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The renormalization group approach to critical dynamics near the nematic-smectic-A phase transition in liquid crystals (*)

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Résumé. — Nous utilisons le groupe de renormalisation pour étudier la dynamique au voisinage de la transition nématique-smectique-A. Nous analysons un modèle simplifié basé sur un calcul récent en modes couplés de la transition nématique-smectique-A. Nous considérons un scaling anisotrope et nous obtenons les relations de récurrence au premier ordre en \( \varepsilon = 4 - d \) pour les coefficients de viscosité, dans la jauge du cristal liquide, pour \( K_1 \to \infty \). L'exposant dynamique est \( z = d/2 + \mu_{II}/2 \) où \( d \) est la dimensionnalité du système et \( \mu_{II} \) un exposant statique. Les coefficients cinétiques sont également exprimés entièrement à partir des exposants statiques.

Abstract. — The dynamics near the nematic to smectic-A phase transition is investigated by the renormalization group method. A simplified model based on a recent mode-coupling calculation on NA transition is analysed. Anisotropic scaling is considered and the recursion relations for the viscosity coefficients in the liquid crystal gauge for \( K_1 \to \infty \), are obtained, correct to first order in \( \varepsilon = 4 - d \). The dynamic exponent is found to be \( z = d/2 + \mu_{II}/2 \) where \( d \) is the dimensionality of the system and \( \mu_{II} \) is a static exponent. The kinetic coefficients are also expressed entirely in terms of static exponents.

1. Introduction. — In a very interesting paper [1], Lubensky and Chen have investigated the static properties near the nematic to smectic-A phase transition by the renormalization group method. A review of the static properties (both theoretical and experimental) of this phase transition has been given by them.

The dynamical critical properties of the NA transition have been studied by Jähnig and Brochard [2], who used linear response theory and later on appealed to the analogy between the NA transition and the superfluid transition [3] and to dynamic scaling [4, 5], in order to infer the temperature dependence of the smectic viscosity coefficient, \( \gamma_3 \). Mc Millan [6] has also done a simple mode theory using the Leslie Ericksen [7] stress tensor. More recently, a mode-coupling calculation which is essentially a perturbation method, has been done on the NA and NC transition by the present authors and Chen and Lubensky [8]. But that, too, does not constitute a self consistent calculation as the various dynamical quantities are still expressed in terms of the unknown \( \gamma_3 \).

This work is an application of the renormalization group to the dynamics of the problem. We consider a simplified model for NA transition based on the recent mode-coupling calculation [8]. This model takes into account the couplings among the gauge field (director \( \mathbf{n} \)), the order parameter (centre of mass density, \( m \), corresponding to the smectic planes) and the velocity, \( \mathbf{v} \). A dynamic renormalization group calculation near the NA transition has been done by Shiwa [9]. The essential difference between his model and the one considered here is that, it neglects the reactive coupling between the director, order parameter and the velocity. Also, we have taken into account anisotropy, which appears to be one of the fundamental properties of the de Gennes model. The degree of anisotropy is reflected in the value of the splay elastic constant, \( K_1 \). The larger the value of \( K_1 \), the greater is the anisotropy [1] of the system. We have done our calculation in the liquid crystal gauge which, we believe, should be the correct gauge for studying the dynamics of the problem.

Assuming anisotropic scaling [1], we get the dynamical exponent \( z = d/2 + \mu_{II}/2 \) where \( d \) is the dimensionality of the system and \( \mu_{II} \) is the static exponent given in ref. [1]. Based on de Gennes analogy [3] between the smectic-A phase of a liquid crystal and the superconducting phase of a metal, we would expect the dynamic exponent for the isotropic case [10], \( z = d/2 \).
The outline of the paper is as follows. In section 2, a simplified model for NA transition is introduced and the dynamical equations based on ref. [8] are given.

In section 3, the static renormalization group is discussed briefly and the dynamic R.G. analysis is developed. The recursion relations for the dynamic coefficients, correct to first order in $\varepsilon = 4 - d$, are written down and their fixed points calculated. The dynamic critical exponent, $x$, is also obtained. In Appendix A, a mode coupling calculation for $\gamma_3$ is given and finally in Appendix B, the stability analysis for the fixed points is done.

