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Instabilities in smectics A submitted to an alternative shear flow

I. — Theory

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Résumé. — On présente une analyse théorique de l'instabilité qui se produit dans un smectique A lorsqu'on le soumet à un écoulement de cisaillement alternatif (vitesse perpendiculaire aux couches). Cet écoulement entraîne un mouvement d'inclinaison alternatif des couches, et, au-dessus d'une amplitude critique \( \theta_0_c \), une instabilité se produit et entraîne la formation d'un réseau quasi périodique de coniques focales.

Cette analyse met en évidence deux régimes différents, à basse fréquence et à plus haute fréquence. A basse fréquence, l'angle de tilt critique varie en \( \sqrt{\omega} \). A plus haute fréquence, la variation est en \( \omega^{-3/2} \). On montre également que l'instabilité prend naissance au cœur de l'échantillon, et non près des parois (comme dans le cas statique).

Abstract. — We give a theoretical analysis of the instabilities which occur in a smectic A sample, when submitted to an a.c. shear flow with velocity normal to the layers. This shear flow induces an a.c. tilt of the layers and, above a critical amplitude \( \theta_0_c \), an instability arises and results in the formation of a quasi-periodic network of focal conics.

A result of this analysis is that one can find two different regimes, for lower and higher frequencies. In the low-frequency regime the threshold tilt angle \( \theta_0_c \) depends on frequency as \( \sqrt{\omega} \). At higher frequencies, one gets a \( \omega^{-3/2} \) dependence. It is also shown that this instability arises in the bulk of the sample, and not near the upper and lower plates as in the static case.

1. Introduction. — The effects of a shear flow in a smectic A sample depend strongly on the geometry of the situation. Three basic geometries can be achieved:

   a) velocity and shear are parallel to the layers. The layers are not involved in the resulting process, and the smectic sample behaves like a plain liquid. The situation is in fact a little more complex since a Guyon-Piersanski instability can arise as in nematics. Anyway, the smectic order does not play any rôle;

   b) velocity is parallel to the layers and the shear wave-vector is normal to the layer. This is the situation studied by Kleman and Horn [1]. At first glance, it seems that nothing should occur since the layers can glide very easily over each other [2]. In fact, for high shears, an instability occurs, but up to now the attempts made by several groups to analyse this situation have been unsuccessful;

   c) in the third situation, velocity is normal to the layers. This is the case if one places a smectic sample in a planar configuration (obtained, for example, by oblique SiO coating) between two plates, and if one moves one of the plates, keeping the other one fixed, parallel to the director alignment. This is the situation that we shall now consider.

   The first evidence for an instability in this configuration was obtained by C. E. Williams [3] under rather crude experimental conditions. The experiment was then resumed by the Montpellier group with a more sophisticated apparatus. Preliminary results were
announced at Madonna di Campiglio \cite{4} and more
detailed ones at Bordeaux \cite{5}. The essential result is
that, above the instability threshold, the applied
shear is relaxed by the motion of a double Grandjean
wall moving up and down in the bulk.

More recent experiments are concerned with the
instability threshold in a.c. experiments. The first
result is that, at given frequency and temperature, this
threshold is characterized by a critical amplitude of
the «tilt angle» \(\theta_{oc}\) (displacement of the plate over
the sample width). The second result is that, at a
given temperature, \(\theta_{oc}\) depends only on frequency.

The aim of this paper is to give a theoretical ana-
lysis of the stability of layers under such a shear flow.
We start from the basic equation for elasticity and
hydrodynamics of smectics. As the direction of the
layers changes through the process, it is useful to start
from a covariant form of these equations. Such a
covariant form has already been given for elasticity
by Kleman and Parodi \cite{6} (K. P.). The resulting fund-
amental equations for hydrodynamics are derived
in appendix A. Starting from these fundamental equa-
tions, we derive a set of non-linear hydrodynamic
equations corresponding to our geometry. We then
show that orientational boundary conditions are
relaxed along a curvature boundary layer.

In the third section we study steady-state solutions
for Poiseuille and linear Couette flows. Such an ana-
lysis has already been performed on Poiseuille flow
by the Orsay group \cite{7} by another method, and our
analysis confirms their results. For Couette flow we
show that the steady-state solution is the trivial one.

The fourth section is devoted to the analysis of a.c.
Couette flow. An analytic solution appears to be
impracticable. We derive the stability domain in the
\((N, M)\) plane from a computer analysis and attempt
to give a physical explanation of the results.

