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Résumé. — En utilisant une approximation de tension superficielle forte, on établit l'équation d'évolution de la surface d'un film $\zeta = \Phi(\xi, \eta, \tau)$ tombant le long de la surface d'un cylindre vertical infini. Celle-ci s'écrit en utilisant des variables sans dimension

$$\Phi_\tau + \Phi\Phi_\xi + \Phi_{\xi\xi} + (1/\mu^2)\,\nabla^2\Phi + \nabla^4\Phi = 0,$$

où (ξ, η) sont les coordonnées cartésiennes sur la surface du cylindre, $-\infty < \xi < \infty$, $0 \leqslant \eta \leqslant 2\,\pi\mu$; μ est le rayon du cylindre. Pour $\mu \leqslant \mu_c = 1$, l'écoulement stationnaire du film est formé d'un train d'anneaux s'écoulant vers le bas de façon irrégulière. A $\mu > \mu_c$, la nature unidimensionnelle de l'écoulement disparaît, et pour $\mu \gg \mu_c$ la surface du film se festonne.

Abstract. — Using the strong surface-tension approximation, an asymptotic equation is derived which describes the evolution of the disturbed surface of a film $\zeta = \Phi(\xi, \eta, \tau)$ flowing down an infinite vertical column. In non-dimensional scaled variables this equation is

$$\Phi_\tau + \Phi\Phi_\xi + \Phi_{\xi\xi} + (1/\mu^2)\,\nabla^2\Phi + \nabla^4\Phi = 0\,,$$

where (ξ, η) are cartesian coordinates on the surface of the cylinder, $-\infty < \xi < \infty$, $0 \leqslant \eta \leqslant 2\,\pi\mu$; μ is the scaled radius of the column. For $\mu \leqslant \mu_c = 1$, the steady flow of the film is a one-dimensional train of rings flowing irregularly downward. At $\mu > \mu_c$ the one-dimensional nature of the flow disappears, and at $\mu \gg \mu_c$ the film surface is expected to assume the form of down-flowing drops in a state of irregular splitting and merging.

1. **Introduction.** — The flow of a viscous liquid film down a vertical wall seems to be one of the simplest and best illustratious examples of a deterministic physical system capable of random-type behaviour [1]. The chaotic nature of the flow manifests itself in the formation on the film surfaces of a self-fluctuating wave ; this wave sometimes assumes the form of drops rolling down the film, merging together and splitting in an irregular manner [2] (Fig. 1). The special feature of this system is that the irregular self-fluctuations exist at any Reynolds number (i.e., even at very small ones). However, the nature of the irregularity may depend on external geometric factors. For example, the flow of a film down the surface of a cylindrical column is, generally speaking, more regular than flow down a *flat* vertical wall. In the first case the wave may even be one-dimensional — a train of rings flowing irregularly downward [3, 4]. Thus, the curvature of the wall may be regarded as a parameter controlling the degree of irregularity of the system.

The present paper is devoted to an analysis of the effect of curvature on the stability of flow of the film.

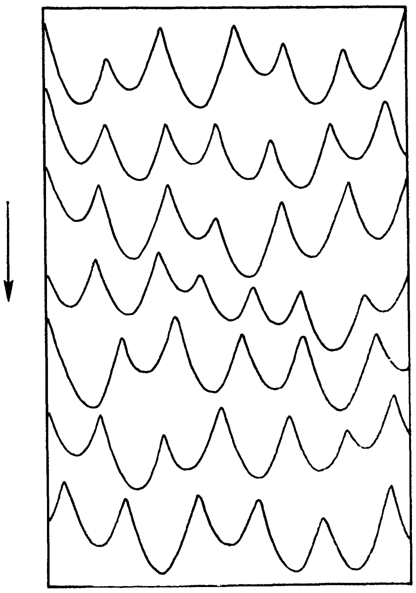

Fig. 1. — Irregular wave motion on water film flowing down a vertical smooth plate.

Before proceeding to the equations for the flow of a film down a vertical cylindrical column, we recall the basic indices of flow down a *flat* wall. The velocity profile of the undisturbed plane-parallel flow is shown in figure 2 :

$$\bar{w}(\bar{x}) = (gh_0^2/2\,\nu)\,[(\bar{x}/h_0)^2 - 2(\bar{x}/h_0)]\,. \quad (1.1)$$

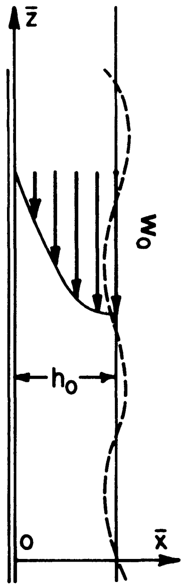

Fig. 2. — Diagram of the undisturbed flow, showing the velocity profile.

