Analysis of ising model critical exponents from high temperature series expansion

J. Zinn-Justin

To cite this version:

J. Zinn-Justin. Analysis of ising model critical exponents from high temperature series expansion. Journal de Physique, 1979, 40 (10), pp.969-975. 10.1051/jphys:019790040010096900 . jpa-00209184

HAL Id: jpa-00209184
https://hal.archives-ouvertes.fr/jpa-00209184

Submitted on 1 Jan 1979

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Analysis of Ising model critical exponents from high temperature series expansion

J. Zinn-Justin

Service de Physique Théorique, CEN, Saclay, B.P. 2, 91190 Gif sur Yvette, France

(Reçu le 19 mars 1979, révisé le 25 juin 1979, accepté le 29 juin 1979)

Abstract. — High temperature series expansion for the critical exponents of the Ising model are reanalysed using a modified ratio method. The analysis shows that a minor modification of the ratio method yields for all lattices a value, for the exponent $\beta$ in three dimensions, close to 1.245, therefore lower than the quoted value 1.250, and much closer to the renormalization group (R.G.) value 1.241. The exponent is analysed in two ways: in one method $\beta$ is estimated directly while in the other one $T_c$, is calculated first. With these new values of $T_c$, the exponent $\alpha$ is recalculated and found to be very close to the R.G. value 0.110. The value of $\gamma$ is not modified (0.638) and is therefore still a problem for hyperscaling, and is in disagreement with the R.G. value (0.630).

1. Introduction. — For a long time high temperature series (H.T.S.) expansions have been the only source of information about the values of critical exponents of ferromagnetic systems in three dimensions [1].

In recent years, renormalization group (R.G.) ideas [2] have led to new methods of calculation, which are based on the study of the small momentum behaviour of the $\phi^4$ continuous field theory. The first method was the $\varepsilon = 4 - d$ expansion [3] which gave values in qualitative agreement with the H.T.S. values but with a poor accuracy. More recently [4, 5, 6] calculations have been performed directly in three dimensions yielding more accurate results, so that some exponents now disagree with the H.T.S. estimates.

For Ising like exponents the situation is the following:

$\gamma_{H.T.S.} = 1.250 \pm 0.003$

$\gamma_{R.G.} = 1.241 \pm 0.002$

$\beta_{H.T.S.} = 0.638 \pm 0.002$

$\beta_{R.G.} = 0.630 \pm 0.0015$

$\alpha_{H.T.S.} = 0.125 \pm 0.010$

$\alpha_{R.G.} = 2 - 3 \beta_{R.G.} = 0.110 \pm 0.0045$.

From these numbers two facts are apparent: On one hand the exponents obtained from both theories are close to an accuracy better than one percent which can be considered to be very good and seems to justify the idea that the $\phi^4$ field theory describes indeed the critical properties of the Ising model.

On the other hand if one takes seriously the guessed error bars on the various results, one discovers a significant discrepancy between the two models. For $\gamma$, for which the H.T. series are the longest and therefore the most significant, the H.T.S. value is much too high compared to the R.G. value.

The exponent $\alpha$ is somewhat higher in the H.T. series but the series are shorter so it is not too trouble-
some. But the exponent \( v \) creates a more serious problem because again the R.G. and H.T.S. values differ by too much, and in addition \( x \) and \( v \) from H.T. series do not satisfy the scaling relation:

\[
x = 2 - 3v
\]
called sometimes a hyperscaling relation because the dimension \( d = 3 \) of space appears explicitly in the relation.

This relation can be proven to be exactly true in the \( \phi^4 \) field theory. So it was tempting for a member of a group involved in the field theoretical calculations, to reanalyse the H.T. series to get an independent feeling of how serious these differences are.

The exponents \( y, v \) and \( x \) were calculated in two and three dimensions for all available lattices.

