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Gauge-dependent critical properties of the nematic to smectic-A phase transition in the 1/N-expansion

S. G. Dunn and T. C. Lubensky

Department of Physics, University of Pennsylvania, Philadelphia, PA 19104, U.S.A.

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Abstract. — A generalized de Gennes model for the nematic to smectic-A transition with an N complex component order parameter, \( \Psi \), is studied in the 1/N expansion. This model is similar to the Landau-Ginzburg model for the superconducting transition with deviations, \( \delta n \), of the director from its uniform equilibrium value, \( n_0 \), playing the role of the vector potential. It is, however, not gauge invariant, and properties of \( \Psi \) change under gauge transformations : \( \Psi = \Psi e^{iQ_L} \), \( A = \delta n + V \). Phase fluctuations in \( \Psi \) are a maximum in the physical (LC) gauge with \( \delta n \perp n_0 \) and minimal in the gauge (SC) with \( V.A = 0 \). A continuum of gauges parameterized by an angle \( \theta \) with \( \theta = \theta_0(\theta = \pi/2) \) corresponding to the LC (SC) gauge is introduced. Isotropic and anisotropic critical behaviour is found as in the \( \epsilon \)-expansion. Thermal critical exponents \( \gamma \) and \( \beta \) are independent of gauge. Magnetic exponents \( \gamma \) and \( \eta \) depend continuously on \( \theta \) for \( 0 < \theta < \pi/2 \). The susceptibility \( \chi_{LC} \) in the LC gauge is related to that, \( \chi_{SC} \), in the SC gauge via \( \chi_{LC} = \chi_{SC}(\exp g) \) where \( g \) is negative and more than logarithmically singular in reduced temperature, \( t \), and wave number \( q \) near criticality because the splay elastic constant \( K^0 \) behaves as a dangerous irrelevant variable. Precursors of the smectic phase Landau-Peierls instability in three dimensions appear in \( g \) in the nematic phase. It is argued that this unusual behaviour of \( \chi_{LC} \) may be responsible for mosaicity seen in X-ray experiments.

1. Introduction. — The nematic to smectic-A phase (NA) transition in liquid crystals [1] is one of the most fascinating and least understood transitions currently under study. Liquid crystals are composed of long bar-like molecules. The nematic phase has the translational symmetry of a fluid but a broken rotational symmetry produced by the alignment of the molecular axes along a common direction specified by a unit vector \( n(x) \), called the director. \( n(x) \) may have fluctuations, depending on spatial co-ordinate \( x \), from its uniform equilibrium value \( n_0 \). The smectic-A phase has a layered structure with layer-planes perpendicular to \( n_0 \). The layer structure is described by a mass density wave [2] with wave number \( q_0 \) and complex amplitude, \( \Psi \), which may be viewed as an order parameter distinguishing the smectic-A phase from the nematic phase. Since \( \Psi \) is a complex number, one might argue that the nematic to smectic-A transition should be analogous to the \( \lambda \)-transition in helium. De Gennes [3] was the first to recognize that
this analogy is imprecise. Because the free energy of the
smectic-A phase is invariant with respect to simulta-
neous rotations of \(n\) and the smectic-A planes but not
of each separately, the smectic-A is more analogous
to a superconductor than to a superfluid with,

\[
\delta \tilde{n}(x) = n(x) - n_0
\]

playing the role of the vector potential, \(A(x)\). There
remain, however, important differences between the
superconductor and the liquid crystal. First and most
striking is that \(\Psi\) does not have long range order in
the smectic-A phase [4, 5] whereas the analogous
order parameter in the superconductor does. Second,
the liquid crystal does not have a gauge invariant
free energy because of the presence in the theory of a
free energy density \(\frac{1}{2} K_1 (\nabla \cdot n)^2\) associated with director
splay distortions, where \(K_1\) is the Frank elastic con-
stant for splay. (Throughout most of this paper, we will
use rescaled elastic constants which we will denote
as \(K_1, K_2, K_3\).) Unrescaled physical elastic con-
stants we will denote with a \(<\) bar \(>\) : \(\bar{K}_1, \bar{K}_2, \bar{K}_3\).

Third, the liquid crystal is quite anisotropic. The ratio
of unrenormalized correlation lengths [6-8], \(\xi_\|\) and \(\xi_\perp\), parallel and perpendicular to \(n_0\), is of order 4 to 8.
The destruction of long range order in the smectic
phase was predicted some time ago by Landau and
Peierls [4] and has been observed experimentally
by X-ray scattering [5]. The lack of gauge invariance
and the large anisotropy lead to anisotropic scaling [9]
in the critical region that has been observed experi-
mentally [6-8]. In analogy with the superconductor,
the nematic to smectic-A transition is predicted
theoretically to be weakly first order [10] though
characteristic latent heats and discontinuities in order
parameters appear to be unobservable under present

Even though the smectic free energy is not gauge
invariant, gauge transformations of the form

\[
\psi = \Psi e^{i\bar{\theta} L}
\]

are possible Contrary to the more familiar situation
encountered in electrodynamics and in supercon-
ductors, experimental observables depend on gauge
because the order parameter, \(\Psi\) (in addition to gauge
invariant currents formed from \(\psi\) and \(\psi^*\)) is obser-
vable. Physical observations, however, are limited
to the gauge in which we will call the Liquid Crystal (LC)
gauge, having \(L = 0\) and \(\delta n \perp n_0\). Since experimental
observations can be made in this gauge only, one
might argue that the introduction of different gauges is
uninteresting. The LC gauge, however, presents
formidable calculational difficulties. There are violent
infrared singularities in perturbation theory [9, 10]
when \(K_1 = 0\). Since \(K_1\) is driven to zero relative to the
divergent bend elastic constant, \(K_3\), as the NA
transition is approached, the \(K_1 = 0\) infrared singularities
make it impossible to carry out a renormalization
group \(\epsilon\)-expansion in the LC gauge. On the other hand,
there are no such problems in a gauge in which
\(\nabla \cdot A = 0\). Since this is analogous to the London gauge
used in superconductors [12], we will refer to this as the
superconducting or SC gauge. This gauge was used to
derive a mean field theory for the NA transition
predicting a first order NA transition [10b]. It was
also used in an \(\epsilon\)-expansion [9] to predict anisotropic
critical behaviour when \(K_1 \to \infty\). Thus the gauge
transformation to the SC gauge seems to avoid many
of the problems encountered in the LC gauge.

It is clear that the problems of calculating in the LC
gauge are a result of violent phase fluctuations in that
gauge leading to the Landau-Peierls destruction of
long range order in the smectic phase. In the
nematic phase, these phase fluctuations have receiv-
ed relatively little attention, though one would
expect them to lead to important modifications of the
order parameter correlation functions presaging the
lack of long range order in the smectic phase. In this
paper, we will study the NA transition in a generalized
model with an order parameter having \(N\) complex
\((2N\) real) components, in an expansion to first order
in \(1/N\) for large \(N\). Our motivation for using the
\(1/N\)-expansion [13, 14] is that it is well controlled and
can be applied in all dimensions greater than 2 (not
just near 4 dimensions as in the \(\epsilon\)-expansion).
Furthermore, the pathologies produced by phase fluctu-
ations in the LC gauge should be manifest at order
\(1/N\). Finally, for large \(N\), the NA transition is second
order [9, 10] so that we do not have to worry about the
predicted weak first order transition for \(N = 1\). Thus
the hope was that the \(1/N\)-expansion would suggest
to us how to think about the more interesting problem
with \(N = 1\). Our program has been a success. Not
only does the \(1/N\)-expansion predict unusual beha-
viour for experimentally observable quantities such as
the scattered X-ray intensity, but some general fea-
tures of this behaviour seem to be independent of \(N\)
for \(N \gg 1\). The discussion of \(N = 1\) will be presented
in a future paper.

1.1 SUMMARY OF RESULTS. — Even within the
\(1/N\)-expansion, the singularities in the LC gauge
are difficult to handle, and we found it convenient to
approach the LC gauge continuously. To do this, we
introduce a continuum of intermediate gauges para-
meterized by an angle \(\theta\) such that \(\theta = 0\) corre-
ponds to the LC gauge and \(\theta = \pi/2\) to the SC gauge. We then
study critical properties in the nematic phase for all
gauges \(0 \leq \theta \leq \pi/2\). We find, as required, that ther-
mododynamic quantities such as the free energy, the
specific heat, and the specific heat exponent \(x\) are
independent of gauge. The Frank elastic constants are
also independent of gauge. The lowest order in \(1/N\)
the rescaled and renormalized elastic constants satisfy

\[
K_1 = K_1^\circ
\]

\[
K_{2,3}(\theta) = K_{2,3}^\circ + Nq_0^2 \xi^4 \Delta (1, r^2, \xi^2)
\]
where \( p \) is a wave number, the function \( \Delta(1, x^2) \) is defined in equation (4.4) and \( \zeta = t^{-1/d-2} \) is the correlation length in the \( N = \infty \) limit, and \( t \) is the reduced temperature appropriate to the second order NA transition. \( q_0^2 \) is chosen to be of order \( 1/N \) so that equation (1.2) represents the \( N = \infty \) form of \( K_2,3 \). Note that in this limit, there is no anisotropy in the correlation lengths.

The order parameter depends explicitly on gauge \( \theta \). We find, as expected, that phase fluctuations are minimum in the SC gauge. In this gauge, the order parameter wave number dependent susceptibility is denoted by \( \chi_{SC}(q, t) \). The universality class of the NA transition and thus properties of \( \chi_{SC}(q, t) \) depend on the values of the bare elastic constants \( K_1, K_2, K_3 \) and \( K_0^2 \) as follows:

- \( K_0^2 = \infty \) and/or \( K_1 = K_2 = K_3 = \infty \) : Isotropic N-vector model.
- \( K_1, K_2 \) and \( K_0^2 \neq \infty \) : N-complex component generalized superconductor discussed in reference [10].
- \( K_1 = \infty \) : Universality class with anisotropic scaling discussed in reference [9].
- \( K_2, K_3 = \infty \) : Another class with anisotropic critical properties, probably corresponding to an inaccessible fixed point discussed in reference [9].

In cases c) and d), the correlation length and critical point exponents \( \eta_\| \) and \( \eta_\perp \) for directions parallel to \( n_\| \) and \( n_\perp \), and \( \eta_\perp \) for the SC (\( \theta = \frac{\pi}{2} \)) gauge to order \( 1/N \) are summarized in table I. Note that in all cases the scaling laws

\[
\gamma_{SC} = v_\| (2 - \eta_\|) = v_\perp (2 - \eta_\perp)
\]

are satisfied.

The susceptibility \( \chi_{SC}(q, t) \) in a gauge \( \theta \neq \frac{\pi}{2} \) is related to \( \chi_{SC}(q, t) \) via

\[
\chi_{SC}(q, t) = \chi_{SC}(q, t) e^{\theta g_0(q, t, K_1, K_2, K_3, K_0^2)}.
\]

The function \( g_0(q, t, K_1) \) does not depend on \( K_2 \) and is zero if either \( K_1 \) or \( K_0^2 \) is infinite. We have displayed the variable \( K_1 \) explicitly in \( g_0 \) because of the very special role, to be discussed shortly, that it plays. \( g_\theta \) is explicitly of order \( 1/N \) so that to the order of the present calculations, it displays no effects of critical anisotropy in the correlation lengths, and satisfies the scaling relation

\[
g_{\theta}(q, t, K_1 ; A) = g_{\theta}(q^2, 1, K_1, \xi^{-\epsilon} ; A\xi) .
\]

The dependence on \( A\xi \), where \( A \approx q_0 \) is the upper momentum cutoff, is necessary to produce logarithmic singularities to be discussed shortly. Note that there is no prefactor proportional to some power of \( \xi \) in this expression as is the case for correlation functions of marginal operators [15]. In section 5, we identify the marginal operator as the generator of rotations in the gauge space. Notice that \( K_1 \) is an irrelevant variable since \( \xi^{-\epsilon} \to 0 \) as \( t \to 0 \). The leading singularity in \( g_\theta \) depends on whether \( 0 < \theta < \frac{\pi}{2} \) or \( \theta = 0 \). In the former case, these singularities are logarithmic and independent of \( K_1 \):

\[
g_{\theta}(0, t, K_1) \sim -\delta \gamma_\theta \log t \quad g_{\theta}(0, 0, K_1) \sim -\delta \eta_\theta \log q
\]

where \( \delta \gamma_\theta \) and \( \delta \eta_\theta \) depend continuously on \( \theta \). The logarithmic singularity in \( g_{\theta}(0, 0, K_1) \) is independent of the direction of \( q \) to order \( 1/N \). These relations imply

\[
\gamma_{\|}(0, t) \sim t^{-\gamma_\|} \quad \gamma_{\parallel}(q_{\|,\perp} = 0) \sim q_{\|,\perp}^{-2+\eta_\|,\perp}
\]

where \( \gamma_\| = \gamma_{SC} + \delta \gamma_\| \) and \( \eta_{\|,\perp} = \eta_{\|,\perp}^{SC} + \delta \eta_{\|,\perp} \) depend continuously on \( \theta \). The values of \( \gamma_\|, \eta_{\|,\perp} \) and \( \eta_{\|,\perp}^{SC} \) are listed in equations (5.12) and (5.23). Note that \( \delta \gamma_\| < 0 \) and \( \delta \eta_{\|,\perp} > 0 \). The continuous dependence of critical exponents on \( \theta \) results from the presence of

| Table I. — Critical exponents in SC gauge to O(1/N). |
|-----------------|-----------------|-----------------|-----------------|
| \( K_0^2, K_1^2, K_2, K_3^2 \) | \( K_0^2 = \infty \) | \( K_0^2 = \infty \) | \( K_0^2 = \infty \) |
| All finite | \( K_1^2 = \infty \) | \( K_1^2 = \infty \) | \( K_1^2 = \infty \) |
| \( \frac{(d-2)}{2} \gamma_{SC} - 1 \) | \( -(3+4(d-1)^2) S_d \) | \( -(3+4(d-1)(d-2)) S_d \) | \( -(3+4(d-1)) S_d \) |
| \( \eta_{\|,\perp} - 2 \) \( \frac{4-d}{d} S_d \) | \( -8 \frac{(d-1)^2}{d} S_d \) | \( -8 \frac{(d-1)^2}{d} S_d \) | \( -8 \frac{(d-1)^2}{d} S_d \) |
| \( \eta_{\|,\perp} - 2 \) \( \frac{4-d}{d} S_d \) | \( -8(d-2) S_d \) | \( -8(d-2) S_d \) | \( -8(d-2) S_d \) |
| \( (d-2) v_{\|,\perp} - 1 \) | \( -4(d-1) S_d \) | \( -4(d-1) S_d \) | \( -4(d-1) S_d \) |
| \( (d-2) v_{\perp,\perp} - 1 \) | \( -4(d-1) S_d \) | \( -4(d-1) S_d \) | \( -4(d-1) S_d \) |

\[
(\frac{d-2}{2}) \gamma_{SC} - 1 = -(3+4(d-1)^2) S_d \frac{d}{N} - (3+4(d-1)(d-2)) S_d \frac{d}{N} - (3+4(d-1)) S_d \frac{d}{N}
\]

where \( p \) is a wave number, the function \( \Delta(1, x^2) \) is defined in equation (4.4) and \( \xi = t^{-1/(d-2)} \) is the correlation length in the \( N = \infty \) limit, and \( t \) is the reduced temperature appropriate to the second order NA transition. \( q_0^2 \) is chosen to be of order \( 1/N \) so that equation (1.2) represents the \( N = \infty \) form of \( K_2,3 \). Note that in this limit, there is no anisotropy in the correlation lengths.
a marginal operator [15]. Correlation lengths calculated as a function of θ show that the correlation length exponents are independent of θ. This is required if hyperscaling $2 - \alpha = (d - 1) \nu_\perp + \nu_\parallel$ is to hold since α is independent of θ. Since $\eta_\parallel$ and $\eta_\perp$ are $O(1/N)$ and $\nu_\parallel - \nu_\perp$ is $O(1/N)$, the scaling relation $\gamma_\theta = (2 - \eta_\theta) \nu_\parallel = (2 - \eta_\theta) \nu_\perp$ holds in all gauges.

When $\theta = 0$, $K_1$ becomes a dangerous irrelevant variable [16], i.e. $g_0 \equiv g_{LC}$ is explicitly proportional to some inverse power of $K_1$ and by the scaling relation is more than logarithmically singular.

We find

$$g_{LC}(q = 0, t, K_1) = -\frac{1}{2} B(d) q_0^2 \frac{q^2}{\sqrt{K_1 K_3}} - \delta \gamma_{LC} \log t \quad (1.7a)$$

$$g_{LC}(q_\parallel, 0, K_1) = -\frac{1}{2} C(d) q_0^2 \frac{1}{\sqrt{K_1 q_\parallel}} + \delta \eta_{LC} \log q_\parallel \quad (1.7b)$$

$$g_{LC}(q_\perp, 0, K_1) = \left\{ \begin{array}{ll}
-\frac{1}{2} E_1(q_\parallel) q_0^2 \frac{1}{\sqrt{K_1 q_\perp}} + \delta \eta_{LC} \log q_\perp, & d > 3 \\
-\frac{1}{2} E_2(q_0) q_0^2 \frac{1}{\sqrt{K_1 q_\parallel}} \log \left( \frac{a}{K_1 q_\perp} \right) + \delta \eta_{LC} \log q_\perp, & d = 3 \\
-\frac{1}{2} E_3(q_\perp) q_0^2 \frac{1}{(K_1 q_\parallel)^2} + \delta \eta_{LC} \log q_\perp, & d < 3, 
\end{array} \right. \quad (1.7c)$$

where the expression for $\delta \gamma_{LC} > 0$ to order $1/N$ can be found from equation (5.18).

Equation (1.7a) is only valid for $K_1/K_3 << 1$. A somewhat different form is valid in the opposite limit. As for $\theta \neq 0$, $\delta \gamma_{LC} = -\nu_\parallel \delta \eta_{LC} = -\nu_\parallel \delta \eta_{LC}$.

