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Résumé. — Nous présentons une réalisation pratique d'une variante du modèle d'accélération de Fermi. Nous mettons en évidence trois bifurcations successives sur le chemin vers le comportement chaotique.

Abstract. — An experimental model of a modification of the Fermi acceleration problem is described. Evidence is presented for three consecutive bifurcations on the period doubling route of the system from regular to chaotic behaviour.

1. **Introduction.** — The discovery of strange attractors has stimulated interest in the physics of turbulence in the last few years. Three scenarios according to which the phenomenon should develop have been proposed [1]. Experimental evidence exists for all three scenarios. For instance, the period doubling scenario of Feigenbaum [2] has been already observed in a number of physical systems : Rayleigh-Bénard convection cell [3], PLL models of the Josephson junction [4], non-linear RLC resonance circuits [5], and a mechanical (compass in an alternating field) model [6].

It is the last of the above-mentioned experiments that stimulated the present author to search for the period doubling cascade in an even simpler mechanical system : an elastic particle jumping on a vibrating surface. The idea of the experiment can be introduced in a number of alternative ways. In the following we have chosen the way in which it has been discovered, i.e., *via* an analogy with the discrete sine-Gordon chain.

2. **The discrete sine-Gordon chain. A cascade of bifurcations of the central island.** — The expression :

$$\mathcal{H}(\theta_1, \theta_2, ..., \theta_N) = \frac{1}{2} \sum_i (\theta_{i+1} - \theta_i)^2 - \lambda \sum_i \cos \theta_i \tag{1}$$

describes the potential energy of a one-dimensional chain of particles coupled by harmonic forces and submerged in an external periodic field. The model known as the discrete sine-Gordon chain serves as a first order approximation for a whole variety of physical systems among which the coupled pendulum chain designed by Scott occupies a special place due to its laboratory scale dimensions [7]. We shall use the vocabulary of the experimental device in the description presented below of the essential properties of the theoretical model. Stationary configurations of the chain described by (1) are given as solutions of a recurrence equation :

$$\theta_{i+1} = 2\,\theta_i - \theta_{i-1} + \lambda \sin \theta_i \tag{2}$$

obtained when condition of equilibrium of forces is applied to (1).

Solutions of the second order difference equation can be conveniently presented within a (θ_i, θ_{i+1}) map. The map for equation (2) displays a translational symmetry and can be seen to consist of identical $2\pi \times 2\pi$ squares. Thus, results of the numerical analysis of the equation are represented by the landscape of a single $(\tilde{\theta}_i, \tilde{\theta}_{i+1})$ square, where $\tilde{\theta}_i = \theta_i[\text{mod } 2\pi]$. See figure 1. The landscape can be described as : islands of order submerged in the sea of chaos.

The shape, area and position of the islands depend on the external field parameter λ. A rigorous analysis of the landscape and its physical meaning for the discrete sine-Gordon and similar models has been performed by Aubry [8] ; here, we shall limit ourselves to but one of its most visible landmarks — the central island located around the (π, π) point.

Without getting into some unnecessary details, the evolution of the island with increasing λ can be described as follows :

For $\lambda \ll 1$ (see Fig. 1*a*) the island is elongated and

(*) Work carried out under Project MR-I.9.