Here $\bar{x}$, $\bar{z}$ are dimensional space coordinates, $\bar{w}$ is the vertical component of the velocity of the liquid, g the gravitational acceleration, ν the kinematic viscosity, h_0 the thickness of the undisturbed film,

$$w_0 = gh_0^2/2\,\nu \quad (1.2)$$

is the velocity of the undisturbed film at the gas-liquid interface ($\bar{x} = h_0$). The Reynolds number associated with the flow (1.1) is defined as

$$R = h_0\,w_0/\nu\,. \quad (1.3)$$

It is known that the flow (1.1) is unstable, instability being represented by spontaneous formation of waves on the film surface (dotted line in figure 2).

The dispersion relation corresponding to the linear stability problem is [5] :

$$\bar{\omega} = (w_0/h_0)\,[2\,i(h_0\,\bar{k}) + (8\,R/15)\,(h_0\,\bar{k})^2 - (\sigma R/3\,\rho h_0\,w_0)\,(h_0\,\bar{k})^4] \quad (1.4)$$

$\bar{\omega}$ is the rate of instability parameter, $\bar{k}$ the disturbance wavenumber and ρ the density of the liquid. Thus, the flow is always unstable ($\mathrm{Re}\,\bar{\omega} > 0$) to disturbances of sufficiently long wavelength. Short-wave disturbances are damped out because of the stabilizing effect of the surface tension σ.

The wavelength corresponding to the maximum amplification rate of small disturbances (Fig. 3) is

$$\bar{\lambda}_c = 2\,\pi/\bar{k}_c = \pi h_0 \sqrt{5\,\sigma/\rho h_0\,w_0^2}\,. \quad (1.5)$$

This quantity is usually related to the experimentally observed wavelengths (Rayleight principle). The imaginary part $\bar{\omega}$ of (1.4) indicates that the wave moves downward at velocity $2\,w_0$. The structure of (1.5) suggests that

$$\gamma = \sigma/\rho h_0\,w_0^2 \quad (1.6)$$

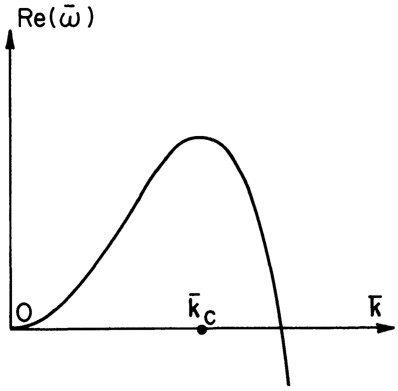

Fig. 3. — Rate of stability parameter $\mathrm{Re}\,(\bar{\omega})$ as a function of disturbance wavenumber $\bar{k}$.

should be taken as the non-dimensional surface tension. The characteristic formation time of the wavy structure of the film may be defined as

$$\bar{\tau}_c = h_0/w_0\,\max\,(\mathrm{Re}\,\bar{\omega}) = 75\,\gamma h_0/16\,w_0\,. \quad (1.7)$$

In order to obtain some idea of the order of magnitude of the quantities involved, we consider the flow of a thin film of water : $h_0 = 0.01$ cm, $\sigma = 72.5$ dyn./cm, $\rho = 1$ g/cm^3, $\nu = 0.01$ cm^2/s.

In this case

$$\bar{\lambda}_c = 1.2\text{ cm}\,, \quad \bar{\tau}_c = 2.9\text{ s}\,, \quad w_0 = 4.9\text{ cm/s}\,,$$
$$R = 4.9\,, \quad \gamma = 301\,. \quad (1.8)$$

These figures are in good agreement with experimental observations [4, 5]. As we see, if the film is sufficiently thin ($\gamma \gg 1$), the wave flow is quasi-one-dimensional ($\bar{\lambda}_c \sim \sqrt{\gamma}h_0 \gg h_0$), while the formation of the wave structure is a quasi-steady process

$$(\bar{\tau}_c \sim \gamma h_0/w_0 \gg h_0/w_0)\,.$$

This observation is the key to the nonlinear asymptotic analysis of the flow of the disturbed film, presented below. At $\gamma \gg 1$ (thin-film or large-surface-tension approximation), it is possible effectively to lower the dimensionality of the problem, and thereby to derive an asymptotic quasi-linear equation which directly describes the dynamics of evolution of the disturbed film surface. Formally, the situation is largely analogous to that obtaining in the recently developed nonlinear theories of chemical instability [6] and cellular flames [7].