We used a typical ratio method with two ideas in mind:

i) To analyse first the exponents, when possible, directly without using an estimated, and possibly biased, value of the critical temperature.

ii) To use extrapolation methods which do not involve the order \( n \) of the series explicitly and which involve a minimal number of manipulations on the series, because the exact structure of the series as a function of \( n \) may be complicated and is not very well-known.

Indeed each manipulation involves new assumptions on the unknown structure of the series and we wish to make as few assumptions as possible. For the same reason, if we introduce explicitly the order \( n \) of the series, we shall have for instance to make the choice of using \( n \) rather than \( n + a \) as a variable. An inappropriate choice will lead to big higher order corrections in \( 1/n \). To get rid of these corrections we will again require additional assumptions on the form of the series.

The direct analysis is possible for \( y \) and \( v \), while for \( x \) the new values of \( T_c \) are necessary to get a reasonable estimate.

The article is organized as follows:

- In section 2 we explain our method.
- In section 3 we analyse \( y \), and in section 4 \( v \) and \( x \), in two and three dimensions for various lattices.
- Section 5 contains our conclusions.

2. The method. — We shall first explain the method, based on an analysis of the coefficients of the series which we have used, and give the explicit extrapolation formulae.

2.1 The susceptibility exponent \( y \). — Following Gaunt and Sykes [7], we have decided to make first a direct analysis of \( y \) which does not involve an estimate of the critical temperature \( T_c \). But the method used here differs on two points.

1) We have not made a conformal mapping to eliminate the antiferromagnetic singularity.

The reason is the following: Such a mapping may bring nearer other unknown complex singularities making the analysis more difficult. The large oscillations observed in the results of reference [7] may be due to such singularities.

Instead we have separated odd and even terms when necessary i.e. for the square, S.C. and B.C.C. lattices and even separated the terms modulo four for the honeycomb and diamond, because the odd and even terms separately have still an oscillation.

At the end of the calculation we have obtained two or four series and we have averaged the terms two by two (four by four respectively).

2) We have used different extrapolation formulae in which in particular the order \( n \) of the series does not appear explicitly. The reason for this is that we do not know if we should use \( n \) or \( n + a \). To correct for this fact one needs a series of Neville extrapolations which rely on the assumption that the series has an expansion in \( 1/n \), and we wanted to avoid as much as possible such assumptions.

Starting from the susceptibility \( \chi \) series expansion:

\[
\chi(V) = \sum_{n=0}^{N} a_n V^n
\]

we shall obtain a sequence for \( y \) by the formulae:

\[
b_n = \left[ 1 - \frac{a_{n+1} a_{n-1}}{a_n^2} \right] \quad (1)
\]

\[
g_n = 1 + \frac{b_n b_{n+1}}{(\sqrt{b_n} - \sqrt{b_{n+1}})^2}. \quad (2)
\]

Indeed if, near the critical point \( V_c \), \( \chi(V) \) behaves like:

\[
\chi(V) \sim (V - V_c)^{-y}.
\]

Then for \( n \) large:

\[
a_n \sim A V_c^{-n} \quad (3)
\]

Therefore:

\[
b_n \sim \frac{g_n - 1}{n^2}. \quad (4)
\]

In this analysis of \( y \) no value of \( V_c \) has to be assumed. Now in a second step we have decided to calculate also \( V_c \) and then to recalculate \( y \) with our new value of \( V_c \).

We have used the following extrapolation formulae:

\[
a_n = \frac{a_{n+1}}{a_n} \quad (5)
\]

\[
(V_c^{-1})_n = \frac{2 C_{n-1} C_{n+1} - C_n (C_{n+1} + C_{n-1})}{C_{n+1} + C_{n-1} - 2 C_n}. \quad (6)
\]
The second formula is similar to a one step Neville extrapolation. Then the exponent $\gamma$ has been obtained from:

$$d_n = \left[ \frac{a_{n+1}}{a_n} V_c - 1 \right] \quad (9)$$

$$\gamma_n = 1 + \frac{d_n d_{n+1}}{d_n - d_{n+1}}. \quad (10)$$

Notice that the formulae are such that corrections of order $1/n$ are automatically suppressed.