Equations (1.7) are expressed in terms of the rescaled elastic constants. Furthermore, because of the limitations of the $1/N$ expansion, the isotropic correlation length $\xi$ rather than the anisotropic correlation lengths $\xi_\parallel$ and $\xi_\perp$ appear in equation (1.7). To make comparisons with experiments, we re-express equation (1.7a) in terms of the experimental elastic constants and correlation lengths denoted by bars over the variables. We find

$$g_{LC}(0, t, K_1) \sim -\frac{1}{2} B(d) q_0^2 \frac{k_B T \xi_{\parallel}}{\sqrt{K_1 K_3}} \quad (1.8)$$

where $k_B$ is Boltzmann’s constant and $T$ is the temperature. The factor of the unrescaled but renormalized correlation length, $\xi_{\parallel}$, was obtained using the anisotropic scaling relation discussed in reference [9] and assuming that $g_{LC} \sim K_1^{-1/2}$. This assumption may break down if anisotropic scaling is allowed at all levels of the calculation. Nevertheless, we will use equation (1.8) in the numerical discussion to follow.

1.2 DISCUSSION. — The results just presented are extremely striking, and it is important to know which of them result from the approximations of the $1/N$-expansion and which might be more generally true. We note that the $1/N$-expansion is strictly speaking valid only as long as $\gamma_\theta$ and $\delta \eta_\theta$ are small. As can be seen from equations (5.12) and (5.23), the coefficients of $1/N$ in these quantities diverge as $1/\theta$ as $\theta \to 0$. Thus, the minimum value of $N$ for which the expansion is valid tends to infinity as $\theta \to 0$. At $\theta = 0$, the constraint that $\gamma_{LC}$ be small is particularly relevant. If equation (1.7a) were valid all of the way down to $t = 0$, it would predict that $\chi_{LC}$ would go to zero rather than diverge at the critical point. Clearly we cannot take this result seriously. It is our belief that the following will hold for the case of general $N$:

1) There will be $\theta$ dependent critical exponents for $\theta_c < \theta < \pi/2$ with $\theta_c$ determined by $\gamma(\theta_c) = 0.2$.) For $0 < \theta < \theta_c$, $\chi_{LC}(q, t, K_1)$ will display new singularities controlled partially by the irrelevant variable $K_1$.

3) For $\theta = 0$, $g_{LC}$ will have approximately the form of equations (1.7) for $q$ and $t$ such that $g_{LC} \ll 1$. For $t$ closer to the critical point, there will be important corrections to $g_{LC}$ leading to a divergent $\chi_{LC}$ at the critical point. Thus a function of decreasing $t$ we expect $\chi_{LC}$ to fall below the value predicted by a pure power law as $g_{LC}$ increases from zero, followed by a crossover to another divergent behaviour as $t \to 0$.

We will discuss these conjectures about general $N$ in more detail in a future publication.
We believe that there is experimental evidence for at least the initial stages of the crossover in $x_{LC}(0, t, K_1)$ discussed above. All of the X-ray scattering measurements [6-8] of $x_{LC}$ and correlation lengths $\xi_1$ and $\xi_2$ fall below the straight line on a log-log plot that would be predicted by a pure power law as $t$ decreases. The agreement with a power law straight line is improved if the external magnetic field is increased. This effect is attributed to the mosaicity of the sample in low magnetic field. The low and high field data fall on the same straight line if they are deconvoluted with respect to Gaussian line-width functions which approximate the angular spread in the director due to mosaicity. This deconvolution leads to an increase in $x_{LC}$ by a factor of about 2 at $t \sim 10^{-4}$ in 80 CB and 8 CB, the most carefully studied materials. This behaviour is qualitatively similar to that predicted by equation (1.8). In fact, taking $K_1 \sim 10^{-6}$ dynes [17], $K_3 \sim 300 \times 10^{-6}$ dynes, $q_0 \xi_2 \sim 800$ and $T \sim 300$ K and $B(d=3) \sim 1.8/(2\pi)$ (obtained by numerical integration), we find from equation (1.7) $g_{LC} \sim -0.55$ and $e^{\omega_{LC}} \sim 0.58$ in remarkable agreement with the factor of 1/2 below power law that is observed experimentally. Note that since $|g_{LC}| < 1$, we may expect equation (1.7) to be approximately valid. Further work is, of course, necessary to determine if the present treatment of director fluctuations does in fact explain experimentally observed mosaicity. We are very optimistic at the moment though we are as yet uncertain if inclusion of magnetic fields as small as 2000 G can lead to a significant reduction in $g_{LC}$.

It is instructive to compare the theory presented here with the Kosterlitz-Thouless theory [18, 19] of a two dimensional superfluid. In both cases phase fluctuations play a very important role. In the 2d superfluid, there is a complex order parameter $\psi = |\psi| e^{i\phi}$. The superfluid transition occurs at a temperature well below the mean field transition temperature so that $|\psi|$ is essentially constant, and phase fluctuations are dominant. In the liquid crystal, we have $\psi_{LC} = \psi_{SC} e^{-i\omega L}$, so that $q_0 L$ and $\phi$ play a similar role. In this case, $\psi_{SC}$ as well as $L$, has critical fluctuations. Thus singularities in the LC gauge near the NA transition are a result of both amplitude and phase fluctuations, whereas those in the 2d superfluid are dominated by phase fluctuations. In the 2d superfluid, it is necessary to consider vortex unbinding to get a complete picture of the transition. A consideration of dislocations in the smectic phase appears to be unnecessary to understand the NA transition. A description of the transition in terms of dislocations is, of course, possible [20, 21]. We have not yet had a chance to compare our theory with the recent dislocation theory of Nelson and Toner [21].

This paper is divided into four sections in addition to the introduction. Section 2 defines the generalized de Gennes model and introduces the continuum of gauges connecting the LC and SC gauges. Section 3 discusses the gauge dependence of various operators. Section 4 presents the details of the $1/N$ calculations in the SC gauge, and section 5 the calculations for the other gauges. In addition there are several appendices dealing with various technical details. This paper is quite technical. Important results, however, can be understood by reading the introduction and section 2.

2. The model. — We begin by defining the de Gennes model [3] for the NA transition, and performing an anisotropic volume-preserving rescaling, as in references [9] and [10]. In the nematic phase the rodlike molecules of which the liquid crystal consists align parallel to one another. The orientation is given by a unit vector $n(x)$, called the director, which may have local variations $\delta n(x)$ from its uniform equilibrium value $n_0$:

$$n(x) = n_0 + \delta n(x).$$

In figure 1, we indicate that the coordinate direction $\hat{e}_1$ is chosen to be $n_0$, so that for small deviations $\delta n(x)$ one has the important constraint

$$\hat{e}_1 \cdot \delta n(x) = 0.$$  

Thus, if there are $d$ spatial dimensions, $\delta n$ must lie in the $d-1$ dimensional subspace orthogonal to $\hat{e}_1$. The constraint is regarded as analogous to a gauge condition in electrodynamics in that it leaves the field $\delta n(x)$ with $d-1$ dynamically independent components. As one cools down into the smectic-A phase the molecules organize into planar layers perpendicular to $n$, so that if $\rho(x)$ is the centre-of-mass molecular density then a Fourier decomposition of $\rho$ has the form

$$\rho(x) = \rho_0 \left\{ 1 + \frac{1}{\sqrt{2}} \left[ \Psi(x) e^{i\psi_{sc}x} + (C.C.) \right] + \cdots \right\}.$$  

(2.3)

![Fig. 1. — Geometry in p-space, showing 1-2 plane and t-subspace.](image)
Here the layer spacing is 2π/q₀. In the de Gennes model, one neglects the higher Fourier components. The field Ψ(x) is the (position dependent) complex amplitude of the mass density wave of wavevector q₀ n, and contains spatial variations on length scales greater than the layer spacing.

The de Gennes Hamiltonian in three spatial dimensions is given by

\[
H = H_S \{ \Psi, \delta n \} + H_N \{ n \}
\]

\[
H_S = \int \left\{ \frac{1}{2} A^0 |\Psi|^2 + \frac{1}{2} U^0 |\Psi|^4 + C^0_\perp |\nabla_\perp \Psi|^2 + C^0_\parallel (\nabla_\perp - i q_0 \delta n) \Psi \right\} \mathrm{d}^3x
\]

\[
H_N = \int \left\{ \frac{1}{2} \vec{K}_{\perp}(\nabla \cdot n)^2 + \frac{1}{2} \vec{K}_{\parallel}(n \cdot \nabla \times n)^2 + \frac{1}{2} \vec{K}_{\parallel}(n \times (\nabla \times n))^2 \right\} \mathrm{d}^3x,
\]

(2.4)

where \( A^0 = a(T - T_{e0})/T \) and \( T_{e0} \) is the mean field transition temperature. The \( \vec{K} \) appearing in \( H_N \) are the bare values of the Frank elastic constants appropriate to the nematic phase, and \( \nabla_\perp = \nabla - \hat{e}_1 \nabla_1 \). The bare correlation lengths \( \xi_\perp^0 \) and \( \xi_\parallel^0 \) are equal to \( \sqrt{C^0_\perp/a} \) and \( \sqrt{C^0_\parallel/a} \), respectively. We now perform an anisotropic rescaling of lengths and of \( \delta n \) so that \( \xi_\perp^0 \) and \( \xi_\parallel^0 \) are both replaced by a common value \( C^0_\perp \), while \( q_0 \) remains unchanged. We choose the Brillouin zones for both \( \Psi \) and \( \delta n \) to be spheres of radius \( \Lambda \). Universal quantities such as critical exponents will not depend on the numerical value of \( \Lambda \approx q_0 \) so lengths are further rescaled to make \( \Lambda \to 1 \).

In addition, we rescale \( \Psi \) so that the coefficient of \( |\nabla \Psi|^2 \) in \( \beta H = H/\beta \) is unity. Finally, we generalize \( \Psi \) to a vector \( \Psi \), with \( N \) complex (2N real) components. Thus in arbitrary spatial dimensions, \( d \), we obtain

\[
\beta H_S = \int \left\{ r_0 |\Psi|^2 + \frac{1}{2} u_0 |\Psi|^4 + (\nabla - i q_0 \delta n) \Psi \right\} \mathrm{d}^3x
\]

\[
\beta H_N = \int \left\{ \frac{1}{2} K_\perp^0 (\nabla \cdot n)^2 + \frac{1}{2} K_\parallel^0 \sum_{i>j=1} (\nabla n_i - \nabla n_j)^2 + \frac{1}{2} K_\parallel^0 \sum_{i>j=1} (\nabla n_i n_j)^2 \right\} \mathrm{d}^3x,
\]

(2.5)

where

\[
|\Psi|^2 = \sum_{i=1}^N \Psi_i^* \Psi_i, \quad (\nabla - i q_0 \delta n) \Psi = \sum_{i=1}^N [(\nabla + i q_0 \delta n) \Psi_i^*].[(\nabla - i q_0 \delta n) \Psi_i],
\]

and where the new coupling constants are defined by

\[
r_0 = A^0/C^0 \Lambda^2
\]

\[
u_0 = a/\beta A^0(C^0)^2
\]

\[
K_\perp^0 = \beta K_\perp^0[\Lambda(C^0)^{2-d}/d]
\]

\[
K_\parallel^0 = \beta K_\parallel^0[\Lambda(C^0)^{2-d}/d]
\]

\[
(C^0)^d = C^0/(C^0)^{d-1}
\]

\[
\xi_\perp = \frac{\Lambda}{\xi_\perp^0}, \quad \xi_\parallel = \frac{\Lambda}{\xi_\parallel^0}
\]

(2.6)

We observe that the model (2.5) is very similar to the Landau-Ginzburg model of a superconductor. That model may be written in the form [12]

\[
\beta H_{LG} = \beta H_S \{ \Psi, A \} + \frac{1}{2} K \int \sum_{i<j}(\nabla A_j - \nabla A_i)^2 \mathrm{d}^3x,
\]

(2.7)

where the generalized smectic field \( \Psi \) is now replaced by the superconducting order parameter \( \Psi \),

\[
q_0 = 2 e/hc, \quad \text{and} \quad K = \beta \mu_0/4 \pi
\]

The electromagnetic vector potential \( A \), which has \( d-1 \) dynamically independent components by virtue of the gauge condition imposed upon it, replaces the director variation \( \delta n \) which is similarly constrained by equation (2.2). Comparing equations (2.5) and (2.7) we see that \( \beta H_{LG} \) has exactly the same form as \( \beta H_{1,0} \), but the constraints on \( A \) and \( \delta n \) may be different. In general the liquid crystal may have anisotropy in the elastic field, \( K_\perp^0 \neq K_\parallel^0 \), and possesses the splay term \( 1/2 K_\parallel^0 (\nabla \cdot n)^2 \) which destroys the gauge invariance which otherwise would be present as in the superconductor. Specifically, \( \beta H_{LG} \) obeys, for any function \( L(x) \) of the coordinates \( x \),

\[
\beta H_{LG} \{ \Psi(x) e^{i q_0 L(x)}, A(x) + \nabla L(x) \} = \beta H_{LG} \{ \Psi(x), A(x) \},
\]

(2.8)
This provides the freedom to choose whatever gauge is most convenient for performing calculations. Below we shall reformulate the de Gennes model (2.5) in a fashion that provides similar freedom in the liquid crystal.

We define gauge changes with the substitution

\[ \psi_{\rho}(x) = \Psi_{\rho}(x) e^{i\Omega_{\rho}(x)} \]

\[ A_{\rho}(x) = \phi n(x) + \nabla L_{\rho}(x), \]  

where the real function \( L_{\rho}(x) \) is determined by the choice of gauge condition:

\[ v(\theta, p).A_{\rho}(p) = 0 \]

\[ v(\theta, p) = \hat{e}_1 \cos \theta + i \hat{p} \sin \theta. \]  

(2.9)

where \( A_{\rho}(p) \) is the spatial Fourier transform of \( A_{\rho}(x) \), \( v(\theta, p) \) is a unit vector, and \( \hat{p} = p/|p| \). The vector \( v(\theta, p) \), as discussed more fully below, is chosen so that \( \theta = 0 \) corresponds to the original gauge condition, equation (2.2), which we refer to as the liquid crystal (LC) gauge. Furthermore, for \( \theta = \pi/2, v(\theta, p) \) is equal to \( i\hat{p} \). This gauge, with \( A_{\theta=\pi/2} \) transverse to \( i\hat{p} \), will be called the superconducting (SC) gauge as discussed in the introduction. Such gauge changes have been used [10] to facilitate comparison with superconductors, and to develop a mean field theory for \( \Omega_{1/2} \). The result is that while there is long range order (LRO) in the SC gauge, it is destroyed by fluctuations in the phase function \( L_{1/2}(x) \) used to connect the LC gauge with the SC gauge. Thus the LC gauge lacks LRO. Gauge changes have also been incorporated in each iteration of a renormalization group (RG) transformation [9] in order that the thinning of degrees of freedom always be performed in the SC gauge. Our intention is to investigate the gauge change in detail by introducing, in equation (2.10), a continuum of gauges intermediate between LC and SC, parameterized by the number \( \theta \) which runs from 0 to \( \pi/2 \). In the present paper the \( 1/N \)-expansion is chosen as a well-controlled laboratory for carrying out the investigation, but we will discuss some general properties of our gauge changes, and will offer some general remarks on the gauge dependence of correlation functions of \( \Omega_{1/2} \) and \( A_{\rho} \).

We now discuss how \( A_{\rho}(p) \) is related to \( \phi n(p) \). We begin with the geometry indicated in figure 1. The 1-2 plane is determined by the vectors \( \hat{e}_1 \) and \( i\hat{p} \); calling \( p_2 = |p - \hat{p}|, \hat{e}_2 \) we define

\[ \hat{e}_2 = (\hat{p} - \hat{p}_1 \hat{e}_1)/\hat{p}_2. \]  

(2.11)

We define a projection operator onto the directions orthogonal to the 1-2 plane by

\[ \mathcal{P}_1 = 1 - \hat{e}_1 \hat{e}_1 - \hat{e}_2 \hat{e}_2. \]  

(2.12)

As indicated in figure 2, the vector \( v(\theta, p) \) defining the

Fig. 2. — Bases used in the 1-2 plane.

\[ v(\theta, p) = \hat{e}_1 \cos \theta + i \hat{p} \sin \theta \]

gauge \( \theta \) also lies in the 1-2 plane. It is useful to define a vector \( \hat{e}_1(\theta, p) \) in the 1-2 plane, orthogonal to \( v(\theta, p) \):

\[ v(\theta, p) \cdot \hat{e}_1(\theta, p)^* = 0 \]

\[ \hat{e}_1(\theta, p) \cdot \hat{e}_1(\theta, p)^* = 1 \]  

(2.13)

One then easily obtains

\[ \hat{e}_1(\theta, p) = \frac{i}{p_2} \left[ (\sin \theta + i\hat{p}_1 \cos \theta) \hat{e}_1 + \right. \]

\[ + \left. (\cos \theta + i\hat{p}_1 \sin \theta) i\hat{p}_2 \right], \]  

(2.14)

where the phase of \( \hat{e}_1 \) has been chosen to give

\[ \hat{e}_1(\theta = 0, p) = \hat{e}_2. \]

Clearly for any \( \theta \), the unit-modulus vectors \{ \( v(\theta, p), \hat{e}_1(\theta, p) \) \} provide a basis in the 1-2 plane, and

\[ \mathcal{P}_1 = 1 - v(\theta, p)^* v(\theta, p) - \hat{e}_1(\theta, p)^* \hat{e}_1(\theta, p) \]

is the projection operator onto the \( t \)-subspace expressed in this basis. Now one finds from equation (2.9) that

\[ A_{\theta=\pi/2} = \mathcal{P}_1 A_{\theta=\pi/2} \hat{\phi} n(\theta), \]  

(2.15)

The gauge condition (2.10) only permits \( A_{\theta} \) to have components in the \( t \)-subspace and in the \( \hat{e}_1^* \) direction:

\[ A_{\theta} = A_{\theta=\pi/2} + A_{\theta=\pi/2} \hat{e}_1(\theta, p) \]

\[ A_{\theta=\pi/2} = \hat{e}_1(\theta, p)^* \hat{e}_1(\theta, p) \]  

(2.16)

Using \( v(\theta).A_{\theta=\pi/2} = 0 = \hat{e}_1, \hat{\phi} n_2 \) and equation (2.9) leads to

\[ L_{\phi}(p) = \frac{1}{i\hat{p}_1} \hat{e}_1(\theta, p)^* A_{\theta=\pi/2}(p) = \frac{1}{p \cdot v(\theta)} \hat{e}_2 \]  

(2.17)
Using the definitions of $e_1$, $e_2$, and $v$ this reduces to

$$A_{\phi\perp}(p) = \frac{\hat{p}_1}{(\hat{p}_1 \cos \theta + i \sin \theta) \delta n_2(p)}. \quad (2.18)$$

Returning to the de Gennes model in the form of equation (2.5) we observe that $\beta H_N$ is gauge invariant:

$$\beta H_N \left\{ \psi, \delta n \right\} = \beta H_N \left\{ \psi_\theta, A_\theta \right\}. \quad (2.19)$$

On the other hand, for $\beta H_N$ we write

$$\beta H_N = \frac{1}{2} \int \left\{ (K_0^0 p_1^2 + K_0^0 p_1^3) | \delta n_2(p) |^2 + (K_0^0 p_2^2 + K_0^0 p_3^3) | \delta n(p) |^2 \right\}$$

$$= \frac{1}{2} \int \left\{ (K_0^0 p_1^2 + K_0^0 p_3^3) \left(1 - \frac{\hat{p}_1^2 \cos^2 \theta}{\hat{p}_1^2}\right) | A_{\phi\perp}(p) |^2 + (K_0^0 p_2^2 + K_0^0 p_3^3) | A_\theta(p) |^2 \right\}, \quad (2.20)$$

where in the last step equations (2.15) and (2.18) have been used. Here \( \int p \equiv \int d^dp/(2 \pi)^d \). In the final line of equation (2.20) explicit $\theta$-dependence has appeared in $\beta H_N$.