2. Fundamental equations in planar geometry. —

Our geometry is shown in figure 1: a smectic sample
is placed between two plates and a strong surface
anchorage forces the director to be parallel to the
plate (along the \(X\)-axis). The smectic layers are parallel
to the \(Z-Y\) plane. A velocity is produced in the \(X\)-direc-
tion, either by a pressure gradient (Poiseuille flow) or
by moving the two plates with opposite velocities.
The sample is supposed to be infinite along the \(X\)-
and \(Y\)-directions (translational symmetry), and the smectic
to be incompressible.

\[
\sigma_{ij} = -p\delta_{ij} + B_0(1 - \varepsilon)n_i n_j - K_1 \text{div } n n_{ij} + K_1 n_j P_{ik} V_k \text{div } n
\]
\[\div S = -\text{div } S\]
\[S_i = -B n_i - \frac{K_1}{1 - \varepsilon} P_{ij} V_j \text{div } n\]
\[P_{ij} = \delta_{ij} - n_i n_j\]
\[\sigma'_{ij} = \sigma_i (n_i n_j n_k n_l A_{kl} + \alpha_A A_{ij} + \alpha_{56} (n_i n_k A_{kj} + A_{ik} n_k n_j))\]
\[A_{ij} = \frac{1}{2}(V_{ij} + V_{ji})\]

For symmetry reasons one has \(V_z = 0\) and \(V_{xz} = V_{x,z}(z)\). Using now the incompressibility condition
\[
\text{div } V = 0, \]
one gets
\[
V_z = 0 \quad V_x = V(z). \quad (2.1)
\]

Several authors have shown that the natural variable
for describing smectic layers is the «phase» \(\phi\) \cite{8}.
Layers are isophase surfaces and, in the minimum
energy configuration (i.e. for a completely aligned
sample with equilibrium lattice parameter), the direc-
tor \(n = \nabla \phi\). For such a sample, \(\phi = \pi\). More gene-
 rally, one has \cite{6}
\[
\nabla \phi = (1 - \varepsilon)n, \quad (2.2)
\]
where \(\varepsilon\) is the relative dilatation of the layers.

Hydrodynamic equations for smectics have been
developed by many authors for small deviations of the
aligned configuration. The M.P.P. formalism \cite{9}
can easily be extended (see appendix A) and a covariant
version is:
\[
\frac{D\phi}{Dt} = -\zeta g + \frac{\varepsilon}{T} n \cdot VT
\]
\[
J_s = \rho_s V - \frac{1}{T} [K_1 \bar{\sigma} + (K_|| - K_z) \mathbf{nm}] \times
\]
\[
\frac{\partial V}{\partial t} = -\text{div } [\sigma VV - \bar{\sigma} - \bar{\sigma}']
\]
where \(\bar{\sigma}\) is the unit second rank tensor and \(\mathbf{nn}\) a dyadic.
\(g, \bar{\sigma}\) and \(\bar{\sigma}'\) have been defined in K.P. and M.P.P. \footnote{1},
\(J_s\) is the entropy flux and \(\rho_s V\) the convective part of
this flux,

\footnote{1 We use below the well-known notation : \(a_{ij} = \partial a_i/\partial x_j\)}
This set of equations simplifies very much for small deviations from the planar structure. Following de Gennes [10] we call \( u \) the \( X \)-displacement of layers and then

\[
\phi = x - u(z) \tag{2.5}
\]

\( \mathbf{n} \) has components \( \left(1 - \frac{u'^2}{2}, 0, -u' \right) \); the angle \( \theta \) between \( \mathbf{n} \) and the \( X \)-axis is

\[ \theta = -u' \]

when \( u' \) denotes the \( z \)-derivative of \( u \).

The relative dilatation of layers is

\[ \epsilon = -\frac{u'^2}{2} \tag{2.6} \]

and a straightforward calculation gives the following results:

\[
S_z = -\frac{B}{2} u^3 + K_1 u''
\]

\[
g = \frac{3}{2} Bu'^2 u'' - K_1 u'''
\]

\[
\sigma_{xx} = -p + f(z)
\]

\[
\sigma_{zz} = B u'^3 - K_1 u'
\]

\[
\sigma_{zz} = -p - \frac{Bu'^4}{2} + K_1 (u'' - u'')
\]

In these equations we have kept only the higher-order terms in layer compression or curvature. The terms involving compression of layers are of higher-order in \( u \) than the curvature terms. However, since the ratio of first-order and second-order elastic constants, \( B/K \), is of order \( 10^{14} \) cm\(^{-2} \) compression terms must be kept. In fact, as will be shown below, these terms appear as dominant except in curvature boundary layers.

