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Applications of energy theorems to surface waves in a plasma

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Résumé. — En utilisant les relations entre les diverses énergies de l’onde et le théorème des variations, les auteurs déduisent des expressions simples permettant de calculer l’influence de divers paramètres sur la courbe de dispersion. Dans une seconde partie, une application est faite pour le mode axisymétrique (onde de surface) se propageant le long d’une colonne de plasma. En particulier, il est montré qu’il y a un lien entre l’atténuation de l’onde et la répartition des énergies.

Abstract. — Using relations between the various energies of a wave and the variation theorem, the authors deduce equations from which the influence of parameters on the dispersion curve can be easily calculated. In the second part, the above results are applied to the axisymmetrical mode (surface wave) which is propagated along a plasma column. In particular, it is shown that the wave damping depends on the repartition of the energies.

1. Introduction. — Since the article [1] of A. V. Trivelpiece and R. W. Gould, the propagation of waves in bounded plasmas has been extensively studied [2]. However, to gain some understanding of the main properties of plasmas, incorporating a travelling wave [3], we must know the effect on the wave propagation of such parameters as the radial profile of the electron density, the collision frequency and the losses in the dielectrics surrounding the discharge. Such studies have only been made in semi-infinite plasmas, and in bounded plasmas within the quasistatic approximation [1]. Usually, we might solve the dispersion equation

\[ D(\omega, \gamma) = 0 \]

(where \( \omega \) is the excitation frequency and \( \gamma \) the propagation constant) by taking into account the above mentioned parameters. Although this is a difficult problem, its solution will give rise to simple laws on the effect of these parameters. Hence, we use the following methods: the dispersion equation is solved in the case of a collisionless plasma, then the energy relations and the variational method enable us to calculate the influence of a single parameter. Thus, in the first part, we give relations between electric, magnetic and kinetic energies for travelling waves along an unmagnetized plasma column. Then, we recall the variation theorem and apply it to the study of the influence of some parameters (density profile, collision frequency, dielectric losses) on the dispersion curve. In the last part, we give numerical results in the case of the azimuthally symmetrical mode \( m = 0 \) (surface wave).

2. Fundamental relations. — 2.1 Simplified model of the propagation structure. — We consider a plasma column (radius \( a \)) surrounded by a cylindrical dielectric of outer radius \( b \) and dielectric constant \( \varepsilon_0 \varepsilon_r = \varepsilon_0 (\varepsilon_r - j\varepsilon_i) \), where the whole system is in a vacuum. We choose \( z \) to be the cylinder axis, so that field dependences are of the form \( \exp(j\omega t - \gamma z) \) with \( \gamma = \alpha + j\beta \). The dispersion curve \( D(\omega, \gamma) = 0 \) is obtained by solving Maxwell’s equations in the three media and applying the boundary conditions to the field components [11]. If we assume that the plasma is homogeneous and without collisions, its relative dielectric constant is

\[ \varepsilon_p = 1 - \frac{\omega_p^2}{\omega^2} \]

where \( \omega_p \) is the plasma frequency. Then, if the surrounding dielectric is a lossless one, the dispersion equation can be easily solved. Furthermore, it is possible to calculate the energy repartition.

2.2 Relations between energies. — From the Maxwell equations, Bers [4] obtains relations between the various energies in the general case of bounded or unbounded plasmas. Following the same method,
we have sought simple relations between the energies in the case of a multilayered, unmagnetized structure (plasma, dielectric, vacuum).

With a lossless system, the conservation law of the power flow along the structure gives a relation called a "resonance relation"

\[ U_m - U_e + U_k = 0 \] (2)

where:
- \( U_m \) is the magnetic energy per unit length
  \[ U_m = \int s W_m \, ds \quad \text{with} \quad W_m = \frac{1}{4} \mu_0 |H|^2 \] (3)
- \( U_e \) is the electric energy per unit length
  \[ U_e = \int s W_e \, ds \] (4)
- \( U_k \) is the electron kinetic energy per unit length
  \[ U_k = \int s W_k \, ds \quad \text{with} \quad W_k = \frac{1}{4} \varepsilon_0 \frac{\omega_p^2}{\omega^2} |E|^2 \] (5)

All the integrals are over appropriate cross sections \( s \).