2. The model. — The equations of motion for the dynamic variables, i.e., the order parameter, $m(r, t)$, the director, $n(r, t)$, and the velocity, $v(r, t)$, are given in ref. [8]. The equilibrium value of the director, $n^0$, is taken to be along the 3-direction and the deviation from equilibrium, $n_i$, in the 1 and 2 directions. We shall now show that it is possible to construct a simplified model of NA transition in which only $\gamma_3$ couples to $n_i$. Since the density wave propagates mainly along the third axis, the order parameter can be written as [11]

$$ m(r, t) = 2^{-1/2} [e^{i q_0 z} \psi(r, t) + e^{-i q_0 z} \psi^*(r, t)], \quad (1) $$

where $z = n \cdot r$, $q_0 = (2 \pi/d)$ and $d$ is the interplanar spacing. Therefore, the term $V m \delta F/\delta m$ which appears [8] in the equation for $\dot{b}_j$ contributes essentially only to $\gamma_3$. Also, of the five viscosities which enter the stress tensor $\sigma^D_{ik}$ only $v_1$ diverges [8] and gives a term proportional to $\gamma_3$. The mode coupling calculation [8] gives the reactive coefficient between the shear and the director $\gamma = -1$. We can, therefore, choose $\lambda_0 = -1$ in our model. If we now look [8] at the term

$$ \frac{1}{2} (\lambda_0 - 1) n^0 \sum_i \frac{\delta F}{\delta n_i} $$

in $\dot{b}_j$ we see that only $v_3$ couples to $n_i$. Finally, the equation for the director when simplified, gives for $\lambda_0 = -1$, a term proportional to $n^0 \cdot v$, indicating once again that $v_3$ couples to $n_i$.

We now consider a system in which there are $n$ components of the order parameter. We work in the liquid crystal gauge near 4 dimensions in which $n^0$ is in the 3-direction while its fluctuating components lie along the 1, 2 and 4 axes. The components of wave vector $k$, parallel and perpendicular to $n^0$ are $k_1$ and $k_2$. The component of $n$ in the direction of $k_1$ is $n_{11}$, and $n_{12}$ and $n_{13}$ are the components transverse to $k_1$. The dynamical equations in ref. [8] become (with $j = \rho v$),

\begin{align}
\gamma_3^{ij} \left( \frac{\partial}{\partial t} + i q_0 \frac{j_3}{\rho} (r, t) \right) \psi_i(r, t) &= - \frac{\delta F}{\delta \psi_i^*}(r, t) + \eta_i(r, t), \quad \alpha = 1, 2 \ldots \frac{n}{2} \quad (2) \\
\gamma_1^{ij} \frac{\partial n_i}{\partial t}(r, t) &= - \gamma_1^{ij} \nabla \cdot \frac{j_3}{\rho}(r, t) - \frac{\delta F}{\delta n_j}(r, t) + \theta_i(r, t), \quad (3a) \\
\gamma_1^{ij} \frac{\partial n_i}{\partial t}(r, t) &= - \frac{\delta F}{\delta n_j}(r, t) + \theta_i(r, t), \quad i = 1, 2 \quad (3b) \\
\frac{\partial j_3}{\partial t}(r, t) &= - \nabla_3 P(r, t) - 2 i q_0 \sum_k \text{Im} \left( \psi_i \frac{\delta F}{\delta \psi_i^*} - \nabla_i \sigma^D_{3k} - \sum_k \frac{\delta F}{\delta n_k}(r, t) + \zeta_3(r, t) \right), \quad (4a) \\
\frac{\partial j_i}{\partial t}(r, t) &= - \nabla_3 P(r, t) + \zeta_m(r, t), \quad m = 1, 2, 4 \quad (4b)
\end{align}

where $\rho$ is the average mass density, $\gamma_3^{ij}$ the bare viscosity which sets the time scale for the order parameter dynamics, $\gamma_1^{ij}$ the bare twist viscosity, $P(r, t)$ the pressure, $\sigma^D_{ik}$ the dissipative part of the stress tensor and $\eta(r, t)$, $\theta(r, t)$ and $\zeta(r, t)$ the noise terms whose correlations are short-ranged in space and time [8].

