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Analysis of high temperature series of the spin $S$ Ising model on the body-centred cubic lattice

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Abstract. — Nickel has generated 21 term series for the susceptibility and the correlation length on the B.C.C. lattice, for various values of the spin. We analyse them here using a variant of the ratio method. Our results fully support Nickel's conclusions i.e. that is no apparent disagreement any longer for the exponents $\gamma$ and $\nu$ between the renormalization group and high temperature series estimates. We believe that these results shed serious doubts about any claim, based on shorter series, of evidence for hyper-scaling violations.

1. Introduction. — In a recent article [1], we emphasized the fact that an analysis of the high temperature (H.T.) series for the susceptibility exponent $\gamma$ of the 3D Ising model, using unbiased ratio methods, yields values of $\gamma$ (~ 1.245) systematically smaller than 1.250, and not too different from the renormalization group (R.G.) value 1.241. In addition, for the three F.C.C., B.C.C. and S.C. lattices the last estimates were all decreasing with the order. Concerned mainly by the too high value of previous estimates, we only underestimated by about a factor two, the possible decrease of the estimates at higher orders, by uncritically accepting the point of view that the amplitude of the confluent singularities was very small for spin 1/2.

Also we noted that although the series gave estimates for the correlation length exponent $\nu$ (0.638) which were large compared to the R.G. value (0.630), the series were rather short, and the apparent convergence was surprisingly fast compared to other series at the same order. This discrepancy was of course the most annoying since it lead to a violation of the hyper-scaling relation between the exponents $\alpha$ and $\nu$:

$$\alpha = 2 - 3 \nu .$$

At the 1980 Cargèse summer institute Nickel [2] presented new series calculations, which add many terms to the susceptibility and correlation length series of the spin $S$ B.C.C. lattice. He has made a preliminary analysis of these series using Dlog Padé approximants and integral approximants. His latest estimates are:

$$\gamma = 1.239 \pm 0.002$$
$$\nu = 0.631 \pm 0.003$$

and thus in remarkable agreement with R.G. values [3]. We have analysed the same series using ratio methods similar to those discussed in reference [1].

Our analysis supports completely, up to minor details, the results of Nickel.

As a consequence we believe that the main discrepancies between R.G. and H.T. series estimates have now been removed and that series analysis no longer provides any reason to suspect the theoretical arguments [4, 5] which lead to the identification of the Ising model with the continuous $S^4$ field theory in the critical domain.

A last point is worth mentioning. A crucial role is played in the whole analysis by the existence of series of the same length for various values of the spin. If we accept the idea that all series should yield the same value of the corresponding exponent, we are
forced to admit the existence of confluent singularities with amplitudes depending on the spin. In particular there seems to exist a value of the spin between 1 and 2, for which the amplitude of the leading confluent singularity vanishes, and from which it is easiest to calculate the exponents.

Let us remind the reader that there exists a natural family of spin distributions first introduced by D. J. Wallace to study such effects. One starts from a fixed spin distribution for an elementary spin, and one then considers the spin at each site to be the sum of \( l \) independent elementary spins. For instance if the elementary spin takes \( \pm 1 \) values, then the spin has the distribution

\[
S^k = -l + 2k \quad \text{with a weight } C_l^k
\]

\[0 \leq k \leq l.\]

It can be shown that, if one chooses the two-spin interaction proportional to \( l \), then a \( 1/l \) power series expansion is a systematic expansion around mean-field theory. This expansion is a close analogue on the lattice of the perturbative expansion of the continuous field theory. In particular one can find in general a value \( l^* \) of \( l \), calculable for instance in an \( \epsilon \)-expansion, for which the amplitude of the leading confluent singularity vanishes. We believe that it would be useful to make calculations with such spin distributions.

An outline of our article is the following. In section 2 we explain the variant of the ratio method we have used. Section 3 contains a discussion of our numerical results. In section 4 we make some concluding remarks.

2. The method. — 2.1 Some remarks about series analysis. — A few simple observations can be made concerning the analysis made on series to find the nature of the singularity of a function on the circle of convergence of the series, from the knowledge of a finite number of terms.