2. **Fundamental equations and basic solution.** — Taking h_0, h_0/w_0, w_0, ρw_0^2 as the units of length, time, velocity and pressure, respectively, one can write the equations for the film hydrodynamics in cylindrical coordinates as follows :

Momentum equations :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{R}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2}\right) \quad (2.1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{R}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right) \quad (2.2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{2}{R} - \frac{\partial p}{\partial z} + \frac{1}{R}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right). \quad (2.3)$$

Continuity equation :

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \quad (2.4)$$

The flow takes place in the layer

$$a \leqslant r \leqslant a + h(z, \theta, t), \qquad -\infty < z < \infty, \quad -\pi \leqslant \theta \leqslant \pi. \quad (2.5)$$

Here a, h are the non-dimendional radius of the column and the thickness of the disturbed film, in units of the thickness h_0 of the undisturbed film. On the cylinder surface ($r = a$) the viscous liquid must satisfy the adhesion condition

$$u = v = w = 0 \quad (r = a). \quad (2.6)$$

On the gas-liquid interface we demand the kinematic impermeability condition

$$\frac{\partial h}{\partial t} - u + \frac{v}{r}\frac{\partial h}{\partial \theta} + w\frac{\partial h}{\partial z} = 0. \quad (2.7)$$

If surface tension is ignored, the forces acting on the free boundaries of both fluids must be equal. In our problem, the atmosphere surrounding the film is assumed to be a weightless, non-viscous, stationary gas at pressure p_0. Thus (in the absence of surface tension), the force acting on the free surface of the liquid film is equal to $-p_0\,\mathbf{n}$, where $\mathbf{n}$ is the unit normal vector.

To include surface tension, $p_0\,\mathbf{n}$ must be augmented by $\gamma(1/R_1 + 1/R_2)\,\mathbf{n}$, where γ is the non-dimensional surface tension (1.6), $1/R_1 + 1/R_2$ is the sum of reciprocals of the radii of curvature of the interface. Thus, in cartesian coordinates, the equilibrium of forces condition is written as follows (see, e.g. [8]) :

$$p_0 n_i + \gamma(1/R_1 + 1/R_2)\,n_i + \sum_k p_{ik} n_k = 0, \qquad i, k = 1, 2, 3. \quad (2.8)$$

where

$$p_{ik} = -p\delta_{ik} + (1/R)(\partial v_i/\partial x_k + \partial v_k/\partial x_i),$$
$$(n_i) = \mathbf{n}, \quad (v_i) = \mathbf{v}, \quad (x_i) = \mathbf{x}.$$

In cylindrical coordinates, equations (2.8) are written as the following three boundary conditions on the liquid film surface :

Zero shearing stress :

$$\frac{p_{rr}}{r}\frac{\partial h}{\partial \theta} + p_{r\theta}\left[1 - \frac{1}{r^2}\left(\frac{\partial h}{\partial \theta}\right)^2\right] - \frac{p_{rz}}{r}\frac{\partial h}{\partial \theta}\frac{\partial h}{\partial z} - \frac{p_{\theta z}}{r}\frac{\partial h}{\partial \theta} - p_{\theta z}\frac{\partial h}{\partial z} = 0 \quad (2.9)$$

$$p_{rr}\frac{\partial h}{\partial z} - \frac{p_{r\theta}}{r}\frac{\partial h}{\partial \theta}\frac{\partial h}{\partial z} + p_{rz}\left[1 - \left(\frac{\partial h}{\partial z}\right)^2\right] - \frac{p_{r\theta}}{r}\frac{\partial h}{\partial \theta} - p_{zz}\frac{\partial h}{\partial z} = 0. \quad (2.10)$$

Condition relating pressure jump to surface tension (γ) :

$$\left[p_{rr} + \frac{p_{\theta\theta}}{r^2}\left(\frac{\partial h}{\partial \theta}\right)^2 + p_{zz}\left(\frac{\partial h}{\partial z}\right)^2 - \frac{2p_{r\theta}}{r}\frac{\partial h}{\partial \theta} + \frac{2p_{\theta z}}{r}\frac{\partial h}{\partial \theta}\frac{\partial h}{\partial z} - 2p_{rz}\frac{\partial h}{\partial z}\right]\left[1 + \frac{1}{r^2}\left(\frac{\partial h}{\partial \theta}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2\right]^{-1} =$$

$$= -p_0 - \gamma\left\{\frac{2}{r^3}\left(\frac{\partial h}{\partial \theta}\right)^2 + \frac{1}{r}\left[1 + \left(\frac{\partial h}{\partial z}\right)^2\right] - \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2}\left[1 + \left(\frac{\partial h}{\partial z}\right)^2\right] + \frac{2}{r^2}\frac{\partial h}{\partial \theta}\frac{\partial h}{\partial z}\frac{\partial^2 h}{\partial \theta\,\partial z} - \right.$$

$$\left. - \frac{\partial^2 h}{\partial z^2}\left[1 + \frac{1}{r^2}\left(\frac{\partial h}{\partial \theta}\right)^2\right]\right\}\left[1 + \frac{1}{r^2}\left(\frac{\partial h}{\partial \theta}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2\right]^{-3/2}. \quad (2.11)$$

