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Thermal instability in a sample of nematic liquid crystal contained in a rotating annulus

P. J. Barratt
Department of Mathematics, University of Strathclyde Glasgow, G1 1XH, U.K.

and I. Zuniga
U.N.E.D., Facultad Ciencias, Apartado Correos 50 487, Madrid, Spain

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Résumé. — Nous employons la théorie des milieux continus pour étudier la stabilité d'un échantillon de cristal liquide nématique sous faible épaisseur contenu entre deux longs cylindres concentriques en rotation à vitesse angulaire constante le long de leur axe, et soumis à un gradient de température radial. Nous considérons le cas d'un matériau initialement aligné parallèlement à l'axe des cylindres. Utilisant les méthodes de Galerkin et d'orthonormalisation, nous déterminons les seuils positif et négatif des gradients thermiques pour lesquels l'apparition d'une instabilité de convection est possible. Les effets de la vitesse angulaire et d'un champ magnétique appliqué sont également examinés. Nous montrons qu'il est possible d'observer l'instabilité à petit nombre d'onde que Velarde et Zuniga [1] ont proposée dans le cas du problème de Rayleigh-Bénard pour les nématiques.

Abstract. — Continuum theory is employed to investigate the stability of a sample of nematic liquid crystal contained between two long, concentric, circular cylinders, rotating with constant angular velocity about the cylindrical axis, when subjected to a radial thermal gradient. We consider the arrangement in which the anisotropic axis of the material is initially uniformly aligned parallel to the axis of the cylinders. Using Galerkin and orthonormalization methods, both positive and negative thermal gradient thresholds are predicted at which the onset of a convective instability is possible. The effects of angular velocity and the strength of an applied magnetic field are also examined. These theoretical results suggest an experimental possibility of observing the small wavenumber instability predicted by Velarde and Zuniga [1] in the Rayleigh-Bénard problem for nematics.

1. Introduction. — The occurrence of stationary convective instabilities in thin horizontal layers of nematic liquid crystal under vertical temperature gradients has been the subject of several theoretical and experimental investigations [1-7] over the past few years. Two particular arrangements have received special attention. In one the anisotropic axis of these transversely isotropic materials is initially uniformly aligned parallel (planar orientation) to the bounding plates while in the other it is aligned perpendicular (homeotropic orientation) to the plates. It is found that stationary convection is possible if one heats the lower plate in the former configuration, as is the case in isotropic fluids, but in the latter configuration the upper plate must be heated.

In these cases typical critical thresholds are greatly reduced compared with those associated with isotropic fluids having similar mean properties. Velarde and Zuniga [1] show that steady convection is also possible in a planar sample when heating is from above. However the critical threshold is such that this is unlikely to be realized under laboratory conditions.

The purpose of this paper is to present a linear analysis of possible steady convective instabilities in a thin sample of nematic liquid crystal which is contained between two long, concentric, circular cylinders rotating about their common vertical axis with constant angular velocity \(\omega_0\), when subjected to a radial thermal gradient. Here we analyse the particular problem when the anisotropic axis is initially uniformly aligned parallel to the axis of rotation. In this problem, one expects the effect of the
centrifugal buoyancy force to be somewhat similar
to that of the gravitational buoyancy force in the
Rayleigh-Bénard problem. However one also has the
additional complication of possible effects due to the
presence of the Coriolis force. After linearizing the
continuum equations about the configuration de-
scribed above, we consider the stability of the uni-
formly aligned configuration with respect to two
particular types of roll solution. In one referred to
here as the Taylor-Proudman instability (axial rolls),
the axis of the rolls is parallel to the axis of rotation
and this is the roll instability observed in isotropic
liquids when the Coriolis force is important. The
other is the Taylor-Couette instability (azimuthal
rolls) in which the perturbed variables are independent
of the azimuthal angle \( \phi \). The analysis indicates that
the latter solution is the one that is most likely to
display the effects due to the coupling between flow
and molecular orientation, peculiar to nematic liquid
crystals, and for this reason we only proceed to obtain
solutions for the Taylor-Couette instability.

We formulate the problem and derive the set of
governing linear equations in section 2. Section 3
presents a qualitative one-dimensional model of the
Taylor-Couette instability and it is subsequently
shown to yield results that agree tolerably well with
those found using the two-dimensional solution. In
section 4, we obtain critical threshold values for the
onset of the Taylor-Couette instability by both
Galerkin and orthonormalization methods, the former
yielding an approximate solution and the latter an
accurate numerical solution. A discussion of the
results is contained in section 5. In particular we
describe the effects of angular velocity \( \omega_0 \) and the
strength \( H \) of a magnetic field, applied parallel to
the initial orientation of the anisotropic axis, upon
the instability thresholds. In addition, we make some
comments concerning the theoretical results presented
in this paper and the observations of Guyon and
Carrigan [8] in an experiment related to the problem
under discussion.

Since detailed accounts of continuum theory and
relevant physical properties of nematic liquid crystals
are to be found in the books by de Gennes [9] and
Chandrasekhar [10] and the reviews by Stephen and
Straley [11], Ericksen [12] and Leslie [13], no further
account is given here. However for completeness a
brief summary of the continuum equations using the
notation employed in reference [13] is contained in
an appendix to this paper.