First (of course) no analysis is possible without some assumptions. Often the use of a particular method implies some hidden assumptions which it is important to understand. Certainly, incorrect assumptions will eventually manifest themselves in the form of poor convergence, but one may need very long series to see such an effect. Conversely, good apparent convergence with short series may sometimes be deceptive. Indeed by systematically trying many methods, one is likely to find one for which the apparent convergence is very good. If one has no \textit{a priori} reasons to prefer such a method, one has to be very careful. Generally by trying more methods, one might find another one with the same apparent convergence rate but with a different limit.

We make these obvious remarks only to emphasize that one should try many methods, to eliminate biases specific to each of them, and accidental good apparent convergence.

Let us now discuss more specifically the problem of interest here. We shall be faced with essentially two kinds of difficulties.

(i) \textit{The function has other singularities.}

This effect is very important for short series. Indeed for longer series, the influence of any singularity not on the circle of convergence dies exponentially with the order. Singularities on the circle of convergence may make any analysis very difficult. Fortunately we shall have to consider only the problem of the anti-ferromagnetic (A.F.) singularity which can be dealt with.

(ii) \textit{The function has confluent singularities.}

This effect is very difficult to detect in short series. For longer series it becomes the dominant one. But it is only easy to deal with it if the series is long enough, so that the influence of other singularities has become negligible.

This suggests the following strategy. For short series one should use simple approximations of the type of Diophant Padé approximants to analyse power law singularities, since such approximations take care of other possible singularities of the function. For longer series many other methods become competitive and even better in the case of confluent singularities. In particular one can use various forms of the ratio method as we shall do in the present article. Since we expect the confluent singularities to consist in a sum of powers, other methods seem although very useful, like the Padé Mellin method as introduced by Baker and Hunter.

Actually many of these methods, including the Padé Diophant and Padé Mellin method, are particular examples of a more general method sometimes called integral approximants (although the name differential approximants would seem more appropriate), suggested first by Joyce and Guttmann. One writes for the function \( f(z) \) of interest a linear differential equation with polynomial coefficients:

\[
\sum_{n=0}^{N} P_n(z) f^{(n)}(z) = 0. \tag{1}
\]

The coefficients \( P_n(z) \) are determined from the Taylor series coefficients of \( f(z) \) by demanding that equation (1) be satisfied up to a maximal order in the expansion variable.

In the case of a function having a confluent singularity at a point \( z_c \), one further chooses the polynomials \( P_n(z) \) of the form:

\[
P_n(z) = (z - z_c)^n A_n(z).
\]

Nickel has studied the case \( n = 2 \). The Padé Mellin method corresponds to the case in which all the polynomials are chosen to be constant.

A natural generalization of the Padé Mellin method, adapted to the case in which a weak anti-ferromagnetic singularity is present at \( z = -z_c \) would seem to be:
2.2 THE RATIO METHOD. — 2.2.1 The basic idea.

We shall now describe the variant of the ratio method we have used here. With respect to the method discussed in reference [1], we have introduced a few modifications allowed by the increased length of the series. We shall always limit ourselves to unbiased estimates, since the biased estimates are extremely sensitive to the assumed value of the critical temperature.

We shall be concerned mainly with two functions, the magnetic susceptibility $\chi(K)$, and the square of the correlation length $\xi^2(K)$, which yield the two exponents $\gamma$ and $2\nu$ respectively. We shall take the example of $\chi(K)$:

$$\chi(K) = \sum_{n=0}^{\infty} \chi_n K^n. \quad (3)$$

We know that for $n$ large, $\chi_n$ behaves like:

$$\chi_n \sim An^{n-1} K_c^{-n} \quad n \to +\infty. \quad (4)$$

Therefore:

$$a_n = - \left[ \ln \frac{\chi_n}{\chi_n^{-1}} \right]^{-1} \sim \frac{n^2}{\gamma - 1}. \quad (5)$$

and we can obtain an estimate $\gamma_n$ for $\gamma$ from the expression:

$$\gamma_n = 1 + 2 \frac{(a_n + a_{n-1})}{(a_n - a_{n-1})^2}. \quad (6)$$

Actually the B.C.C. lattice susceptibility has an A.F. singularity at $-K_c$. Therefore we shall separately analyse the odd and even terms. For instance we shall define:

$$a_n = - \left[ \ln \frac{\chi_n \chi_{n-4}}{\chi_n^{-2}} \right]^{-1}. \quad (7)$$