The free energy functional to be inserted in eqs. (2), (3) and (4) is the usual de Gennes [3] free energy in $d$-dimensions, supplemented by terms associated with the velocity and external fields [8]. Thus

$$ F = \int d^d r \left\{ \frac{1}{2} |\psi|^2 + |(\nabla - i q_0 n) \psi|^2 + \frac{1}{2} U_0 |\psi|^4 + \frac{1}{2} K_{ij}^{n0} (\nabla \cdot n)^2 + \frac{1}{2} K_{ij}^{n0} \sum_{l \neq j} (\nabla_i n_j - \nabla_j n_i)^2 + \frac{1}{2} K_{ij}^{n0} \sum_{l} (\nabla_i n_l)^2 + \frac{j_3^2}{2 \rho} (r) - \sum_{l, j} \psi_j(r) h_{ij}(r) - \sum_{l} b_3(r) n_l(r) - A(r) \cdot \frac{j(r)}{\rho} \right\}, \quad (5) $$

where $\psi$ has $n/2$ complex components,

$$ a^0 = a'(T - T_\Phi)/T_\Phi, $$

and $T_\Phi$ is the mean field transition temperature, $K_0^{n0}$ the bare Frank constants, and $h$, $b$ and $A$ the external fields which couple to the density, the fluctuating
components of the director and the velocity, respectively.

3. Renormalization group analyses. — The static renormalization group (R.G.) recursion relations for the NA transition in the superconduction gauge are given by Lubensky and Chen [1]. Since we are doing the dynamics, it is essential that we work in the physical liquid crystal gauge. The difference between the two gauges has been explained in ref. [1]. There are two fixed points for \( K_i = 0, \infty \). But \( K_i = 0 \), creates a problem in the liquid crystal gauge and we find that we cannot work in this limit. When \( K_i = \infty \) and \( n \geq 238.17 \), there is a stable fixed point with anisotropic scaling (\( k_i \to b^{-1} k_i, k_i \to b^{-1} k_i + k_i \)). In the liquid crystal gauge, we find the static exponents to be [12],

\[
\eta_s = \frac{q_0^2}{12 \pi^2} \frac{\sqrt{K_3(3 \sqrt{K_3} + 4 \sqrt{K_1})}}{\sqrt{K_1(\sqrt{K_3} + \sqrt{K_1})^3}} - \frac{2}{3 \pi^2} \frac{1}{\sqrt{K_2(\sqrt{K_3} + \sqrt{K_2})}},
\]

\[
\mu_\parallel = -\frac{q_0^2}{6 \pi^2} \frac{1}{\sqrt{K_3 + K_1}} + \frac{q_0^2}{3 \pi^2} \frac{1}{\sqrt{K_2(\sqrt{K_3} + \sqrt{K_2})}},
\]

\[
\eta_s = \epsilon - \mu_\parallel.
\]

In the limit \( K_i = \infty \), the fixed points reduce to those in the superconducting gauge [1].

To apply the renormalization group to the dynamics, we follow a method similar to the one used by Halperin, Hohenberg and Siggia [10]. We do a scale transformation on the frequencies as well as the momenta

\[
\omega \to \omega' = b^z \omega,
\]

where \( b \) is an arbitrary constant greater than unity and \( z \) is the dynamic exponent. The R.G. transformations of the dynamic variables consist of integrating out the short wavelength fluctuations (\( b^{-1} A < k < A \)) and rescaling the fields,

\[
\psi(b^{-1} k, t) \to b^{d-z+\eta_1+\eta_2} \psi(k', b^{-z} t),
\]

\[
n(b^{-1} k, t) \to b^{d-z+\eta_1+\eta_2} n(k', b^{-z} t),
\]

\[
j(b^{-1} k, t) \to b^{d-\eta_1+\mu_\parallel} j(k', b^{-z} t).
\]

The recursion relations for the dynamic coefficients \( v_i, \gamma_i \) and \( \gamma_3 \) are obtained by using the above scale transformations to eqs. (2), (3) and (4). The correction terms for \( v \) and \( \gamma_1 \) are taken from the mode coupling calculation [8] in which the integrals have to be performed near 4-dimensions for momenta between \( b^{-1} A \) and \( A \). The correction term for \( \gamma_3 \) is given in Appendix A.