2. Formulation of the problem. — Here we investi-
gate the stability of a sample of nematic liquid crystal
which occupies the space between two long, concent-
ric, circular cylinders, the axis of the cylinders being
in the vertical direction. The cylinders are considered
to rotate with a common angular velocity \( \omega_0 \) about
the vertical axis while the internal cylinder is kept
at a constant temperature \( T_1 \) and the external cylinder
at a constant temperature \( T_2 \). Cylindrical polar
coordinates \((r, \phi, z)\) are chosen so that the \( z \)-axis
coincides with the common axis of the cylinders. The
director is assumed to be initially uniformly aligned
parallel to the \( z \)-axis and we allow for the effect of a
uniform magnetic field applied parallel to this initial
director orientation.

One simple solution of the continuum equations
(A.1)-(A.4) is the uniformly aligned configuration
with the director, velocity and temperature fields
having physical components of the form:

\[
\mathbf{n}^0 = (0, 0, 1), \quad \mathbf{v}^0 = (0, \omega_0, 0),
\]

\[
T = T(r) = \left\{ T_1 \ln \left( R_2/r \right) - T_2 \ln \left( R_1/r \right) \right\}/\ln (R_2/R_1)
\]

where \( R_2 \) and \( R_1 \) are the radii of the outer and inner
cylinders, respectively. In addition, one finds

\[
p = \int \rho g \, dz + p_0, \quad \gamma = - \chi_s H^2,
\]

where \( p_0 \) is a constant and \( H \) is the magnetic field
strength.

We now consider the stability of this uniformly
aligned state when subjected to a small amplitude
perturbation \( \mathbf{n} \) in the director field, associated with
which are small perturbations \( \mathbf{v}, \theta, \tilde{p} \) and \( \tilde{\gamma} \) in the
velocity, temperature, pressure and director tension
fields, respectively. Writing the physical component
forms for fluctuations in the velocity and director
fields as

\[
\mathbf{v} = (v_r, v_\phi, v_z), \quad \mathbf{n} = (n_r, n_\phi, 0),
\]

where the components are functions of the coordi-
nates \( r, \phi, z \) and time, and adopting the Boussinesq
approximation [14], one readily obtains the linearized
equations

\[
\left( \frac{\partial \mathbf{v}}{\partial t} + \omega_0 \left( \frac{\partial \mathbf{v}}{\partial \phi} - 2 \mathbf{v}_\phi \right) \right) + \alpha_4 \left( \frac{\partial^2 \mathbf{v}_r}{\partial r^2} + \frac{1}{r} \frac{\partial \mathbf{v}_r}{\partial r} - \frac{\mathbf{v}_r}{r^2} \right) +
\alpha_4 \left( \frac{\partial^2 \mathbf{v}_\phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \mathbf{v}_\phi}{\partial r} \right) +
\alpha_2 \frac{\partial^2 n_r}{\partial \phi \partial \phi} + \alpha_2 \omega_0 \frac{\partial^2 n_\phi}{\partial \phi \partial \phi} +
\frac{(\alpha_2 + \alpha_4 + \alpha_5)}{2} \frac{\partial^2 v_z}{\partial r \partial z} + \frac{(\alpha_4 + \alpha_5 - \alpha_2)}{2} \frac{\partial^2 v_z}{\partial z^2},
\]

(2.4)
where the $\kappa$'s in (2.9) are thermal diffusivities. Also $\alpha$ is the thermal expansion coefficient. In deriving the linearized torque equations (2.7) and (2.8), the director inertia term $\sigma n_r$ in (A.2) has, as is common, been neglected.

To ease calculations in the remainder of the paper, two simplifying assumptions are introduced. The first is to assume that the principle of exchange of stabilities holds (see for example Chandrasekhar [15]). This effectively means that one sets all terms containing a partial time derivative in equations (2.4)-(2.9) identically equal to zero. The second is to employ the narrow gap approximation in which one assumes

$$f_r = f(\theta, r) + f_0 \delta r,$$

$$r = r_m$$

and $$(\partial T/\partial r) = \phi_m = (\partial T/\partial r)_m,$$ (2.11)

where

$$r_m = (R_1 + R_2)/2, \quad d = R_2 - R_1,$$ (2.12)

$f$ is any function of $r$ and $\phi_m$ is the mean temperature gradient across the sample. This narrow gap limit effectively renders the coefficients in the linear system of equations (2.4)-(2.10) independent of the polar variable $r$. With the aid of these two simplifications, one now seeks solutions of the resultant linearized equations which satisfy the boundary conditions

$$v_r = v_\phi = v_r = n_r = n_\phi = \theta = 0,$$ (2.13)

on the surfaces $r = R_1$ and $r = R_2$.