But, and this is a modification with respect to reference [1], we shall average two consecutive $a_n$'s before calculating $\gamma_n$:

$$\bar{a}_n = \frac{1}{2} (a_n + a_{n-1}). \quad (8)$$

Let us examine what is the effect, at leading order, of this averaging procedure. Taking into account the A.F. singularity we can write [8]

$$\chi_n = An^{n-1} K_c^{-n} (1 + B(-1)^n n^{2-\gamma+1} + \cdots). \quad (9)$$

For $a_n$ this yields a correction term proportional to $(-1)^n n^{1+\gamma-\nu}$, so that the average $\bar{a}_n$ has a correction proportional to $(1 + \alpha - \gamma) n^{1-\nu}$.

Since in three dimensions we expect:

$$1 + \alpha - \gamma \approx 0.13$$

we see that the correction is then not only smaller by a factor $1/n$, but has a smaller coefficient. Since our extrapolation method amplifies the corresponding oscillations, this operation is very useful. Finally we calculate $\gamma_n$ by:

$$\gamma_n = 1 + 2 \frac{(\bar{a}_n + \bar{a}_{n-2})}{(\bar{a}_n - \bar{a}_{n-2})^2}. \quad (10)$$

At the same time we can get an estimate of the critical temperature $K_c^{-1}$ by:

$$(K_c^{-1})_n = \left( \frac{\chi_n}{\chi_{n+1}} \right)^{1/4} \exp \left( \frac{2}{\bar{a}_n (\bar{a}_n - \bar{a}_{n-2})} \right). \quad (11)$$

It should be noted that these extrapolation formulae are reasonable only as long as $(\gamma - 1)$ is not too big. Otherwise they have to be modified to avoid the generation of too large higher order corrections.

2.2.2 Confluent singularities. — Eventually we shall be interested in testing the idea that the series are affected by the existence of a confluent singularity, with an exponent $\gamma - A_1$. The value of $A_1$ is predicted by the R.G. [3]:

$$A_1 = \frac{\omega}{c} = 0.50 \pm 0.02. \quad (12)$$

This means that $\chi_n$ should have the form:

$$\chi_n = An^{n-1} K_c^{-n} \left[ 1 + \frac{c}{n^{A_1}} + \cdots \right]. \quad (12)$$

Since we shall be forced to make a more refined extrapolation to deal with this correction term, we need to decrease even further the influence of the A.F. singularity. Therefore we shall average three consecutive terms:

$$\bar{a}_n = \frac{1}{4} (a_n + 2a_{n-1} + a_{n-2}). \quad (13)$$

Then we define:

$$b_n = \frac{1}{A_1} \left[ (\bar{a}_n)^{3/2} - (\bar{a}_{n-2})^{3/2} \right]. \quad (13)$$

This operation eliminates completely the $n^{-A_1}$ correction term. We get then a new estimate for $\gamma$ by:

$$\gamma_n = 1 + \frac{(c_n - c_{n-2})^2}{2(c_n + c_{n-2})^{4/(A_1-1)}}. \quad (15)$$

If $A_1$ is not too different from 0.5, as predicted by
the R.G., then the correction term proportional to \( n^{-2d_1} \) will be almost completely eliminated together with the regular \( 1/n \) correction, so that \( \gamma_n \) should differ from \( \gamma \) by terms of order \( n^{-3d_1} \) close to \( n^{-1} \).