Here

$$p_{rr} = -p + \frac{2}{R}\frac{\partial u}{\partial r}, \quad p_{\theta\theta} = -p + \frac{2}{R}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right),$$
$$p_{zz} = -p + \frac{2}{R}\frac{\partial w}{\partial z}, \quad p_{r\theta} = \frac{1}{R}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right), \tag{2.12}$$
$$p_{\theta z} = \frac{1}{R}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right), \quad p_{rz} = \frac{1}{R}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$$

u, v, w — non-dimensional radial, angular and vertical components of velocity, respectively.

The solution of problem (2.1)-(2.11) corresponding to undisturbed flow of the film (the basic solution) is

$$u^{(0)} = v^{(0)} = 0\,, \quad w^{(0)} = \tfrac{1}{2}(r^2 - a^2) - (1 + a)^2 \ln (r/a) \tag{2.13}$$
$$p^{(0)} = p_0 + \gamma/(1 + a)\,, \quad h^{(0)} = 1\,.$$

3. **Large surface tension approximation ($\gamma \gg 1$).** — The crucial point in asymptotic nonlinear analysis of stability is correct choice of the small parameter and the corresponding space-time scalings. To this end we appeal to the two-dimensional version of the dispersion relation (1.4) for the flow of a film down a *flat* vertical wall [9] :

$$\omega = -2\,ik + R\left[\frac{8}{15}k^2 - \frac{1}{3}\gamma(k^2 + l^2)^2\right], \tag{3.1}$$

$$\delta h \sim \exp(\omega t + ikz + ily)\,.$$

Here y, z are horizontal and vertical coordinates, respectively, on the wall surface.

The dispersion relation (3.1) shows that at $\gamma \gg 1$ the instability region ($\operatorname{Re}\omega > 0$) is concentrated in the zone of long-wave perturbations, where

$$k \sim l \sim 1/\sqrt{\gamma}\,, \quad \operatorname{Re}\omega \sim 1/\gamma\,.$$

Since the wavelength of the transverse disturbance ($2\,\pi/l$) is of the order of $\sqrt{\gamma}$, we may expect the nature of the flow to be most sensitive to the curvature of the column when the number of transverse waves

$$(2\,\pi a)\,(l/2\,\pi) = al$$

is finite, i.e., when $a \sim \sqrt{\gamma}$. Since $\operatorname{Im}\omega \sim 1/\sqrt{\gamma}$ in the instability zone (i.e., $\operatorname{Im}\omega$ is less than $\operatorname{Re}\omega$ in order of magnitude), our problem involves, as it were, two characteristic times. However, this complicating factor may be eliminated if we transform to a coordinate system moving at velocity $\beta = \operatorname{Im}(\omega/k)$ along the column. In case of a flat wall, $\beta = 2$ (3.1). Thus, we apply the transformation

$$z' = z + \beta t\,. \tag{3.2}$$

Putting $\gamma = 1/\varepsilon^2$ and relying on the above estimates, we introduce the following scaled coordinates and parameters :

$$X = r - a\,, \quad Z = \varepsilon z'\,, \quad T = \varepsilon^2 t$$
$$\alpha = a\varepsilon\,, \quad Y = \alpha\theta \quad (|\,Y\,| \leqslant \alpha\pi)\,. \tag{3.3}$$

The parameter α is thus the scaled radius of the column.

After transformation to the new variables (3.3), the solution of problem (2.1)-(2.9) is sought in the form of an asymptotic expansion :

$$u = \sum_{n=1} \varepsilon^n\, U_n(X, Y, Z, T)\,,$$
$$v = \sum_{n=1} \varepsilon^n\, V_n(X, Y, Z, T)\,,$$
$$w = w^{(0)}(X, \varepsilon) + \sum_{n=1} \varepsilon^n\, W_n(X, Y, Z, T)\,, \tag{3.4}$$
$$p = p^{(0)}(\varepsilon) + \sum_{n=1} \varepsilon^n\, P_n(X, Y, Z, T)\,,$$
$$h = 1 + \sum_{n=1} \varepsilon^n\, H_n(Y, Z, T)\,.$$

The required equation for the disturbed film surface (principal term) is obtained in the solution of the third-approximation problem (see Appendix) :

$$\frac{\partial H_1}{\partial T} - \frac{2}{3\,\alpha}\frac{\partial H_1}{\partial Z} - 4\,H_1\frac{\partial H_1}{\partial Z} + \frac{8}{15}\frac{R}{}\frac{\partial^2 H_1}{\partial Z^2} + \frac{R}{3\,\alpha^2}\nabla^2 H_1 + \frac{R}{3}\nabla^4 H_1 = 0\,. \tag{3.5}$$

In the process of the solution we also obtain $\beta = 2$.