2.2 THE EXPONENT $v$. — In order to calculate $v$ one can use the series [8] for the second moment of the two-point correlation function $m^{(2)}$:

$$m^{(2)}(V) \sim (V_c - V)^{-\gamma - 2v}. \quad (11)$$

The natural procedure to calculate $v$ is to calculate the ratios of the coefficients of the series of $\chi(V)$ and $m^{(2)}(V)$

$$m^{(2)}(V) = \sum m_n V^n \quad (12)$$

$$V_n = \frac{m_n}{a_n} \sim M n^{2v} \quad (13)$$

We have calculated the quantities:

$$S_n = \frac{V_{n+1}}{V_n} - 1 \quad (14)$$

$$v = \frac{1}{2} \frac{S_n S_{n+1}}{S_n - S_{n+1}}. \quad (15)$$

2.3 THE EXPONENT $\alpha$ [9]. — For the exponent $\alpha$, in order to get a reasonable accuracy it is necessary to use the critical temperature as obtained from the susceptibility series. The method is then identical to the method used for the second calculation of $\gamma$.

3. The results for $\gamma$. — 3.1 THE TWO DIMENSIONAL ISING MODEL. — To check our methods we have applied them first to the two dimensional Ising model [10].

The lattice with the highest coordination number having the smallest number of singularities, we expect to obtain the best results from the triangular lattice. This is exactly what happens. Table I shows our result for the direct method. Tables II and III show the result for the square and honeycomb lattice respectively. In all cases the deviation from the exact result 1.75 is smaller than $2 \times 10^{-3}$ for the last terms.

Table II. — The exponent $\gamma$ for the square lattice.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\gamma_n$</th>
<th>$n$</th>
<th>$\gamma_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1.687 383</td>
<td>15</td>
<td>1.746 342</td>
</tr>
<tr>
<td>8</td>
<td>1.644 574</td>
<td>16</td>
<td>1.758 167</td>
</tr>
<tr>
<td>9</td>
<td>1.796 052</td>
<td>17</td>
<td>1.748 928</td>
</tr>
<tr>
<td>10</td>
<td>1.902 710</td>
<td>18</td>
<td>1.751 581</td>
</tr>
<tr>
<td>11</td>
<td>1.790 525</td>
<td>19</td>
<td>1.751 440</td>
</tr>
<tr>
<td>12</td>
<td>1.726 409</td>
<td>20</td>
<td>1.750 090</td>
</tr>
<tr>
<td>13</td>
<td>1.757 062</td>
<td>21</td>
<td>1.751 479</td>
</tr>
<tr>
<td>14</td>
<td>1.759 143</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table III. — The exponent $\gamma$ for the honeycomb lattice.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\gamma_n$</th>
<th>$n$</th>
<th>$\gamma_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>1.734 903</td>
<td>24</td>
<td>1.754 348</td>
</tr>
<tr>
<td>16</td>
<td>1.811 143</td>
<td>25</td>
<td>1.755 486</td>
</tr>
<tr>
<td>17</td>
<td>1.806 116</td>
<td>26</td>
<td>1.752 624</td>
</tr>
<tr>
<td>18</td>
<td>1.799 413</td>
<td>27</td>
<td>1.751 046</td>
</tr>
<tr>
<td>19</td>
<td>1.782 250</td>
<td>28</td>
<td>1.752 788</td>
</tr>
<tr>
<td>20</td>
<td>1.761 663</td>
<td>29</td>
<td>1.751 820</td>
</tr>
<tr>
<td>21</td>
<td>1.756 796</td>
<td>30</td>
<td>1.750 591</td>
</tr>
<tr>
<td>22</td>
<td>1.761 985</td>
<td>31</td>
<td>1.750 678</td>
</tr>
<tr>
<td>23</td>
<td>1.751 949</td>
<td>32</td>
<td>1.751 551</td>
</tr>
</tbody>
</table>

Only Padé approximants for the square lattice give a somewhat better convergence. But it is clear that one cannot rely on only one method specially a method with well-known problem of instabilities like Padé approximants.

