The « vertical resolution » of our model is thus rather poor. In addition several approximations have been made in order to simplify the problem while in a numerical simulation the horizontal dependence can be treated exactly at a given level of numerical precision. Nevertheless, the modes that have been retained in the model are likely to contain the most universal features of convection close to the threshold (at least for stress-free boundaries) and we think that having an explicit equation in physical space for these modes is a great advantage over treating them implicitly on an equal footing with other less important modes as in the full numerical simulation.

The basic idea of eliminating some irrelevant features of convection linked to the vertical dependence of the fluctuations is not an original approach since it already underlies the amplitude equation formalism. But let us emphasize that models such as the one derived here should be considered as simplified versions of the primitive Boussinesq equations. As a matter of fact all qualitatively important features of Rayleigh-Bénard convection are present. The linearized operator accounts for unstable modes with a well defined preferred wave-length and no preferred orientation. Nonlinearities are formally cubic in order to preserve the supercritical character of the instability. Moreover these nonlinearities include some important nonvariational corrections automatically. Finally large scale flows induced by long wave-length modulations of the intensity of convection are taken into account in a way which preserves the symmetry properties of the fluid layer and restore the non-trivial role of the Prandtl number. The simplifications which have been made are well justified at small \( \varepsilon \) except for the last one which drastically reduces the complexity of nonlinear
interactions. Quantitatively speaking this simplification implies an incorrect account of the evolution of the structure every time several distinct wave-packets interact. In the same way the motion of isolated defects can be slightly altered. But preliminary simulations have shown that as far as the qualitative aspects of pattern selection are involved, the precise form of the cubic coupling does not matter much as long as it remains local in physical space \[9b\], which is the case since (13) can be solved explicitly (see 2.2.2). Since the main original point of the model is the coupling with large scale secondary flows we can still expect valuable results even with the simplified nonlinear term. Indeed, preliminary results of simulations on model (19) already present interesting novel features \[28\] when compared to earlier numerical \[9, 10\] and theoretical \[2\] studies of textures. This could be expected since the nature and the arrangement of defects present in a given texture are likely to be sensitive to the large scale drift flow. Of course the series of approximations used restrict the validity of the model to small values of \( \varepsilon \) but we hope that qualitative answers drawn from theoretical and numerical analysis of such models will bring some progress in the understanding of the transition to turbulence in large aspect ratio systems.

References

[19] At finite and small — but not vanishingly small — distances from the threshold for axisymmetric convection, simulations \[20\] shows that with non-linear terms \( \psi^3 \) and \( \chi \psi \) the solution remains stationary while with \( \psi^3 (\nabla \cdot \psi)^2 \) the solution becomes time dependent at \( \varepsilon = 0.065 \) in a box of radius \( R = 100 \), the motion corresponding to a regular slow drift of the rolls towards the centre. This contrasts with what one would infer from the amplitude equation at lowest order and definitely shows the motivation for including in a simple way a large class of nonvariational corrections.
[26] In their paper Dislocation motion in layered structures (Philos. Mag. A, to appear) E. Dubois-Violette, E. Guazelli and J. Prost, introduce for the first time the large scale secondary flows in the problem of dislocation motions. Our semi-microscopic model could be a good starting point for a quantitative check of their analytical approach.
[27] In particular it should be noted that the fastest growing mode of the linearized model always corresponds to \( q_{\text{max}} = q_c \) while it can be shown from the Boussinesq equations that \( (q_{\text{max}} - q_c)/q_c \sim \varepsilon \).