The second term of this equation indicates the *acceleration* of the wave moving down the surface of the column, in comparison with the case of flat wall. The wave moves downward with velocity

$$2\,w_0(1 + 1/3\,\alpha\sqrt{\gamma})\,. \tag{3.6}$$

The fifth term of equation (3.5) implies that the wavelength corresponding to the maximum velocity of small disturbances diminishes with increasing curvature of the wall (cf. Eq. (1.5)) :

$$\bar{\lambda}_c = \pi h_0 \sqrt{5\,\gamma(1 + 5/8\,\alpha^2)}\,. \tag{3.7}$$

Thus, at $\gamma \gg 1$ the wavelength is considerably more sensitive to the wall curvature than the propagation velocity.

At $\alpha = 0.5$, the wavelength is approximately half the wavelength corresponding to a flat wall ($\alpha = \infty$).

Using the transformations

$$Z + (2/3\,\alpha)\,T = -\,(\sqrt{10}/4)\,\xi\,,\quad Y = (\sqrt{10}/4)\,\eta\,,\quad T = (75/64\,R)\,\tau\,,\quad H_1 = (4\sqrt{10}\,R/75)\,\Phi\,,\quad \alpha = (\sqrt{10}/4)\,\mu\,. \tag{3.8}$$

We bring equation (3.5) to the following one-parameter form, which is more convenient for analysis

$$\Phi_\tau + \Phi\Phi_\xi + \Phi_{\xi\xi} + (1/\mu^2)\,\nabla^2\Phi + \nabla^4\Phi = 0 \tag{3.9}$$

$$(\Phi = \Phi(\xi, \eta, \tau), \qquad -\pi\mu \leqslant \eta \leqslant \pi\mu)\,.$$

Since the problem is periodic with respect to the angle $\theta = \eta/\mu$, the following smoothness conditions must be satisfied on the boundary of the strip $|\,\eta\,| \leqslant \pi\mu$:

$$\left.\frac{\partial^k\Phi}{\partial\eta^k}\right|_{\eta=-\pi\mu} = \left.\frac{\partial^k\Phi}{\partial\eta^k}\right|_{\eta=\pi\mu}\,, \quad k = 0, 1, 2, 3\,. \tag{3.10}$$

In [1] a numerical solution was undertaken of the one-dimensional version of equation (3.9), $\Phi = \Phi(\xi, \tau)$: it was shown that there exist solutions in the form of irregularly self-fluctuating waves (Fig. 4). Figures 4*a* and 4*b* show the disturbed surface at times close to the initial disturbance; 4*c*-4*e* correspond to the developed quasi-periodic wavy surface of the film. The problem was solved in the interval $0 \leqslant \xi \leqslant 100$ with periodic boundary conditions. The initial disturbance was the function $\Phi(\xi, 0) = \sin(\pi\xi/100)$. Interesting results in the analysis of these chaotic solutions were recently obtained by Manneville [10].

Let us see what happens if the disturbance of the film is assumed to depend only on the transverse space coordinate (i.e., $\Phi = \Phi(\eta, \tau)$). The nonlinear term $\Phi\Phi_\xi$ then disappears, and the result is a linear equation :

$$\Phi_\tau + \Phi_{\eta\eta\eta\eta} + (1/\mu^2)\,\Phi_{\eta\eta} = 0\,. \tag{3.11}$$

This equation admits solutions of the form

$$\Phi = A\exp(\omega t + ik\eta) \quad \text{where} \quad \omega = (1/\mu^2)\,k^2 - k^4\,. \tag{3.12}$$

At $k < 1/\mu$, the solution (3.12) is an exponentially increasing function of time. At first glance this would seem to indicate some defect in the asymptotic behaviour described by the nonlinear equation (3.9). We note, however, that the condition $|\,\eta\,| \leqslant \pi\mu$ excludes the possibility of disturbances with $k < 1/\mu$, and so eliminates solutions (3.12) with positive increment ω. Thus, the fact that the domain of solutions of the problem is limited to the strip $|\,\eta\,| \leqslant \pi\mu$ is an essential factor in ensuring that problem (3.9)-(3.10) is well-posed.