The viscous stress tensor has the following non-vanishing components

\[
\sigma'_{xx} = \bar{\eta} V'
\]

\[
\sigma'_{zz} = -\alpha_{56} \theta V'
\]

with

\[ \bar{\eta} = \frac{1}{2}(\alpha_4 + \alpha_{56}). \]

Finally, neglecting the coupling with thermal diffusion, one obtains

\[
V = \frac{\partial u}{\partial t} - \frac{3}{2} Bu'^2 u'' - K_1 u''' \tag{2.8a}
\]

\[
\rho \frac{\partial V}{\partial t} = -\frac{\partial p}{\partial x} + \frac{3}{2} B u'^2 u'' - K_1 u'''' + \bar{\eta} V'' \tag{2.8b}
\]

\[
0 = -\frac{\partial p}{\partial z} - 2 Bu'^3 u' + K_1(u'' - u') - \alpha_{56} (u'' V' + u' V'').
\]

Differentiating equations (2.8b, c), one gets

\[
\frac{\partial^2 p}{\partial x \partial z} = 0 \quad \frac{\partial^2 p}{\partial x^2} = 0. \tag{2.9}
\]

Therefore \( \frac{\partial p}{\partial x} = p' = \text{const.} \)

After some algebra, equations (2.8a, b) can be rewritten as

\[
V = \frac{\partial u}{\partial t} - \zeta \left(p' + \rho \frac{\partial V}{\partial t} + \bar{\eta} V''\right). \tag{2.10}
\]

We shall now show that, in this expression, the last two terms can be neglected for low-frequency experiments. At frequency \( \omega \), equation (2.10) becomes

\[
V \left(1 - \rho \omega \zeta - \frac{\bar{\eta} \zeta}{R^2}\right) = \frac{\partial u}{\partial t} - \zeta p', \tag{2.11}
\]

where \( R \) is a typical curvature radius. \( \zeta \) [11] is of order \( \lambda^2/\eta \) where \( \lambda = (K/B)^{1/2} \) is the curvature penetration depth, of order \( 10^{-7} \) cm. Then, with \( \eta \approx 10^{-11} \), \( \rho \approx 1 \), \( g = 10^{-13} \), one gets \( \zeta \rho \approx 10^{-13} \) and, for any practicable frequency, \( \zeta \rho \omega \ll 1 \). The same argument applies to the following term: \( \bar{\eta} \zeta/R^2 \) is of order \( \lambda^2/R^2 \) and must be much smaller than 1 anywhere except near the core of dislocations, where linear elasticity cannot be applied. Equation (2.10) can therefore be rewritten as

\[
V = \frac{\partial u}{\partial t} - \zeta p' \tag{2.12}
\]

which is the equation for permeation introduced by de Gennes and the Orsay group [12].

The second basic equation is now obtained from equations (2.8a, b) and (2.12):

\[
\rho \frac{\partial^2 u}{\partial t^2} = -p' + \left(\frac{3}{2} Bu'^2 u'' - K_1 u''\right) + \bar{\eta} \frac{\partial u''}{\partial t}. \tag{2.13}
\]

The three first terms in the r.h.s. of (2.13) correspond to the elastic force. In this sense, this equation is very similar to the Navier-Stokes equation. The last two terms are due to the effect of the permeation force on the elastic stress tensor.

It seems useful at this point to make some comments on our new set of fundamental equations, equations (2.12) and (2.13). At first glance it seems surprising that an expansion of \( u \) gradients up to the fourth-order should be used in the frame of a very simplified model (rigid-model for smectics — linear hydrodynamics). It would be questionable for simple fluids and seems more doubtful for smectics. This can be better understood if we remember that a smectic material looks very much like a solid. It is well known that, in a solid, elasticity leads to fourth-order differential equations. Thus the appearance of fourth-
order derivatives of the displacement in the elastic force is consistent with the solid-like behaviour of the smectic material along a direction normal to the layers.

Another questionable point is the use of linear hydrodynamic equations in a non-linear equation like (2.13). Non-linear hydrodynamics have recently been established by Brand and Pleiner [13]. But it must be pointed out that, in our equations, \( u \) and \( u' \) must not be considered as small. In so far as we are dealing with configurations which can be derived from the initial one \( (u = 0) \) by large distortions, the displacement \( u \) and the layer inclination \( \theta \) are not expected to remain small. On the other hand, the elastic force will essentially depend on the layer curvature \( u'' \) which is expected to remain small. In that sense, equation (2.13) is a linearized equation in \( u'' \) and its derivatives.

As will now be shown, this equation can be simplified in the bulk and reduced to a second-order differential equation in \( u \).