In addition, if one should require from any method that it converges reasonably well for two dimensional models, the best method for one model will not automatically be the best one for any model, as at some stage specific properties start playing a role.

We now calculate as explained in the previous section $V_c^{-1}$. The exact result for the triangular lattice is:

$$V_c^{-1} = 2 + \sqrt{3} = 3.732 050... \quad (16)$$

The last terms differ only from the exact result by a few parts in $10^5$ (Table IV).

We then calculate $\gamma$ (Table IV). One sees that the final accuracy is similar to the accuracy obtained from the direct method. But due to the uncertainty on the value of $V_c^{-1}$ which translates in an additional uncertainty over $\gamma$. The method is somewhat less accurate as:

$$\frac{d\gamma}{dV_c^{-1}} = -9 \quad (17)$$
Table IV. — The critical temperature $V_{c}^{-1}$ and the exponent $\gamma$ for the triangular lattice

$V_{c}^{-1}$ exact = $2 + \sqrt{3} = 3.732\, 050\, 8$

$\frac{d\gamma}{dV_{c}^{-1}} = - 9$

| $n$ | $V_{c}^{-1}$ | $\gamma$
|-----|--------------|------
| 10  | 3.733 078    | 1.751 266 |
| 11  | 3.730 884    | 1.750 666 |
| 12  | 3.731 588    | 1.750 441 |
| 13  | 3.732 021    | 1.750 447 |
| 14  | 3.732 067    | 1.750 477 |
| 15  | 3.732 032    | 1.750 488 |
| 16  | 3.732 011    | 1.750 489 |

Table V. — The critical temperature $V_{c}^{-1}$ and the exponent $\gamma$ for the square lattice

$V_{c}^{-1}$ exact = $1 + \sqrt{2} = 2.414\, 213\, 5$

$\frac{d\gamma}{dV_{c}^{-1}} = - 17$

| $n$ | $V_{c}^{-1}$ | $\gamma$
|-----|--------------|------
| 15  | 2.414 617    | 1.751 292 |
| 16  | 2.413 540    | 1.751 565 |
| 17  | 2.413 131    | 1.751 612 |
| 18  | 2.414 106    | 1.751 542 |
| 19  | 2.414 118    | 1.751 608 |
| 20  | 2.414 213    | 1.751 667 |
| 21  | 2.414 123    | 1.751 614 |

Table VI. — The critical temperature $V_{c}^{-1}$ and the exponent $\gamma$ for the honeycomb lattice

$V_{c}^{-1}$ exact = $\sqrt{3} = 1.732\, 050\, 8$

$\frac{d\gamma}{dV_{c}^{-1}} = - 34.5$

| $n$ | $V_{c}^{-1}$ | $\gamma$
|-----|--------------|------
| 26  | 1.731 947    | 1.751 431 |
| 27  | 1.732 015    | 1.751 660 |
| 28  | 1.731 947    | 1.751 937 |
| 29  | 1.731 984    | 1.751 638 |
| 30  | 1.732 029    | 1.751 569 |
| 31  | 1.732 027    | 1.751 789 |
| 32  | 1.731 999    | 1.751 934 |

Similar observations are valid for the square lattice (Table V) and the honeycomb lattice (Table VI).

For the square:

\[ \frac{d\gamma}{dV_{c}^{-1}} = - 17 \]  \hspace{1cm} (18)

and the honeycomb:

\[ \frac{d\gamma}{dV_{c}^{-1}} = - 34.5 \]  \hspace{1cm} (19)

Notice the very important fact that the exponents are the least sensitive to an error in $V_{c}$ for the most compact lattice.