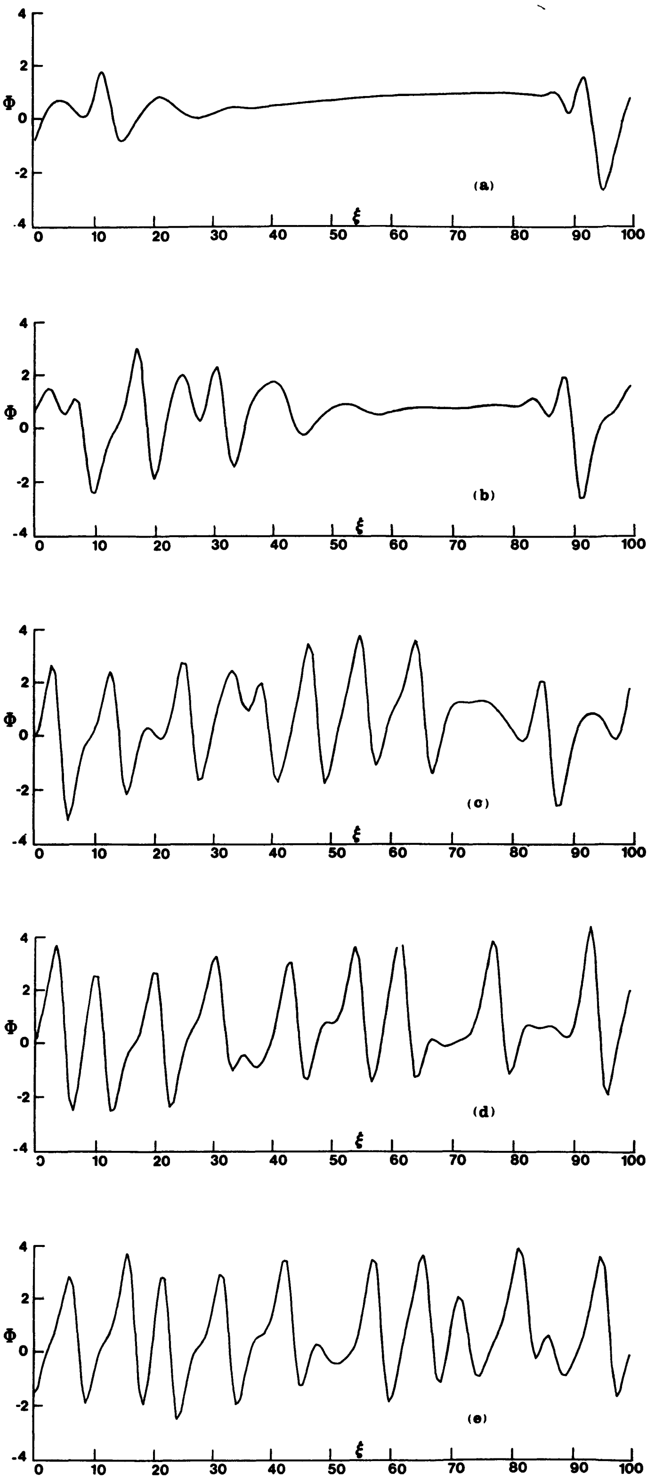

Fig. 4. — Result of numerical solution of one-dimensional version of equation (3.9) ($\mu = \infty$).

At $\mu = \infty$ (i.e., flow down a *flat* vertical wall) problem (3.9)-(3.10) assumes the form

$$\Phi_\tau + \Phi\Phi_\xi + \Phi_{\xi\xi} + \nabla^4\Phi = 0\,, \quad -\infty < \xi, \eta < \infty\,. \tag{3.13}$$

The nonlinear dynamic problem of a disturbed film flowing down a flat vertical (or inclined) wall has been considered by many authors, starting from the pioneering work of Benney [11]. The bifurcational approach proposed above leads to simpler and more tractable asymptotic relations than the usual method of long-wave expansions. Recall that in that last-named method the amplitude of the disturbance is assumed to be of the same order of magnitude as the film thickness.