2.2.3 Direct determination of the exponent of the leading confluent singularity. — The simplest method consists in calculating first the ratio between the terms of two series having the same behaviour:

\[
\frac{\chi_1(K)}{\chi_2(K)} = \sum_{n=0}^{\infty} \frac{\chi_{1,n} K^n}{\chi_{2,n} K^n}.
\]

Taking the ratios of the terms of the two series we get:

\[
a_n = \frac{\chi_{2,n}}{\chi_{1,n}} = \frac{A_2}{A_1} \left( \frac{K_{c_1}}{K_{c_2}} \right)^n \left( 1 + \frac{C_2 - C_1}{n^{d_1}} + \cdots \right). \tag{16}
\]

Calculating the quantity \( b_n \), we find:

\[
b_n = \ln \left( \frac{a_n}{a_{n-1}} \right) \sim A_1 (d_1 + 1) \left( \frac{C_2 - C_1}{n^{d_1+2}} \right). \tag{17}
\]

The quantity \( b_n \) is nothing but the discrete second logarithmic derivative of \( a_n \).

We can then take again a logarithmic derivative:

\[
c_n = -\left[ \ln \frac{b_n}{b_{n-1}} \right]^{-1} \sim \frac{n}{d_1}.
\]

and then finally obtain an estimate \((d_1)_n\) for \( d_1 \):

\[
(d_1)_n = (c_n - c_{n-1})^{-1} - 2. \tag{19}
\]

Of course we have also to deal with the A.F. singularity here, so that we shall consider again the odd and even terms separately, and average the \( c_n \)'s.

An alternative method is to take the ratio:

\[
d_n = \frac{a_n(\chi)}{a_n(\xi^2)} \tag{20}
\]

By doing this we eliminate the critical temperature, so that we can take one ratio less.

2.2.4 The exponent of the A.F. singularity. — As we have already mentioned the B.C.C. lattice series have at \( K = -K_c \) a singularity of the form:

\[
\chi(K) \sim (K_c + K)^{1-x}
\]

which yield a correction to \( \chi_n \) of the form:

\[
\chi_n = A n^{n-1} K_c^{n-1} (1 + B (-1)^n n^{2-\gamma+1} + \cdots) \tag{21}
\]

It is therefore possible to estimate the combination \( \alpha - \gamma \) from the series, with probably the same kind of accuracy as in the case of \( d_1 \).

We have proceeded in the following way. We have first calculated \( a_n \):

\[
a_n = -\ln \frac{\chi_n}{\chi_{n-2}} = -\frac{(\gamma - 1)}{n^2} + 4 B (-1)^n n^{2-\gamma+1} + \cdots \tag{23}
\]

Then the quantity \( b_n \):

\[
b_n = -\frac{(a_n - a_{n-1})}{8 B n^{2-\gamma+1}} + 2 (-1)^n \frac{(\gamma - 1)}{n^3} \tag{24}
\]

Since we expect:

\[
\alpha - \gamma - 1 \approx -2.13
\]

we see that now the A.F. singularity gives the leading contribution. From now on we proceed in the same way as for \( d_1 \), after equation (17).

Needless to say, we have tried these different methods on specific examples constructed to imitate the B.C.C. lattice series in some of their properties.

2.2.5 The exponent \( \eta \). — To check in some respect the correlations between \( \gamma \) and \( \nu \) we have also analysed the ratio:

\[
a_n = \frac{\chi_n}{\xi_n^2} \sim D n^{-\alpha} \tag{26}
\]

The method is quite similar to the method used to calculate \( d_1 \) and \( \gamma - \alpha \).

Having calculated an estimate \((\xi(\eta))_n\) for \( \eta \nu \), we have divided it by the corresponding estimate for \( \nu \) at the same order, and obtained a series of estimates for \( \eta \).

3. Results. — 3.1 The Square Lattice. — Nickel has also generated long series (up to order 34) for the square lattice. Although almost any method seems to work well for the spin 1/2 square lattice, nevertheless we have used it to test some aspects of our methods. Of course we cannot try the forms constructed to deal with a confluent singularity since it is absent in this case. On the other hand we can see the effect of the modifications introduced to eliminate more systematically the A.F. singularity (Eq. (8)). Table I gives our results for \( \gamma \) and \( \nu \), and the inverse critical coupling constant \( V_c^{-1} \) in the variable:

\[
V = \tanh (K)
\]

as given by the series \( \chi(v) \) and \( \xi^2(v) \).