3.2 THE THREE DIMENSIONAL ISING MODEL. — We know now series for four lattices [10, 11, 12, 7] the face centred cubic (F.C.C.), the body centred cubic (B.C.C.), the simple cubic (S.C.), and the diamond (D.) with decreasing coordination numbers. From our two dimensional experience we expect the best results to come from the F.C.C. lattice which has always been known to give a smaller value of $\gamma$ [12].

The results by direct analysis are given on tables VII, VIII, IX, X respectively. It is remarkable that the three first lattices give values smaller than the commonly quoted value [13], 1.250. For the last one the diamond, the convergence is too poor and nothing can be said.

Table VII. — The exponent $\gamma$ for the F.C.C. lattice.

| $n$ | $\gamma$
|-----|-------
| 3   | 1.241 261 |
| 4   | 1.314 167 |
| 5   | 1.261 279 |
| 6   | 1.238 297 |
| 7   | 1.235 119 |
| 8   | 1.246 048 |
| 9   | 1.249 038 |

Table VIII. — The exponent $\gamma$ for the B.C.C. lattice.

| $n$ | $\gamma$
|-----|-------
| 3   | 1.278 651 |
| 4   | 1.247 252 |
| 5   | 1.232 332 |
| 6   | 1.239 451 |
| 7   | 1.243 039 |

Table IX. — The exponent $\gamma$ for the S.C. lattice.

| $n$ | $\gamma$
|-----|-------
| 7   | 1.288 339 |
| 8   | 1.333 335 |
| 9   | 1.259 283 |
| 10  | 1.227 105 |
| 11  | 1.235 466 |
| 12  | 1.245 328 |
| 13  | 1.248 973 |

Table X. — The exponent $\gamma$ for the diamond lattice.

| $n$ | $\gamma$
|-----|-------
| 15  | 1.250 935 |
| 16  | 1.254 882 |
| 17  | 1.240 640 |
| 18  | 1.237 913 |

Notice the very important fact that the exponents are the least sensitive to an error in $V_{c}$ for the most compact lattice.
From all results, one can probably estimate:

\[ \gamma = 1.245 \pm 0.002 \text{ or } 0.003 \quad (20) \]

a result which is inconsistent with previous estimates except the analysis of references [12, 14].

To discover the reason for such a deviation and to check the consistency of our results we have then estimated \( V_c \) for all lattices (Tables XI, II, XIII, XIV respectively).

Table XI. — The critical temperature \( V_c^{-1} \) and the exponent \( \gamma \) for the F.C.C. lattice

\[
\begin{array}{ccc}
 n & V_c^{-1} & \gamma_n \\
 9 & 9.828007 & 1.245602 \\
 10 & 9.828685 & 1.245287 \\
 11 & 9.829447 & 1.245126 \\
 12 & 9.829913 & 1.245052 \\
 13 & 9.830078 & 1.245019 \\
 14 & 9.830153 & 1.245004 \\
 15 & 9.830200 & 1.245002 \\
\end{array}
\]

Table XII. — The critical temperature \( V_c^{-1} \) and the exponent \( \gamma \) for the B.C.C. lattice

\[
\begin{array}{ccc}
 n & V_c^{-1} & \gamma_n \\
 9 & 6.41028 & 1.244778 \\
 10 & 6.40733 & 1.245191 \\
 11 & 6.40612 & 1.244339 \\
 12 & 6.40561 & 1.244829 \\
 13 & 6.40592 & 1.244558 \\
 14 & 6.40598 & 1.244692 \\
 15 & 6.40621 & 1.244565 \\
\end{array}
\]

Table XIII. — The critical temperature \( V_c^{-1} \) and the exponent \( \gamma \) for the S.C. lattice

\[
\begin{array}{ccc}
 n & V_c^{-1} & \gamma_n \\
 13 & 4.584103 & 1.246751 \\
 14 & 4.583840 & 1.246621 \\
 15 & 4.584273 & 1.246294 \\
 16 & 4.584463 & 1.246330 \\
 17 & 4.584710 & 1.246216 \\
 18 & 4.584721 & 1.246261 \\
 19 & 4.584785 & 1.246203 \\
\end{array}
\]

Our estimates differ from the previous admitted values for the B.C.C. and S.C. lattices.