We note that in the case of a single medium (plasma filled guide), equation (2) is true only if \( \omega_p < \omega \); for some other media, this condition can be verified even if \( \omega_p > \omega \), but in these conditions the electrical energy outside the plasma can balance the kinetic energy in the plasma.

It is also possible to calculate the total electromagnetic energy per unit length

\[ U = U_m + U_e + U_k \] (6)

which can be written, using (2):

\[ U = 2U_e = 2(U_m + U_k) \] (7)

2.3 VARIATIONAL THEOREM. — From the Maxwell equations, an equation [4] between variations in the propagation parameters (\( \gamma \), \( \omega \)) and those of the propagation structure (\( \varepsilon_{pr} \), \( \varepsilon_p \), ...) can be obtained

\[ \delta \beta, P = \delta \omega \int s W_m \, ds + \int s \delta(\omega \varepsilon_p) W_e \, ds + \int s \delta(\omega \varepsilon_e) W_e \, ds \] (8)

where \( W_m \) and \( W_e \) are the energy densities per unit length and \( P \) the power flow

\[ P = \int s E \wedge H^* \cdot \mathbf{i} \, ds \] (9)

where \( \mathbf{i} \) is the unit vector of the propagation direction. From equation (8), we can deduce the group velocity of the wave, and the variations of \( \beta \) or \( \omega \) versus those of \( \varepsilon_p \) or \( \varepsilon_e \). Thus we can easily find the group velocity of the wave

\[ V_g = \frac{\partial \omega}{\partial \beta} = \frac{P}{\int_s (W_m + W_e + W_k) \, ds} = \frac{P}{U} \] (10)

where \( W_k \) is the kinetic energy density per unit length.

3. Applications of the variational theorem. — The use of the variational theorem enables us to study the deformation of the dispersion curve due to the following parameters: radial density distribution, collisions, dielectric losses.

3.1 INFLUENCE OF DENSITY VARIATIONS. — If the density depends on \( r \), the variation of the wave vector is given by:

\[ \delta \beta = \frac{\omega}{P} \int_s W_k(s) \delta \omega^2(s) \, ds \] (11)

where \( \delta \omega^2(s) \) is due to the local density variation and \( \omega_p^2 \) the mean (unperturbed) value.

3.2 INFLUENCE OF COLLISIONS. — As the initial calculation is performed without taking into account collisions, we cannot find the damping associated with the collision frequency \( \nu \); however from equation (11), we can calculate the variation of the wave vector since the dielectric constant of the plasma (weakly modified by the collisions) is of the form [4]:

\[ \varepsilon_p(v) = 1 - \frac{\omega^2_p}{\omega^2 + v^2} - j \frac{v}{\omega} \frac{\omega_p^2}{\omega^2 + v^2} \] (12)

or

\[ \varepsilon_p(v) = \varepsilon_p(v = 0) + \delta \varepsilon_p \] (13)

with

\[ \delta \varepsilon_p = \frac{\omega_p^2}{\omega_p^2 + v^2} \left( \frac{v^2}{\omega^2} - j \frac{v}{\omega} \right) \] (14)

Thus, we obtain, if \( v \ll \omega \)

\[ \alpha_p = + \frac{v}{\omega} \cdot \frac{\omega U_k}{P} \] (15)

Equation (15) shows that so long as \( v \) is small, the wave damping is nearly proportional to \( v \) and the kinetic energy.