In table II we show the results for the combination \( \gamma - \alpha \) coming from the A.F. singularity, and for \( \eta \) as explained in section 2.2.4 and 2.2.5.

It appears from these tables that the convergence towards \( \gamma \) is somewhat better that one would normally expect, at least up to order 26,27.

3.2 The B.C.C. Lattice. — We have then applied the method described in section 2.2.1 to the series expansions of \( \chi(K) \) and \( \xi^2(K) \) on the B.C.C. lattice,
Table I. — Estimates for the exponents $\gamma$ and $v$, and the critical coupling constant $V_{c}$ of the spin 1/2 Ising model on the square lattice. The exact values are:

\[
\begin{align*}
\gamma &= 1.75, & v &= 1, & V_{c}^{-1} &= 1 + \sqrt{2} = 2.41421356...
\end{align*}
\]

For four values of the spin $S$ i.e. 1/2, 1, 2 and $\infty$. For comparison we have also made the same calculation for the spin 1/2 F.C.C. [9] and S.C. [10] lattices.

Tables III and IV give the results for the exponents $\gamma$ and $v$ respectively. One sees that the estimates given by the different spins slowly become closer with increasing order. At order 21, the difference for $\gamma$ between the spin 1/2 and spin $\infty$ is about $10^{-2}$. We have also plotted against $1/n$ ($n$ being the order) the estimates for $\gamma$ and $v$ in figures 1 and 2. Although all series show oscillations due other distant singularities, the effect of these singularities seems to be somewhat stronger for the spin 1/2. Nevertheless, apparently

Table II. — Estimates for $\gamma - \alpha$ and $\eta$ for the spin 1/2 Ising model on the square lattice. The exact values are:

\[\eta = 1/4; \quad \gamma - \alpha = 7/4\].

\[
\begin{array}{|c|c|c|}
\hline
n & \gamma - \alpha & \left(\eta - \frac{1}{4}\right) \times 10^{3} \\
\hline
21 & 1.624 & 1.841 \\
22 & 1.233 & 0.711 \\
23 & 1.258 & 0.624 \\
24 & 1.563 & 1.176 \\
25 & 1.747 & 0.365 \\
26 & 1.957 & 0.730 \\
27 & 1.941 & 0.731 \\
28 & 1.782 & 0.391 \\
29 & 1.709 & 0.652 \\
30 & 1.645 & 0.465 \\
31 & 1.650 & 0.421 \\
32 & 1.706 & 0.501 \\
33 & 1.734 & 0.368 \\
34 & 1.761 & 0.394 \\
\hline
\end{array}
\]

Table III. — Estimates for the exponent $\gamma$ for 4 values of the spin on the B.C.C. lattice, not taking into account the possibility of a confluent singularity.

\[
\begin{array}{|c|c|c|c|c|}
\hline
n & Spin 1/2 & Spin 1 & Spin 2 & Spin $\infty$ \\
\hline
7 & 1.265 61 & 1.269 10 & 1.256 15 & 1.245 76 \\
8 & 1.233 26 & 1.247 54 & 1.242 75 & 1.236 69 \\
9 & 1.235 52 & 1.240 87 & 1.237 79 & 1.233 43 \\
10 & 1.241 02 & 1.238 12 & 1.235 41 & 1.231 99 \\
11 & 1.245 96 & 1.239 12 & 1.235 62 & 1.232 38 \\
12 & 1.246 18 & 1.239 08 & 1.235 30 & 1.232 28 \\
13 & 1.245 85 & 1.239 59 & 1.235 61 & 1.232 65 \\
14 & 1.245 01 & 1.239 39 & 1.235 45 & 1.232 62 \\
15 & 1.244 64 & 1.239 39 & 1.235 52 & 1.232 76 \\
16 & 1.244 50 & 1.239 25 & 1.235 47 & 1.232 87 \\
17 & 1.244 39 & 1.239 20 & 1.235 47 & 1.232 87 \\
18 & 1.244 34 & 1.239 15 & 1.235 48 & 1.232 94 \\
19 & 1.244 17 & 1.239 11 & 1.235 51 & 1.233 02 \\
20 & 1.244 05 & 1.239 09 & 1.235 54 & 1.233 10 \\
21 & 1.243 89 & 1.239 04 & 1.235 56 & 1.233 18 \\
\hline
\end{array}
\]