3. A qualitative model. — Here we examine a simple form of perturbation in which velocity, director and temperature have the physical components

$$v = (v^0, v^\theta, 0) \exp(i q z), \quad n = (n^0, n^\theta, 0) \exp(i q z),$$

$$\theta = \theta_0 \exp(i q z),$$ (3.1)

where $v^0, v^\theta, n^0, n^\theta, \theta_0$ are constants. One observes that the continuity equation (2.10) implies that $v_\phi$ must be identically zero. Substituting the expressions (3.1) into the equations (2.4)-(2.9), one obtains the set of five algebraic equations

$$- (\alpha_4 + \alpha_5 - \alpha_2) q^2 v^0 +$$

$$+ 2 \rho \omega_0 v_\phi^0 - \rho \alpha \omega_0^2 \theta_0 = 0,$$ (3.2)

$$- (\alpha_4 + \alpha_5 - \alpha_2) q^2 v_\phi^0 - 2 \rho \omega_0 v_\phi^0 = 0,$$ (3.3)

$$- (k_3 q^2 + \chi_2 H^2) n_\phi^0 - \alpha_2 i q v_\phi^0 = 0,$$ (3.4)

$$- (k_3 q^2 + \chi_2 H^2) n_\phi^0 - \alpha_2 i q v_\phi^0 = 0,$$ (3.5)

$$- (\kappa_1 + \kappa_2) q^2 \theta_0 +$$

$$+ i k_2 \phi_m q n_\phi^0 - \phi_m v_\phi^0 = 0.$$ (3.6)
For a non-trivial solution of this set of equations, the determinant of the coefficients of terms on the left hand side of the equations must be zero and this leads to the result

\[
\phi_m = \frac{\left( \kappa_1 + \kappa_2 \right)}{\rho_\alpha(r_m^0) \omega_0^2} \left\{ (\alpha_4 + \alpha_5 - \alpha_2) q^2 \quad \right. \frac{+ 4 \rho^2 \omega_3^2}{1 - \alpha_2 \kappa_2 (k_3 + \alpha_6 \beta^2 / q^2)} \left. \right\}.
\] (3.7)

In the absence of any applied magnetic field, there are several points of interest to be noted from equation (3.7). Since for nematic materials \( \alpha_2 \kappa_2 / k_3 \) is typically of the order \( 10^2 \), one expects threshold values for \( \phi_m \) to be smaller than those appropriate for conventional isotropic fluids. For sufficiently small angular velocities, the Coriolis force is negligible and we obtain the planar Bénard convection result that \( \phi_m d^4 \) is a universal function. On the other hand, if \( \omega_0 \) is sufficiently large so that the Coriolis force dominates the viscous force, one finds that \( r_m \phi_m \) is almost a constant. The effect of an applied magnetic field with positive anisotropic susceptibility is obviously to increase the threshold and for sufficiently large fields the threshold value of \( \phi_m \) approaches its value for isotropic fluids and eventually exceeds it.

Of course it must be remembered that the expressions (3.1) cannot possibly satisfy the boundary conditions (2.13), except in the case of the trivial zero solution. However, as we shall see later, the result (3.7) gives a reasonably good qualitative description of the instability.

4. Solution of the problem. — Here we consider three particular types of perturbation solution for equations (2.4)-(2.10) subject to the boundary conditions (2.13). The first is that in which the perturbation variables depend only upon the polar variable \( r \). An inspection of the appropriate equations and boundary conditions readily indicates that no such non-trivial solution is possible. This result contrasts with that for Couette flow, where the Pieranski-Guyon homogeneous instability [16] is observable.

Secondly we consider solutions which depend only upon \( r \) and \( \phi \) (see Fig. 1a). Neglecting the gravitational term in equation (2.6) (1), one observes that the system uncouples into two distinct sets of equations. One set consists of equations which determine the functions \( v_r, v_\phi, \) and \( \theta \) and the other of equations which yield expressions for \( v_\theta, n_1, \) and \( n_2 \). However since this uncoupling essentially reduces the problem to one which is very similar to the corresponding problem in the theory of isotropic fluids, we do not pursue this solution any further in the present paper but prefer to investigate effects which are peculiar to the physical structure of nematic materials.

The remainder of this section is concerned with the third type of solution in which the perturbation variables are functions of \( r \) and \( z \) only (see Fig. 1b). Employing a normal mode analysis, we therefore seek solutions of the form

\[
(v_r, v_\phi, v_z) = (v_1, v_2, v_3) \exp ilz, \quad (n_r, n_\phi) = (n_1, n_2) \exp ilz, \quad \theta = \theta \exp ilz,
\] (4.1)

where \( v_1, v_2, v_3, n_1, n_2, \theta \) are functions of \( r \) alone.

Eliminating \( \phi \) and \( v_3 \) from the equations results in the reduced set

\[
\eta_\phi \frac{d^4 v_1}{dr^4} - (2 \alpha_1 + \eta_\phi + \eta_\beta) l^2 \frac{d^2 v_1}{dr^2} + \eta_\phi l^4 v_1 - 4 \rho \omega_0 l^2 v_2 + 2 \rho r_m \omega_0^2 l^2 \theta - 2 i \rho g \theta \frac{d \theta}{dr} = 0,
\] (4.2)

(1) Walowit, Tsao and Di Prima [17] suggest that this term has a negligible effect in isotropic liquids when a shear flow is also present. Of course, if the experiment is carried out in a zero gravitational field this term vanishes.
where
\[ \eta_a = \alpha_a + \alpha_3 + \alpha_6, \quad \eta_b = \alpha_4 + \alpha_5 - \alpha_2. \]

For convenience, one now introduces the following scales:

- time: \( \tau = (\gamma_1 + \gamma_2) d^2/k_3 \); length: \( L = d \); temperature: \( \theta = \phi_m d k_3 / k_1 \).