For convenience, we define the following quantities,

\[
f_{v_i} = \frac{\gamma_3}{\gamma_1} q_0^2,
\]

\[
W_{v_i} = \frac{\rho_1}{\nu_1 \gamma_1},
\]

\[
Y_{v_i} = \frac{\gamma_3}{\gamma_1} q_0^2,
\]

\[
X_i = \frac{1}{\gamma_3^2} (L + F + G) - P,
\]

\[
\eta_1 = \frac{q_0^2}{12 \pi^2} \frac{\sqrt{K_3(3 \sqrt{K_3} + 4 \sqrt{K_1})}}{\sqrt{K_1(\sqrt{K_3} + \sqrt{K_1})^3}} - \frac{2}{3 \pi^2} \frac{1}{\sqrt{K_2(\sqrt{K_3} + \sqrt{K_2})}},
\]

\[
\mu_\parallel = -\frac{q_0^2}{6 \pi^2} \frac{1}{\sqrt{K_3 + K_1}} + \frac{q_0^2}{3 \pi^2} \frac{1}{\sqrt{K_2(\sqrt{K_3} + \sqrt{K_2})}},
\]

\[
\eta_s = \epsilon - \mu_\parallel.
\]

where \( L, F, G \) and \( P \) are defined in eq. (A.14). From static scaling, \( \rho_1 \) is given by

\[
\rho_{i+1} = b^{\mu} \rho_i.
\]

The dynamic recursion relations correct to first order in \( \epsilon = 4 - d \) are, therefore, given by

\[
v_{i+1} = b^{z-2+\eta_2-2\eta_0} v_i \left( 1 + \frac{n}{2} \frac{f_{v_i}}{64 \pi^2} \ln b \right),
\]

\[
\gamma_{i+1} = b^{z-2-\eta_1} \gamma_i \left( 1 + \frac{n}{2} \frac{Y_{v_i}}{16 \pi^2} \ln b \right),
\]

\[
\gamma_{3+1} = b^{2z-2-\eta_0} \gamma_3 (1 + X_i \ln b).
\]

Making use of the scaling for \( \rho_i \) in eq. (2), we get

\[
q_{0i+1} = b^{2z-2+\eta_2-2\eta_0} q_{0i}.
\]

The exponent \( \eta_1 \) is now obtained from the requirement that \( q_0 \) remains a constant in eq. (13),

\[
\eta_1 = d - 2z + \mu_\parallel.
\]

Using eqs. (10), (11) and (12), the recursion relations for \( f_{v_i}, W_{v_i} \) and \( Y_{v_i} \), become

\[
f_{v_{i+1}} = b^{z-\eta_2+\mu_1} f_{v_i} \left[ 1 + X_i \ln b - \frac{n}{2} \frac{f_{v_i}}{64 \pi^2} \ln b \right],
\]

\[
W_{v_{i+1}} = b^{\eta_2+2\eta_0} W_{v_i} \left[ 1 - \frac{n}{2} \frac{f_{v_i}}{64 \pi^2} \ln b - X_i \ln b \right],
\]

\[
Y_{v_{i+1}} = b^{\eta_2-\eta_0} Y_{v_i} \left[ 1 + X_i \ln b - \frac{n}{2} \frac{Y_{v_i}}{16 \pi^2} \ln b \right].
\]

We then solve for \( f_{v_i}, X_i \) and \( Y_{v_i} \), from the condition that each of \( f_{v_i}, W_{v_i} \) and \( Y_{v_i} \) approach a finite fixed point. The trivial fixed points for these equations are
The non trivial fixed points are found to be
\[ f_2^* = \frac{64 n^2}{\varepsilon + \frac{8 f_2^*}{1 + R^*}}, \]
\[ X^* = \frac{-\varepsilon}{2} - \frac{4 f_2^*}{1 + R^*}, \]
\[ Y_1^* = \frac{32 n^2}{3} \frac{\varepsilon}{2} - \frac{4 f_2^*}{1 + R^*}. \]

where \( f_2^* \) and \( R^* \) are given \([1]\) by
\[ f_2 = q_3^2/8 \pi^2 K_i. \]

The dynamic exponent \( z \) is obtained by using the above values in eqs. (10), (11) and (12) and requiring that \( v_1, \gamma_1, \) and \( \gamma_3 \) also reach their fixed points. From all three equations, we get
\[ z = \frac{d}{2} + \frac{4 f_2^*}{3} + \frac{1}{1 + R^*} \]
\[ = \frac{1}{2}(d + \mu_i). \]