3.3 INFLUENCE OF THE DIELECTRIC LOSSES. — In the same way, we can calculate the influence of the dielectric losses (attenuation coefficient \( \alpha_d \)) of a die-
lectric for which the dielectric permittivity is 
\( \varepsilon_a = \varepsilon_0 (\varepsilon_r - j\varepsilon_i) \):

\[
\alpha_d = \frac{\omega \tan \delta \cdot U_{ed}}{P} \tag{16}
\]

where \( \tan \delta = \varepsilon_i / \varepsilon_r \) and \( U_{ed} \) is the electrical energy per unit length in the dielectric.

From equations (15) and (16), we obtain the wave damping:

\[
\alpha = \frac{\omega \tan \delta \cdot U_{ed}}{P} + \frac{v}{\omega} \cdot \frac{\omega U_k}{P} \tag{17}
\]

3.4 IMPORTANCE OF THE KINETIC ENERGY. — All the variations which we calculated above depend on the electron kinetic energy. With equations (6) and (10), we can write:

\[
\frac{U_k}{P} = \frac{V_s}{U} = \left( 1 - \frac{U_m}{U} \right) \frac{1}{V_s} \tag{18}
\]

When the magnetic energy is much smaller than the total energy \( (U_m \ll U) \), the kinetic energy of the electrons tends to half the total energy \( (U_k/U \rightarrow 1/2) \). Such a case corresponds to surface waves where the phase velocity tends to zero (quasistatic approximation). Then:

\[
\frac{U_k}{U} = \frac{1}{2} \tag{19}
\]

So, we find for the damping:

\[
\alpha' = \frac{v}{2 V_s} \text{ when } v \ll \omega \tag{20}
\]

which is often used in propagation studies [1].

In the opposite case, where the phase velocity of the wave is near the velocity of light, the magnetic energy tends to half of the total energy and the kinetic energy tends to zero, so that damping due to collisions also tends to zero.

All these results are valid for every mode propagated in a plasma without a magnetic field (guide mode, plasma eigenmodes). As an example, we shall give some results for the azimuthally symmetrical mode being propagated along a plasma column (surface wave).

4. Surface wave \((m = 0)\). — The propagation of surface waves along of plasma column was shown in 1959 [1]. Nevertheless, the use of these modes to generate a plasma and to characterize the discharges (densities and collision frequency measurements in inhomogeneous plasmas) is recent. Consequently, it is interesting to calculate the dispersion curve and also the damping and energy distribution [5]. As an example, numerical computations have been performed in the case of a given structure:

- plasma radius \( a = 7.5 \times 10^{-2} \text{ cm} \)
- dielectric thickness \((b - a) = 17.5 \times 10^{-2} \text{ cm} \)
- dielectric constant \( \varepsilon_r = 4 \) driven at a frequency \( f = 2450 \text{ MHz} \).

Figure 1 gives:

- the phase \( \beta = 2\pi / \lambda \) \textit{versus} \( \omega / \omega_p \) (\( \omega = \text{constant} \)) in the case of an homogeneous lossless plasma;
- the phase and group velocities \textit{versus} \( \omega / \omega_p \).

The following figures give \textit{versus} \( \omega / \omega_p \):

- the ratio: magnetic energy to total energy (Fig. 2a);
- plasma kinetic energy and dielectric electric energy (Fig. 2b);
- the wave attenuation \( \alpha \) calculated from equation (15) and the usual damping coefficient \( \alpha' = v / 2 V_s \) [6] with \( v = 6 \times 10^8 \text{ rad. s}^{-1} \) (Fig. 3).

All these results show the importance of the energy distribution on the attenuation: \( \alpha \) and \( \alpha' \) are in agreement when the magnetic energy is negligible, i.e. when \( V_m/c \) is smaller than 1; in the opposite case, if the magnetic energy is greater than \( U/10, \alpha \) is lower than \( \alpha' \) (\( \alpha \) tends to zero whereas \( \alpha' \) tends to \( v / 2 c \)).