For the F.C.C. we get:

\[ V_c^{-1} = 9.8302 \text{ instead of } 9.8300 \quad [12]. \quad (21) \]

Table XIV. — The critical temperature \( V_c^{-1} \) and the exponent \( \gamma \) for the D. lattice

\[
\begin{array}{ccc}
 n & V_c^{-1} & \gamma_n \\
 16 & 2.835266 & 1.2493664 \\
 17 & 2.829583 & 1.2486977 \\
 18 & 2.824766 & 1.2482367 \\
 19 & 2.824637 & 1.2487703 \\
 20 & 2.826630 & 1.2490684 \\
 21 & 2.827110 & 1.2482858 \\
 22 & 2.826893 & 1.2483763 \\
\end{array}
\]

For the B.C.C.:

\[ V_c^{-1} = 6.4062 \text{ instead of } 6.4055 \quad [1]. \quad (22) \]

For the S.C.:

\[ V_c^{-1} = 4.5848 \text{ instead of } 4.5844 \quad [1]. \quad (23) \]

When we then use these new values in order to calculate \( \gamma \) we have results completely consistent with the direct estimates as can be seen from table XV.

Taking into account the probable error on \( V_c^{-1} \) these values are completely compatible although they have some tendency to be systematically slightly higher.

So our analysis by both methods gives results which are below the accepted value of 1.250 and suggest a value much closer to and almost compatible with the R.G. value of 1.241 ± 0.0015.

Table XV. — Comparison between the values of \( \gamma \) from direct estimates \( \gamma^{(1)} \), and those using the value of \( V_c^{-1} \) \( \gamma^{(2)} \).

\[
\begin{array}{ccc}
\text{Lattice} & \gamma^{(1)} & \gamma^{(2)} \\
\text{F.C.C.} & 1.244 & 1.245 \\
\text{B.C.C.} & 1.243 & 1.244 \\
\text{S.C.} & 1.245 & 1.246 \\
\text{D.} & 1.246 & 1.248 \\
\end{array}
\]

4. The exponents \( \nu \) and \( \alpha \). — We shall be rather brief in the description of our results concerning these exponents because our results for \( \nu \) are very similar to those previously obtained, while for \( \alpha \) we obtain smaller values partially because we use new values of \( V_c^{-1} \).

Series for the 2-D Ising model are rather short and the convergence very slow. For example with 8 terms for the triangular lattice one obtains

\[ \nu = 0.963. \quad (24) \]

For the three dimensional lattices, specially the F.C.C. and B.C.C., by contrast, the convergence seems...
to be very rapid as already noticed by other authors [8].

Table XVI gives our results which agree completely with the standard value:

\[
v = 0.638 \pm 0.002 \quad (25)
\]

instead of

\[
v_{\text{R.G.}} = 0.630 \pm 0.0015. \quad (26)
\]

The value suggested by the S.C. is higher (0.643) but the convergence seems poorer.

Using our new values of \( V_c \) we analyse the series for \( \alpha \) (Table XVII).