To obtain the temperature dependence of the physical kinetic coefficients, we assume that \( v_1, \gamma_1, \) and \( \gamma_3 \) approach their fixed point values such that
\[ l = \log_k \frac{A}{K_\perp} = \log_{\mu_i} \frac{A}{K_\parallel} \]
where \( b' = b(\varepsilon + \mu_i) \) and \( K_\perp \) and \( K_\parallel \) are the inverse correlation lengths perpendicular and parallel to the director. After \( l \) iterations, the correction terms can be neglected \([10]\) and eqs. (10), (11) and (12) give us
\[ v_1(T) = b^{-(e-2+\mu_i)} v_1^* \sim \kappa_\perp^{-(3d/2+\mu_i/2)} v_1^* \]
\[ \gamma_1(T) = b^{-(e-2+\mu_i)} \gamma_1^* \sim \kappa_\perp^{-(3d/2+\mu_i/2)} \gamma_1^* \]
\[ \gamma_3(T) = b^{-(2-\varepsilon+\mu_i)} \gamma_3^* \sim \kappa_\perp^{-(3d/2+\mu_i/2)} \gamma_3^*. \]

These results are in accordance with dynamical scaling \([10]\).

4. Conclusion. — We have studied the dynamics of the NA transition in the liquid crystal gauge assuming anisotropic scaling. One of the central results of this work is the dynamic critical exponent \( z = d/2 + \mu_i/2 \). This is a generalization of the result \( z = d/2 \), one would expect from a simple analogy of the NA transition with the \( \lambda \) transition in superfluid helium. We have also found the temperature dependence of the dynamic coefficients. It is seen that if the anisotropy is small, there is a critical divergence of the viscosities \( \gamma_1 \) and \( \gamma_1 \) while \( \gamma_3 \) goes to zero. The critical exponents for \( \gamma_1 \) and \( \gamma_1 \) reduce to the mode-coupling values \([8]\) in the limit \( \varepsilon \to 1 \). The exponents we have obtained are the ones that would be observed at \( K_\parallel \to \infty \) which is unstable with respect to temperature and \( K_\parallel \) only. If we now approach this fixed point with \( K_\parallel \) small but not zero, there will be a crossover from mean-field to isotropic-critical behaviour \([1]\).

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Appendix A

In this Appendix, we present the mode coupling calculation for \( \gamma_3 \) in the liquid crystal gauge near \( \varepsilon = 4 - d \) dimensions. To find the correction to \( \gamma_3 \) from eq. (2a), we must first solve for \( \bar{n} \) and \( \bar{n} \) from eqs. (3) and (4).

Rewriting these two equations, we get
\[ \gamma_3^0 \frac{\partial n_3}{\partial t} (k, t) + i k_\perp \frac{\gamma_3^0}{\rho} v_3(k, t) + \left( K_3^0 k_\parallel^2 + K_3^0 k_\perp^2 \right) n_3(k, t) = f_{n_3}(k, t) + \theta_3(k, t), \]
\[ \gamma_3^0 \frac{\partial n_3}{\partial t} (k, t) + \left( K_3^0 k_\parallel^2 + K_3^0 k_\perp^2 \right) n_3(k, t) = f_{n_3}(k, t) + \theta_3(k, t), \]
\[ \frac{\partial v_3}{\partial t} (k, t) + i k_\parallel \frac{\partial}{\rho} P(k, t) + 2 v_1 k_\parallel^2 \frac{\partial}{\rho} v_3(k, t) + i k_\perp (K_3^0 k_\parallel^2 + K_3^0 k_\perp^2) n_3(k, t) = f_{v_3}(k, t) + \zeta_3(k, t), \]
\[ \frac{\partial v_m}{\partial t} (k, t) = i k_{m} P(k, t) + \zeta_m(k, t), \quad m = 1, 2, 4. \]

Here \( f_{n_3}, f_{n_3}, \) and \( f_{v_3} \) contain the non-linear terms arising due to the different couplings in the free energy and are given by
\[ f_n(k, t) = -\frac{1}{\gamma_3^0} \int \frac{d^d q'}{(2\pi)^d} \Gamma_3(k, q) \psi(-q', t) \psi^*(-q' - k, t), \]
\[ f_{n_3}(k, t) = -\frac{1}{\gamma_3^0} \int \frac{d^d q'}{(2\pi)^d} \Gamma_3(k, q) \psi(-q', t) \psi^*(-q' - k, t), \]
\[ f_{v_3}(k, t) = -\frac{1}{\gamma_3^0} \int \frac{d^d q'}{(2\pi)^d} \Gamma_3(k, q) \psi(-q', t) \psi^*(-q' - k, t). \]
and \(f_{n_1}\), are obtained by replacing \(\Gamma_{\perp}\) by \(\Gamma_{\parallel}\) on the R.H.S. of eq. (A.5). The vertices \(\Gamma_{\perp}\) and \(\Gamma_{\parallel}\) are the perpendicular and transverse components of \(\Gamma_i(k, q) = -q_0(2q + k)\).