The importance of the magnetic energy is related to the following fact: in the case of propagation along unmagnetized plasmas, the magnetic field of the wave does not transfer energy to the electrons. A consequence of these results: for the high values of \( \omega / \omega_a \), i.e., for dense plasmas, the surface wave propagation is possible even if the collisions are strong. This result is easily explained: the kinetic energy decreases rapidly...
with $\omega/\omega_p$ (Fig. 2b); the plasma becomes a conductor and the electromagnetic field does not enter the plasma. On figure 4, we have plotted the argument $\Gamma a$ of the Bessel function $I_0(E_z = E_0 I_0(\Gamma a))$. $\Gamma$ is defined by the relation $\Gamma^2 = \beta^2 - \epsilon_0 \omega^2/c^2$ which shows that $\Gamma$ depends on $\omega/\omega_p$ and also $\beta$. In the studies based on the quasistatic assumption, dimensionless parameters are employed, but unfortunately this is impossible here [9].

On figure 5, we have plotted the penetration of the field $E_z$ into the plasma. This penetration is defined by a length $\delta$ such that:

$$\frac{E_z(a)}{E_z(a-\delta)} = 2.$$  \hspace{1cm} (21)

The following remarks must be made:

1) For high values of $\omega/\omega_p$ ($\omega/\omega_p \rightarrow 0.447$), the argument $\Gamma a$ increases rapidly and the field does not penetrate. A detailed study [10] shows that the field decreases very rapidly outside the plasma: in these conditions, we really have a surface wave where the energy is localized near the plasma-dielectric boundary, approximately the same in the plasma and the dielectric.

2) For low values of $\omega/\omega_p$ (i.e. for high densities) $\Gamma a$ increases rapidly and the field does not penetrate but in contrast with the previous case, the field outside the plasma decays very slightly; since the energy is mainly outside the plasma, the wave is not damped by collisions in the plasma.

3) $\Gamma a$ passes through a minimum value for which the field $E_z$ is almost independent of $r$ in the plasma. Near this minimum, we can assume that the field has a constant value in the plasma. Hence, such a created discharge must look like a positive column discharge whereas this is not the case of plasmas generated under the conditions of 1 and 2.
4) Figure 5 gives the kinetic energy in the plasma and the electrical energy in the dielectric. One sees (Fig. 2b) that, except for the low values of \( \omega_0 / \omega_p \), the kinetic energy is always greater than the electrical energy. Consequently, taking into account equation (17), the dielectric losses are generally negligible in the wave propagation. Table I gives some damping constants due to the collisions and to the dielectric in the case of a plasma column generated in a thick capillary tube. In most cases \( \alpha_p > \alpha_d \). However this inequality can be inverted if

\[
\frac{v}{\omega} \cdot \frac{1}{\lg \delta} \leq \frac{U_{ed}}{U_k}
\]

and we give in the last column of table I, the values of \( \frac{v}{\omega} \cdot \frac{1}{\lg \delta} \) such as \( \alpha_d = \alpha_p \).

5. Conclusion. — The use of relations between the various energies of the wave and the variational theorem allows one to calculate the propagation conditions and their variations due to various parameters: structure dimensions, density, collision frequency. Furthermore, this study gives us an explanation of the surface wave damping conditions. Thus we can understand why such waves can be propagated along strongly collisional plasmas (plasma at atmospheric pressure [8]).

Table I. — Damping of the azimuthally symmetrical mode \((m = 0)\) being propagated along a plasma column: 2a \(= 1.5 \times 10^{-3} \) m; 2b \(= 5 \times 10^{-3} \) m, \( \varepsilon_0 = 4 \).

<table>
<thead>
<tr>
<th>( \omega_0 / \omega_p )</th>
<th>( \xi_p (N/m^3) ) ( (v = 10^9) )</th>
<th>( \xi_d (N/m^3) ) ( (lg \delta = 10^{-3}) )</th>
<th>( \frac{v}{\omega} \cdot \frac{1}{\lg \delta} )</th>
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<td>0.002</td>
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References