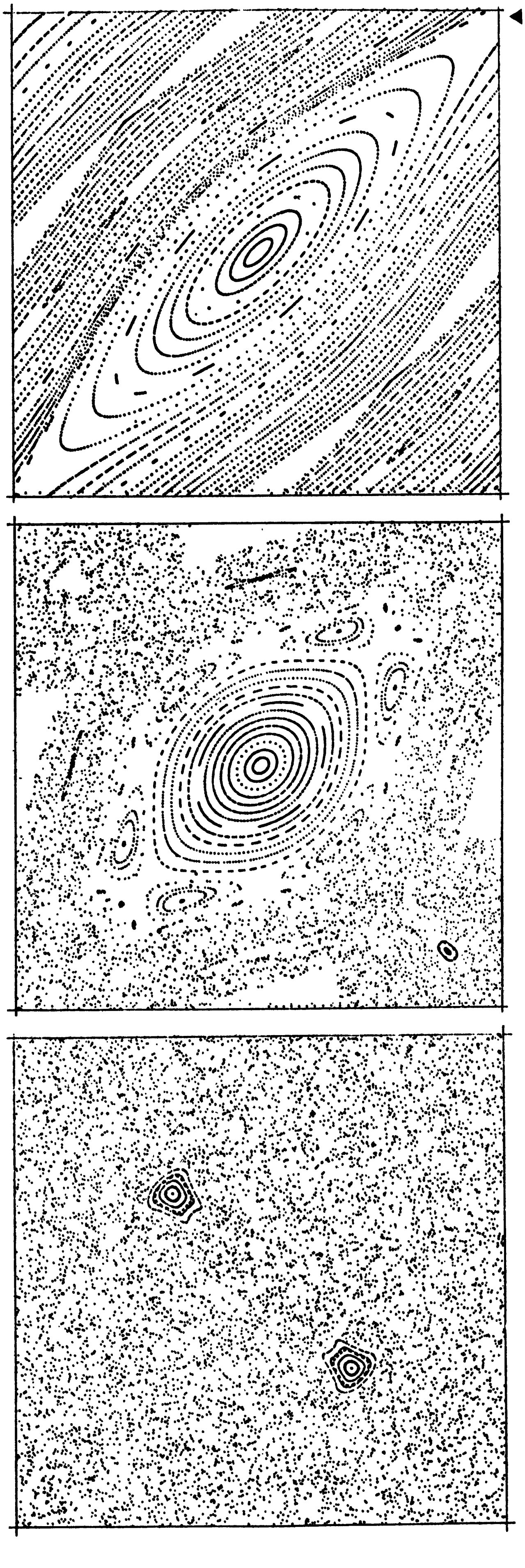

Fig. 1. — $(\tilde{\theta}_i, \tilde{\theta}_{i+1})$ map for the discrete sine-Gordon equation (2). $\lambda = 0.5$, 1.5 and 5 in figures *a*, *b* and *c*, respectively.

stretches almost from (0, 0) to the $(2\pi, 2\pi)$ corner of the map. At this stage the island is filled with smooth elliptic orbits, whose discrete rotation induced by consecutive iterations of (2) is very slow even at the centre of the island where it is fastest. As λ increases (see Fig. 1*b*), the island shrinks changing to a more square shape. Orbits of a given period « flow down » the island; those already at the shore enter the sea of chaos while new, of a shorter period are generated by the (π, π) top of the island. To be more precise, orbits of commensurate periods do not cover smooth curves but are seen as chains of discrete points which, as λ increases, prove to be located within smaller islands (within which a similar process takes place). Consequently, the island at first sight monolithic proves to be by no means such. It must be rather seen as a collection of concentric belts of smaller islands (which on their own display similar morphology) separated by thin strips of chaos. « Consecutive » commensurate belts are separated by dams of incommensurate orbits. The belts, very tight at the top of the island, become looser as with increasing λ they shift towards the island's shore. At a certain $\lambda = \lambda_1$ (see Fig. 1*c*) the period of rotation at the top of the island reaches 2 — the island bifurcates into two smaller and the story repeats itself. $\lambda = \lambda_2$ makes the two islands bifurcate simultaneously, and so on. Consecutive bifurcations make the central island disperse into a well organized archipelago of more and more tiny islands. Values $\{\lambda_i\}_{i=1}^{\infty}$ at which the bifurcations are observed are arranged in a sequence accumulating at a λ_∞ limit in which the archipelago becomes non-denumerable while its total area shrinks to zero — a Cantor set is formed. It has been shown recently [9] that like in the original one-dimensional case [2] convergence of the $\{\lambda_i\}$ sequence becomes geometrical at high stages of the bifurcation cascade and is governed by a universal constant : $\delta^{2D} = 8.72...$, different from that found for one-dimensional mappings ($\delta^{1D} = 4.67...$).