Hence one may rewrite equations (4.2)-(4.6) in terms of non-dimensional variables \( v', n, \) and \( \theta \) in the form

\[
(D^4 - \eta_a a^2 D^2 + \eta_2 a^4) v'_i - Y_1 a^2 v'_2 + R \theta - i R' D \theta = 0, \\
(D^2 - \eta_a a^2) v'_2 - Y_2 a^2 v'_1 = 0, \\
(D^2 + a^2) v'_1 + A (D^2 - k'a^2) n_1 = 0, \\
\eta_4 a v'_2 + (D^2 - k'a^2) n_2 = 0, \\
(D^2 - k'a^2) \theta = A n_1 + k' v'_1 = 0,
\]

where
\[ r = R_1 + xd, \quad \frac{d}{dr} = \frac{1}{d} D, \quad l = a/d, \quad v'_i = k_3 v_i/(\gamma_1 + \gamma_2) d, \quad i = 1, 2. \]

In addition
\[ \eta_1 = (2 \alpha_1 + \alpha_3 + \alpha_6)/\eta_a; \quad \eta_2 = \eta_5/\eta_a; \quad Y_1 = 4 \rho d^2 \omega_0/\eta_a; \quad Y_2 = \eta_a Y_1/\alpha_4; \]
\[ R = \rho \omega_0 \alpha_5 \phi_m \kappa_2 (\gamma_1 + \gamma_2) d^4/\eta_a k_1 \kappa_1; \quad R' = R g/\rho \omega_0 d^2; \quad \eta_3 = \eta_5/\alpha_4; \]
\[ \eta_4 = (\gamma_2 - \gamma_1) k_1; \quad k = k_3 + \frac{\kappa_3}{k_2} + \frac{\eta_4 \kappa_3}{k_2 a}; \quad \kappa' = \frac{\kappa_3}{k_2} + \frac{\eta_4 \kappa_3}{k_2 a}; \quad \gamma_i = (\kappa_1 + \kappa_2)/\kappa_1; \]
\[ \kappa' = \kappa_1/\kappa_2 (\gamma_1 + \gamma_2). \]

Here we note that \( R \) is a non-dimensional parameter akin to the Rayleigh number. If one neglects the gravitational term which contains \( R' \), there are several interesting observations that are in agreement with the qualitative model of section 3. In the event that \( H \) is zero and the effect of the Coriolis terms \( Y_1 \) and \( Y_2 \) is relatively small, \( \phi_m a^2 \) is almost a constant. Again if \( H \) is zero and \( \omega_0 d^2 \) is fixed then \( \phi_m a^2 \) is a universal function of \( a \). Finally if \( \kappa_3 \) is positive, the magnetic torque obviously strengthens the elastic torque and hence acts as a stabilizing force.

We now proceed to solve the set of equations (4.8)-(4.12) subject to the boundary conditions
\[ v'_i = v'_2 = n_1 = n_2 = \theta = 0, \]

on the boundaries \( x = 0 \) and \( x = 1 \) by two distinct methods. The first utilizes a first order Galerkin method to obtain approximate results and hence a qualitative description of the onset of instability. The second employs an orthonormalization procedure to obtain accurate numerical results.

4.1 THE GALERKIN METHOD. — Employing a complete set of trial functions, one first writes expansion representations for the functions \( v_1, v_2, n_1, n_2, \) and \( \theta \) as is described by Finlayson [18] and Velarde and Zuniga [1]. It is then possible to determine the qualitative behaviour of the mean thermal gradient \( \phi_m \) as a function of the wavenumber \( a \) by keeping only the first terms in the expansions. Although there are several different sets of trial functions available, we employ the polynomial expansions
\[ v'_i = \sum_{n=1}^{\infty} A_n u_n(x), \quad n_1 = \sum_{n=1}^{\infty} B_n n_n(x), \]
\[ \theta = \sum_{n=1}^{\infty} C_n \theta_n(x), \]

where
where
\[ u_n = \theta_n = (1 - x)^{2n} x^{2n}, \quad (4.17) \]
are chosen so as to satisfy the boundary conditions.

Substitution of these expansions into equations (4.8)-(4.12), with the requirement that the residuals are orthogonal to the trial functions, leads to the set of linear algebraic equations

\[ \begin{align*}
\beta_1 & = \langle v_1, (D^4 - \eta_1 a^2 D^2 - \eta_2 a^4) v_1 \rangle, \\
\beta_2 & = -a^2 \langle v_1, v_2 \rangle, \\
\beta_3 & = \langle v_1, \theta \rangle + iaR' \langle v_1, D\theta \rangle, \\
\beta_4 & = -\frac{\eta_2}{\eta_4} \langle v_2, v_1 \rangle, \\
\beta_5 & = -\langle v_2, (D^2 - \eta_3 a^2) v_2 \rangle, \\
\beta_6 & = \langle n_1, (D^2 + a^2) v_1 \rangle, \\
\beta_7 & = a \langle n_1, (D^2 - ka^2) n_1 \rangle, \\
\beta_8 & = \eta_4 a \langle n_2, v_2 \rangle, \\
\beta_9 & = \langle n_2, (D^2 - k' a^2) n_2 \rangle, \\
\beta_{10} & = k' \langle \theta, v_1 \rangle, \\
\beta_{11} & = -a \langle \theta, n_1 \rangle, \\
\beta_{12} & = \langle \theta, (D^2 - ka^2) \theta \rangle.
\end{align*} \]

Here

\[ \langle f_1, f_2 \rangle = \int_0^1 f_1 f_2 \, dx, \quad (4.20) \]

and \( A_1, A_2, B_1, B_2, C_1 \) are the coefficients of the first terms in the expansions (4.16). For the above choice of trial functions, one notes that all integrals involving odd derivatives must vanish which means that the gravitational term does not contribute to the first order terms.