3. General properties of gauge changes. — The model in gauge $\theta$ is described by $\beta H_\phi \left\{ \psi_\theta, A_\theta \right\}$, where the subscript on $H$ indicates explicit $\theta$-dependence. The explicit and implicit dependences must cancel because $\beta H_\phi \left\{ \psi_\theta, A_\theta \right\} = \beta H \left\{ \Psi, \delta n \right\}$ arose from a substitution, so that

$$\frac{d}{d\theta} \beta H_\phi \left\{ \psi_\theta, A_\theta \right\} = 0. \quad (3.1)$$

The thermodynamic properties of the system as described in gauge $\theta$ are determined by the partition function

$$Z_\theta = \mathcal{D}\psi_\theta \mathcal{D}A_\theta e^{-\beta H_\phi \left\{ \psi_\theta, A_\theta \right\}}. \quad (3.2)$$

While $\beta H_\phi \left\{ \psi_\theta, A_\theta \right\} = \beta H \left\{ \Psi, \delta n \right\}$, the functional integration indicated in equation (3.2) is different from that appearing in

$$Z_{LC} = \mathcal{D}\Psi \mathcal{D}\delta n e^{-\beta H \left\{ \Psi, \delta n \right\}}. \quad (3.3)$$

In Appendix A, we calculate the Jacobian for the change of variables from $\left\{ \Psi, \delta n \right\}$ to $\left\{ \psi_\theta, A_\theta \right\}$. The result is given by

$$\int \mathcal{D}\psi_\theta \mathcal{D}A_\theta = J_\theta^{-1} \int \mathcal{D}\Psi \mathcal{D}\delta n \quad (3.4a)$$

As an example we shall derive a formula for $dG_\theta/d\theta$, where

$$G_\theta(x, x') = \langle \psi_\theta(x) \psi_\theta(x')^* \rangle = \langle \Psi(x) \Psi(x')^* e^{i\phi(x) - i\phi(x')} \rangle \quad (3.8)$$

is the smectic correlation function in gauge $\theta$. In section 4 we shall calculate $G_{sc} = G_\theta = \delta \phi/\pi$, so that from the

$$\log J_\theta = \frac{1}{2} N_0 K_d^{-1} \int_0^\theta \tan \theta' I_d(\cos^2 \theta') d\theta', \quad (3.4b)$$

where $N_0$ is the total number of states, and

$$K_d^{-1} = 2d^{-1} \pi^{d/2} \Gamma(d/2)$$

as in reference [13]. The function $I_d(\cos^2 \theta)$, which we shall encounter frequently, is discussed in Appendix B. The result (3.4b) for $J_\theta$ is a non-singular function of $\theta$. The important point is that the Jacobian is independent of the fields $\psi_\theta, A_\theta$ so that it factors out of the functional integral, as indicated in equation (3.4a), and one has

$$F_\theta = F_{LC} - T \log J_\theta \quad (3.5)$$

for the free energy. Evidently such thermodynamic quantities as the specific heat are $\theta$-independent and may be calculated using whatever $\theta$ is most convenient. Furthermore, the statistical average of any operator $Q$ may be evaluated in any gauge $\theta$ merely by expressing $Q$ via equations (2.9) and (2.17) in terms of operators appropriate to the gauge $\theta$. Thus, we have

$$\langle Q \rangle_\theta = \int \mathcal{D}\psi_\theta \mathcal{D}A_\theta Q \exp(-\beta H_\phi \left\{ \psi_\theta, A_\theta \right\}) \quad (3.6)$$

$$= \langle Q \rangle_{LC}. \quad (3.7)$$

Hence no subscript is needed on the averaging, and one writes simply $\langle Q \rangle$. In general the operator $Q$ can depend explicitly and implicitly on $\theta$, and one has

$$\frac{d}{d\theta} \langle Q \rangle_\theta = \left\langle \frac{d}{d\theta} Q_\theta \right\rangle. \quad (3.7)$$

where the functional integration indicated in equation (3.2) is different from that appearing in

$$Z = \mathcal{D}\psi_\theta \mathcal{D}A_\theta e^{-\beta H_\phi \left\{ \psi_\theta, A_\theta \right\}}. \quad (3.2)$$

While $\beta H_\phi \left\{ \psi_\theta, A_\theta \right\} = \beta H \left\{ \Psi, \delta n \right\}$, the functional integration indicated in equation (3.2) is different from that appearing in

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In Appendix A, we calculate the Jacobian for the change of variables from $\left\{ \Psi, \delta n \right\}$ to $\left\{ \psi_\theta, A_\theta \right\}$. The result is given by

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As an example we shall derive a formula for $dG_\theta/d\theta$, where

$$G_\theta(x, x') = \langle \psi_\theta(x) \psi_\theta(x')^* \rangle = \langle \Psi(x) \Psi(x')^* e^{i\phi(x) - i\phi(x')} \rangle \quad (3.8)$$

is the smectic correlation function in gauge $\theta$. In section 4 we shall calculate $G_{sc} = G_\theta = \delta \phi/\pi$, so that from the
formula for \( \frac{dG}{d\theta} \) one may find \( G_\theta \) in an arbitrary gauge \( \theta \). Consider an infinitesimal gauge change from \( \theta \) to \( \theta + d\theta \)

\[
G_{\theta + d\theta}(x, x') = G_{\theta}(x, x') + iq_0 \left\langle \psi_{\theta}(x) \bar{\psi}_{\theta}(x') (L^\theta_{\theta}(x) - L^\theta_{\theta}(x')) \right\rangle \delta \theta + O(\delta \theta^2)
\]

where the last equality follows from equation (2.17), and \( v' = \frac{d}{d\theta} v(\theta, p) \). \( L^\theta_{\theta}(x) \) is the generator of local gauge transformations in \( \psi_{\theta}(x) \). Because \( L^\theta_{\theta} \) is linear in \( A_\theta \), we encounter averages of the form \( \left\langle \psi_{\theta} \bar{\psi}_{\theta} A_{\theta} \right\rangle \). Therefore, we introduce the three-point vertex function \( \Gamma_{ij} \):

\[
\Gamma_{ij}(x, x_2, y) = \frac{\delta G_{\theta}^{-1}(x, x_2)}{\delta \langle A_{\theta}(y) \rangle}
\]

Upon insertion in equation (3.9) and Fourier transformation, the result is

\[
\frac{d}{d\theta} \log G_{\theta}(q) = q_0 \int_p v(\theta, p) \frac{\partial}{\partial \langle A_{\theta}(y) \rangle} D_{\theta}(p) \big[ \Gamma_{ij}(q - p, q - p) G_{\theta}(q - p) - \Gamma_{ij}(q, q + p, p) G_{\theta}(q + p) \big].
\]

In order to evaluate the gauge dependence of \( G_{\theta}(q) \) to leading order in the 1/N-expansion in section 5, we shall insert in equation (3.12) the leading order form of \( \Gamma_{ij} \). Integration of (3.12) with respect to \( \theta \) then allows one to find \( G_{\theta}(q) \), provided one knows \( G_{\theta_0}(q) \) for some gauge \( \theta_0 \).
We notice that this function decreases as $\theta$ increases, and is smallest in the SC gauge. Returning to $\langle \Psi_\theta \rangle$ we have

$$\langle \Psi_\theta (x) \rangle = \int \langle u_\theta (p) \rangle \langle \psi_{\theta}(x) \rangle$$

and

$$\langle \Psi_\theta \rangle \approx \langle \Psi_{SC} \rangle \exp \left[ -\frac{1}{2} q_0^2 \int \langle u_\theta (p) \rangle^2 \right]$$

Equation (3.16b) follows from equation (3.16a) because the Hamiltonian (3.14) is quadratic in $u$. We conclude that $\langle \Psi_\theta \rangle$ is largest in the SC gauge, where $\theta = \pi/2$, and decreases monotonically as $\theta$ decreases. The smallest value of $\langle \Psi_\theta \rangle$ occurs in the LC gauge, $\theta = 0$. Focussing on the small-p regime of the cumulant, in the LC gauge, we find

From Appendix B we have $I_d (1 - \alpha^2) \approx 1/2 \alpha$ for small $\alpha$, and so we conclude that the $p$ integral in equation (3.17) has an IR divergence for $d \geq 3$. Thus $\langle \Psi_{LC} \rangle = 0$ for $d \leq 3$, a reflection of the Landau-Peierls instability [5] of the smectic-A structure for $d \geq 3$. We also find that for any $d$, $\langle \Psi_{LC} \rangle$ vanishes as $\exp\left[-\text{constant} \times K_0^{-1/2}\right]$ for $K_0 \to 0$, indicating that a non-zero value of $K_0$ is necessary to maintain the stability of the smectic-A phase. In all other cases (i.e. $K_0 \neq 0$ and $d > 3$, or for $\theta \neq 0$) $\langle \Psi_\theta \rangle$ does not vanish, and it is largest in the SC gauge.

$\langle \Psi_{LC} \rangle \approx \langle \Psi_{SC} \rangle \exp\left[-\frac{K_0^{-1}}{4} \int p^2 \int \langle u_\theta (p) \rangle^2 \right]$.

From Appendix B we have $I_d (1 - \alpha^2) \approx 1/2 \alpha$ for small $\alpha$, and so we conclude that the $p$ integral in equation (3.17) has an IR divergence for $d \geq 3$. Thus $\langle \Psi_{LC} \rangle = 0$ for $d \leq 3$, a reflection of the Landau-Peierls instability [5] of the smectic-A structure for $d \geq 3$. We also find that for any $d$, $\langle \Psi_{LC} \rangle$ vanishes as $\exp\left[-\text{constant} \times K_0^{-1/2}\right]$ for $K_0 \to 0$, indicating that a non-zero value of $K_0$ is necessary to maintain the stability of the smectic-A phase. In all other cases (i.e. $K_0 \neq 0$ and $d > 3$, or for $\theta \neq 0$) $\langle \Psi_\theta \rangle$ does not vanish, and it is largest in the SC gauge.

3.2 DECOUPLING APPROXIMATION. — The SC gauge apparently best describes the order emerging at the transition, so it is natural to try « decoupling in the SC gauge » as mentioned in reference [9]:

$$G_d (x, x') = \langle \psi_{SC} (x) \psi_{SC} (x') \rangle$$

where $L_\theta = L_\theta \pm 1/2$.

The decoupling approximation will agree with the low-temperature version of the model provided we assume that the $1-2$ plane component of the SC gauge director correlation function acquires a « mass » of $(2 q_0^2 - \Psi^2)^{1/2}$ at temperatures well below the transition. At temperatures closer to the transition this expression for the mass (i.e. $\lim_{|p| \to 0} \langle \psi_{SC} (x) \rangle^2$) may not be correct, but that subject will not be pursued here.

We may derive from equation (3.18) an approximate formula for $d \log G(q)/d\theta$. Differentiating with respect to $\theta$, Fourier transforming, and using the Ward identity obeyed by $\Gamma_{\theta}^d$ (discussed in Appendix C) we find

$$\frac{d}{d\theta} \log G_d (q) = q_0 \int \left[ \frac{v(\theta, p)}{v(\theta, p_\theta)} D_d (p_\theta, p) D_d (p) \right] \left[ \Gamma_q (q, p, q, p) - \Gamma_q (q, q + p, q, -p) \right]$$

Obviously equation (3.19) is very similar to equation (3.12). In section 5 we shall employ the exact formula, equation (3.12), in the $1/N$-expansion, but shall comment briefly on the consequences of using the decoupling formula, equation (3.19) in the $1/N$-expansion.

3.3 DIRECTOR CORRELATIONS. — We may easily leads to discuss the gauge dependence of the director correlation function in gauge $\theta$ by considering a gauge change from $\theta$ to $\theta'$. Applying $v(\theta', p)$ to

$$A_\theta (p) = A_0 (p) + i p \Delta L(p)$$

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and we find

$$D_{\theta}(p)_{ij} = \left( \delta_{ik} - \frac{p_i \nu^{(\theta', p)}_{ik}}{p \nu^{(\theta', p)}} \right) \times D_{\theta}(p)_{kk} \left( \delta_{kj} - \frac{\nu^{(\theta', p)}_{kj}}{\nu^{(\theta', p)^*}} \right).$$

(3.22)

If one knows $D_{\theta_0}$ in some gauge $\theta_0$, one can then find $D_\theta$ for any $\theta$. For instance, one observes immediately from the form of $\nu$ equation (2.20), that in the LC gauge

$$D_{\theta}(p)_{ij} = D_{\theta}(p) (\xi')_{ij} + D_{\theta}(p) (\xi_2)_{ij}. \quad (3.23)$$

In lowest order perturbation theory one has, for $T > T_c$,

$$D_1(p) = (K_1^2 p_1^2 + K_3^3 p_3^2)^{-1} \quad (K_2^2 + K_3^3 p_3^2)^{-1}.$$

(3.24)

One may easily make symmetry arguments that $D_1$ and $D_2$ have this form more generally [22], with $K_2$ and $K_3$ replaced by fluctuation renormalized elastic constants $K_2^2$ and $K_3^3$ which depend on $p$ and on the reduced temperature $t = \frac{T - T_c}{T_c}$. In addition, there have been numerous observations [6-8, 17] that in the nematic phase $K_2$ and $K_3$ display pretransitional enhancements as one approaches the N-A transition, while $K_1$ remains essentially constant through the transition. In section 4 we shall discuss the $1/N$-approximation to these renormalizations. Insertion of equation (3.23) into equation (3.22) leads to

$$D_{\theta}(p)_{ij} = D_1(p) (\xi')_{ij} + D_2(p) (\xi_2)_{ij}.$$

(3.25)

We conclude that the gauge dependence of $D_\theta$ is rather simple: The part acting on the $t$-subspace is gauge invariant, while the acting on the 1-2 plane obeys a covariance rule. Once the coefficient $D_1(p)$ of

$$\frac{\hat{p}_1^*}{1 - \hat{p}_1^2 \cos^2 \theta} \epsilon_{1}(\theta, p)^* \epsilon_{1}(\theta, p)$$

is known in some gauge $\theta_0$, the 1-2 part of $D_\theta(p)$ is known in every other gauge $\theta$.

4. 1/N-expansion in SC gauge for $T > T_c$. — In reference [13] Ma argues that because critical exponents are universal, the numerical values of coupling constants may be chosen for expedience in calculations. Here we choose $u_0 = O(1/N)$ and $q_0 = O(1/N)$. The number, $N$, of complex components of the field $\psi_i$ will be treated as a large number. Although the case of experimental interest is $N = 1$, we will use $1/N$ as a small parameter for expansion of critical exponents and thermodynamic functions.

Referring to the self-energy diagrams for $G_a$ illustrated in figure 4, we observe $\Sigma_a$ is of order unity, while $\Sigma_b$ to $\Sigma_f$ are of order $1/N$. Following reference [13], we write Dyson's equation for the Green's function of equation (3.8) for $2 < d < 4$:

$$G_a^{(1)}(r, q) = r + q^2 + \Sigma_a(r, q) - \Sigma_a(r, 0) \quad (4.1)$$

$$r = r_0 - r_{oc} + \Sigma_a(r_0) - \Sigma_a(0, 0) + O(1/N)$$

$$r_0 - r_{oc} = r + \frac{2 u_0 N J}{d - 2} \rho^{d-2} + O(1/N)$$

Fig. 4. — Self-energy diagrams for $G$ to $O(1/N)$. 

where $J = \frac{1}{2} K_2 \pi^{(\frac{1}{2} d - 1)} \csc \pi^{(\frac{1}{2} d - 1)}$. Equations (4.1) are valid for all gauges $\theta$, since the order unity contributions to $\Sigma_a$ are independent of $\theta$. In this section we will be most concerned with $\theta = \pi/2$, though the results for arbitrary $\theta$ to be discussed in the next section can be obtained with proper interpretation by evaluating $\Sigma_a$ directly to order $1/N$ in any gauge $\theta$. We will use the notation $G_{sc}(q) \equiv G_{0 = \pi/2}(q)$. The reduced temperature $t = \frac{T}{T_c} - 1$, where $T_c$
is the renormalized transition temperature, is proportional to $r_0 - r_{oc}$:

$$r_0 - r_{oc} \sim t. \quad (4.2)$$

In order to discuss the contributions to $G_{sc}$ (or, indeed, to $G_\theta$) at $O(1/N)$ we need to evaluate the director correlation function $D$ at order unity.