Can the cascade of bifurcations be illustrated by some simple experiments carried out on Scott's machine ?

To answer the question we must analyse the physical sense of both the elliptic fixed point (π, π) and the orbits which surround it.

The (π, π) point always represents this particular configuration in which all pendula stand *up* the gravitational field. Configurations seen in the map as elliptic orbits surrounding the point represent modulations imposed of the up-standing chain. That the arranged in-line, up-standing chain is stationary, is obvious, since both gravitational and elastic forces vanish in this configuration. That a modulated up-standing chain can be stationary, is less obvious, but calculations show that the wavelength of the modulation can be so adjusted relative to its amplitude that the condition of equilibrium of forces is satisfied. For commensurate wavelengths another factor must be taken into account — the phase of the modulation wave. Only at one particular phase is the modulation wave pure. An endeavour to shift the phase leads to a secondary modulation. This is the physical sense of the belts of secondary islands to which commensurate orbits are transformed as λ increases.

It is in agreement with intuition that the higher the amplitude, the longer the wavelength of the modulation. Consequently, the shortest modulation is always represented by the centre of the island; when its wavelength reaches 2, the island bifurcates. And so on.

Though *stationary*, all the configurations described are *unstable mechanically* — any fluctuation makes them fall down. It would need a demon prestidigitator to use Scott's machine to demonstrate the cascade of bifurcations described. It was the main aim of the present author to find a way in which the instability could be overcome and the cascade visualized in a live experiment. Such a way exists and is described in the next part of the paper.

3. **The jumping particle model.** — The idea lies in a reinterpretation of equation (2). This second order difference equation can be regarded as resulting from a set of two first order difference equations :

$$\theta_{i+1} = \theta_i + v_i \tag{3a}$$

$$v_i = v_{i-1} + \lambda \sin \theta_i . \tag{3b}$$

The two equations make a good approximation for equations of motion of a perfectly elastic particle jumping perpendicularly (within a gravitational field) on a horizontal vibrating plane. The model is, as the present author has found out, already known as the Pustilnikov modification of the Fermi acceleration model [10]. θ_i denotes now the time of the i-th collision in which the particle arriving at the vibrating plane with velocity v_{i-1} departs from it to the next jump with velocity v_i modified by $\lambda \sin \theta_i$. The next jump lasts (in appropriate units of time) v_i and ends at θ_{i+1}.

Let us notice that the model is essentially different from the « kicked pendulum » for which equation (1) can be regarded as Lagrangian [11].

Due to the equivalence of equations (3) and (2) the modes of jumping which are possible in the Pustilnikov model are illustrated by the same set of maps that we analysed above. Each stationary configuration of the sine-Gordon chain has its counterpart in a specific jumping mode of the Pustilnikov system. As before, we shall limit ourselves only to a short analysis of modes represented by the central island; due to the cascade of its bifurcations, the modes should follow the Feigenbaum scenario of period doublings.

Twice in each cycle the vibrating surface crosses the state of zero velocity — in its upper and lower turning point. If the particle is thrown onto the surface in such a moment, its velocity will be preserved during the collision ($\lambda \sin \theta_i = 0$). Thus, if the initial velocity is chosen in such a way that the resulting jump lasts $T = 2\pi$, the next collision will coincide with the next turning point (of the same kind) and so on — a periodic jumping pattern is initiated. Since there are two kinds of zero velocity points, two kinds of the simplest jumping modes are possible. Equations :

$$\theta_i = i\, 2\pi \tag{4a}$$

$$\theta_i = i\, 2\pi + \pi \tag{4b}$$

describe the two modes.