Table IV. — Estimates for the exponent $v$ for 4 values of the spin, not taking into account the possibility of a confluent singularity.

\[
\begin{array}{|c|c|c|c|c|}
\hline
n & Spin 1/2 & Spin 1 & Spin 2 & Spin $\infty$ \\
\hline
7 & 0.660 04 & 0.648 29 & 0.637 53 & 0.630 52 \\
8 & 0.646 42 & 0.640 48 & 0.633 39 & 0.628 26 \\
9 & 0.640 68 & 0.635 67 & 0.630 19 & 0.626 07 \\
10 & 0.640 41 & 0.634 51 & 0.629 68 & 0.626 14 \\
11 & 0.638 52 & 0.632 87 & 0.628 52 & 0.625 39 \\
12 & 0.638 98 & 0.632 99 & 0.628 75 & 0.625 82 \\
13 & 0.638 54 & 0.632 56 & 0.628 51 & 0.625 76 \\
14 & 0.638 39 & 0.632 54 & 0.628 61 & 0.625 98 \\
15 & 0.638 18 & 0.632 44 & 0.628 63 & 0.626 10 \\
16 & 0.637 65 & 0.632 27 & 0.628 61 & 0.626 19 \\
17 & 0.637 49 & 0.632 23 & 0.628 67 & 0.626 32 \\
18 & 0.637 15 & 0.632 09 & 0.628 65 & 0.626 38 \\
19 & 0.636 96 & 0.632 04 & 0.628 69 & 0.626 49 \\
20 & 0.636 77 & 0.631 96 & 0.628 70 & 0.626 56 \\
\hline
\end{array}
\]
after order 12, a general trend dominates the oscillations, suggesting that we are now finally approaching the asymptotic regime.

If we now assume that the limiting value is the same for all spins, and in addition that we are indeed now seeing the asymptotic regime, then we are forced to conclude that all series are affected by the presence of a correction term with an exponent significantly smaller than 1, due to some confluent singularity. In addition it seems that the amplitude of this singularity is largest for $S = 1/2$, and vanishes for a value of $S$ between 1 and 2.

Evidence for such a confluent singularity had already been given, of course [11] from shorter series. But we believe that in these shorter series the influence of other singularities was still very strong, so that the evaluation of the amplitude of the confluent singularity in a given series was somewhat unreliable.

This would explain the main difference between the former analyses, and the analysis presented here or by Nickel, i.e. the fact that the amplitude of the confluent singularity seems to vanish for a value of the spin between 1 and 2, rather than for the spin 1/2.

On figures 1 and 2 we have added for comparison the results obtained for the spin 1/2 F.C.C. and S.C. lattices. One sees that the results for the F.C.C. lattice are comparable to those of B.C.C. lattice at the same order. This is actually also true for other values of the spin both for $\gamma$ and $\nu$. As usual, the results for the S.C. lattice are somewhat higher and seem to be affected strongly by other more distant singularities. It would probably be useful, specially in the case of the F.C.C. lattice, to calculate with higher values of the spin, or other spin distributions, to improve the conver-

<table>
<thead>
<tr>
<th>Spin 1/2</th>
<th>Spin 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$K_c^{-1}(\gamma)$</td>
</tr>
<tr>
<td>7</td>
<td>6.342 526 8</td>
</tr>
<tr>
<td>8</td>
<td>6.360 628 3</td>
</tr>
<tr>
<td>9</td>
<td>6.359 413 1</td>
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<tr>
<td>10</td>
<td>6.355 945 6</td>
</tr>
<tr>
<td>12</td>
<td>6.353 675 0</td>
</tr>
<tr>
<td>13</td>
<td>6.353 818 4</td>
</tr>
<tr>
<td>14</td>
<td>6.353 924 7</td>
</tr>
<tr>
<td>15</td>
<td>6.354 031 8</td>
</tr>
<tr>
<td>21</td>
<td>6.354 049 5</td>
</tr>
</tbody>
</table>

Table V. — Values of the critical temperature $K_c^{-1}$ for 4 values of the spin on the B.C.C. lattice, as derived from the $\chi(K)$ and $\xi^2(K)$ series.