Table XVI. — The critical exponent \( v \) calculated for the three lattices F.C.C., B.C.C. and S.C.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( v ) F.C.C.</th>
<th>( v ) B.C.C.</th>
<th>( v ) S.C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.637 106</td>
<td>0.635 050</td>
<td>0.640 505</td>
</tr>
<tr>
<td>7</td>
<td>0.637 027</td>
<td>0.640 050</td>
<td>0.646 350</td>
</tr>
<tr>
<td>8</td>
<td>0.637 401</td>
<td>0.638 936</td>
<td>0.648 583</td>
</tr>
<tr>
<td>9</td>
<td>0.637 763</td>
<td>0.639 074</td>
<td>0.643 858</td>
</tr>
<tr>
<td>10</td>
<td>0.637 952</td>
<td>0.638 793</td>
<td>0.644 882</td>
</tr>
<tr>
<td>11</td>
<td>0.638 004</td>
<td>0.638 937</td>
<td>0.642 813</td>
</tr>
<tr>
<td>12</td>
<td>0.638 00</td>
<td>0.638 754</td>
<td>0.643 817</td>
</tr>
</tbody>
</table>

Table XVII. — The critical exponent \( \alpha \) calculated for the three lattices F.C.C., B.C.C. and S.C.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha ) F.C.C.</th>
<th>( \alpha ) B.C.C.</th>
<th>( \alpha ) S.C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.155 631</td>
<td>0.018 897</td>
<td>0.283 566</td>
</tr>
<tr>
<td>11</td>
<td>0.123 338</td>
<td>0.191 250</td>
<td>0.252 074</td>
</tr>
<tr>
<td>12</td>
<td>0.111 063</td>
<td>0.139 062</td>
<td>0.105 290</td>
</tr>
<tr>
<td>13</td>
<td>0.108 740</td>
<td>0.111 791</td>
<td>0.105 389</td>
</tr>
<tr>
<td>14</td>
<td>0.109 332</td>
<td>0.110 099</td>
<td>0.109 687</td>
</tr>
</tbody>
</table>

The convergence is poor but the interesting result is that the last calculated terms agree with each other to predict:

\[
\alpha = 0.110 \quad (27)
\]

which is exactly the R.G. value and the value obtained from accurate measurements on binary mixtures [15].

This value is definitively lower than the quoted value [1]:

\[
\alpha = 1/8. \quad (28)
\]

So that the discrepancy in the relation [16]:

\[
\alpha = 2 - \nu d \quad (29)
\]

is reduced from 0.030 to 0.024 but does not disappear.

5. Conclusions. — From our method, as a result of our analysis it seems reasonable to conclude:

\[
\gamma = 1.245 \pm 0.002 \quad \text{or} \quad 0.003 \quad .
\]

This value takes account all lattices and is surprisingly lower than the values coming from most other analysis. We do not know what the reason is for such a discrepancy or any reason why our method should be rejected. We do not claim that it is superior to others used previously but at least it shows that the results are method dependent. As a result the difference between the R.G. value and H.T.S. value has been much reduced and the difference can no longer be considered as a strong indication of the difference between the two models: the Ising-model and the \( \phi^4 \) continuous field theory.

Also with our method we have obtained values of \( \alpha \):

\[
\alpha = 0.110
\]

in complete agreement with the R.G. value. Unfortunately we have obtained a value for \( v \) identical to the values obtained by previous authors:

\[
\alpha = 0.638
\]

which differs from the R.G. value:

\[
v = 0.630
\]

so that the discrepancy in the relation

\[
\alpha = 2 - 3v
\]

has been reduced by only a small amount. Although the series are short the convergence is surprisingly good, compared for instance to the 2.D Ising-model.

Perhaps a few terms more in the series would show some indication of an overestimate of the speed of convergence?

In the same way an increase in the length of the series for \( \gamma \) might solve definitively the problem of its real value. A last comment: in general in our estimates the lattices with highest coordination number seem to give the most accurate results because apparently the singularity at \( T_c \) is the only one close to the circle of convergence, and also because, in a method involving the knowledge of \( T_c \), the results for the exponent are the least sensitive to the exact value of \( T_c \), effect which is not compensated by a more accurate determination of \( T_c \) in the longer series.
References

[1] For a review see


[13] See reference [1]. Also