\[
f_{n_3}(k, t) = i\eta_0 \int \frac{d^dq'}{(2\pi)^d} \left[ \psi(q', t) \left[ G^{0-1}(q' - k) \psi^*(q' - k, t) - h^*(q' - k, t) \right] - \psi^*(q' - k, t) \left[ G^{0-1}(q') \psi(q', t) - h(q', t) \right] \right] - ik_1 \int \frac{d^dq'}{(2\pi)^d} \Gamma_i(k, q') \psi(-q', t) \psi^*(-q' - k, t). \tag{A.6}
\]

The frequencies of the eigenmodes are obtained by setting the R.H.S. of (A.1), (A.2) and (A.3) equal to zero. The eigenfrequencies for \(n_{i_1}\) \((i = 1, 2)\) are trivial and are given by

\[
i\omega_i = \frac{1}{\gamma_i} \left( K_i^0 \gamma_1^0 + K_i^0 \gamma_2^0 \right). \tag{A.7}
\]

The eigenfrequencies corresponding to the coupled equations for \(n_3\) and \(n_{\perp}\) are found by first using the incompressibility conditions in eqs. (A.3) and (A.4) to eliminate the pressure and then solving the \(2 \times 2\) matrix. We get

\[
i\omega_F = \frac{(C + B)}{2} \pm \frac{R}{2}, \tag{A.8}
\]

where

\[
B = \frac{1}{\gamma_1^0} \left( K_3^0 \gamma_1^0 + K_2^0 \gamma_1^0 \right),
\]

\[
C = \frac{2 \gamma_1^0 \gamma_1^0 \gamma_2^0}{\rho k^2},
\]

\[
R = \left[ (C - B)^2 - \frac{4 k^4 B_1^0}{k^2 \rho} \right]^{1/2}. \tag{A.8a}
\]

Using the above eigenfrequencies and working with left eigenvectors, we get

\[
n_{n_3}(k, t) = I(\omega_F) f_{n_3}(k, t'), \tag{A.9}
\]

\[
n_{n_1}(k, t) = \left[ \frac{B - C}{2R} + \frac{1}{2} \right] I(\omega_F) f_{n_3}(k, t') - \left[ \frac{B - C}{2R} - \frac{1}{2} \right] I(\omega_F) f_{n_3}(k, t') - \left[ \frac{B - C}{2R} - \frac{1}{2} \right] I(\omega_F) f_{n_3}(k, t') \tag{A.10}
\]

and

\[
v_3(k, t) = \left[ \frac{B - C}{2R} + \frac{1}{2} \right] I(\omega_F) f_{n_3}(k, t') - \left[ \frac{B - C}{2R} - \frac{1}{2} \right] I(\omega_F) f_{n_3}(k, t') + \frac{ik^2 \gamma_1^0 B}{k^2 \rho} I(\omega_F) f_{n_3}(k, t') - \frac{ik^2 \gamma_1^0 B}{k^2 \rho} I(\omega_F) f_{n_3}(k, t'), \tag{A.11}
\]

where

\[
I(\omega) = \int_{-\infty}^{t} dt' e^{-i\omega(t-t')}.
\]

Writing eq. (2) in \(q\)-space and using \(\int \frac{d^dk}{(2\pi)^d} \rightarrow \int_k\)

\[
\eta_0 \frac{\partial \psi}{\partial t} (q, t) + G^{0-1}(q) \psi(q, t) - h(q, t) = -i\eta_0 \frac{\gamma_3^0}{\rho} \int_k v_3(k, t) \psi(q - k, t) - \int_k \psi(q + k, t) \left[ \Gamma_{\perp}(k, -q - k) n_{1}(-k, t) + \Gamma_{\parallel}(k, -q - k) n_{1}(-k, t) \right] + \eta(q, t). \tag{A.12}
\]
Substituting the values of \( v_3, n_1 \) and \( n_{ti} \) in (A.12), and looking for terms proportional to \(-i\alpha_0^2\) in the limit \( K_1 \to \infty \), we get, after some tedious algebra:

\[
\gamma_3 = \frac{\gamma_0^2 + L + F + G}{1 + P}
\]  