To this end consider the order-unity contributions $\pi_a + \pi_b$ to the polarization function $\pi_{ij}$ where

$$D_{ij}^{-1} = D_{0ij}^{-1} - \pi_{ij},$$

as illustrated in figure 5

$$\pi(p)_{ij} = \pi_a(p)_{ij} + \pi_b(p)_{ij} + O(1/N)$$

$$= Nq_0^2 \int \left[ \frac{1}{p + (k + 1 2)} \right] \left( r + (k - 1 2) p \right)^2 \right] - (p = 0) + O(1/N)$$

$$- Nq_0^2 p^2 \Delta(r, p^2) \Phi(p)_{ij}, \quad (4.3)$$

where for any vector $W$, $\Phi(W)_{ij} = \delta_{ij} - W_i W_j / |W|^2$, and where

$$\Delta(r, p^2) = \int_0^1 2 \alpha(2 - \alpha) \left[ r + p^2 \alpha(1 - \alpha) \right] d\alpha = p^{d-4} \Delta(r/p^2, 1). \quad (4.4)$$

In the second line of equation (4.3) we have drawn on the discussion of Ward identities in Appendix C, and in the final line have used results of Appendix D. Thus we may write

$$D^{-1}(p)_{ij} = D^{(0)}_{ij} + Nq_0^2 p^2 \Delta(r, p^2) \Phi(p)_{ij} + O(1/N) \quad (4.5)$$

a result which holds for any $\theta$. We therefore find in the SC gauge, to leading order,

$$D_{sc}(p)_{ij} = \frac{1}{K_2 p_2^2 + K_3 p_3^2} (\Phi)_{ij} + \frac{p_1^2}{K_1^0 p_2^2 + K_3 p_3^2} \epsilon_{sc}(p)^\star \epsilon_{sc}(p)$$

$$\tilde{K}_{2,3}(r, p^2) = K_2^0 + Nq_0^2 \Delta(r, p^2). \quad (4.6a)$$

Equation (4.6b) implies $\tilde{K}_{2,3}(r, p^2 = 0) \sim -t^{1/2}$ as $t \to 0$. This is in agreement with the general scaling laws (to order zero in the $1/N$-expansion) which predict $\tilde{K}_{2,3} \sim -t^{-\alpha}$. Employing the rules for gauge dependence of $D_\theta$

as discussed in reference to equation (3.25) we conclude that

$$D_\theta(p)_{ij} = \frac{1}{K_2 p_2^2 + K_3 p_3^2} (\Phi)_{ij} + \left( p_1^2 \right) \left( 1 - \frac{1}{p_2^2 \cos^2 \theta} \right) \frac{1}{K_1^0 p_2^2 + K_3 p_3^2} \epsilon_{sc}(0, p) \epsilon_{sc}(0, p). \quad (4.7)$$

For the present we confine our attention to the SC gauge, and so we simply use equation (4.6a).

### 4.1 Calculation of $\gamma_{sc}$

Returning to $G_\theta(q)$ we begin by considering the susceptibility $\chi_\theta = G_\theta(q = 0)$, which is proportional to $t^{-\gamma_\theta}$, and we immediately obtain from equation (4.1) that

$$\gamma_\theta = \frac{2}{d - 2} + O(1/N). \quad (4.8)$$

Thus for $2 < d < 4, \gamma_\theta > 1$, and so $r \propto (r_0 - r_{oc})^{\gamma_\theta} \ll r_0 - r_{oc}$ for $r_0 - r_{oc}$ small. This implies that

$$\Sigma_\theta(0, 0) - \Sigma_\theta(r, 0) \approx r_0 - r_{oc} \sim r^{1/\gamma_\theta}. \quad (4.9)$$

If we expand $\gamma_\theta$ in powers of $1/N$,

$$\gamma_\theta = \frac{2}{d - 2} + \frac{\gamma_1 \theta}{N} + O(1/N^2),$$

$$\Sigma_\theta(0, 0) - \Sigma_\theta(r, 0) \propto r^{d-2/2} \left( 1 - \left( 1 - \frac{1}{2} \right)^{\gamma_1 \theta} \log r + O(1/N^2) \right). \quad (4.10)$$
The leading term arises from $\Sigma_a$, as above. The remaining self-energy diagrams will be investigated at small $r$ for contributions behaving as $r^{d-2}/2 \log r$. $\Sigma_b(r, 0)$ and $\Sigma_c(r, 0)$ have been evaluated by Ma in reference [13] and will not be re-evaluated here. In the SC gauge, $\Sigma_a(r, 0) = 0$. Also one easily shows that $\Sigma_a(r, 0)$ has no terms as large as $r^{d-2}/2 \log r$ for small $r$. The diagram $\Sigma_i$ represents the insertion of $\Sigma_b(r, q') - \Sigma_b(r, 0)$ on the internal $G(q')$ line in the graph of $\Sigma_a$, just as $\Sigma_e$ represents the insertion of $\Sigma_b(r, q') - \Sigma_b(r, 0)$. Because $\Sigma_a(r, 0) = 0$ in the SC gauge, 

$$ \Sigma_i^{SC}(r) = N u_0 q_0^2 \int_{p,k} \frac{(2k)(2k)}{(r + k + p)^2} D_{SC}(p) \cdot \quad (4.11) $$

We may evaluate this by using 

$$ \int \frac{(2k + p)(2k + p)}{(r + k + p)^2} = \frac{1}{2} \frac{\partial}{\partial r} \int \frac{(2k + p)(2k + p)}{(r + k)^2} (r + (p + k)^2) $$

$$ = -2 \frac{\partial}{\partial r} F_{ij}(r, p) + 4 \int_k \frac{k_k k_j}{(r + k^2)^3} $$

$$ = \frac{1}{2} p^2 \delta^{ij} \frac{\partial}{\partial r} A(r, p^2) + J r^{d-4}/2 \delta_{ij} $$

$$ = \delta^{ij} \pi(r, p^2) + \delta^{ij} J r^{d-4}/2 , $$

where $F_{ij}$ is the function discussed in Appendix D, and 

$$ \pi(r, p^2) = J \int_0^1 \frac{1}{r + p^2 + \alpha(1 - \alpha)} d^d x = p^{d-4} \pi(r/p^2, 1) $$

as in reference [13]. Upon insertion of equation (4.12) into equation (4.11) one finds 

$$ \Sigma_i^{SC}(r, 0) = N u_0 q_0^2 \int \pi(r, p^2) \left[ \frac{d - 2}{K^2 p_2^2 + K_3 p_1^2} + \frac{\delta^{ij}}{K^0 p_2^2 + K_3 p_1^2} \right] . $$

(4.14)

One may readily check that for $p^2 < r$ the integral in equation (4.14) contributes terms of order $r^{d-1}$ or smaller to $\Sigma_i^{SC}(r, 0) - \Sigma_i^{SC}(0, 0)$. So we confine our attention to the regime of integration $p^2 > r$, where we may use

$$ \pi(r, p^2) = \pi(0, 1) \left[ 1 + 2(d - 3) \frac{r}{p^2} + O \left( \frac{r^2}{p^4} \right) \right] + \frac{2 J}{1 - d/2} \left( \frac{r}{p^2} \right)^{4 - 1} \left[ 1 - 4 \frac{r}{p} + O \left( \frac{r^2}{p^4} \right) \right] $$

$$ A(r, p^2) = \pi(0, 1) \left[ 1 + 1 - 2 \frac{r}{p} + O \left( \frac{r^2}{p^2} \right) \right] + \frac{2 J}{1 - d/2} \left( \frac{r}{p^2} \right)^{4 - 1} \left[ 1 + 4 \frac{r}{p} + O \left( \frac{r^2}{p^2} \right) \right]. $$

(4.15a)

(4.15b)

It turns out that the leading term for non-zero $r$ and $d < 4$ in each of these expressions, $\frac{2 J}{1 - d/2} \left( \frac{r}{p^2} \right)^{4 - 1}$, will contribute to the exponent $y_{SC}$:

$$ \Sigma_i^{SC}(r, 0) - \Sigma_i^{SC}(0, 0) = N q_0^2 u_0 (d - 2) \int \pi(0, 1) \frac{p^{d-6}}{K_2(0, p^2) \sin^2 \phi + K_3(0, p^2) \cos^2 \phi} \times $$

$$ \times \left[ \frac{\pi(r, p^2)}{\pi(0, p^2)} \frac{K_2(0, p^2) \sin^2 \phi + K_3(0, p^2) \cos^2 \phi}{K^2(0, p^2) \sin^2 \phi + K^3(0, p^2) \cos^2 \phi} - 1 \right] + N q_0^2 u_0 \int \pi(0, 1) \frac{p^{d-6}}{K^2 \sin^2 \phi + K^3(0, p^2) \cos^2 \phi} \times $$

$$ \times \left[ \frac{\pi(r, p^2)}{\pi(0, p^2)} \frac{K^2 \sin^2 \phi + K^3(0, p^2) \cos^2 \phi}{K^2(0, p^2) \sin^2 \phi + K^3(0, p^2) \cos^2 \phi} - 1 \right] \approx \frac{4 J}{\pi(0, 1)} \frac{d - 1}{4 - d} u_0 \frac{1}{2 \pi} \int_0^{\infty} \sin^{d-2} \phi \, d\phi \sin(\sin^2 \phi) $$

(4.16a)

where

$$ f(\mu) = \log \left[ \left( \frac{K^2 \mu + K_3(0, r)(1 - \mu)}{K_0 \mu + K_3(0, r)(1 - \mu)} \left( \frac{K_2(0, r) \mu + K_3(0, r)(1 - \mu)}{K_3(0, r)(1 - \mu)} \right)^{d-2} \right] . $$

(4.16b)
In deriving equation (4.16a) we have restricted the range of $p$ integration to $p^2 > r$, and because of the benign large-$p$ behaviour of the terms retained have allowed the upper cutoff to be infinite. It is clear that for $r$ small, $f(p) \sim \log \left( \frac{r^{d-1}(d-4)}{2} \right)$, except in a small interval in $\varphi$ centred at $\varphi = \pi/2$ in which $f \sim \log \left( \frac{r^{d-2}(d-4)}{2} \right)$. The size of this interval varies as $\sqrt{K_0}^{d-2}$ so that its contribution to the integral in the final line of equation (4.16a) is unimportant for $r \rightarrow 0$. Obviously $\Sigma^SC(r, 0) - \Sigma^SC(0, 0)$ has contributions of order $r^{d-1} \log r$, which will affect the exponent $\gamma_{sc}$ and the $\varphi$-dependence of $f(\sin^2 \varphi)$ is relatively unimportant. In order to extract the $r$-dependence we define the angular average

$$\langle \langle f(\sin^2 \varphi) \rangle \rangle_d = \frac{1}{\pi} \int_0^{\pi} f(\sin^2 \varphi) \sin^{d-2} \varphi \, d\varphi \, d\varphi.$$  

We expand $f(\sin^2 \varphi)$ about the average value of $\sin^2 \varphi$ :

$$\langle \langle \sin^2 \varphi \rangle \rangle_d = \frac{d-1}{d}$$  

$$\langle \langle f(\sin^2 \varphi) \rangle \rangle_d = f(\langle \langle \sin^2 \varphi \rangle \rangle_d) + \frac{1}{2} \langle \langle \sin^2 \varphi - \langle \langle \sin^2 \varphi \rangle \rangle_d \rangle^2 \rangle_d f''(\langle \langle \sin^2 \varphi \rangle \rangle_d) + \cdots$$  

It is straightforward to check that for $f(p)$ given by equation (4.16b), the leading term in equation (4.18b) behaves as $\log \left( \frac{r^{d-1}(d-4)}{2} \right)$, and all the subsequent terms in equation (4.18b) approach finite constants as $r \rightarrow 0$. Thus we find from equation (4.16a),

$$\Sigma^{SC}(r, 0) - \Sigma^{SC}(0, 0) \approx 8JSd \left\{ \frac{d-1}{4-d} \right\} u_0 \rho^{d-2/2} \left\{ \log \left( \frac{K_0^{(d-1)} + K_3^{(0)}}{K_1^{(d-1)} + K_3^{(0)}} \right) + \log \left( \frac{K_2^{(0)}(d-1) + K_3^{(0)}}{K_2^{(d-1)} + K_3^{(0)}} \right) \right\} \rightarrow$$  

$$S_d \equiv \frac{K_2}{2 \pi(0, 1)}.$$  

Using equations (4.6b), (4.10), and (4.19) one can easily find $\gamma_{sc}$ to order $1/N$. The value of $\gamma_{sc}$ depends on the values of the bare elastic constants $K_0^{(0)}$, $K_1^{(0)}$, $K_3^{(0)}$. If all are infinite, there is no contribution from director fluctuations and $\gamma_{sc}$ reduces to its value for a $\phi^4$ theory with $q_0 = 0$. If $K_0^{(0)}$ is infinite, or if $K_1^{(0)}$ and $K_3^{(0)}$ are both infinite, the situation is the same as if all three are infinite. If $K_1^{(0)}$ is infinite but $K_2^{(0)}$ and $K_3^{(0)}$ are finite, only the second term in equation (4.19) contributes to $\gamma_{sc}$ whereas if $K_2^{(0)}$ is infinite but $K_1^{(0)}$ and $K_3^{(0)}$ are finite only the first term contributes. Finally if $K_1^{(0)}$, $K_2^{(0)}$, and $K_3^{(0)}$ are all finite, both terms in equation (4.19) contribute. The values of $\gamma_{sc}$ correct to order $1/N$ are tabulated in table I.

4.2 Calculation of $\eta_{\parallel sc}$ and $\eta_{\perp sc}$. — We now turn to the evaluation of the exponents $\eta_{sc}$ and $v_{sc}$. For this discussion we will label as $\parallel$ and $\perp$ the directions parallel and perpendicular, respectively, to the direction of equilibrium nematic alignment $\hat{e}_1$. Thus $G^{sc\parallel}(r = 0, q_\parallel) \sim q^{2-\eta_{sc}^{\parallel}}$ and $G^{sc\perp}(r = 0, q_\perp) \sim q^{2-\eta_{sc}^{\perp}}$ for small $q_\parallel$, $q_\perp$. Because $\eta$ has no contributions at order unity, we expect $\eta = 0(1/N)$, so that

$$G^{sc\parallel}(r = 0, q) = q^2 + \Sigma^{SC}(0, 0) - \Sigma^{SC}(0, 0) \sim q^2 - \eta_{sc}^{\parallel} q^2 \log q + O(1/N^2).$$  

We wish to extract from $\Sigma^{SC}(0, q) - \Sigma^{SC}(0, 0)$ any parts which vary as $q^2 \log q$ for small $q$. Ma [2] has discussed $\Sigma_b$ and we shall concern ourselves with the only other $q$-dependent diagram $\Sigma_{e}^{SC}$. One easily shows that

$$|q \cdot \hat{e}_{1,sc}(p)|^2 = (p_2^2 q_\parallel - p_1 p_2 q_\perp)^2/p_2^2$$  

$$q \cdot \Sigma_{e}^{SC} q = p_2^2 q_\parallel^2/p_2^2,$$

where we use the notation $q = q_\parallel \hat{e}_1 + q_\perp$ and $p_2 = p_2 \perp q_\perp/q_\perp + p_3$ for performing the $p$ integrations

$$\Sigma_{e}^{SC}(0, q) - \Sigma_{e}^{SC}(0, 0) = \Sigma_{e}^{SC}(0, q) =$$  

$$= -4 q_0^2 \int_{2\pi} dp_1 \int_{2\pi} dp_2 \int_{2\pi} dp_3 \left( p_2^2 + q^2 + 2 p_1 q_\parallel + 2 p_2 q_\perp \right)^{-1} \times$$
where the angles are defined by

\[
\begin{align*}
\phi_1 &= p \cos \varphi_1 \\
|p_2| &= p \sin \varphi_1 \\
\phi_2 &= p \sin \varphi_1 \cos \varphi_2 \\
|p_3| &= p \sin \varphi_1 \sin \varphi_2.
\end{align*}
\]

As usual in such calculations of \( \eta \), the range of \( \mathbf{p} \) integration for \( |\mathbf{p}| < |\mathbf{q}| \) gives contributions of order \( q^2 \) or smaller, which can be neglected relative to the \( q^2 \log q \) terms we seek for small \( |\mathbf{q}| \). We integrate only for \( p^2 > q^2 \), and so may approximate the integrand by writing

\[
\begin{align*}
\Sigma_{c,sc}(0, q) &\approx -4 q_0^2 K_{d-2} (2 \pi)^2 \int_0^{2 \pi} \int_0^{2 \pi} \sin^{d-2} \varphi_1 \, d\varphi_1 \int_0^{2 \pi} \sin^{d-3} \varphi_2 \, d\varphi_2 \times \\
&\times \left\{ \frac{q_2^2 \sin^2 \varphi_2}{K_2(0, p^2) \sin^2 \varphi_1 + K_3(0, p^2) \cos^2 \varphi_1} + \cos^2 \varphi_1 \frac{q_2^2 \sin^2 \varphi_1 - q_1^2 \cos^2 \varphi_1 \cos^2 \varphi_2}{K_2(0, p^2) \sin^2 \varphi_1 + K_3(0, p^2) \cos^2 \varphi_1} \right\} \\
&\approx -4 \frac{1}{4 - d} \frac{K_{d-1} N a(0, 1)}{2 \pi} \int_0^{2 \pi} \sin^{d-2} \varphi_1 \, d\varphi_1 \left\{ \left( -\frac{d-2}{d-1} \right) q_2^2 \log \left( \frac{K_2(0, q^2) \sin^2 \varphi_1 + K_3(0, q^2) \cos^2 \varphi_1}{K_2(0, q^2) \sin^2 \varphi_1 + K_3(0, q^2) \cos^2 \varphi_1} \right) + \\
&\quad + \left( -\frac{d-1}{d} q_2^2 \sin^2 \varphi_1 + \frac{1}{d} q_2^2 \cos^2 \varphi_2 \right) \log \left( \frac{K_2(0, q^2) \sin^2 \varphi_1 + K_3(0, q^2) \cos^2 \varphi_1}{K_2(0, q^2) \sin^2 \varphi_1 + K_3(0, q^2) \cos^2 \varphi_1} \right) \right\},
\end{align*}
\]

where we have taken the upper cutoff to infinity. The discussion is now quite similar to that of equation (4.16a) with the result

\[
\begin{align*}
\Sigma_{c,sc}(0, q) &\approx -8 \frac{S_d}{4 - d} \left\{ (d-2) q_2^2 \log \left( \frac{(d-1) K_2(0, q^2) + K_3(0, q^2)}{(d-1) K_2^0 + K_3^0} \right) + \\
&\quad + \left[ (d-1)^2 q_2^2 \log \left( \frac{(d+1) K_2^0 + K_3(0, q^2)}{(d+1) K_2^0 + K_3^0} \right) \right].
\end{align*}
\]

The values of \( \eta_{sc} \) implied by equation (4.25) again depend on the values of \( K_2^0, K_3^0, K_3^0 \). If all are finite or \( K_3^0 = \infty \) and/or \( K_2^0 \) and \( K_3^0 \) are infinite, \( \eta_{sc} = \eta_{1,sc} \) and there is no anisotropy. If \( K_2^0 \) or \( K_3^0 \) is infinite and the other two elastic constants are finite, \( \eta_{2,sc} \neq \eta_{1,sc} \) and there is critical anisotropy. The results for \( \eta_{1,sc} \) and \( \eta_{1,sc} \) to order \( 1/N \) are summarized in table I. Note that \( \eta_{1,sc} \) depends only on \( K_2^0 \) and \( K_3^0 \) whereas \( \eta_{1,sc} \) depends on all elastic constants.