Since the set of equations (4.18) has a non-trivial solution if and only if the determinant of the matrix of coefficients on the left hand side of the equations vanishes, it follows that

\[ R = \frac{\beta_2 \beta_{12} (\beta_3 \beta_4 Y_1^2 + \beta_1 \beta_3)}{\beta_3 \beta_5 (\beta_6 \beta_{11} - \beta_7 \beta_{10})} = \frac{f_1 Y_1^2 + f_2}{f_3}, \quad (4.21) \]

where \( f_1, f_2, f_3 \) are functions of \( a, \theta, M, H \) as well as the material constants. Without resorting to a computer, it is now an easy calculation to obtain the neutral stability curve.

4.2 Orthogonalization. — One first reformulates the problem as that of solving the set of first order linear differential equations

\[ Df = Lf, \quad (4.22) \]

subject to the boundary conditions

\[ Bf(0) = 0, \quad Bf(1) = 0, \quad (4.23) \]

where \( L \) is a 12 \times 12 matrix, \( f \) is the twelve component vector function \( [v_1', v_2', n_1, n_2, \theta, Dv_1', Dv_2', Dn_1, Dn_2, D\theta, D^2 v_1', D^3 v_1'] \) and \( B \) is the 6 \times 12 matrix \([I, Q_6] \). Here \( I \) is the 6 \times 6 unit matrix and \( Q_6 \) is the 6 \times 6 zero matrix. To obtain accurate results the equations are integrated directly using an orthonormalization procedure and further details are given by Conte [19] and Barratt and Sloan [20].

In evaluating numerical results, this paper uses experimental data obtained for the nematic material MBBA. Assuming the Parodi relation [21], the viscosities are taken to be those reported by Gahwiller [22]. We use the same values as Dubois-Violette [3] for the parameters \( \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \kappa_7, \kappa_8, \kappa_9, \kappa_{10}, \kappa_{11}, \kappa_{12} \) and \( k_3 \) and take \( k_2 = 3 \times 10^{-12} \) dynes (Haller [23]), \( \chi_a = 1.23 \times 10^{-7} \) and \( \rho g a = 1 \) g cm\(^{-2} \) K\(^{-1} \). For these values of the material parameters, one specifies values for \( \omega_0, \theta, M, \) and \( a \) and then evaluates that value of \( \phi_m \) with smallest magnitude for which a non-trivial solution exists. As \( a \) varies, these values of \( \phi_m \) describe the neutral stability curve in the \((a, \phi_m)\) plane and the critical gradient threshold is that having the smallest modulus on this curve.

In figure 2, we give a comparison of the results obtained by both methods when \( H = 0, Y_1 = 3 \) and
Table 1. — Values of the critical thresholds $R^+$, $R^-$, and the corresponding non-dimensional wavenumbers $a^+$, $a^-$ calculated by a first order Galerkin method and by an orthonormalization procedure. $Y_1 = 3$, $r_m = 2$ cm and $d = 0.02$ cm.

<table>
<thead>
<tr>
<th></th>
<th>Galerkin</th>
<th>Orthonormalization</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^+$</td>
<td>3.2</td>
<td>3.0</td>
<td>7%</td>
</tr>
<tr>
<td>$R^+$</td>
<td>167.0</td>
<td>142.2</td>
<td>17%</td>
</tr>
<tr>
<td>$a^-$</td>
<td>0.29</td>
<td>0.26</td>
<td>12%</td>
</tr>
<tr>
<td>$R^-$</td>
<td>$-1.50 \times 10^5$</td>
<td>$1.61 \times 10^5$</td>
<td>7%</td>
</tr>
</tbody>
</table>

$r_m = 2$ cm. The broken line denotes the approximate solution obtained from the first order Galerkin method and the solid line represents the accurate numerical results found using the orthonormalization method. As is seen in table I, the approximate solution for the critical temperature gradient deviates from the true value by less than 17%, which indicates that the Galerkin approach does give a good qualitative description of the instability. An important use of the Galerkin method is that it readily yields a solution in the near vicinity of the true solution which is a helpful guide in finding an accurate numerical solution by means of the orthonormalization procedure.