<table>
<thead>
<tr>
<th>Spin 2</th>
<th>Spin $\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$K_c^{-1}(\gamma)$</td>
</tr>
<tr>
<td>7</td>
<td>3.405 326 9</td>
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<tr>
<td>8</td>
<td>3.408 633 0</td>
</tr>
<tr>
<td>9</td>
<td>3.410 072 6</td>
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<tr>
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<td>11</td>
<td>3.410 301 3</td>
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</tbody>
</table>
gence by decreasing the amplitude of the leading confluent singularity. In table V we give estimates for the value of the critical temperature \( K_c^{-1} \) for various spins, as derived from the series expansions of \( \chi(K) \) and \( \xi^2(K) \). If we assume that the leading correction to \( (K_c^{-1})_n \) is proportional to \( n^{-1.5} \) as predicted by the R.G. (an assumption consistent with the comparison between the results coming from \( \chi(K) \) and \( \xi^2(K) \)), we get some estimates for \( K_c^{-1} \):

\[
\begin{align*}
S & = 1/2 & K_c^{-1} & = 6.3543 \pm 0.0003 \\
S & = 1 & K_c^{-1} & = 4.45123 \pm 0.00007 \\
S & = 2 & K_c^{-1} & = 3.40996 \pm 0.00007 \\
S & = \infty & K_c^{-1} & = 2.29838 \pm 0.00006
\end{align*}
\]

which are in excellent agreement with the values proposed by Nickel:

\[
\begin{align*}
S & = 1/2 & K_c^{-1} & = 6.35422 \pm 0.00008 \\
S & = \infty & K_c^{-1} & = 2.29835 \pm 0.00001
\end{align*}
\]

Only the quoted apparent errors are somewhat different.

We wish to stress here again the danger of making biased estimates of the exponents using estimates of the critical temperature \([12]\). If we use the formulae of reference \([1]\) for example:

\[
\begin{align*}
a_n & = \frac{x_{n+1}}{x_n} \\
\gamma_n & = 1 + \frac{K_c(a_{n-1} - a_n)}{(K_c a_n - 1)(K_c a_{n-1} - 1)}.
\end{align*}
\]

We can derive an estimate for the variation \( \delta \gamma \) of \( \gamma \) induced by a variation \( \delta K_c \) of \( K_c \):

\[
\delta \gamma \approx 2 n \frac{\delta K_c}{K_c}.
\]

If in addition we have to cancel the correction coming from a confluent singularity with an exponent \( \Delta_1 \), the relation becomes:

\[
\delta \gamma \approx \frac{2}{\Delta_1} (1 + \Delta_1) n \frac{\delta K_c}{K_c}.
\]

If \( \Delta_1 \) is close to 0.5, this yields:

\[
\delta \gamma \approx 6 n \frac{\delta K_c}{K_c}.
\]

For example a relative variation of \( 10^{-5} \) of \( K_c \) yields at order 20 a variation of \( 1.2 \times 10^{-3} \) of \( \gamma \).

### 3.3 Determination of the Exponent \( \Delta_1 \) of the Leading Confluent Singularity.

In table VI we give estimates we obtain for the exponent \( \Delta_1 \) with the method exposed in section 2.2.3 applied to the ratios of the coefficients first of the magnetic susceptibility and second of the correlation length of the spin \( \infty \) and the spin 1.

Examining in addition the estimates coming from all other series we conclude:

\[
\Delta_1 = 0.52 \pm 0.07.
\]

All estimates coming from susceptibility and correlation length series lie after order 15 in this range. The situation is even better if one eliminates the spin 1/2. The convergence of the estimates obtained from the ratio of the susceptibility and correlation length series (Eq. (20)) seems poorer. Still, after order 15, among 30 estimates, only 3 are slightly outside this interval. So we believe that the apparent error we quote here is not unreasonable.