**Curvature Boundary Layers.** — In order to achieve a planar configuration one has to anchor smectic molecules parallel to the two plates at the sample boundaries \( (n_x = 1 \text{ for } z = 0 \text{ and } z = d) \). This forces the layers to be normal to these plates. The relative motion of the plates in its turn forces the layers to curve. One must have two regions of opposite curvature near the upper and lower plates.

We will now show that curvature is limited to two layers. These layers will be most efficient for relaxing stresses if they are located close to the surface. We thus arrive at a picture where the orientational boundary condition on the layers is relaxed in thin boundary layers, up and down the sample, while the bulk orientation is not sensitive to these boundary conditions.

We now look again at the two elastic terms in equation (2.13). Let \( \theta_0 \) be the inclination of layers out of the boundary layer \( (\theta = 0 \text{ on the surface}) \). Obviously \( |\theta_0| = |u'| \) and, if \( \theta \) varies from 0 to \( \theta_0 \) on a distance \( l \), \( u'' \) will be of order \( \theta_0/l \) and \( |u'''| \) of order \( |\theta_0||l|^3 \). The second elastic term will be much smaller than the first one except in a layer of thickness \( l \) such that

\[
\frac{3B}{2} \frac{\theta_0^3/l}{l^3} \simeq K_1 \theta_0/l^3
\]

or, dropping the 3/2 factor

\[
l = \lambda/\theta_0
\]

where \( \lambda \), the curvature penetration length, is of order of a molecular length.

For very small \( \theta \), \( l \) is of the order of the sample width and there is no boundary layer. But, for values of \( \theta \) as small as \( 10^{-3} \), one has \( l \approx 1 \mu \) which means that in most actual experimental situations the introduction of curvature boundary layers is valid, and the curvature term in equation (2.13) can be omitted in the bulk. Such a result has already been obtained by the Orsay group on the basis of a free energy analysis, the validity of which can be questionable in a dynamic problem. We think that our analysis gives a safer basis for the introduction of a curvature boundary layer.

The introduction of this boundary layer has two consequences:

i) the boundary condition on the layer orientation is now relaxed;

ii) equation (2.13) was fourth-order in the spatial coordinates. It had to be solved subject to boundary conditions for displacement and its first derivative. The omission of the curvature terms leads to a second-order equation

\[
\rho \frac{\partial^2 u}{\partial t^2} = -p' + \frac{3}{2} B u^2 u'' + \frac{\eta}{\mathcal{T}} \frac{\partial u'}{\partial t} \quad (2.14)
\]

which now has to be solved subject to boundary conditions on the displacement only. This point is therefore consistent with the first one.


3.1 Pure Permeation: Poiseuille Flow. — This case has also been treated by the Orsay group using another formalism [7]. The layers do not move:

\[
\frac{\partial u}{\partial t} = 0 .
\]

Hence, from equation (2.12)

\[
V = -\zeta p'.
\]

The shape of the layers is obtained from (2.14) : out of the curvature boundary layer, one has

\[
3u^2 u'' = 2p'/B .
\]

Taking again the geometry shown on figure 1, we have as boundary conditions

\[
u(\pm d/2) = 0
\]

and, by symmetry considerations

\[
\theta(z = 0) = 0 .
\]

The solution is

\[
u = \frac{3}{4} \left( \frac{2p'}{B} \right)^{1/3} \left[ z^{4/3} - \left( \frac{d}{2} \right)^{4/3} \right] . \quad (3.1)
\]

A similar result has been obtained by the Orsay group by minimization of the elastic free-energy, which is a standard method in the case of an equilibrium configuration but needs some justification in the case of a dynamical steady-state situation. This solution has been fully discussed by the Orsay group, particularly the non-physical meaning of the singularity of \( \theta \) for \( z = 0 \), which is due to the omission of the curvature terms.

The effect of a Poiseuille flow can now be summarized as follows:
i) a smooth curvature of the layers (out of the curvature boundary layers discussed in the last section) is due to the balance between viscous and first-order elastic forces;

ii) a permeation flow results, with a velocity \( V = -\bar{\zeta} \). Except in critical domains, near the N-A transition, \( \bar{\zeta}(\approx \lambda^2/\eta) \) is very small (\( \bar{\zeta} \approx 10^{-13} \) cgs) and, even with pressure gradients as high as \( 10^7 \) cgs, one gets velocities of the order of \( 10^{-2} \) \( \mu \)s. Their observation seems to be impossible.