4. **Linear stability analysis.** — The dispersion relation corresponding to linear stability analysis for the undisturbed film surface ($\Phi \equiv 0$) is

$$\omega = -(k^2 + l^2)^2 + k^2 + (1/\mu^2)(k^2 + l^2), \qquad \delta\Phi \sim \exp(\omega\tau + ik\xi + il\eta). \tag{4.1}$$

In view of the periodicity condition,

$$l = n/\mu \quad (n = 0, 1, 2, 3, ...). \tag{4.2}$$

Analysis of equation (4.1) is made more convenient by introducing the new parameters

$$k = \kappa/\mu, \quad l = \lambda/\mu, \quad \omega = \Omega/\mu^4, \tag{4.3}$$

equation (4.1) then becomes

$$\Omega = -(\kappa^2 + \lambda^2)^2 + (\kappa^2 + \lambda^2) + \mu^2\kappa^2. \tag{4.4}$$

Figure 5 illustrates marginal stability curves ($\Omega = 0$) in the (λ, κ)-plane for various values of μ.

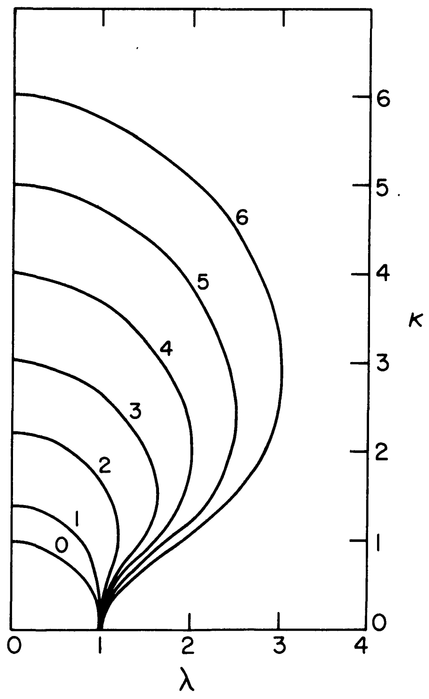

Fig. 5. — Marginal stability curves in plane of scaled wavenumbers (κ, λ) for $\mu = 0, 1, 2, 3, 4, 5, 6$.

It is readily shown that when

$$\mu < \sqrt{2n^2 - 1 + 2n\sqrt{n^2 - 1}} \equiv f(n) \tag{4.5}$$

the disturbance cannot contain unstable (increasing) harmonics of wavelength $\leqslant 2\pi\mu/n$. Thus, when $\mu < f(1) = 1$ the flow is one-dimensional (Fig. 6*a*). We emphasize yet again that the flow, though one-dimensional, is nevertheless not a regular periodic wave.

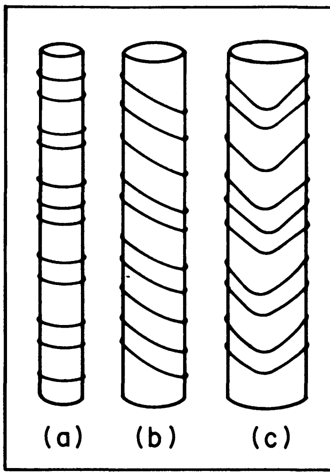

Fig. 6. — Diagram illustrating varying nature of flow of film with increasing radius of cylinder.

When $f(1) \leqslant \mu < f(2) \simeq 3.86$, the flow becomes two-dimensional, in the form of irregularly downward-flowing inclined rings (Fig. 6*b*). It is to this case, apparently, that the photograph reproduced in Binnie [4] corresponds. For water-film thickness $h_0 = 0.01$ cm in the region of inclined-rings instability, the wavelength column-diameter ratio $\bar{\lambda}_c/2ah_0$ is in the interval 0.5-0.75. This estimate is in good agreement with the photograph published in [4].

When $f(2) \leqslant \mu < f(3) \simeq 5.83$ the flow becomes even more complicated (Fig. 6*c*).

Thus, the wall curvature has a significant influence on the transverse disturbance of the film, suppressing harmonics of high *angular* frequency. The qualitative effect of the curvature on the longitudinal harmonics of the disturbance is considerably lower. Basically, it appears as a slight decrease in the wavelengths of the downward-rolling waves.

With increasing μ the number of perturbed harmonics increases, and the flow becomes increasingly irregular with respect to both η and ξ.

Of course, these considerations as to the varying nature of the flow with increasing μ are merely plausible assumptions, dictated by the results of the linear stability analysis. For definitive verification, it would be very interesting to undertake a numerical solution of the initial-value problem for the two-dimensional equation (3.9).