4.3 Calculation of \( v_{sc,sc} \) and \( v_{1,sc} \).—We now consider the calculation of the correlation lengths and the associated exponents. The correlation length \( \xi \) along a given coordinate direction \( \hat{e}_i \) is given by

\[
\xi_{\hat{e}_i}^2 = \left[ \frac{\partial}{\partial (q_i^2)} G_{\hat{e}_i}^{-1}(r, q) \right]_{q=0} = G_{\hat{e}_i}(r, 0) \left[ 1 + \frac{\partial}{\partial (q_i^2)} (\Sigma_{\hat{e}_i}^0, q + \Sigma_{\hat{e}_i}^0, q) \right]_{q=0} + O \left( \frac{1}{N \tau} \right). 
\]

Because \( \Sigma_{\hat{e}_i}^0 \) and \( \Sigma_{\hat{e}_i}^0 \) are \( O(1/N) \), one has for \( N \rightarrow \infty \), \( \xi = r^{-1/2} \sim r^{-1/(d-2)} \), and so \( v = \frac{1}{d-2} + O(1/N) \). Contri-
butions to the exponent $\nu_0$ at order $1/N$ may be extracted from the coefficient of $\log r$ in an asymptotic expansion of $\frac{\partial \Sigma_0}{\partial q_0^2}$ for small $r$. Beginning with $\Sigma_0$ one finds

$$\frac{\partial \Sigma_0(r, q)}{\partial q_0^2}|_{q_0^2=0} = u_0 \int \frac{1}{1 + u_0 N \pi(r, p^2)} \left( \frac{4}{d} \frac{p^2 - r - p^2}{(r + p^2)^3} \right). \tag{4.27}$$

The contribution to this integral from $p^2 < r$ is of order $r^{d/2}$ or smaller and cannot affect $\nu$. In the regime $r < p^2 < 1$, we ignore $r$ compared to $p^2$ and find

$$\frac{\partial \Sigma_0(r, q)}{\partial q_0^2}|_{q_0^2=0} \approx \left( \frac{4-d}{d} \right) \frac{2 S_3}{N} \int_1 r^1 \frac{dp}{p} = - \left( \frac{4-d}{d} \right) \frac{S_3}{N} \log r. \tag{4.28}$$

This is all that is required to find $\nu$ in the « chargeless » case, $q_0 = 0$.

To undertake the case $q_0 \neq 0$, we proceed as in equations (4.22) and (4.23) to write

$$\Sigma_{c}^c(r, q) = \Sigma_{c}^c(r, 0) = -4 q_0^2 \frac{K_2(q_0)}{(2 \pi)^2} \int_0^\infty \frac{p^2 - 3 d \pi}{r + p^2} \sin^{d-2} \varphi_1 \, d\varphi_1 \int_0^{\pi} \sin^{d-3} \varphi_2 \, d\varphi_2 \times \left\{ \frac{q_0^2 \sin^2 \varphi_2}{K_2(r, p^2) \sin^2 \varphi_1 + K_3(r, p^2) \cos^2 \varphi_1} + \cos^2 \varphi_1 \frac{q_0^2 \sin^2 \varphi_1 + q_0^2 \cos^2 \varphi_1 \cos^2 \varphi_2}{K_1^2 \sin^2 \varphi_1 + K_3(r, p^2) \cos^2 \varphi_1} + O(q_0^4) \right\}. \tag{4.29}$$

The regime $p^2 < r$ gives terms which are finite as $r \to 0$, so as before we focus on the regime $r < p^2 < 1$. We then may replace $r + p^2 \to p^2$, $K_2(r, p^2) \to K_2(0, p^2)$, etc. We replace the unit upper cutoff on $p$ with $\infty$, and perform the elementary integration and $p$ integration to obtain

$$\Sigma_{c}^c(r, q) \approx \left( \frac{4-d}{d} \right) \frac{2 S_3}{N} \int_1 r^1 \frac{dp}{p} = - \left( \frac{4-d}{d} \right) \frac{S_3}{N} \log r. \tag{4.30}$$

Once again we use the averaging procedure defined in equations (4.17) and (4.18) to find

$$\Sigma_{c}^c(r, q) \approx -4 q_0^2 \frac{K_2(q_0)}{(2 \pi)^2} \int_0^\infty \frac{p^2 - 3 d \pi}{r + p^2} \sin^{d-2} \varphi_1 \, d\varphi_1 \int_0^{\pi} \sin^{d-3} \varphi_2 \, d\varphi_2 \times \left\{ \frac{q_0^2 \sin^2 \varphi_2}{K_2(r, p^2) \sin^2 \varphi_1 + K_3(r, p^2) \cos^2 \varphi_1} + \cos^2 \varphi_1 \frac{q_0^2 \sin^2 \varphi_1 + q_0^2 \cos^2 \varphi_1 \cos^2 \varphi_2}{K_1^2 \sin^2 \varphi_1 + K_3(r, p^2) \cos^2 \varphi_1} + O(q_0^4) \right\} \tag{4.31}$$

The correlation length exponents to order $1/N$ in the SC gauge for the various possible values of $K_1^0$, $K_2^0$ and $K_3^0$ are summarized in table I. One may easily verify that they satisfy the scaling relation

$$\gamma = \nu(2 - \eta) \tag{4.32}$$

for the cases in which the exponents are isotropic, namely, for $K_1^0$ all finite, for $K_2^0 = \infty$, or equivalently for $q_0 = 0$. For the other cases, $K_1^0 = \infty$ or $K_2^0 = \infty$ (but not both) the exponents are anisotropic. (That is, $\nu_{\parallel SC} \neq \nu_{\perp SC}$ and $\eta_{\parallel SC} \neq \eta_{\perp SC}$), and they satisfy the scaling relations

$$\gamma = \nu(2 - \eta) = \nu(2 - \eta). \tag{4.33}$$

Now that all the SC gauge results are before us we may offer some qualitative comments. Although the $1/N$-expansion states that $K_{2,3}(r, 0)$ actually diverge as $t \to 0$, one observes in experiments [6-8] evolution in the elastic constant $K_3$ by a factor of order twenty and $K_2$ somewhat less for the samples available and temperatures accessible at present. This suggests that these experiments penetrate about two decades into the temperature range in which coupling of $\psi$ to the director fluctuations is significant. One might imagine that more realistic temperature dependent elastic constants $K_2(t)$, $K_3(t)$ ought to replace the $1/N$-renormalized ones in equations (4.19) and (4.31), but the usefulness of such a procedure is unclear. It should be stressed that not only is $N$ not
large in the physically interesting case $N = 1$, but also that the SC gauge susceptibility has yet to be connected with the physical LC gauge. In section 5 we shall develop such a connection.

4.4 COMPARISON WITH PREVIOUS WORK. — Although the above critical exponents do not have direct relevance to experiment, they may be compared to the results of the renormalization group [9]. There one finds, for sufficiently large $N$, an isotropic fixed point having $K^*_1 = 0$, stable with respect to $K_1$, and an anisotropic fixed point with $K^*_1 = \infty$, unstable with respect to $K_1^{-1}$. For trajectories beginning with large finite $K_1$ one predicts crossover from exponents appropriate to the anisotropic fixed point to those appropriate to the isotropic fixed point, as the irrelevant parameter $K_1$ is driven towards zero.

In the $1/N$-expansion, one may make analogous descriptions of the situation. In this case $K^*_0$ is fixed and $\tilde{K}_0$ and $\tilde{K}_3$ grow as we proceed to smaller $t$ or $q^2$. For $K^*_0$ very large and $t$ or $q^2$ not too small, the term involving $K^*_0$ is approximately equal to its $K^*_0 = \infty$ limit and the effective critical exponents are anisotropic. As we proceed to smaller $t$ or $q^2$, these terms contribute in a substantial way to the exponents, giving isotropic critical exponents.

We may also note that in the RG there are physically inaccessible anisotropic fixed points, one with $K^*_2 = \infty$, one with $K^*_2 \sim -N^2$, and one with $K^*_2 \sim -1$ for large $N$. The $1/N$-expansion to this order produces an anisotropic fixed point with $K^*_2 = \infty$. We believe that this corresponds to the fixed point with $K^*_2 \sim -N^2$ found in the $\varepsilon$-expansion.

It is clear that the RG and $1/N$-expansion provide qualitatively similar descriptions of the critical behavior. Notice that the thinning of degrees of freedom in the RG is performed in the SC gauge, and it is in the SC gauge that the $1/N$-expansion has been performed so far. We may, in fact, compare the results quantitatively. We read the critical exponents listed in table 1 to order $\varepsilon$, where $\varepsilon = 4 - d$, and find

\[
v_{sc}(K^*_0 \neq \infty) = \frac{1}{2}(1 + \frac{1}{2} \varepsilon - 24 \varepsilon/N) + O(\varepsilon^2, 1/N^2)
\]

\[
\eta_{sc}(K^*_0 \neq \infty) = -9 \varepsilon/N + O(\varepsilon^2, 1/N^2)
\] (4.34)

\[
v_{||sc}(K^*_0 \neq \infty) = \frac{1}{2}(1 + \frac{1}{2} \varepsilon - 27 \varepsilon/2 N) + O(\varepsilon^2, 1/N^2)
\]

\[
v_{\perp sc}(K^*_0 \neq \infty) = \frac{1}{2}(1 + \frac{1}{2} \varepsilon - 35 \varepsilon/2 N) + O(\varepsilon^2, 1/N^2)
\]

\[
\eta_{|| sc}(K^*_0 \neq \infty) = O(\varepsilon^2, 1/N^2)
\]

\[
\eta_{\perp sc}(K^*_0 \neq \infty) = -8 \varepsilon/N + O(\varepsilon^2, 1/N^2).
\] (4.35)

For large $N$ one may extract from the recursion relations [23] in reference [9] the exponents $v$ and $\eta$ corresponding to the appropriate fixed points. For the isotropic fixed point having $K^*_1 = 0$ one finds exactly the exponents in equations (4.34), and for the anisotropic fixed point having $K^*_1 = \infty$ one finds exactly those given in equations (4.35).

5. 1/N-expansion in intermediate and LC gauges for $T > T_c$. — We now wish to investigate the behaviour of the order parameter correlation function $G_\theta(p, q)$ for other values of $\theta$ than the SC gauge value of $\pi/2$. One approach would be the straightforward use of the self-energy diagrams in figure 4 for arbitrary $\theta$, as we did for $\theta = \pi/2$ in the previous section. A less laborious method is to employ the general formula for the $\theta$-dependence of $G_\theta$, equation (3.12), inserting for the vertex function $V_\theta$ its leading order in $1/N$ value, and find

\[
\Gamma_\theta(p + q, q ; p) = -q_0(p_i + 2q_i).
\] (5.1)

When this is done, the expression for $d/d\theta \log G_\theta(q)$ is manifestly order $1/N$, so we insert in the integral the order unity form of $D_\theta(p)$, and $G(p) = (r + p^2)^{-1}$, with $r \sim t^{2 \varepsilon d - 2}$. The only terms we need to keep are those which are even functions of the integration variable, and so we find

\[
\frac{d}{d\theta} \log G_\theta(q) = -2q_0^2 \int \frac{p_i^2}{(1 - \vec{p}_i^2 \cos^2 \theta)^2} \frac{\sin \theta \cos \theta}{p^2(K^*_0 p_2^2 + \tilde{K}(r, p^2 p_i^2))} \times
\]

\[
\times \frac{(2p_i \hat{e}_i - (1 + \hat{p}_i^2) p_i)(p(r + p^2 + q^2) - 4q(p,q))}{[r + (p - q)^2][r + (p + q)^2]}.
\] (5.2)

Because the leading-order forms of $\Gamma_\theta$ and $G$ are $\theta$-independent, we may integrate with respect to $\theta$. The result is

\[
G_\theta(q) = G_{sc}(q) \exp g_\theta(q)
\] (5.3a)

\[
g_\theta(r, q, K^*_0) = q_0 \int \frac{\cos^2 \theta}{(1 - \vec{p}_i^2 \cos^2 \theta)^2} \times
\]

\[
\times \frac{p_i^2}{p^2(K^*_0 p_2^2 + \tilde{K}(r, p^2 p_i^2))} \times
\]

\[
\times \frac{(2p_i \hat{e}_i - (1 + \hat{p}_i^2) p_i)(p(r + p^2 + q^2) - 4q(p,q))}{[r + (p - q)^2][r + (p + q)^2]}.
\] (5.3b)
Remembering that $K_3$ is given by equation (4.6b), it is easy to verify that $g_8$ satisfies the scaling relation
\[ g_8(r, q; K_1^0) = g_8(b_r^2 r, bq; b^{-e} K_1^0) = g_8(1, r^{-1/2} q, r^{1/2} K_1^0), \]
i.e., $g_8$ scales like the correlation function of marginal operators. Expressed in terms of $r$ and $\xi = r^{-1/2}$, this scaling relation is identical to equation (1.4). Thus one would expect $g_8(r, q = 0, K_1^0) \sim \log r$ and $g_8(0, q, K_1^0) \sim \log q$ since $K_1^0$ is irrelevant. However, when $\theta = 0$, $g_{\theta=0}$ diverges as $K_1^0 \to 0$ for all values of $r$ and $q$ because
\[ 1 - q^2 \cos^2 \theta = \xi^2 \quad \text{when} \quad \theta = 0. \]

By rescaling $p_2$, in this case, one can see that $g_{\theta=0}(q = 0) \sim K_1^{-1/2}$ for small $K_1^0$. Thus $K_1^0$ is a dangerous irrelevant variable when $\theta = 0$.

5.1 EVALUATION OF $g_8(r, q = 0)$ FOR $\theta > 0$. We will now use equation (5.3) to evaluate critical exponents as a function of $\theta$. We begin with the calculation of the susceptibility $\chi(t) = G_d(t, q = 0)$. If we are to have the usual power law behaviour $\chi \sim t^{-\nu}$, then the most singular portions of $g_8(q = 0)$ at small $t$ must behave as $\log t \propto \log r$. So let us examine the small-$r$ behaviour of
\[ g_8(r, q = 0) = - q_0^2 \int_0^{\infty} \frac{\cos^2 \theta}{(1 - p_2^2 \cos^2 \theta)} \frac{p_1^2 p_2^2}{(K_1^0 p_2^2 + K_3(r, p_2^2) p_1^2)} \frac{1}{r + p_2^2}. \]

We now define $s = K_1^0/\bar{K}_3(r, p_2^2)$ and
\[ P_d = \int_0^{\infty} \frac{\cos^2 \theta \cos^2 \varphi \sin^4 \varphi}{(1 - \cos^2 \theta \sin^2 \varphi) (1 - [1 - s] \sin^2 \varphi)} \frac{d \varphi}{2 \pi}. \]

Using the identity
\[ P_d = \frac{\sin^2 \theta}{\sin^2 \theta - s} I_d(\cos^2 \theta) - \frac{\cos^2 \theta}{\sin^2 \theta - s} \frac{s}{1 - s} I_d(1 - s), \]
the expression (5.5) for $g_8(r, q = 0)$ becomes
\[ g_8(r, q = 0) = - q_0^2 K_{d-1} \int_0^{1} \frac{p^{d-3} dp}{(r + p^2) \bar{K}_3(r, p^2)} \left[ \frac{\sin^2 \theta}{\sin^2 \theta - s} I_d(\cos^2 \theta) - \frac{\cos^2 \theta}{\sin^2 \theta - s} \frac{s}{1 - s} I_d(1 - s) \right]. \]

This will be seen to effect a separation of the behaviour for $\theta = 0$ and $\theta \neq 0$, and we shall discuss these two cases separately.