5. Results and discussion. — As is seen in figure 2 two instabilities are possible. One occurs when the outer cylinder is heated and the other when the inner cylinder is heated. Taking $Y_1 = 3$ and $r_m = 2$ cm and heating from the outside, one finds that the sample becomes unstable as $R$ exceeds 142.2. For a gap of 0.075 cm the corresponding critical temperature gradient is $\phi_m = 9.6$ °C/cm. Since the angular velocity $\omega_0 = 66.13$ rad/s, the buoyancy force $\alpha^2 \omega_0 \phi_m = 8.39 \times 10^4$. This result agrees well with the threshold for Benard convection in a planar configuration with heating from below. In that case Dubois-Violette [3] and Barratt and Sloan [7] find, for the same sample thickness, that $\phi_m = 85.3$ and hence $g \phi_m = 8.36 \times 10^4$. As is discussed later, the difference between the two thresholds is due to the slightly stabilizing effect of the Coriolis force. We also note that the direction of the rolls in both cases is perpendicular to the initial molecular orientation and the critical wavenumber is $a = 3.0$.

The anisotropic mechanism which drives the instability is as described by Dubois-Violette [2, 3] for the corresponding Rayleigh-Bénard problem. In a typical nematic liquid crystal the heat conductivity along the molecules is about twice as large as that in a perpendicular direction and thus the heat fluxes are directed along the molecules. The coupling between the flow and molecular orientation is such that convective motions induce a molecular distortion. If this distortion of the director is produced in such a manner that the heat is focussed into the initially warmer region the system is strongly destabilizing and vice versa. The coupling between flow and molecular orientation is due to the viscous torques

$$I_2 = a_3 \frac{\partial v_r}{\partial r} \quad \text{and} \quad I_1 = a_2 \frac{\partial v_z}{\partial z}. \quad (5.1)$$

In the event of circular rolls, $\partial v_r/\partial r$ and $\partial v_z/\partial z$ are of the same order of magnitude and hence it follows that $| I_2 | \ll | I_1 |$, since $| a_2 |$ is approximately $| a_3 | \times 10^2$ in nematics. This explains the occurrence of plane Benard convective instabilities when a planar sample is heated from below and a homeotropic sample is heated from above. However this situation can be inverted for very small wave numbers, where $| \partial v_z/\partial r |$ becomes very much greater than $| \partial v_z/\partial z |$. Velarde and Zuniga [1] first found this mechanism responsible for driving the gravitational instability when a planar sample is heated from above and it is essentially the same mechanism which drives the instability when the annulus is heated from the inside. This type of instability corresponds to the negative part in figure 2 and the critical threshold is $R_c = 1.613 \times 10^3$ with the corresponding wave number $a = 0.29$. Thus the magnitude of this threshold is $10^3$ times greater than that found when heating is from the outside. Since the associated temperature difference required across the sample is so high, one can hardly expect to observe this instability in a planar sample heated from above. However in the cylindrical geometry one can increase the centrifugal buoyancy force and therefore reduce the size of temperature gradient necessary for the onset of instability.

Fig. 3. — Neutral stability curve when $a_3$ is positive ($a_3 = | a_3 |$). The small wavenumber instability mode is no longer possible. The unbroken line corresponds to the results obtained by orthonormalization and the broken line to the results obtained by Galerkin.
In figure 3 we illustrate the neutral stability curve for a nematic whose material parameters are identical with those of MBBA but with $a_3 = |a_3|$. In this case one observes that the small wavenumber instability is no longer possible. The reason for this is that the torques $T_1$ and $T_2$ are no longer in competition but rather strengthen each other. Hence the system is stable to heating from the inside while the threshold for instability is lower when heating is from the outside than that found for $a_3 < 0$.

The relation (4.21) enables us to distinguish between three different regimes. A very slow rotation regime where the convection has the same properties as in the Rayleigh-Bénard problem. A second regime for comparatively high rotations where the Coriolis force has an important stabilizing effect, thus increasing the instability threshold. The third regime is for very high angular velocities where one might expect a transition from the Taylor-Couette to the Taylor-Proudman roll instability.

When the angular velocity is small the Coriolis forces $\beta_2 Y_1$ and $\beta_4 Y_1$ are much smaller than the corresponding viscous force component $\beta_1 \beta_5$. In this case (4.21) reduces to

$$B = f_2/f_3 \ Y_1^2, \quad (5.2)$$

where $R \equiv BY_1^2$, with the result that the centrifugal buoyancy force is equivalent to the gravitational buoyancy force in the corresponding Rayleigh-Bénard problem, except for the influence of the cylindrical geometry. In the present problem there are two particularly distinct arrangements which are analogous to the planar configuration between parallel plates. In one the director alignment is horizontal and in the azimuthal direction and in the other, which is the case under investigation here, it is parallel to the axis of rotation. It is thus the geometry that gives rise to the Taylor-Couette instability studied in this paper and the rolls develop in such a way that they are perpendicular to the director. At this point we note that in the corresponding problem for isotropic liquids the rolls are aligned with their axis parallel to the rotation axis as is demonstrated by Busse and Carrigan [24]. This also appears to be the case in the experiment of Carrigan and Guyon [8] with nematics when the molecular alignment is horizontal and the convective rolls are observed to be vertical.

Increasing the angular velocity means that the effect of the Coriolis terms becomes more significant and the threshold is given by the full expression (4.21). The wavenumber increases with $\omega_0$, and hence makes the heat transport between the two cylinders more efficient and consequently the system is more stable. This stabilizing effect of the Coriolis force on the instability threshold resulting from heating the outer cylinder is illustrated by the broken line in figure 4. The unbroken line in figure 4 shows the corresponding effect when heating is applied to the inside. However in this case the effect is much smaller because the rolls are now largely elongated along the $z$-direction and the associated Coriolis force is greatly reduced. Hence it is possible that this mode could be observed even when comparatively high angular velocities are employed.