where

\[
L = -\frac{q_0 \gamma_3}{\varrho} \int \left( \frac{1}{2} + \frac{B - C}{2 R} \right) \frac{k^2}{W^2} \Gamma(k, -q) G^0(q - k),
\]

\[
F = \frac{q_0 \gamma_3}{\varrho} \int \frac{\Gamma(k, -q) G^0(q - k)}{k^2 R W^2},
\]

\[
G = -\frac{q_0^2 \gamma_3}{\varrho} \int \left( \frac{1}{2} + \frac{B - C}{2 R} \right) \frac{1}{W^2},
\]

\[
P = \frac{q_0^2 \gamma_3}{\varrho} \int \left( \frac{1}{2} + \frac{B - C}{2 R} \right) \frac{G^0(q - k)}{W}.
\]

where we have used

\[
\Gamma_0(q) = [\gamma_3 G^0(q)]^{-1}, \quad W = i\alpha_0(k) + \Gamma_0(q - k).
\]

**Appendix B**

**Stability analyses.** — Here we shall investigate the stability of the fixed points for \( f_0, W, Y_1, f_2 \) and \( f_3 \). The recursion relations for these quantities in the differential form are

\[
\frac{df}{dl} = f_0 \left( \varepsilon + \frac{8 f}{3 + R} + X_1 - \frac{n}{2} \left( \frac{f}{64 \pi^2} \right) \right)
\]

\[
\frac{dW}{dl} = W_1 \left( -X_1 - \frac{n}{2} \left( \frac{f}{64 \pi^2} \right) \right)
\]

\[
\frac{dY_1}{dl} = Y_1 \left( \varepsilon + \frac{8 f_2}{3 + R} + X_1 - \frac{n}{2} \left( \frac{Y_1}{16 \pi^2} \right) \right)
\]

\[
\frac{df_2}{dl} = f_2 \left( \varepsilon - \mu_{\perp} - \frac{n}{6} f_2 \right),
\]

\[
\frac{df_3}{dl} = f_3 \left( \varepsilon - \mu_{\perp} - \frac{n}{6} f_3 \right),
\]

where \( X_1 \) is given by

\[
X_1 = \frac{f_0}{16 \pi^2 \sqrt{2} W_1 (4 W + 1)^{3/2}} \left[ 4(W_1 - 1)(4 W_1 + 1) \times \right.
\]

\[
\times \left. \left( 2 W_1 + 1 + \sqrt{4 W_1 + 1} \right)^{1/2} - \left( 2 W_1 + 1 - \sqrt{4 W_1 + 1} \right)^{1/2} \right] - 2 \sqrt{2(8 W_1 + 1)} (W_1 + 1).
\]

The fixed points are obtained by putting the L.H.S. of the above equations equal to zero and then solving them. This gives the following values to leading order in \( \varepsilon \), for \( n = 238.17 \),

\[
W_0 = 0.038
\]

\[
f_0^* = 0.295 \pi^2 \varepsilon
\]

\[
Y_0^* = 0.065 \pi^2 \varepsilon
\]

\[
f_2^* = 0.024 \varepsilon
\]

\[
f_3^* = 0.026 \varepsilon.
\]
We linearize (B.1) by expanding about the fixed points (e.g. \( f_* = f_*^0 + \delta f_* \)) and use (B.3) to obtain

\[
\frac{d(\delta f_1)}{dt} = -0.11 \pi^2 \omega \delta f_1 + 0.91 \pi^2 \omega \delta f_2 + 0.28 \pi^2 \omega \delta f_3
\]

\[
\frac{d(\delta W)}{dt} = -0.01 \delta f_1 - 0.03 \pi^2 \omega \delta W
\]

\[
\frac{d(\delta Y_1)}{dt} = -0.01 \pi^2 \omega \delta f_1 - 0.05 \pi^2 \omega \delta Y_1 + 0.07 \pi^2 \omega \delta f_2 + 0.02 \pi^2 \omega \delta f_3
\]

\[
\frac{d(\delta f_2)}{dt} = -0.99 \omega \delta f_2 - 0.01 \omega \delta f_3
\]

\[
\frac{d(\delta f_3)}{dt} = 0.03 \omega \delta f_2 - 1.02 \omega \delta f_3
\]

The eigenvalues of these equations are obtained by solving the 5 \times 5 matrix. We find that all five eigenvalues are negative definite indicating that the fixed points are stable.

References

[12] Eq. (6) agrees with the static exponents calculated in the liquid crystal gauge by Jing-Huei Chen (private communications).