For non-zero values of $\theta$ we argue that the second term on the RHS of equation (5.8) is not singular enough at small $r$ to affect the critical properties of $g_8(r, 0)$. To see this we examine two regimes of $p$-integration:
\[ p^2 < r = \xi^{-2} \quad \text{and} \quad p^2 > r = \xi^{-2}. \]

For $p^2 < r, s \approx K_1^0/\bar{K}_3(r, 0) \sim r^{1/2}$, so we employ the small-$s$ approximation to $I_d(1 - s)$,
\[ I_d(1 - s) \approx \frac{1}{2 \sqrt{s}} + \frac{d}{2 \pi} B \left( \frac{d + 1/2}{2} \right) + O(\sqrt{s}), \]
taken from Appendix B. One then easily finds that the segment of the momentum range having $0 < p^2 < r$ contributes at most an $r$-independent constant. Similarly, for $p^2 > r, s \approx K_1^0/\bar{K}_3(0, p^2) \sim p^d$ and one again finds an $r$-independent constant for the segment $r < p^2 < 1$. Thus for non-zero values of $\theta$, only the first term on the RHS of equation (5.8) can contribute to any singularities in $g_{\theta}$. For this term as well, the regime $0 < p^2 < r$ gives a constant. However, from the range $r < p^2$ one obtains approximately
\[ g_{\theta=0}(r, q = 0) \approx - q_0^2 K_{d-1} \int_0^{1} \frac{p^{d-5} dp}{K_{d-1} \bar{K}_3(0, p^2)} \frac{\sin^2 \theta I_d(\cos^2 \theta)}{\sin^2 \theta - K_1^0/\bar{K}_3(0, p^2)} \approx (d - 1) \frac{K_{d-1}}{N} S_0 I_d(\cos^2 \theta) \log r. \]
In the first line of equation (5.10) we have approximated \((r + p^2)^{-1} \approx p^{-2}\) and \(K_3(r, p^2) \approx K_3(0, p^2)\), making possible the integration in the second line. It is clear that \(g_{\theta=0}(r, q = 0)\) vanishes if \(K_1^0\) or \(K_3^0\) is infinite or if \(\theta\) vanishes. Assuming that \(K_1^0\) and \(K_3^0\) are finite and that \(\theta\) is non-zero, the second line of equation (5.10) follows for sufficiently small \(r\). Notice that this result for \(g_{\theta=0}(r, q = 0)\) does not depend on \(q_0\). Employing the exponentiation indicated in equation (5.3a) we find

\[
\chi_d(r, q = 0) \sim \chi_{SC}(r, q = 0) \frac{K_{k-1}^{d+1}}{K_{k-1}^{d}} |d\cos^2 \theta|
\]

for \(\theta \neq 0\) and \(r\) sufficiently small. Equation (5.11) implies the gauge-dependent susceptibility exponent

\[
\gamma(\theta) = \frac{2}{d-2} \left\{ 1 - 3 \frac{S_d}{N} - 4(d-1)^2 \frac{S_d}{N^2} \right\}
\]

The same expression for the exponent \(\gamma(\theta)\) is obtained if we instead use the decoupling approximation, equations (3.18) and (3.19), read to order \(1/N\). This implies that the decoupling is exact to first order in \(1/N\), at least as far as the susceptibility for \(\theta \neq 0\) is concerned. This expression may also be obtained by direct calculation of the self-energy diagrams of figure 4 in gauge \(\theta\). The exponent \(\eta\) calculated below will display a similar gauge dependence to that of \(\gamma(\theta)\).

5.2 Evaluation of \(g_{LC}(r, 0)\). — Because \(I_d(d\cos^2 \theta)\) is singular at \(\theta = 0\), the expression for \(\gamma(\theta)\) is not well behaved for small \(\theta\), and a simple power law, \(x \sim t^{-\tau}\), will not be obtained. Although the \(\theta\)-dependent term in \(\gamma(\theta)\) is formally smaller by a power of \(1/N\) than the leading term, for finite \(N\) and sufficiently small \(\theta\) it appears that \(\gamma(\theta)\) could change sign. Clearly the \(1/N\)-expansion breaks down as we approach \(\theta_c\), where \(\gamma(\theta_c) = 0\). The details of the regime \(0 < \theta < \theta_c\) will be explored in the more general paper, but it is instructive to examine the behaviour of \(g_{LC}(r, q = 0) = g_{\theta=0}(r, q = 0)\) in the \(1/N\)-expansion. To do so, we notice that for \(\theta = 0\) the first term in equation (5.8) vanishes, while the second term gives

\[
g_{LC}(r, 0) \approx -q_0^2 K_{d-1} \int_0^1 \frac{p^{d-3} dp}{(r + p^2) K_3(r, p^2)} \left[ 1 - \frac{d}{2} B\left(\frac{d + 1}{2}, \frac{1}{2}\right) \right].
\]

Again we are making the small-s approximation. For sufficiently small \(r\), it is a good approximation for \(p^2 \ll r\), while for \(p^2 \sim 1\) it might not be good but it is harmless. If we had \(K_1^1 \ll K_3^1\) then we would always have \(s < 1\), but that is not always the situation in the liquid crystals one has available. Nonetheless proceeding with the small-s approximation we find that in the second term of equation (5.13) only the range \(r < p^2 < 1\) is important, and the same sort of approximations used in equation (5.10) lead to

\[
-g_0^2 K_{d-1} \left[ -\frac{d}{2} B\left(\frac{d + 1}{2}, \frac{1}{2}\right) \right] \int_0^1 \frac{p^{d-3} dp}{(r + p^2) K_3(r, p^2)} \approx - (d - 1)^2 \frac{S_d}{N} \log r.
\]

Equation (5.14) is correct for \(K_1^1\) and \(K_3^1\) finite and very small \(r\). In order to discuss the first term in equation (5.13), we let \(x = p\xi\), where \(\xi = r^{-\frac{1}{2}} + \mathcal{O}(1/N)\), and write

\[
K_3(\xi) = K_3(\xi) + N q_0^2 A(1, 0) \xi^x
\]

\[
\mathcal{Q}(x) = (A(1, x^2) - A(1, 0))/A(1, 0).
\]

For small \(r, 1 - K_3^0/K_3(\xi) \approx 1\) and so we find

\[
g_{LC}(r, q = 0) \approx -\frac{1}{2} K_{d-1} \frac{K_1^1}{\sqrt{K_1^0 K_3(\xi)}} \int_0^\xi \frac{x^{d-3} dx}{(1 + x^2)(1 + \mathcal{Q}(x)^{1/2})} - (d - 1)^2 \frac{S_d}{N} \log r.
\]

We may replace the upper limit of integration in equation (5.16) with \(\infty\), the difference appearing as contributions which for \(d < 4\) are at most finite constants as \(r \to 0\). The result is therefore

\[
g_{LC}(r, q = 0) \approx -\frac{1}{2} K_{d-1} C(d) \frac{K_1^1 q_0^2}{\sqrt{K_1^0 K_3(\xi)}} - (d - 1)^2 \frac{S_d}{N} \log r
\]
The number \( C_j(d) \) depends only on \( d \), is finite for \( 2 < d < 4 \), and one finds numerically that \( C_1(d = 3) \sim 1.8 \).

Using the exponentiation indicated in equation (5.3a) we have

\[
\chi_{LC}(t) \sim t^{-\gamma_{SC}} e^{-\frac{2\gamma_{SC}}{d-2}} \beta^2 \frac{t^{\gamma_{SC}}}{N} \exp\left[-\frac{1}{2} K_{d-1} C_1(d) \frac{q^2 \beta^2}{\sqrt{K_0^3 K_3(\xi)}}\right],
\]

(5.18)

which vanishes faster than any power of \( t \) as \( t \to 0 \). We believe that this \( t \)-dependence is an artifact of the \( 1/N \)-expansion: although \( g_{LC}(\xi) = 0(1/N) \) is formally small, for very small \( t \) it grows quite large, and the \( 1/N \)-expansion is not applicable. The investigation of this situation will not be presented here.

It may be objected that in deriving equations (5.11) and (5.18) we made use of the exponentiation, equation (5.3a), and because \( g_{\theta} = 0(1/N) \) we have only really shown that

\[
\frac{\chi_\theta}{\chi_{SC}} = 1 + g_{\theta}(r, q = 0) + O(1/N^2).
\]

(5.19)

Certainly the RHS of equation (5.19) may be interpreted as giving the leading terms of an expansion in powers of \( 1/N \) for any of a wide variety of functions. Nonetheless we believe that the RHS of equation (5.19) is correctly interpreted as \( \exp(g_{\theta}(r, q = 0)) \). Observe that an exponential structure emerges naturally in the decoupling approximation, equation (3.18) (admittedly for \( G_0(x, x') \), not \( G_0(q) = 0 \)). Provided that one can perform the \( \theta \)-integrations involved, what one obtains from equation (3.12) or (3.19) is an expression for \( \log(G_0(q)/G_{SC}(q)) \), so that the exponentiation of equation (5.3a) comes about without a \( 1/N \)-expansion. We further note that in this theory gauge-dependent exponents should arise because of the existence of a marginally relevant operator associated with gauge changes, to be discussed below. The exponentiation is required to yield \( \chi_\theta \sim t^{-\gamma(\theta)} \) for \( \theta \neq 0 \). A more general investigation of this theory, beyond the scope of the \( 1/N \)-expansion, will be presented elsewhere.

5.3 Evaluation of \( g_\theta(r = 0, q) \) for \( \theta \neq 0 \).— In order to investigate the exponent \( \eta(\theta) \) one considers \( g_\theta(r = 0, q) \) for small \( q \). The usual power law behaviour, \( G_\theta(r = 0, q) \sim q^{-2+\eta(\theta)} \), will be manifested as \( \log q \) behaviour of the most singular portions of \( g_\theta(r = 0, q) \). Putting \( r = 0 \) in equation (5.3b), we find

\[
g_\theta(r = 0, q) = q_0^2 \int_{p_1} \frac{\cos^2 \theta}{p_2^2 \cos^2 \theta} \left( \frac{p_1^2}{p_2^2(K_0^2 p_2^2 + K_3(0, p^2)) - (2 p_1 
abla_1 -(1 + p_1^2)p_1)(p(p^2 + q^2) - 4 q(p,q))}{(p-q)^2 (p+q)^2} \right).
\]

(5.20)

As in the discussion of \( g_{\theta_{0}}(r, 0) \) one easily shows that the integration range \( 0 < p^2 < q^2 \) is not important to the critical behaviour for \( \theta \neq 0 \). Focussing on the range \( q^2 < p^2 < 1 \), we may approximate \( (p \pm q)^2 \approx p^2 \) to find

\[
g_{\theta_{0}}(0, q) \approx - q_0^2 \int_{p_2 > q} \left( \frac{\cos^2 \theta}{1 - p_2^2 \cos^2 \theta} \right) \frac{p_2^2}{p_2^2(K_0^2 p_2^2 + K_3(0, p^2))} = - q_0^2 K_{d-1} I_d(0, p^2) \frac{dp}{K_3(0, p^2)} \int_{p_2 > q} \frac{p_2^2}{p_2^2(K_0^2 p_2^2 + K_3(0, p^2))}.
\]

(5.21)

Once again making use of the identity (5.7) we obtain for non-zero \( \theta \),

\[
g_{\theta_{0}}(0, q) \approx (d - 1) \frac{K_{d-1}}{K_d} \frac{S_d}{N} I_d(\cos^2 \theta) \log q^2.
\]

(5.22)

In obtaining equation (5.22) we kept only the contribution which for \( \theta \) non-zero is singular enough as \( q \to 0 \) to affect the exponent \( \eta \):

\[
\eta(\theta) = \eta_{SC} + 2(1 - d) \frac{K_{d-1}}{K_d} \frac{S_d}{N} I_d(\cos^2 \theta).
\]

(5.23)

Equations (5.22) and (5.23) are correct only if \( K_0^2 \) and \( K_3^0 \) are finite. If either of these two elastic constants is infinite then \( \eta(\theta) = \eta_{SC}. \) For \( K_0^2 \) finite and \( q^2 \) sufficiently small the SC gauge exponents are isotropic (that is \( \eta_\perp = \eta_\parallel \) and this isotropy is passed along to all the intermediate gauges \( \theta \)).

5.4 Evaluation of \( g_{LC}(r = 0, q). \)— In discussing the LC gauge, \( \theta = 0 \), we must be careful to distinguish between the cases \( g_{LC}(r = 0, q = q_{\parallel}) \) (in which \( q = q_{\parallel} \hat{e}_1 \)) and \( g_{LC}(r = 0, q = q_{\perp}) \) (in which \( q \cdot \hat{e}_1 = 0 \)). In the case \( q = q_{\parallel} \) we have
Letting $p = qx$ and $p_1 = \cos \varphi$, equation (5.24) may be manipulated to read

$$g_{\text{LC}}(0, q_\parallel) = -q_\parallel^2 \int_0^1 \frac{1}{p^2(K_0^0 p_1^2 + K_3(0, p^2) p_1^2)} \cdot \frac{p_1^2 (p^2 + q_\parallel^2(1 + 4 p_1^2))}{(p - q_\parallel)^2 (p + q_\parallel)^2}.$$  (5.24)

In this expression we use the definition $s = K_0^0 / K_3(0, x^2 q_1^2)$. One may easily check that the integrand of $\tilde{g}_\parallel(x)$ is not singular at $(x = 1, \varphi = 0)$ (i.e. $p = \pm q_\parallel$) in spite of the fact that the denominator vanishes at that point. We will analyse the expression using the small-$s$ approximation again, with the same apologies as before. The first factor in the integrand is sharply peaked at $\cos^2 \varphi = 0$, so we expand the second factor in powers of $\cos^2 \varphi$. The second and successive terms in this expansion turn out to be too rapidly decreasing for large $x$ to be of importance in the critical behaviour. Keeping only the leading term implies that

$$\tilde{g}_\parallel(x) \approx \frac{1}{1 + x^2} \left\{ \int_0^1 \frac{dy}{s + y^2} + \int_0^1 \left[ (1 - y^2)^{d-1} - 1 \right] \frac{dy}{y^2} + O(s) \right\} \approx \frac{\pi}{1 + x^2} \left\{ \frac{1}{2 \sqrt{s}} - \frac{d}{2 \pi} B \left( \frac{d + 1}{2}, \frac{1}{2} \right) + O(\sqrt{s}) \right\}.  \quad (5.26)$$

Inserting equation (5.26) into equation (5.25a), and observing that $(1 + \frac{K_3^0 q_\parallel^2}{Nq_\parallel^2 K_3(0, 1)})^{-1} = 1 + O(s)$, we finally obtain

$$g_{\text{LC}}(0, q_\parallel) \approx -\frac{1}{4} K_{d-1} q_\parallel^2 B \left( \frac{d}{2}, 1 - \frac{d}{4} \right) \frac{q_\parallel^{-1} q_\parallel^{-1}}{\sqrt{K_0^0 K_3(0, q_\parallel^2)}} - 2(d - 1)^2 \frac{S_d}{N} \log q_\parallel. \quad (5.27)$$

For the case $q = q_\parallel$, we have

$$g_{\text{LC}}(0, q_\parallel) = -q_\parallel^2 \int_0^1 \frac{p_1^2 (p^2 + q_\parallel^2) - 4(1 + p_1^2)(p - q_\parallel)^2}{p^2(K_0^0 p_1^2 + K_3(0, p^2) p_1^2) (p + q_\parallel)^2 (p - q_\parallel)^2}. \quad (5.28)$$

One easily checks that for $p^2 \gg q^2$ the integrands in equations (5.24) and (5.28) are identical, so that $g_{\text{LC}}(0, q_\parallel)$ will contain a term $-2(d - 1)^2 \frac{S_d}{N} \log q_\parallel$ for small $q_\parallel$. It will also contain a term which behaves as

$$q_\parallel^{-1} q_\parallel^{-1} \sqrt{K_0^0 K_3(0, q_\parallel^2)},$$

but the numerical coefficient of this term depends on the entire range of $p$-integration, and is affected in particular by the singularity of the integrand at $p = \pm q$. That singularity is also responsible for a new singular behaviour for $d = 3$. A detailed calculation (outlined in Appendix E) gives the result

$$g_{\text{LC}}(0, q_\parallel) \approx \frac{1}{4} K_{d-1} q_\parallel^2 \frac{q_\parallel^{-1} q_\parallel^{-1}}{\sqrt{K_0^0 K_3(0, q_\parallel^2)}} \cdot 2^{-1} B \left( \frac{d-1}{2}, \frac{3-d}{2} \right) \times \left\{ 1 - \left( \frac{4 K_0^0}{K_3(0, q_\parallel^2)} \right)^{d-3} \frac{1}{\pi} B \left( \frac{d-2}{2}, \frac{4-d}{2} \right) \right\} - 2(d - 1)^2 \frac{S_d}{N} \log q_\parallel. \quad (5.29)$$

This expression has the remarkable feature that the leading singular behaviour is different for $d > 3$, $d < 3$, and $d = 3$. For $d < 3$, we have $g_{\text{LC}}(0, q_\parallel) \sim -q_\parallel^{-d/2}$ for very small $q_\parallel$. For $d > 3$, we have $g_{\text{LC}}(0, q_\parallel) \sim -q_\parallel^{-d/2}$. Most significantly, for $d = 3$ equation (5.29) gives

$$g_{\text{LC}}(0, q_\parallel) \mid_{d=3} \approx 2 \frac{\sqrt{\bar{K}_3(0, q_\parallel^2)}}{\sqrt{K_0^0}} \log \left( \frac{4 K_0^0}{\bar{K}_3(0, q_\parallel^2)} \right) - 16 \frac{\pi^2}{N} \log q_\parallel. \quad (5.30)$$
Therefore in three dimensions the exponentiation equation (5.3a) leads to the highly singular result

$$\chi_{LC}(t = 0, q) \mid_{d=3} \sim q^2$$

for very small $q$. We observe that equations (5.27) and (5.30) imply that $\chi_{LC}(0, q || 0)$ and $\chi_{LC}(0, 0, q \perp)$ both vanish as $q \to 0$, faster than any power of $q$. This is regarded as an artifact of the $1/N$-expansion, as in the discussion of equation (5.18). We observe that the singular behaviours of $g_{LC}(0, q || 0)$ and $g_{LC}(0, 0, q \perp)$ are qualitatively different, reflecting the fundamental anisotropy of the theory present even when $K_1^0$ and $K_3^0$ are both finite, resulting from the renormalization of $K_3$ relative to $K_1$. Furthermore, the fact that the leading singular behaviour of $g_{LC}(0, 0, q \perp)$ is different for spatial dimension above, below, or equal to 3 is regarded as a reflection of the incipient Landau-Peierls instability of the ordered phase. This is the only example we have found for $T > T_c$ of qualitatively different leading critical behaviour above and below three dimensions — the hallmark of the Landau-Peierls instability.