### 3.2 Linear Couette Flow.

We are now looking for a steady-state solution in the case where the relative velocity of the upper and lower plates is constant. We have just seen that, even in the case of a Poiseuille flow, the permeation velocity is extremely weak. We can neglect it in the present case. On the other side, in a steady-state solution, the velocity \( V \) and the pressure gradient are not time dependent. Our basic equations (2.12-14) now give

\[
V = \frac{\partial u}{\partial t} \quad (3.2)
\]

\[
-p' + \bar{\eta} \frac{\partial u}{\partial t} + \frac{3B}{2} u^2 u' = 0 \quad (3.3)
\]

Differentiating these two equations with respect to \( t \), one gets

\[
\frac{\partial^2}{\partial t \partial z} (u^3) = 0 \quad (3.4)
\]

\[
\frac{\partial^2 u}{\partial t^2} = 0 \quad (3.5)
\]

The only solution compatible with the boundary conditions satisfies

\[ u'' = 0 \quad p' = 0 \quad u' = C(t - t_0) \]

If the two plates are at \( (z = 0 \) and \( z = d ) \) and if the motion of the plates starts at time \( t = 0 \), the solution is

\[ u = V_0 \frac{zt}{d} \]

\[ V = V_0 \frac{z}{d} \]

where \( V_0 \) is the velocity of the upper plate, the lower one being at rest. This is obviously the trivial solution obtained in a case of a Newtonian isotropic liquid. This is not very surprising since:

i) the orientational boundary conditions do not hold outside of the boundary layer;

ii) an uniform tilt involves an uniform compression of layers which does not result in any permeation force.

### 4. Extension to alternative solutions.

From an experimental point of view, steady-state experiments are very difficult to perform on smectics A in this geometry. The usual rotating apparatus, like Couette flows, implies a cylindrical geometry with a resulting distortion of layers. On the other hand, a linear geometry, which is compatible with a good alignment of layers, can be used only in the case of extremely slow motions.

One is therefore led to the use of apparatus where the sample is placed between a fixed plate and a parallel plate moving with an a.c. linear motion. The amplitude is slowly increased until an instability is obtained.

Such a simple apparatus leads to mathematical difficulties in the theoretical analysis: non-linearities in equation (2.14) introduce all the harmonics of the fundamental frequency (that of the plate motion). Any attempt to obtain an analytic solution involves unphysical approximations, except perhaps for very small perturbations to the state solutions. In this latter case, one is very far from the instability threshold, and the situation seems hopeless. For these reasons, we have been led to perform a computer analysis.

In order to draw physical results from such a numerical analysis, it is convenient to introduce dimensionless numbers characteristic of the regime; this is indeed a classical procedure in hydrodynamics. We shall use the geometry defined in figure 2, which corresponds to our experimental conditions. An a.c. motion is applied to the upper plate, while the lower one is kept at rest:

\[ u(d) = -\theta_0 d \sin \tau \]

\[ u(0) = 0 \]

\( d \) is the sample thickness and \( \theta_0 \) the tilt amplitude. We define now the following dimensionless variables

\[ Z = z/d \; \quad \tau = \omega t \; \quad U = -u/d\theta_0 \quad (4.1) \]

which characterize height, time and distortion amplitude. Equation (2.14) can now be written as

\[ M \frac{\partial^2 U}{\partial \tau^2} - NU' U'' - \frac{\partial U''}{\partial \tau} = 0 \quad (4.2) \]

where \( U' \) et \( U'' \) denote \( \partial U/\partial Z \) and \( \partial^2 U/\partial Z^2 \), and \( M \) and \( N \) are the following numbers

\[ M = \rho d^2 \omega \bar{\eta} = \omega \tau H \]

\[ N = (3/2) B \theta_0^2/\omega \bar{\eta} = (3/2) \theta_0^2/\omega \tau \phi \quad (4.3) \]

![Fig. 2. — Geometry for the alternative shear flow.](image_url)
Fig. 3. — $U$ as a function of $\tau$ (for fixed $M$ and increasing values of $N$). a) $M = 6 \times 10^{-2}$, full line $N = 7.8$, dotted line $N = 7.5 (Z = 7/12)$; b) $M = 3$, full line $N = 1.3$, dotted line $N = 1.1 (A : Z = 5/12, B : Z = 7/12, C : Z = 3/4)$.

Here $\tau_\psi$ and $\tau_H$ are respectively the smectic order parameter [14] and the hydrodynamic relaxation time [10]. Note that, for a given sample, $M$ depends only on the frequency while $N$ depends both on the frequency and on the distortion amplitude.

The numerical solution of equation (4.2) is obtained as follows:

i) we use a finite difference method [15]. The analysis is performed on discrete values of $Z$ and $\tau$ and (4.2) is replaced by a set of difference equations;

ii) this system is then solved using a relaxation method [16]. The first trial function corresponds to an uniform tilt of layers $U_0 = Z \sin \tau$. (4.4)

After each step a new trial function $U(Z, \tau)$ is obtained. Successive calculations are performed until $U(Z, \tau)$ remains stationary. A detailed description of this method is given in appendix B.