The first one (4*a*) is represented by the (0, 0) or (2π, 2π) corner of the map. The second one (4*b*) by the (π, π) centre surrounded by central island. A perturbation destroys the first mode. The second one is stable — a perturbation leads only to a phase modulation of it. The phase modulation is equivalent to the angular modulation analysed in the case of the coupled pendule chain.

Bifurcation of the central island results in doubling of the period of the jumping mode. Which after the first bifurcation is given by :

$$\theta_i = i\, 2\pi + \pi + (-1)^i \Delta , \qquad \Delta < \pi/2 \tag{5}$$

and can be described in plain words as a mode in which a jump shorter than 2π is followed by a longer one. Altogether, the periodicity is preserved in 4π.

Consecutive bifurcations double the period up to infinity.

To the best of the present author's knowledge, the jumping particle model has been always regarded as a convenient thought experiment, and no efforts have been made to construct it as a real experimental system. Below, a working design is described.

4. **Experimental set-up.** — Figure 2 presents block diagram of the experimental system. The vibrating plane is provided by the surface of a concave lens fixed to the membrane of a small loudspeaker supplied from an audio generator. Frequency of the vibration equals about 100 Hz. The slight curvature of the lens surface stabilizes the trajectory of the jumping particle. Jumping modes of a steel sphere of a diameter $\sigma \cong 4$ mm are monitored by an oscilloscope whose x-axis sweep is triggered by the audio generator. Since collisions of the steel sphere with the lens surface produce clear click sounds, a microphone placed above the system makes it possible to observe the collisions as regular wave packets. Position of the front edge of the collision sound indicates directly its phase in relation to the vibration of the lens surface. Consequently, a periodic jumping mode is seen as a standing image of the wave packets produced by consecutive collisions.

Since at low levels of the vibration amplitude a trivial solution in which the sphere moves together (in contact) with the vibrating surface is also possible,

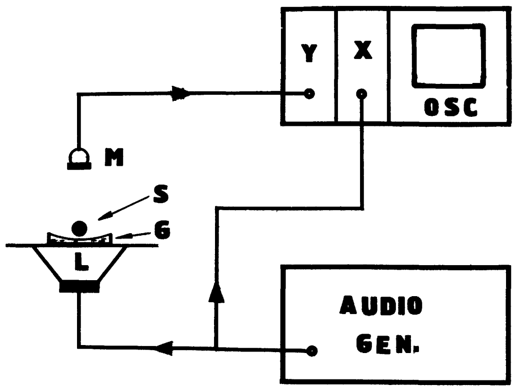

Fig. 2. — Experimental set-up. M : microphone, S : steel sphere, G : lens, L : loudspeaker.

the periodic jumping mode must be initiated by an external perturbation — a delicate knocking at the lens does the job.

5. **Results.** — A threshold λ_0 exists below which the 2π-periodic jumping mode is not possible. Knocking at the lens produces only a short sequence of jumps and the sphere stops i.e. gets into the trivial solution. Above λ_0, the particle can be easily put into an endless regular sequence of collisions heard as a clear rattling sound and visible as a standing image on the screen of the monitoring oscilloscope. Figure 3A presents the case. Exposure time was about 1 s. Thus, the image is made from about 100 of identical traces. A slow, continuous increase of the vibration amplitude makes the phase of the jumping mode shift towards π — the standing image shifts along the x-axis.

At a well defined λ_1 the tune of the rattling sound changes. At the same time the image seen on the oscilloscope screen splits in two. Figure 3B presents the first splitting. The period of the jumping mode doubles. Further increase of the vibration amplitude λ makes the two images separate more and more up to another critical value λ_2 in which each of the two images splits in two on its own. The second splitting presented in figure 3C results in a subtle change of the rattling sound. The third bifurcation observed at a λ_3 value of the vibration amplitude is very delicate and easily perturbed by any fluctuation. Figure 3D presents a clear image of the event.

What happens next is difficult to say without a more sophisticated analysis of the observed chaotic at the first sight jumping mode. Anyway, next steps of the period doubling route seem to be too delicate to be observed in a real experiment due to the effects of the unavoidable noise.