### 5.5 Evaluation of $v(\theta)$

The exponent $v(\theta)$ is investigated by computing the correlation length $\xi_i(\theta)$ in gauge $\theta$ along the coordinate direction $\hat{e}_i$ defined by equation (4.26). We have

$$\Delta \xi_i^2(\theta) \equiv \xi_i^2(\theta) - \frac{\xi_{SC,i}^2}{\xi_{SC,i}^2} = - \frac{\partial g_{d}(r, q)}{\partial (q_i^2)} \bigg|_{q=0}.$$  \hspace{1cm} (5.32)

When the $O(1/N)$ approximation for $g_d(r, q)$, equation (5.3b), is inserted in equation (5.32) we must distinguish the two cases: $\Delta \xi_i^2 = \Delta \xi_{-1}^2$, and $\Delta \xi_i^2 = \Delta \xi_{-1}^2$ (i.e. for any of the $d - 1$ equivalent coordinate directions orthogonal to $\hat{e}_1$). In order to find $O(1/N)$ corrections to the exponent $v(\theta)$, we must seek temperature dependence in $\Delta \xi_i^2$ more singular than $r^{-1} \sim t^{-2/(d-2)}$, the $t$-dependence of the order-unity form of $\xi_i^2$. One finds the expressions

$$\Delta \xi_i^2 = - \frac{q_0^2 K_{d-1}}{2 \pi} \int_0^\infty \sin^2 \varphi \sin^2 \varphi \sin^2 \varphi \frac{1}{1 - \cos^2 \theta \sin^2 \varphi} \int_0^1 \frac{\lambda_i p^d - 3 dp}{(r + p^2)^2} \left( K_1^0 \sin^2 \varphi + K_3^0 (r, p^2) \cos^2 \varphi \right)$$

$$\lambda_\parallel = 1 - 4 \hat{p}_1^2 - \frac{4 \hat{p}_2^2}{r + p^2},$$

$$\lambda_\perp = 1 + \frac{d-1}{2} \left( 1 + \hat{p}_1^2 - \frac{p_2^2}{r + p^2} \right).$$  \hspace{1cm} (5.33)

The analysis of equation (5.33) proceeds in exactly the same fashion as that of equation (5.5). Because $\lambda_\parallel \neq \lambda_\perp$, there is anisotropy present. However, for $\theta \neq 0$, $\Delta \xi_i^2(\theta)$ is never more singular than $r^{-1} \sim t^{-2/(d-2)}$, and so the exponent $v_i$ is unaffected to order $1/N$. For $K_1^0$ and $K_3^0$ finite,

$$v_\parallel(\theta) = v_\perp(\theta) = v_{SC} = - \frac{1}{d-2} \left[ 1 - 4 d(d-1) S_d \frac{4}{N} + O(1/N^2) \right].$$  \hspace{1cm} (5.34)

If $K_1^0$ or $K_3^0$ if infinite, of course, $\Delta \xi_i^2 = 0$ and we again have the same correlation length exponents from the SC gauge listed in table I. As mentioned in section 3, the specific heat and its associated exponent $\alpha$ are $\theta$-independent. The hyperscaling relation,

$$2 - \alpha = v_\parallel + (d-1) v_\perp,$$  \hspace{1cm} (5.35)

then implies that the correlation length exponents should be independent of $\theta$. One may easily check that the critical exponents $\gamma(\theta)$, $\eta(\theta)$, and $\nu$ derived above for $\theta \neq 0$ obey the scaling relations

$$\gamma(K_1^0 = \infty) = (2 - \eta(K_1^0 = \infty)) v_\parallel(K_1^0 = \infty)$$

$$= (2 - \eta(K_1^0 = \infty)) v_\perp(K_1^0 = \infty)$$

$$\gamma(\theta, K_1^0 = \infty) = (2 - \eta(\theta, K_1^0 = \infty)) v_\parallel(K_1^0 = \infty).$$  \hspace{1cm} (5.36)

Furthermore these exponents may also be obtained directly from the self-energy diagrams of figure 4 interpreted in the appropriate gauge $\theta$. 

Turning to the behaviour of equations (5.33) at \( \theta = 0 \), we obtain

\[
\Delta \xi_0^2(\theta = 0) \approx - \frac{1}{2} \tilde{g}_0^2 K_{d-1} C_2(d) \frac{\xi^2 + \epsilon}{\sqrt{K_0^0 K_{\xi}(\xi)}} \\
\Delta \xi_1^2(\theta = 0) \approx - \frac{1}{2} \tilde{g}_0^2 K_{d-1} \left( C_2(d) + \frac{4}{d-1} C_3(d) \right) \frac{\xi^2 + \epsilon}{\sqrt{K_0^0 K_{\xi}(\xi)}},
\]

where \( C_j(d) \) is defined by equation (5.17b). One finds numerically that \( C_2(d = 3) \approx 1.2 \) and \( C_3(d = 3) \approx 1.1 \), so that in three dimensions the coefficient of \( \Delta \xi_1^2(\text{LC}) \) is about three times as large as that of \( \Delta \xi_2^2(\text{LC}) \). Anisotropy is fundamental in \( \Delta \xi_1^2(\text{LC}) \) but is manifested only in this mild fashion. On the RHS of equations (5.37) \( \xi = r^{-\frac{1}{2} + O(1/N)} \), so that \( \Delta \xi_2^2(\text{LC}) \approx r^{-1}\xi^2 \sim t^{-(8-d)/(2d-2)} \) for small \( t \). Had we found \( r^{-1} \log r \), we would simply have deduced that \( v_{\text{LC}} \neq v_{\text{SC}} \). Instead \( \Delta \xi_2^2(\text{LC}) \) is much more singular than the order-unity form of \( \xi \), and is negative. Although \( \Delta \xi_2^2(\text{LC}) \) is formally smaller than the order-unity part, for finite \( N \) and sufficiently small \( t \) it appears that \( | \Delta \xi_2^2(\text{LC}) | \) can exceed \( \xi^2_{\text{SC}} \), so that \( \xi^2(\text{LC}) \) would become negative. This unacceptable result may be discarded by observing that, as in the earlier discussions of \( g_{\text{LC}}(r, 0) \) and \( \varrho_{\text{LC}}(0, \xi) \), the \( 1/N \)-expansion ceases to be applicable when we look at \( t \) small enough that \( | \Delta \xi_2^2(\text{LC}) | \sim 1 \).

To summarize, we have found in equations (5.17a), (5.27), (5.29), and (5.37) that \( g_{\text{LC}}(r, \xi) \) possesses highly singular dependence on \( r \sim t^{(d-2)/(2d-2)} \) and on \( \xi \). All of these results behave as negative powers of \( K_0^0 \) for \( K_0^0 \) small. To use the language of the RG, \( K_0^0 \) is a dangerous irrelevant parameter in the calculation of \( g_{\text{LC}}(r, \xi) \). In the \( \epsilon \)-expansion, the stable isotropic fixed point at \( K_0 = 0 \) is approached by any trajectory which begins with finite elastic constants. As a result of the dependence of \( g_{\text{LC}} \) on a negative power of \( K_0^0 \), \( g_{\text{LC}} \) displays essential singularities to which we have computed \( O(1/N) \) approximations above.

Another example of the limits of validity of the \( 1/N \)-expansion is seen in the exponents \( \gamma(\theta) \) and \( \eta(\theta) \), given by equations (5.12) and (5.23). The \( \theta \)-dependent terms which are present in the exponents for sufficiently small \( t \) or \( \varrho \) involve the function \( I_{d}(\cos^2 \theta) \), which grows as \( \theta \) decreases. Thus there exists a value \( \theta_0 \) of \( \theta \) for which

\[
\gamma(\theta_0) = \eta(2 - \eta(\theta_0)) = 0. \quad (5.38)
\]

For \( \theta \leq \theta_0 \), the order \( 1/N \) terms would apparently overwhelm the leading-order terms. We interpret this to mean that for \( \theta \leq \theta_0 \) the functional forms of \( \gamma(\theta, t, \xi, \varrho) \) and \( \gamma(\theta, t = 0, \xi, \varrho) \) will be qualitatively different than for \( \theta > \theta_0 \). In fact, in a more general treatment \( K_0^0 \) will be a dangerous irrelevant parameter in the calculation of \( g(\theta) \) for all \( \theta \leq \theta_0 \).

5.6 Marginal Operator. — As mentioned above, continuous dependence of critical exponents on some parameter in the Hamiltonian is usually associated with the presence of a marginally relevant operator [15]. We may display such an operator in the present context by considering the effect of an infinitesimal gauge change \( \delta \theta \) on the quantity of greatest interest, \( G_\theta(x, x') \), as in equation (3.9a). A similar argument could be constructed for \( \langle \psi_\theta(x) \rangle \). The calculation of \( G_{\theta + \delta \theta}(x, x') \) may be represented as resulting from the addition of a new term to \( \beta H \) :

\[
G_{\theta + \delta \theta}(x, x') = \langle \psi_\theta(x) \psi_\theta(x')^* \rangle \sim \langle e^{iq_0(L_d(x) - L_d(x'))} \delta \theta \rangle, \quad (5.39)
\]

where \( \langle \cdots \rangle \) indicates an equilibrium average with respect to an effective nonlocal Hamiltonian (not a Hamiltonian density)

\[
\beta H'(x, x') = \beta H - iq_0(L_d(x) - L_d(x')) \delta \theta. \quad (5.40)
\]

We first note that by the use of a cumulant expansion the second factor on the RHS of equation (5.39) may be written as

\[
\exp[- \frac{1}{2} q_0^2 (L_d(x) - L_d(x'))^2 \delta \theta^2 + O(\delta \theta^4)]
\]

which may be neglected for \( \delta \theta \) infinitesimal. It follows that to leading order in \( \delta \theta \), \( G_{\theta + \delta \theta}(x, x') \) may be calculated by averaging \( \psi_\theta(x) \psi_\theta(x')^* \) over the ensemble defined by \( \beta H'(x, x') \). We will argue that for \( K_0^0 \) finite,

\[
iq_0(L_d(x) - L_d(x'))
\]

is a marginal operator. The structure of the novel term in \( \beta H'(x, x') \) is somewhat different from that usually
associated with the existence of a marginal operator. Typically one encounters a term of the form \( \int \lambda Q(x) \, dx \) in the Hamiltonian. \( Q \) is a marginal operator and \( \lambda \) a marginal parameter provided that the scaling behaviour of \( Q \) is given by \( Q(x/b) = b^\eta Q(x) \). Thus \( \int Q(x) \, dx \) scales as a constant, which is the behaviour we shall seek for \( i q_0 (L_\alpha(x) - L_\alpha(x')) \).

Following the discussion of anisotropic scaling in reference [9], we allow different scaling behaviour for directions along and perpendicular to \( \hat{e}_1 \), by writing

\[
p_1 \rightarrow b^{1 + \eta_1} p_1 \\
p_2 \rightarrow b p_2 \\
\mu_\parallel = (v_\parallel - v_\perp)/v_\perp.
\]

We then find

\[
\delta n(p) = b^{1/2(2 + d + \eta_\parallel - \eta_\perp)} \delta n(b^{1 + \eta_1} p_1, b p_2).
\]

In order that \( q_0 \) should behave as a constant under scaling one chooses \( \eta_\alpha = 4 - d - \mu_\parallel \) as in reference [9], and concludes that for \( \mu_\parallel = 0 \)

\[
i q_0 (L_\alpha(x) - L_\alpha(x')) = i q_0 (L_\alpha(x/b) - L_\alpha(x'/b)) \tag{5.43}
\]

In the \( 1/N \)-expansion, \( v_\parallel = v_\perp \) for \( K_1^o \) finite and so \( \mu_\parallel = 0 \). In the \( e \)-expansion the stable fixed point at \( K_1 = 0 \) has \( \mu_\parallel = 0 \) also. Thus for \( K_1^o \) finite we reach a regime in which \( i q_0 (L_\alpha(x) - L_\alpha(x')) \) is a marginal operator, and \( \delta \theta \) is a marginal parameter in the modified model \( \beta H'(x, x') \). For \( K_1^o = \infty, v_\parallel > v_\perp \) in the \( 1/N \)-expansion, and similarly at the \( K_1 = \infty \) fixed point in the \( e \)-expansion \( \mu_\parallel > 0 \), and so \( \delta \theta \) is expected to be irrelevant. This indicates, as we have found in the \( 1/N \)-expansion, that critical exponents should not depend on gauge when \( K_1^o \) is infinite.

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Appendix A. Jacobian for gauge changes. — In this appendix we present details of the derivation of equations (3.4). Consider the effect of an infinitesimal gauge change from \( \theta \) to \( \theta + \delta \theta \) on the measure \( \theta_{\theta_0} \, D\theta_\theta \) in functional integrals such as in equation (3.2). As in equation (3.9b) we have

\[
L_{\theta + \delta \theta}(p) - L_{\theta}(p) = \frac{dL_\theta(p)}{d \theta} \delta \theta \\
= \frac{v(\theta, p).A_\theta(p)}{v(\theta, p).p} \delta \theta + O(\delta \theta^2),
\]

and so we may write

\[
dA_{\theta + \delta \theta}(p) = \left( \delta^{ij} - \frac{p^i v(\theta, p)^j}{p \cdot v(\theta, p)} \delta \theta \right) dA_\theta(p) + O(\delta \theta^2).
\]

Because \( A_\theta(x) \) is a real field, we have

\[
D A_{\theta + \delta \theta} = \prod_p d A_{\theta + \delta \theta}(p) \\
= \left[ \prod_p d A_{\theta + \delta \theta}(p) \right]^{1/2}, \tag{A.3}
\]

where in the first line we have restricted the wavevectors \( p \) to half \( p \)-space, and in the second line have removed
the restriction. The amplitudes $A_{\psi + \delta \phi}(p)$ may vary over all complex values. Inserting equation (A.2) into equation (A.3) we find

$$
\mathcal{D} A_{\psi + \delta \phi} = \left[ \prod_p \left( 1 - \frac{v(\theta, p).p}{v(\theta, p).p} \delta \theta + O(\delta \theta^2) \right) \prod_p \mathcal{D} A_d(p) \right]^{1/2}
$$

$$
= \left[ 1 - \frac{1}{2} \delta \theta + O(\delta \theta^2) \right] \mathcal{D} A_\theta
$$

$$
= \left[ 1 - N_0 \frac{K_d^{-1}}{2} \int d^2 \theta \left( \tan \theta \delta \theta + O(\delta \theta^2) \right) \right] \mathcal{D} A_\theta.
\tag{A.4}
$$

The symbols $N_0$ and $K_d$ are defined following equations (3.4) in the text, and the function $L_d(\cos^2 \theta)$ is discussed in Appendix B. Turning now to $\psi_\theta(x)$ we find

$$
\psi_\theta(x) = \psi_\theta(x) (1 + iq_0 L_d(x) \delta \theta + O(\delta \theta^2))
$$

$$
\psi_{\psi + \delta \phi}(q) = \psi_\theta(q) - q_0 \int_p \frac{v(\theta, p).A_d(p)}{v(\theta, p).p} \psi_\theta(q - p) \delta \theta + O(\delta \theta^2).
\tag{A.5}
$$

It follows that

$$
\mathcal{D} \psi_{\psi + \delta \phi} = \left[ \prod_q \left\{ \frac{1}{2} \cdot \sum_p \left[ \frac{v(\theta, p).A_d(p)}{v(\theta, p).p} \frac{\partial \psi_\theta(q - p)}{\partial \psi_\theta(q)} + \psi_\theta(q - p) \frac{\partial A_d(p)}{\partial \psi_\theta(q)} \right] \delta \theta + O(\delta \theta^2) \right\} \mathcal{D} \psi_\theta.
\tag{A.6}
$$

The second term in square brackets vanishes, and the first only contributes for $p = 0$. As a consequence, we have

$$
\mathcal{D} \psi_{\psi + \delta \phi} = \left[ 1 + iq_0 N_0 \frac{1}{\Omega} \int \delta L(x) d^4 x + O(\delta \theta^2) \right] \mathcal{D} \psi_\theta,
\tag{A.7}
$$

where $\Omega = \text{volume of system}$. Our gauge choice does not govern $L(p = 0)$, and a uniform shift $L(x) \to L(x) + \text{constant}$ cannot affect physical results. Exploiting this phase freedom we may write

$$
\mathcal{D} \psi_{\psi + \delta \phi} = \mathcal{D} \psi_\theta.
\tag{A.8}
$$

Considering the gauge change to $\theta$ from the LC gauge, $\theta = 0$, to be made by a succession of infinitesimal gauge changes, one derives equations (3.4).

Appendix B. Properties of the functions $L_d(\cos^2 \theta)$ and $A(r, p^2)$. — The function $A(r, p^2)$ is defined by equation (4.4), and satisfies $A(r, p^2) = p^{d-4} A(r/p^2, 1)$. To extract the small-$r$ behaviour of $A(r, 1)$, we first integrate by parts to obtain

$$
A(r, 1) = \frac{4J}{2 - d} r^{d-2}/2 + \frac{4J}{d - 2} \int_0^1 \left[ r + \alpha(1 - \alpha) \right]^{d-2}/2 d\alpha,
\tag{B.1}
$$

and we observe that

$$
\pi(r, 1) = \frac{d}{dr} \left[ \frac{2J}{d - 2} \int_0^1 \left[ r + \alpha(1 - \alpha) \right]^{d-2}/2 d\alpha \right].
\tag{B.2}
$$

Integrating term-by-term the small-$r$ expression, equation (4.15a), one finds

$$
\frac{2J}{d - 2} \int_0^1 \left[ r + \alpha(1 - \alpha) \right]^{d-2}/2 d\alpha = \pi(0, 1) \left[ \frac{1}{2(d - 1)} + r + (d - 3) r^2 + O(r^3) \right] +
\tag{B.3}

+ \frac{4J}{2 - d} \int_0^1 \left[ 2 - \frac{8}{d(d - 2)} r^2 + O(r^3) \right].
Inserting equation (B.3) into (B.1) leads to the expression for the small-$r$ behaviour for $A(r,1)$, equation (4.15b).