We obtained the following results (see Figs. 3 and 4):

i) for small values of $N$, the solution is very close to the unperturbed one $U_0(Z, \tau)$;

ii) when $N$ increases, $U(Z, \tau)$ gradually departs from $U_0$ and shows a distortion of the smectic layers in the bulk. This distortion increases as $N$ grows;

iii) when $N$ reaches a critical value $N_c(M)$, $U(Z, \tau)$ diverges. We thus obtain in the $(N, M)$ plane a critical curve $N_c(M)$ (Fig. 5) delimitating two regions; that is:

— region I ($N < N_c$) where the applied motion results in smooth distortion of the smectic layers;

— region II ($N > N_c$) where an instability occurs.

Let us look again at curve $N_c(M)$. For small values of $M$ ($M < 10^{-1}$) this curve has a plateau. Then $N_c(M)$ has a steep decrease roughly corresponding to an $M^{-4}$ law for $M > 2.5$. The difference between these two behaviours can be explained by the relative magnitude of the inertial term, as will now be shown. In order to get some physical understanding of what occurs, let us start again from equation (4.2).

For small values of $M$, the first term (involving inertial effects) can be considered as a perturbation term. Let $\psi = U''$ the curvature of layers. $U'$ can be taken as the zero-order solution $U' = \sin \tau$

and the unperturbed equation describes the relaxation of curvature with a relaxation time $\tau_R = (N \sin^2 \tau)^{-1}$.
For small values of $N$ ($N \ll 1$) this relaxation time is much smaller than the period of the oscillating system and the effective relaxation time can be taken as the time average value of $\tau_r$, i.e. $2/N$. Under this approximation, (4.2) is easily solved and one finds the following results

$$\psi = -\frac{MZ}{1 + N^2/4} \left[ \frac{N}{2} \sin \tau - \cos \tau \right] \quad (4.5)$$

$$U = Z \left\{ \sin \tau - \frac{M(Z^2 - 1)}{6(1 + N^2/4)} \left[ \frac{N}{2} \sin \tau - \cos \tau \right] \right\} \quad (4.6)$$

The effect of the perturbation corresponding to the inertial term is to introduce a small distortion in the bulk. This distortion suffers a phase delay $\phi = \tan^{-1}(2IN)$.

Consider now the opposite case where $N \gg 1$. $\tau_r$ is now much smaller than the oscillating period, except when $\sin \tau = 0$. In order to get some insight into the behaviour of the system, let us solve (4.2) using the following very crude approximation. The relaxation equation

$$MZ \sin \tau - N \sin^2 \tau \psi - \dot{\psi} = 0$$

is solved taking $\sin \tau$ as a constant. One then gets

$$\psi = -\frac{MZ}{N \sin \tau} \left[ 1 - e^{-(N \sin^2 \tau) \tau} \right]. \quad (4.7)$$

This solution does not diverge for $\tau = n\pi$, since one then has $\psi \sim \tau \sin \tau$, but leads at each half period to higher and higher maxima. One thus soon reaches the limit of elasticity and instability must occur.

This conclusion is indeed based upon a rather crude approximation. This calculation emphasizes a very important point: the stiffness of the stabilizing elastic force decreases faster than the inertial destabilizing force when $\sin \tau \to 0$. As long as the relaxation time for curvature is much longer than the period, this has no practical consequence. But, when the relaxation rate gets much smaller than the period, this becomes the source of the instability mechanism. It is easier now to understand why, for small values of $M$, the limit between the two regimes and therefore the instability threshold depend only on the relaxation time, i.e. on $N$.

On the other hand, for values of $M$ of the order of or greater than 1, the first term in equation (4.2) can no longer be considered as a perturbation or, in other words, the inertial forces play a central rôle in the instability mechanism. As this rôle is a destabilizing one, one expects the instability threshold to appear at lower relaxation times, i.e. at lower values of $N$. This is exactly what occurs.

Let us now return to physical quantities (Fig. 6):

![Fig. 6. — The product $N_c M$ plotted versus $M$. This curve represents the variation of $\theta_{0c}^2$ (square of the critical tilt angle) with the frequency.](image)

i) in the low frequency regime ($M < 1$, i.e. $\omega \tau_H \ll 1$), one finds that

$$\theta_{0c}^2/\omega = N_c B/\eta$$

depends only on the temperature $T$. $\theta_{0c}$ is the critical value for $\theta_0$;

ii) at high frequencies ($M \gg 1$, $\omega \tau_H \gg 1$), one gets $\theta_{0c}^2 \sim \omega^{-3}$ at fixed temperature.