The effect of a magnetic field applied parallel to the molecular alignment has also been studied and the results are shown in figure 5. One observes that the results are very similar to those found in the planar configuration for the corresponding Rayleigh-Bénard problem by Guyon and Pieranski [4], Pieranski, Dubois-Violette and Guyon [5] and Velarde and Zuniga [1]. They show that for small field strengths the critical temperature difference $\Delta T_c$ satisfies the empirical law

$$\Delta T_c(H) = \Delta T_c(H = 0) [1 + (H/H_0)^2], \quad (5.3)$$

Fig. 4. — Variation of the Rayleigh number with rotation. The dashed line represents the threshold when heating is from the outside and the continuous line represents the threshold when heating is from the inside.

Fig. 5. — Variation of the threshold with the magnetic field strength. The heavy line describes results when heating is from the outside and the broken line describes results when heating is from the inside. Here $\Delta R = \{ R(H) - R(H = 0) \}/R(H = 0)$. 

$\text{Fig. } 4$. Variation of the Rayleigh number with rotation. The dashed line represents the threshold when heating is from the outside and the continuous line represents the threshold when heating is from the inside.

$\text{Fig. } 5$. Variation of the threshold with the magnetic field strength. The heavy line describes results when heating is from the outside and the broken line describes results when heating is from the inside. Here $\Delta R = \{ R(H) - R(H = 0) \}/R(H = 0)$. 

$\text{Fig. 5}$. Variation of the threshold with the magnetic field strength. The heavy line describes results when heating is from the outside and the broken line describes results when heating is from the inside. Here $\Delta R = \{ R(H) - R(H = 0) \}/R(H = 0)$.
where $H_0$ is approximately 20 G, for a sample thickness of 5 mm, irrespective of the sign of the thermal gradient. Figure 5 indicates that our results are in good agreement with this relationship. As is shown in figure 5 the effect of the magnetic field is greater on the small wavenumber instability than on the other, the reason for this being as follows. The magnetic field exerts a torque on the director which always opposes any director perturbation, since it attempts to force the molecules to align parallel to the initial orientation. Thus when heating is from the inside the magnetic torque acts so as to increase the effect of $\Gamma_1$, but when heating is from the outside it strengthens the viscous torque $\Gamma_2$. Obviously this stabilizing effect is stronger when heating is from the inside than it is when heating is from the outside, since $|\Gamma_1| \lesssim |\Gamma_2|$ in the former case while $|\Gamma_1| \gg |\Gamma_2|$ in the latter. For large enough magnetic fields the induced torque eventually overcomes the viscous torques. In this event the small wavenumber instability disappears while the threshold for the circular roll instability becomes field-independent and tends to the corresponding values for isotropic liquids as found in the planar case by Velarde and Zuniga [1].

For very high angular velocities, the expression (4.21) approximates $\mathbf{B} = f_1 f_3$, and so $B$ becomes independent of $Y_1$ and would lead to a convective instability with very small wavelength. However this striking property is far away from laboratory conditions and it is more likely that a z-independent mode has a lower threshold magnitude than the axisymmetric mode, as a consequence of the Taylor-Proudman theorem for isotropic liquids.

We now compare our theoretical results with the observations of an experiment performed in this cylindrical geometry by Carrigan and Guyon [8]. However is must be noted that their preliminary experiments were more concerned with the qualitative effect of the Coriolis force on the onset of convective instabilities in two samples with different initial orientation patterns rather than with a quantitative study of its effect on the values of the thresholds. They examined the situations in which the director is initially uniformly aligned in either the azimuthal or vertical directions in the same experiment by having an azimuthal orientation in the upper part of the apparatus and a vertical orientation in the lower part, the upper and lower parts being independent compartments. Fixing the temperature gradient, they find that the first instability motions are initiated in the upper compartment in the form of axial rolls. For this configuration, experiments using different temperature gradients, but a constant mean temperature, indicate that the critical temperature difference $\Delta T$ varies as $\omega_0^{-2}$. This corresponds to the situation where the Coriolis effect is negligible and is in agreement with the corresponding result for the gravitational instability in the plane Rayleigh-Bénard problem. In the lower region they found a loosely defined threshold value for the onset of poorly defined axial rolls when $\omega_0 \approx 65 s^{-1}$ and $\Delta T = 3^\circ C$ and this gives a threshold value of about four times that corresponding to the gravitational instability in the Rayleigh-Bénard problem. Upon decreasing the rotation rate the threshold required for the complete disappearance of the rolls is observed to be the much smaller value of $\omega_0 = 35 s^{-1}$, thus exhibiting a hysteretic phenomenon.

We obtain the result that at the onset of instability $\omega_0^2 r_m \Phi_m = 0.90 \times 10^7$ which, for $\Delta T = 3^\circ C$ and the dimensions of the Carrigan and Guyon experiment, yields a critical angular velocity of 34 s$^{-1}$. This is obviously a much lower value than their observations and is in fact almost the threshold value for the complete disappearance of the rolls. Of course the temperature at which the experiment was performed could be of significance here. However our result for the threshold magnitude agrees well with the results for the plane Rayleigh-Bénard problem as well as the observations in the azimuthal configuration.