The function $I_d(\cos^2 \vartheta)$ is defined by

$$I_d(\cos^2 \vartheta) = \frac{1}{2 \pi} \int_{0}^{\pi} \cos^2 \vartheta \sin^4 \vartheta \, d\vartheta$$

(B.4)

In order to extract the small-$\vartheta$ behaviour of $I_d(\cos^2 \vartheta)$ we let $y = \cos \varphi$ in the integrand, to re-write it in the form

$$I_d(\cos^2 \vartheta) = \frac{1}{\pi} \int_{0}^{1} \frac{dy}{y^2 + \tan^2 \vartheta} + \frac{1}{\pi} \int_{0}^{1} \frac{dy}{y^2 + \tan^2 \vartheta} \left[ (1 - y^2)^{\frac{d-1}{2}} - 1 \right].$$

(B.5)

The first integral in equation (B.5) is elementary, and in the second we may expand in powers of $\vartheta$. The result is

$$I_d(\cos^2 \vartheta) \approx \frac{1}{2 \vartheta} \left[ 1 + \frac{3}{6} \vartheta^2 + O(\vartheta^4) \right] - \frac{d}{2 \pi} B \left( \frac{d+1}{2}, \frac{1}{2} \right) \left[ 1 + \frac{1}{3} (d - 2) \vartheta^2 + O(\vartheta^4) \right].$$

(B.6)

Appendix C. Ward identities. — For any value of $\vartheta$ we may generalize the model by writing

$$\beta H = \beta H_N \{ \psi_\vartheta, A_\vartheta \} + \beta H_N \{ A_\vartheta \} + \beta H_{EXT},$$

where $\beta H_N$ is generalized by the insertion of a « gauge fixing term »

$$\beta H_N \{ A_\vartheta \} = \frac{1}{2} \int p D_0^{-1}(p). A_\vartheta(p). A_\vartheta(-p)$$

(C.1)

$$D_0^{-1}(k)_{kl} = \frac{\delta_{kl}}{2} + \frac{1}{2} \sum_{ij} \delta_{ij} \cos \theta \frac{\delta_{kl}}{p_i^2} = \frac{1}{2} \sum_{ij} \delta_{ij} \cos \theta \frac{\delta_{kl}}{p_i^2}$$

(C.2)

The model as discussed in the text corresponds to $w \to 0$, and to $J \to 0$ and $\eta = \eta^* \to 0$, where these are fields that couple linearly to the fields $A_\vartheta, \psi_\vartheta$, and $\psi_\vartheta$:

$$\beta H_{EXT} = - \int (J(x). A_\vartheta(x) + \eta(x) \psi_\vartheta^*(x) + \eta^*(x) \psi_\vartheta(x)) \, d^4x.$$  

(C.3)

We now perform an arbitrary infinitesimal inhomogeneous gauge change $\mu(x)$ (suppressing $\vartheta$ subscripts)

$$\psi(x) \to \psi'(x) = \psi(x) e^{q_\vartheta(x)}$$

$$A(x) \to A'(x) = A(x) + V_\vartheta(x),$$

(C.4)

and point out that the partition function $Z$ must be independent of $\mu(x)$ because equation (C.4) is simply a change of variables under the integral in the functional integral which defines $Z$. Expanding $\beta H \{ \psi, A' \}$ to order $\mu$, inserting in $Z$, and expanding it to order $\mu$, one finds

$$\int \nabla \cdot D_0^{-1}(x). A'_\vartheta(x) \, d^4x - \nabla \cdot J(x) + i \eta_\vartheta(x) \psi_\vartheta^*(x) - \eta(x) \psi_\vartheta(x) = 0.$$  

(C.5)

To justify the second line of equation (4.3) we set $\eta = \eta^* = 0$ and use $D_0^{-1}(x') \delta(x') \delta A'(x')$ to find

$$\nabla \cdot D_0^{-1}(x') \delta(x') = 0.$$  

(C.6)

This implies that the polarization function is transverse:

$$p_i \pi_{\vartheta i}(p) = 0.$$  

(C.7)

We observe that $\pi_{\vartheta i}(p)$ (as depicted in figure 5b) is $p$-independent, and proportional to $\delta_{ij}$. On the other hand $\pi_{\vartheta}(p = 0) \propto \delta_{ij}$. Reading equation (C.7) to leading order in $p$ for small $p$, we find

$$\pi_{\vartheta i} = - \pi_{\vartheta}(p = 0)_{ij}.$$  

(C.8)
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The result, equation (C.8), is used in the second line of equation (4.3). It can also be obtained by direct manipulation of the analytical expressions to which the diagrams in figure 5 correspond. It also can be obtained by the use of the Ward identities derived below. The above derivation holds for any $\theta$, although in the text we only use it in the SC gauge, $\theta = \pi/2$. If one calculates $D$ directly in gauge $\theta$ then equation (C.8) is needed.

In order to derive Ward identities we define

$$\Gamma = - \log Z + \int \{ J(x) \langle A(x) \rangle + \eta(x) \langle \psi(x) \rangle + \eta^*(x) \langle \psi^*(x) \rangle \} d^4x, \quad (C.9)$$

so that one has

$$J_i(x) = \delta \Gamma / \delta \langle A_i(x) \rangle \quad (C.10)$$

$$\eta(x) = \delta \Gamma / \delta \langle \psi(x) \rangle \quad \eta^*(x) = \delta \Gamma / \delta \langle \psi^*(x) \rangle .$$

Using equations (C.10) to replace $J_i$, $\eta$, and $\eta^*$ in equation (C.5), and applying $\delta^2 \langle \psi(y) \rangle / \delta \langle \psi^*(y) \rangle$ we find (for $J = \eta = \eta^* = 0$)

$$\nabla \cdot \frac{\delta^2 \Gamma}{\delta \langle A_i(x) \rangle \delta \langle \psi(y) \rangle \delta \langle \psi^*(y) \rangle} = -iq_0 \left( \frac{\delta^2 \Gamma}{\delta \langle \psi(x) \rangle \delta \langle \psi^*(y) \rangle} - \frac{\delta^2 \Gamma}{\delta \langle \psi(y) \rangle \delta \langle \psi^*(x) \rangle} \right) . \quad (C.11)$$

Making use of

$$\frac{\delta^2 \Gamma}{\delta \langle \psi(x) \rangle \delta \langle \psi^*(y) \rangle} = G^{-1}(y, x) \quad (C.12)$$

and of equation (3.10), we find

$$q_i \Gamma_i(y', y, x) = iq_0 \left( \delta(x - y) - \delta(x - y') \right) G^{-1}(y', y) . \quad (C.13)$$

After Fourier transforming one obtains the Ward identity, valid for all $\theta$,

$$q, \Gamma_i(p + q, p, q) = q_0(G^{-1}(p) - G^{-1}(p + q)) . \quad (C.14)$$

which is used in the derivation of equation (3.19).

One can derive further Ward identities from equation (C.5). For instance, applying

$$\frac{\delta^3}{\delta \langle A_j(x') \rangle \delta \langle \psi(y) \rangle \delta \langle \psi^*(y') \rangle}$$

letting the external fields vanish, and Fourier transforming yields a Ward identity for $\Gamma_i(pp', qq')$ quite similar to that for $\Gamma_i(pp', q)$:

$$q_i \Gamma_i(p + q + q', p ; q, q') = q_0(\Gamma_i(p + q' ; p ; q, q') - \Gamma_i(p + q + q', p + q ; q')) . \quad (C.15)$$

Appendix D. Integrals involved in polarization function. — In this appendix we evaluate the function

$$F_{ij}(p) = \int \left\{ \frac{1}{[r + (k + \frac{1}{2} p)^2] [r + (k - \frac{1}{2} p)^2]} - (p = 0) \right\} \quad (D.1)$$

encountered in equation (4.3). For the purposes of this appendix choose coordinates so that $p = p \hat{e}_1$. $F_{ij}$ obviously vanishes unless $i = j$, so we consider $F_{jj}$ in the cases $j = 1$ and $j \neq 1$.

Begin by considering a case of $j \neq 1$, say, $F_{22}$. We will employ the formula

$$\frac{1}{AB} = \int_0^1 [xA + (1 - x) B]^{-2} \, dx . \quad (D.2)$$
Then we find
\[ F_{22} = \int_k k^2 \int_0^{\hat{v}} dx \left\{ \left[ r + \alpha(1 - \alpha) p^2 + (k + \frac{1}{2}(2 \alpha - 1) p)^2 \right]^{-2} - (p = 0) \right\}. \] (D.3)

Shifting the origin in k-space we obtain
\[ F_{22} = \frac{1}{d} \int_k k^2 \int_0^{\hat{v}} dx \left\{ \left[ r + k^2 + \alpha(1 - \alpha) p^2 \right]^{-2} - (p = 0) \right\} \]
\[ = -\frac{2}{d} K_d \int_0^{\hat{v}} p^2 \alpha(1 - \alpha) \frac{d\beta}{\alpha} \int_0^{\hat{v}} \frac{k^{d+1} dk}{\left[ r + \alpha(1 - \alpha) \beta p^2 + k^2 \right]^3} \]
\[ = -\frac{1}{d} J \int_0^{\hat{v}} p^2 \alpha(1 - \alpha) \frac{d\beta}{\alpha} \int_0^{\hat{v}} \left[ r + \alpha(1 - \alpha) \beta p^2 \right]^{(d-4)/2} d\beta \]
\[ = \frac{J}{d - 2} \int_0^{\hat{v}} p^2 \alpha(1 - \alpha) \frac{d\beta}{\alpha} \left[ r + p^2 \alpha(1 - \alpha) \right]^{(d-2)/2} d\beta. \] (D.4)

Comparing to equation (B.1) we conclude that
\[ F_{22} = -\frac{1}{d} p^2 \Delta(r, p^2). \] (D.5)

Returning to the case \( j = 1 \), we proceed similarly, using equation (D.2) and shifting the origin in k-space to find
\[ F_{11} = \int_k k^2 \int_0^{\hat{v}} dx \left\{ (k^2 + \frac{1}{2} p^2(2 \alpha - 1)^2) \left[ r + k^2 + p^2 \alpha(1 - \alpha) \right]^{-2} - (p = 0) \right\} \]
\[ = F_{22} + \int_0^{\hat{v}} \frac{d\alpha}{\alpha} \frac{1}{2} p^2(2 \alpha - 1)^2 \int_k \left[ r + k^2 + p^2 \alpha(1 - \alpha) \right]^{-2}. \] (D.6)

With the substitution \( y = k^2/(r + p^2 \alpha(1 - \alpha)) \), we find
\[ F_{11} - F_{22} = \frac{1}{4} p^2 \int_0^{\hat{v}} (2 \alpha - 1) \left[ r + p^2 \alpha(1 - \alpha) \right]^{(d-4)/2} \frac{1}{2} K_d \int_0^{\hat{v}} \frac{y^{d-2}/2 \ dy}{(1 + y)^2} \]
\[ = \frac{1}{4} J p^2 \int_0^{\hat{v}} 2 \alpha(2 \alpha - 1) \left[ r + p^2 \alpha(1 - \alpha) \right]^{(d-4)/2} d\alpha - \frac{1}{4} J p^2 \int_0^{\hat{v}} (2 \alpha - 1) \left[ r + p^2 \alpha(1 - \alpha) \right]^{(d-4)/2} d\alpha. \] (D.7)

In equation (D.7) the second term on the RHS vanishes because the integrand is antisymmetric about \( \alpha = \frac{1}{2} \). The first term is identified as \( \frac{1}{4} p^2 \Delta(r, p^2) \). Recalling equation (D.5), we conclude that \( F_{11} = 0 \). It follows that
\[ F_{ij}(p) = -\frac{1}{d} p^2 \Delta(r, p^2) \delta(p)_{ij}, \] (D.8)
where the projection operator \( \delta(p) \) is defined following equation (4.3).

**Appendix E : Asymptotic behaviour of \( g_{1c}(0, q_1) \).** — In this appendix we sketch the derivation of equation (5.29). As in equation (5.25) we write
\[ g_{1c}(0, q_1) = -\frac{2}{N \Delta(0, 1)} K_{d-2} \int_0^{s \hat{v}} \frac{x \ dx}{N q_0^d \Delta(0, 1)} \mathcal{J}(x, s) \] (E.1)
Once again we work in the limit in which \( s \) is small, so that the integrand in equation (E. 2a) is sharply peaked at \( y = 0 \). Simply replacing \( L_i(x, y) \) by \( L_i(x, 0) \) in this integrand would introduce a non-integrable singularity at \( x = 1 \). Instead we rescale the integration variable \( z \) in equation (E. 2b) by the factor

\[
\lambda(x, y) \equiv \left( \frac{4 x^2(1 - y^2)}{(x^2 - 1)^2 + 4 x^2 y^2} \right)^{1/2},
\]

and we find

\[
L_i(x, y) = 2[(x^2 - 1)^2 + 4 x^2 y^2]^{-d/2} \left(1 - y^2\right)^{1 - d/2} \mathcal{F}(x, y)
\]

(E. 4a)

\[
\mathcal{F}(x, y) \equiv (2 x)^{2-d} \int_0^{\lambda(x,y)} \frac{x^2 + 1 - 4(1 + i^2)(1 - z^2/\lambda^2)}{(1 + z^2)^2} \frac{dz}{\sqrt{1 - z^2}}.
\]

(E. 4b)

We approximate \( L_i(x, y) \) by replacing \( (1 - y^2)^{1 - d/2} \mathcal{F}(x, y) \) with \( \mathcal{F}(x, 0) \) in equation (E. 4a). Letting \( u = x^2 - 1 \) and \( A \equiv 4 K_{d-2}/Nd(0, 1) \) \((2 \pi)^2\), we find

\[
g_{cc}(0, q_4) \approx -\frac{1}{2} A \int_{-1}^{1} \frac{\mathcal{F}(u)}{\sqrt{s(x)}} \frac{du}{(u^2 + 4 s(1 + u))^{d/2}}
\]

(E. 5)

\[
\text{The upper limit of the } t \text{ integration may be replaced with } \infty. \text{ The large } u \text{ regime gives contributions behaving as } \log q_4, \text{ which we may remove by writing}
\]

\[
g_{cc}(0, q_4) \approx -\frac{\pi}{4} A \int_{-1}^{1} \frac{\mathcal{F}(u)}{\sqrt{s(x)}} \frac{du}{(u^2 + 4 s(1 + u))^{d/2}}
\]

(E. 6)

The first term in equation (E. 6) gives a contribution which at most behaves as \( \log q_4 \), so we identify it as part of

\[
-2(d - 1)^2 \frac{\pi}{N} \log q_4,
\]

as is argued in the text following equation (5.28). In the second term of equation (E. 6) we may replace the upper cutoff of the \( u \) integral with \( \infty \). Thus we encounter the function

\[
F(\sigma) \equiv \int_{-1}^{\infty} \frac{\mathcal{F}(u)}{\sqrt{s(x)}} \left\{ \frac{1}{(u^2 + 4 s(1 + u))^{d/2}} - \frac{1}{(u^2 + 1)^{d/2}} \right\} du
\]

(E. 7)

where the case of interest is \( \sigma = 1 \). For small-\( s \), we have approximated

\[
\left[ u + 4 s(x) i^2(1 + u) \right]^{-d/2} \to \left[ u^2 + 4 s(1) i^2 \right]^{-d/2}
\]

in equation (E. 6). In this limit, the integrand of \( \partial F/\partial \sigma \) is sharply peaked at \( u = 0 \) (i.e. \( x = 1 \)) so that

\[
\frac{\partial F}{\partial \sigma} \approx -\frac{\varepsilon}{2} \frac{\mathcal{F}(0)}{\sqrt{s(1)}} \left( 2 t \sqrt{s(1)} \right)^{1-t} \sigma^{-\frac{3}{2}(1+\varepsilon)} 2 \int_{0}^{\infty} \frac{dv}{(1 + v^2)^{1+\varepsilon/2}}.
\]

(E. 8)

The remaining integral in equation (E. 8) is

\[
\frac{1}{2} B \left( \frac{1 + \varepsilon}{2}, \frac{1}{2} \right)
\]

and the remaining \( \sigma \) dependence is readily integrated.
Choosing as a reference value $\sigma_0 = 1/4 t^2 s(1)$ we find

$$F(\sigma = 1) = \int_{\sigma_0}^{1} \frac{\partial F}{\partial \sigma} d\sigma = -\frac{F(0)}{\sqrt{s(1)}} B\left(\frac{e - 1}{2}, \frac{1}{2}\right) + \frac{F(0)}{s(1)^{1/2}} (2t)^{-1} B\left(\frac{e - 1}{2}, \frac{1}{2}\right),$$

(E.9)

because $F(\sigma_0) = 0$. The remaining $t$ integration in equation (E.6) is now simple. Making use of

$$F(0) = -2e^{-2} B\left(\frac{2 + e}{2}, \frac{-e}{2}\right),$$

(E.10)

one derives equation (5.29).

References

[22] See, e.g., Forster, D., Hydrodynamic Fluctuations, Broken Symmetry and Correlation Functions (Benjamin, Reading Massachusetts) 1975.
[23] The following minor corrections should be made in reference [9]: (1) In equation (8b), the second argument of the function $D_{ij}$ should read $b_{q_i}$. (2) In equation (16), $C_{u}$ should not be squared in the recursion relation for $u$. (3) Equation (22c) may be corrected by replacing $f_2$ by $f_{2}/\alpha$ everywhere it appears. (4) In the numerator of the integrand of equation (31), $q_{2}$ should be replaced by $q_{2}$. (5) In the equation for $I_{3}$ in equation (A.7), $L_{000}$ should be replaced by $L_{100}$, and $d/dK_{1}$ should be replaced by $d/dK_{2}$. 