5. Conclusion. — After having established a covariant version of the linear hydrodynamics for smectics A, and having confirmed the results of the Orsay group for steady state flows, we found two regimes in the case of a.c. linear shears.

In the low-frequency case ($\omega \tau_H \ll 1$), at low shear amplitude, the steady-state solution (uniform tilt of the layers) is only slightly perturbed. Outside of a curvature boundary layer, the smectic layers remain rigid. This regime is very similar to the hydroelastic regime introduced by Bartolino and Durand in the case of an a.c. compression of layers [17].

For higher shears, the relaxation time for the curvature of the smectic layers becomes smaller than the shear period. One then reaches an instability threshold. The ratio $\theta_{0c}^2/\omega$, where $\theta_{0c}$ is the critical tilt amplitude, depends only on temperature.

At higher frequencies ($\omega \tau_H > 1$), the inertial terms play a central rôle. The instability threshold is lowered, and is now proportional to $\omega^{-3/2}$ at fixed temperature. The present analysis was performed in a continuum model below the instability threshold. This means:

i) that we have assumed defect-free sample;

ii) that we are unable to predict what will occur above the instability threshold. Let us discuss these two points a little further.

a) Dislocations. — In the case of an a.c. compression (or dilation) applied to a smectic sample in a parallel configuration, Bartolino and Durand have shown that the motion of dislocations parallel to the
plates relaxes the elastic stress. Our situation is quite different:

i) in parallel configuration the layers end on a free surface. Dislocations can easily be created or destroyed on this free surface. In our configuration, the smectic layers are anchored on the plate surfaces, and this does not allow for an easy creation or destruction of dislocations;

ii) our strain amplitude is much higher than the one applied by Bartolino and Durand through piezoelectric ceramics. The density of dislocations needed for relaxing stresses is much larger than the one existing in the sample at rest.

b) Above the threshold our instability mechanism should lead to the creation of dislocations in the bulk. Our previous experimental results show that one then gets a plastic regime where strains are relaxed by the motion of a double wall of focal conics. Why focal conics and not dislocations? Probably because the mobility of dislocations is too small and does not allow them to follow the quick motion needed for relaxing strains. A detailed discussion of this point is given in the paper devoted to the description of experimental results.

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Appendix A. — A COVARIANT VERSION OF HYDRODYNAMIC EQUATIONS FOR SMECTICS A. — We start from the K.P. formalism: the smectic order parameter is described by a phase $\phi$. Layers correspond to isophase surfaces. The local symmetry of the medium is described by a director $\mathbf{n}$ normal to the layers:

$$
\mathbf{n} = (1 - \varepsilon) \nabla \phi . \tag{A.1}
$$

In a state problem, the equilibrium condition is derived from the minimization of

$$
F = \int \rho F d^3r
$$

where

$$
\rho F = \rho f + \mu [\phi_{,i} - (1 - \varepsilon) n_i] + \lambda (n_i^2 - 1) . \tag{A.2}
$$

Here $f$ is the specific free-energy and $\mu$ and $\lambda$ are Lagrange multipliers. $\phi_{,i}$ denotes $\partial \phi / \partial x_i$. This minimization process leads to

$$
\delta F = \int \left[ \sigma_{ij} \delta u_{j,i} + g \delta \phi - (h_i - \lambda n_i) \delta n_i \right] d^3r + \text{surface terms} \tag{A.3}
$$

where $h$ is the molecular field, $\delta u$ is an elementary displacement of the fluid, $g$ and $\sigma$ are given in equation (2.4).

In equation (A.3), $\phi$ and $\mathbf{n}$ appear as dynamical variables. As shown in K.P., when one is not very close to the $A \leftrightarrow C$ or $A \leftrightarrow N$ transition temperatures, one can use a rigid smectic model: $\mathbf{n}$ then must be normal to the layers. The hydrodynamic version of this constraint is based upon the non-hydrodynamic character of the variable $\mathbf{n}$: $\mathbf{n}$ is characterized by a finite relaxation time for vanishing wave-vector. As a result, $\mathbf{n}$ must disappear as a variable from the hydrodynamic equations. This can be easily achieved by choosing the Lagrange multipliers vector $\mu$ such that $h$ remains parallel to $\mathbf{n}$. The resulting equation will be true for frequencies lower than the inverse relaxation time for $n, B_1/\gamma_1$, which is of order $10^6$ Hz.