Amplitudes λ_1, λ_2, λ_3 have been measured and in units in which $\lambda_1 = 1$, are equal 1.097 and 1.117 respectively. Thus the Feigenbaum factor :

$$\delta_1 = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_2} = 4.8 \pm 0.6 . \tag{6}$$

A systematic error in the above-presented value cannot, however, be excluded due to non-linear distortions introduced by the loudspeaker. In the simple experimental equipment used during the preliminary study the factor could not be eliminated.

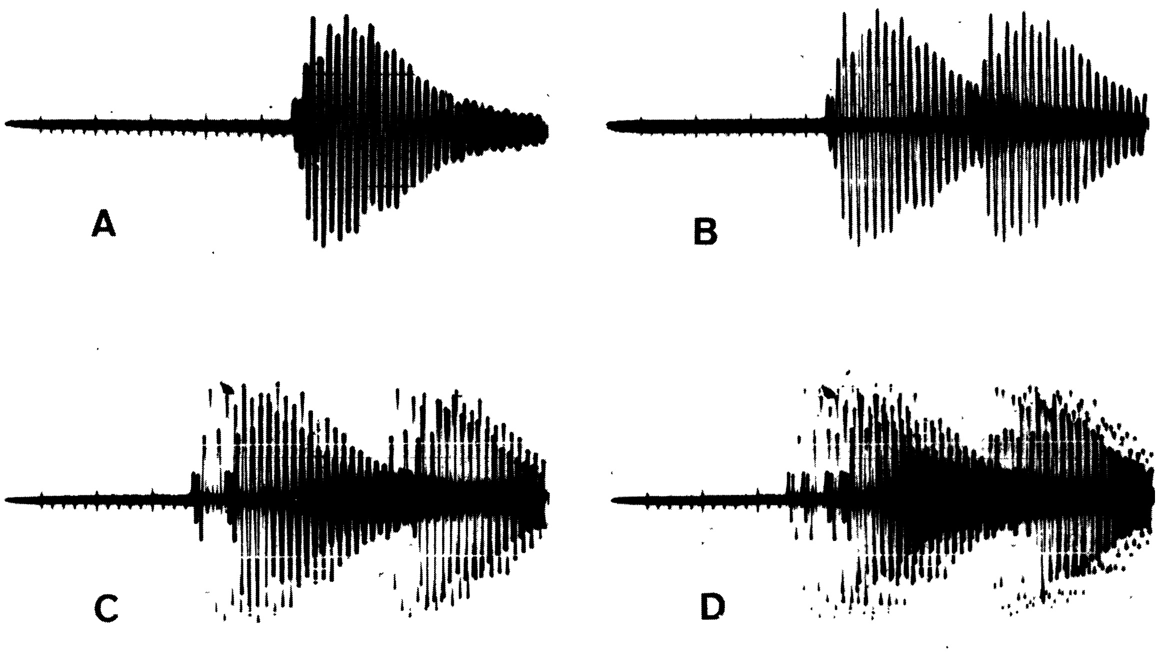

Fig. 3. — Oscilloscope recordings of three consecutive bifurcations in the jumping particle model. A : before the first bifurcation ; B : after the first and before the second bifurcation ; C : after the second and before the third bifurcation ; D : just after the third bifurcation.

6. **Discussion.** — Equations ($3a$, b) describe an ideal system in which the « vibrating » surface provides momentum to the colliding particle without changing position itself. This is certainly not the case in the experimental system we designed.

This inconsistence matters, of course, but it seems that results presented are determined by still another factor : dissipation. At each collision the jumping particle looses a part of its energy. Consequently, equation ($3b$) should rather read :

$$v_i = kv_{i-1} + \lambda \sin \theta_i , \qquad 0 \leqslant k \leqslant 1 . \qquad (7)$$

Exact, quantitative properties of this dissipative model (already considered by others in connection with a strange attractor it may produce [12]) cannot be determined without numerical analysis, but its most essential qualitative features can be deduced by intuition.