Appendix. — The equation system and boundary conditions for the first-approximation problem is

$$\frac{\partial U_1}{\partial X} = 0, \quad \frac{\partial^2 V_1}{\partial X^2} = 0, \quad \frac{\partial^2 W_1}{\partial X^2} = 0, \quad \frac{\partial P_1}{\partial X} = 0 \tag{A.1}$$

$$U_1 = V_1 = W_1 = 0 \quad (\text{for } X = 0) \tag{A.2}$$

$$U_1 = 0, \quad 2H_1 + \frac{\partial W_1}{\partial X} = 0, \quad \frac{\partial V_1}{\partial X} = 0,$$

$$P_1 = -\frac{1}{\alpha^2} H_1 - \nabla^2 H_1 \quad (\text{for } X = 1).$$

The solution of problem (A.1-A.2) has the form

$$U_1 = 0, \quad V_1 = 0, \quad W_1 = -2H_1 X,$$
$$P_1 = -(1/\alpha^2) H_1 - \nabla^2 H_1 . \tag{A.3}$$

At this stage, therefore, the solution of the first-approximation problem is defined up to an as yet unknown function H_1. To determine H_1, one must go on to the next approximation :

The second approximation problem is

$$\frac{\partial U_2}{\partial X} + \frac{\partial W_1}{\partial Z} = 0, \qquad -\frac{\partial P_1}{\partial Y} + \frac{1}{R}\frac{\partial^2 V_2}{\partial X^2} = 0,$$
$$\beta \frac{\partial W_1}{\partial Z} + 2(X-1)U_2 + X(X-2)\frac{\partial W_1}{\partial Z} = -\frac{\partial P_1}{\partial Z} + \frac{1}{R}\frac{\partial^2 W_2}{\partial X^2} + \frac{1}{\alpha R}\frac{\partial W_1}{\partial X} \tag{A.4}$$

$$\frac{\partial V_2}{\partial X} = 0, \qquad 2H_2 + \frac{\partial W_2}{\partial X} = 0, \qquad (\beta - 1)\frac{\partial H_1}{\partial Z} - U_2 = 0 \qquad (\text{for } X = 1)$$
$$U_2 = V_2 = W_2 \qquad (\text{for } X = 0). \tag{A.5}$$

The solution of problem (A.4-A.5) has the form

$$U_2 = X^2 \frac{\partial H_1}{\partial Z}, \qquad V_2 = -\frac{1}{2} RX(X-2)\left(\frac{1}{\alpha^2}\frac{\partial H_1}{\partial Y} + \nabla^2 \frac{\partial H_1}{\partial Y}\right),$$
$$W_2 = R\frac{\partial H_1}{\partial Z}\left(\frac{1}{6}X^4 - \frac{2}{3}X^3 + \frac{4}{3}X\right) - \frac{1}{2}RX(X-2)\left(\frac{1}{\alpha^2}\frac{\partial H_1}{\partial Z} + \nabla^2\frac{\partial H_1}{\partial Z}\right) + \frac{1}{\alpha}X(X-2)H_1 - 2XH_2, \tag{A.6}$$
$$\beta = 2.$$

As we see the second approximation is still insufficient to determine the function H_1.

Proceeding to the third approximation, and noting that here it is quite sufficient to consider only the continuity equation (2.4) and conditions (2.6), (2.9), we obtain

$$\frac{\partial U_3}{\partial X} + \frac{1}{\alpha}U_2 + \frac{\partial V_2}{\partial Y} + \frac{\partial W_2}{\partial Z} = 0 \tag{A.7}$$

$$U_3 = 0 \qquad (\text{for } X = 0) \tag{A.8}$$

and

$$\frac{\partial H_1}{\partial T} + 2\frac{\partial H_2}{\partial Z} - 2H_1\frac{\partial H_1}{\partial Z} - U_3 - \frac{\partial H_2}{\partial Z} + \left(W_1 - \frac{1}{3\alpha}\right)\frac{\partial H_1}{\partial Z} = 0 \qquad (\text{for } X = 1).$$

Hence

$$U_3 = -\frac{2}{\alpha}\frac{\partial H_1}{\partial Z}\left(\frac{1}{3}X^3 - \frac{1}{2}X^2\right) + \frac{1}{2}R\left(\frac{1}{\alpha^2}\nabla^2 H_1 + \nabla^4 H_1\right)\left(\frac{1}{3}X^3 - X^2\right) -$$
$$- R\frac{\partial^2 H_1}{\partial Z^2}\left(\frac{1}{30}X^5 - \frac{1}{6}X^4 + \frac{2}{3}X^3\right) + \frac{\partial H_2}{\partial Z}X^2 \tag{A.9}$$

and, finally,

$$\frac{\partial H_1}{\partial T} - \frac{2}{3\alpha}\frac{\partial H_1}{\partial Z} - 4H_1\frac{\partial H_1}{\partial Z} + \frac{8R}{15}\frac{\partial^2 H_1}{\partial Z^2} + \frac{R}{3\alpha^2}\nabla^2 H_1 + \frac{R}{3}\nabla^4 H_1 = 0. \tag{A.10}$$

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