This choice was made in writing equations (2.4), and with such a choice, since $\mathbf{n}$ is a unit vector, the third term in the r.h.s. of (A.3) vanishes. The consequence is that no molecular field, or bulk-torque, acts on the director, and it is well known that, in such a case, the viscous stress tensor is symmetric.

The entropy source $\Sigma$ is defined by

$$
\rho \frac{ds}{dt} = - \nabla \cdot \mathbf{J}_s + \Sigma ,
$$

where $s$ is the specific entropy and $\mathbf{J}_s$ the entropy flux. It is a general result that $\Sigma$ satisfies

$$
T \Sigma = - \mathbf{J}_s \cdot \mathbf{V}T + (\sigma_{ij} + \sigma'_{ij}) V_{j,i} - \rho \frac{df}{dt} . \tag{A.4}
$$

Hence, using (A.3),

$$
T \Sigma = - \mathbf{J}_s \cdot \mathbf{V}T + \sigma'_{ij} V_{j,i} - g \dot{\phi} . \tag{A.5}
$$

From this expression, and taking into account the symmetry of the medium and Onsager relationships, one easily gets equations (2.3) as phenomenological equations between thermodynamic fluxes and forces.

Appendix B. — COMPUTER ANALYSIS. — We had to solve the following equation

$$
NU^{1/2} U'' + \dot{U} - M \dot{U} = 0 \tag{4.2}
$$

and we used a finite-differences method. We worked in the $(Z, \tau)$ plane, and a set of difference equations was derived for discrete points $(Z = p \Delta Z, \tau = q \Delta \tau)$; derivatives were replaced by differences and equation (4.2) by the following one:

$$
2 U_{p,q} = \frac{N}{(\Delta Z)^2} \left[ U_{p-1,q-1}^2 U_{p+1,q-1} - 2 U_{p,q-1}^3 - \frac{U_{p,q-1}^2 - 4 U_{p-1,q-1} U_{p,q-1} + U_{p-1,q-1}^3}{2} - 2 U_{p-1,q-1} U_{p,q-1} U_{p+1,q-1} + 5 U_{p-1,q-1}^2 - U_{p,q-1} \right] + U_{p+1,q-1} + U_{p-1,q-1} - U_{p,q-1} + 2 U_{p,q-1} - U_{p-1,q-1} \tag{B.1}
$$

$$
- M \frac{(\Delta Z)^2}{\Delta \tau} \left[ U_{p-1,q-1} - 2 U_{p-1,q} + U_{p-1,q+1} \right] .
$$
with the following boundary conditions
\[ U_{0,q} = U_{p,0} = 0, \]
\[ U_{N_x,q} = \sin \tau. \]
At each step we used the values previously computed in the r.h.s. of equation (B.1). This procedure was followed until the mean value \( \delta^2 \) of the square of the variation of the \( U_{p,q} \)'s between two successive steps was less than \( 10^{-6} \).

A faster convergence was obtained at each step by computing two values: \( U_{p,q}^{(1)} \) starting from \( p = 1 \), and \( U_{p,q}^{(2)} \) starting from \( p = N_z - 1 \). We then took as a basis for the next step the following average value
\[ U_{p,q} = \frac{(N_z - p) U_{p,q}^{(1)} + p U_{p,q}^{(2)}}{N_z}. \]

These calculations were performed at fixed \( M \) (fixed frequency) and for increasing values of \( N \) (increasing tilt amplitudes). This procedure follows the experimental one. For the lowest value of \( N \) and for the first step, we used as initial values for the \( U_{p,q} \)'s the steady state solution
\[ U_{p,q} = (p \sin \tau)/N_z. \]
For the next values of \( N \), we used as initial distribution for the \( U_{p,q} \)'s the one obtained with the previous value of \( N \).

These calculations were performed on the 33.03 IBM computer of the Centre National Universitaire Sud de Calcul (C.N.U.S.C.). For each point of the curve \( N_c(M) \), the C.P.U.-time was of order of 6-10 seconds.

We have verified that the law \( N_c = \text{const.} \) for \( M < 1 \) does not depend on the size of the discretization lattice and that this critical value does not vary when the increments \( \Delta \tau \) and \( \Delta Z \) decrease. For instance, when the number of points on the discretization lattice varies from 481 to 1 045 (\( \Delta Z \) varies from 1/12 to 1/18) \( N_c \) does not change significantly. Identical results were obtained with a value of \( \Delta \tau \) (0.4875) incommensurate with the period. We have obtained, for small values of \( N \), a numerical solution close to the unperturbed one. The approximate solution (Eq. (4.6)) gives a similar result. This tends to confirm that our numerical solution is valid and not the result of some artefact.

References