First of all, let us notice, that dissipation removes the two-dimensional translational symmetry of the (θ_i, θ_{i+1}) map observed for the non-dissipative case.

Having lost its symmetry the (θ_i, θ_{i+1}) map becomes awkward in application. Instead, the (θ_i, v_i) phase space can be used. One must remember, however, that it is now only the θ_i variable that can be expressed modulo 2π; the velocity v_i cannot be reduced like that.

Dissipation damps any phase modulations, consequently elliptic orbits are not possible in their previous form. In general, they must turn into spirals falling down onto centres located near elliptic fixed points — point attractors with their bassins are formed. If, however, λ is sufficiently high and if an elliptic orbit is commensurate of a sufficiently low order, then being itself built as a belt of secondary elliptic fixed points (surrounded by smaller elliptic orbits), it turns into a belt of point attractors (each surrounded by smaller bassins). The jumping mode corresponding to such a belt of attractors can be described as a stable commensurate phase modulation of a given jumping mode. As the dissipation increases this possibility is less and less probable due to another effect — a shift of point attractors from their $k = 1$ positions. The shift is necessary to compensate dissipation. In the simplest case of the 2π-periodic jumping mode represented in the absence of dissipation by the $(\pi, 2\pi)$ elliptic fixed point (we use now the (θ_i, v_i) notation !) corresponding attractor is shifted to $(\pi - \varepsilon, 2\pi)$ position, where $\lambda \sin \theta_i$ term does not vanish. The shift can be easily calculated as :

$$\varepsilon = \arcsin \frac{1-k}{\lambda} 2\pi . \qquad (8)$$

It is obvious that this point attractor must vanish below well defined value λ_0 at which the shift equals $\pi/2$. Similar reasoning is valid for any other attractors formed from elliptic fixed points and concerns in particular the whole family of attractors formed by the cascade of bifurcations of the central island. In consecutive steps of the cascade, the basin of the central attractor disperses into archipelago of smaller and smaller basins; at constant dissipation, the basins become eventually too small to keep the perturbed by noise jumping mode inside — the Feigenbaum scenario is cut short. What happens next, we are not able to answer at present. The jumping mode must leave the period doubling route. Are there any new attractors ready to accept it ?

A few words of comment seem to be necessary concerning the value of δ_1 that we determined experimentally : it coincides (within the error limits) with the universal convergence ratio $\delta^{1D} = 4.67...$ characteristic for one-dimensional mappings which, at first sight, appears to be in contradiction with two-dimensional nature of mapping ($3a$, $3b$) which we have shown to describe the idea of the jumping particle model and which produces a cascade of period doublings with $\delta^{2D} = 8.72...$

The explanation of this discrepancy lies once more in the dissipation present in the experimental realization of the jumping particle model. As shown by Helleman and Zisook [13] even infinitesimal dissipation changes radically convergence of the bifurcation cascade of a two-dimensional mapping at its high stages so that the limit convergence ratio equals δ^{1D} instead of δ^{2D} (the latter being preserved only in the pure area-preserving, i.e., $k = 1$ case).

One must remember, however, that this theoretical result concerns the behaviour of the bifurcation cascade in its limit, while the value we determined describes the cascade only at its very beginning. Consequently, one should rather calculate the first three λ_{-s} of the cascade produced by mapping ($3a$, 7) and find its starting convergence ratio for $k \in (0, 1)$. Such calculations described in detail elsewhere [14] have been performed proving that the value of δ_1 we determined (i.e., 4.8) can be fitted by $k \cong 0.2$, thus, the experimental system proves to be highly dissipative. Experiments, which are being performed at present, indicate that the dissipation factor depends strongly on details of the experimental system (for example materials of which both the jumping particle and the vibrating surface are made, their masses, frequency at which the jumping mode is excited, etc.), so, the measured value of δ_1 may vary in a wide range.

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