At first sight the theoretical results presented here appear to be in disagreement with the experimental observations, since we conclude that for comparatively small angular velocities the first unstable mode is the azimuthal one. However a possible explanation of this apparent conflict between theory and experiment is the following. For isotropic liquids it is well known both theoretically and experimentally [24] that the axis of the convection rolls align parallel to the axis of rotation whenever the Coriolis force is important, this being a consequence of the Taylor-Proudman theorem. On the other hand for a very tall annulus when the effect of the Coriolis force is negligible, Busse [25] remarks that the problem becomes mathematically identical to the plane Bénard problem with the centrifugal force replacing gravity. Thus it is possible in our case that the competition between the alignment induced by rotation and the natural azimuthal mode induced by molecular orientation gives the non-uniform structure observed in the experiment. That is although the azimuthal mode is the first unstable mode whenever the Coriolis effect is sufficiently small, there is a certain angular velocity at which there is a transition to the axial roll mode. It would therefore appear to be possible that Carrigan and Guyon's experiment was performed under such conditions that they could not observe the azimuthal threshold. This could be due to effects such as the ratio of the gap width to height of the cylinder, the threshold magnitude of the angular velocity required for the onset of instability, or even non-linear terms that have been neglected in our analysis. Our calculations therefore ask for a check employing a smaller critical angular velocity which may be achieved by using wider cylinders and greater temperature gradients to see if the azimuthal mode is observable,
provided the effect of the Coriolis force is small enough. An exact analysis of the transition between the azimuthal rolls mode and the axial mode is an interesting non-linear problem beyond the modest scope of this paper.

6. Concluding remarks. — We have shown that the effect of the centrifugal buoyancy force is equivalent to that of the gravitational buoyancy force in the corresponding Rayleigh-Bénard problem for comparatively small angular velocities. Thermal convective instabilities are possible when heating is from both the inside and the outside of the annulus and in both cases the first unstable mode is a Taylor-Couette roll instability. When heating is from the inside the rolls are elongated and the cylindrical geometry presents the possibility of observing this abnormal instability, since one may increase the centrifugal force and so reduce the required critical thermal gradient to a realizable value. When heating is from the outside the azimuthal rolls are circular and should, under suitable conditions, occur at a lower threshold than the axial rolls observed by Carrigan and Guyon [8]. It therefore seems a worthwhile exercise to perform their experiment when a uniform vertical alignment obtains throughout the sample to check if our predictions are observable under favourable conditions.

As regards the methods employed in solving the problem, the benefits of the first order Galerkin method should be noted. In the first place it yields a good qualitative description of the instability phenomena by means of a very simple calculation. Secondly it gives reasonably good quantitative results that are of considerable help in obtaining accurate numerical solutions by the orthonormalization procedure.

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Appendix. — Employing Cartesian tensor notation, the Ericksen-Leslie equations are the balance laws

$$
\rho \dot{v}_i = - p_i - \left( \frac{\partial W}{\partial n_{k,j}} \right) n_{k,j} + \tilde{t}_{ij} + F_i,
$$

(A.1)

$$
\sigma \tilde{n}_i = \gamma n_i + \left( \frac{\partial W}{\partial \tilde{n}_{l,j}} \right)_{,j} - \frac{\partial W}{\partial \tilde{n}_i} + \tilde{g}_i + G_i,
$$

(A.2)

$$
T \dot{S} = r - q_{i,i} + \tilde{t}_{ij} A_{ij} - \tilde{g}_i N_i, \quad S = - \frac{\partial W}{\partial T},
$$

(A.3)

with the constraints

$$
v_{i,i} = 0, \quad n_i n_i = 1,
$$

(A.4)

where

$$
2 W = 2 W_0 + k_2 n_{i,i} n_{j,j} + (k_1 - k_2 - k_4) n_{i,i} n_{j,j} + k_4 n_{i,i} n_{j,j} + (k_3 - k_2) n_i n_j n_k n_l n_{k,l} n_{l,k},
$$

$$
\tilde{t}_{ij} = \alpha_1 n_p A_{kp} n_i n_j + \alpha_2 n_j N_i + \alpha_3 n_i N_j + \alpha_4 A_{ij} + \alpha_5 n_j n_k A_{ki} + \alpha_6 n_i n_k A_{kj},
$$

$$
\tilde{g}_i = - \gamma_1 N_i - \gamma_2 A_{ik} n_k, \quad q_i = - \kappa_1 T_{,i} - \kappa_2 n_i n_k T_{,k},
$$

$$
2 A_{ij} = v_{i,j} + v_{j,i}, \quad 2 N_i = 2 \tilde{n}_i + (v_{i,i} - v_{i,k}) n_k
$$

(A.5)

and a superposed dot denotes a material time derivative. Assuming that external body forces only arise from an applied magnetic field and gravity, the estimates proposed by Ericksen [26] yield

$$
F = - \rho \gamma e, \quad G = \chi_e (H, n) H
$$

(A.6)

where $\chi_e$ is the coefficient denoting the anisotropic part of the magnetic susceptibility and $e$ is a unit vector in the vertical direction.

References