Our estimate is in complete agreement with previous estimates of Saul et al. \([11]\), Camp et al. \([11]\) and Rehr \([7]\). On the other hand it differs from the estimate of Bessis et al. \([6]\)

\[
\Delta_1 = 0.83 \pm 0.11
\]

which used the Padé Mellin method and analysed only the spin 1/2 series. However, we believe that the quoted error is largely underestimated for various reasons:

First the authors had to rely on previous estimates of the critical temperature.

Second they found a very small amplitude for the confluent singularity so that its determination could obviously not be very accurate. A preliminary analysis of the spin \( S \) B.C.C. series by Moussa \([13]\) using the same Padé Mellin method seem to confirm this point of view.

Finally they had to exclude the F.C.C. lattice from their analysis.

Table VI. — Values of \( \Delta_1 \) as obtained from the comparison of the susceptibility and correlation length series for spin 1 and spin \( \infty \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_1 )</td>
<td>1.83</td>
<td>1.01</td>
<td>0.590</td>
<td>0.457</td>
<td>0.464</td>
<td>0.494</td>
<td>0.541</td>
<td>0.544</td>
<td>0.543</td>
<td>0.532</td>
<td>0.527</td>
<td>0.528</td>
<td>0.528</td>
</tr>
<tr>
<td>( \Delta_1 )</td>
<td>1.24</td>
<td>0.905</td>
<td>0.675</td>
<td>0.582</td>
<td>0.482</td>
<td>0.504</td>
<td>0.474</td>
<td>0.508</td>
<td>0.505</td>
<td>0.506</td>
<td>0.513</td>
<td>0.503</td>
<td></td>
</tr>
</tbody>
</table>

For the ratio of amplitudes we obtain:

\[ \frac{A_2}{A_4} = 0.71 \pm 0.07. \]

These results are completely consistent with previous estimates coming from R.G. and H.T. series.

Table VII shows a comparison of values of the ratio of amplitudes obtained from H.T. series [14], calculations based on the \( \varepsilon \)-expansion [15] and R.G. calculations in three dimensions [16].

Table VII. — Values for the amplitude ratio \( A_2/A_4 \) of the leading confluent singularity from various methods.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.65</td>
<td>0.70 ± 0.03</td>
<td>0.65 ± 0.05</td>
<td>0.71 ± 0.07</td>
</tr>
</tbody>
</table>

3.4 The Exponents After Elimination of the Leading Confluent Singularity. — We now use the method described in section 2.2.2 to eliminate corrections coming from the leading confluent singularity. We have allowed the parameter \( \Delta_1 \) to vary in a large range:

\[ 0.35 \leq \Delta_1 \leq 0.65 \]

and calculated \( \gamma, \nu \) and \( \eta \).

Tables VIII, IX show our results for \( \gamma \) and \( \nu \) for \( \Delta_1 = 0.5 \). They have been plotted against \( 1/n \) in figures 3 and 4.

One sees immediately that the difference between the last estimates of order 21 given by the various spins has been reduced by about a factor 3, which justifies \( a \ posteriori \) our assumptions. By varying \( \Delta_1 \) it is possible to arrive to situations in which the final values are even closer.

For instance for \( \Delta_1 = 0.35 \) one finds for \( \gamma \):

\[ 1.239 1 \leq \gamma \leq 1.239 9 \quad \text{for} \quad S \geq \frac{1}{2}. \]

On the other hand for \( \Delta_1 = 0.45 \) one finds for \( \nu \):

\[ 0.630 3 \leq \nu \leq 0.630 6 \quad \text{for} \quad S \geq \frac{1}{2}. \]

This does not mean that these values of \( \Delta_1 \) should be regarded as new estimates for the exponent. Rather these are effective values which cancel not only the difference between the various spins due to the leading confluent correction, but also the effects of other higher order corrections.
The exponent $v$ after elimination of the leading confluent singularity, assuming $\alpha_1 = 0.50$. After order 16, the results for spin 2 and spin $\infty$ are indistinguishable.

One remark can be made. When one varies the value of $\alpha_1$ in the range indicated above, it is for spin 1 that the value of $\gamma$ changes the least. For $v$ the minimal change occurs for a value of the spin between 1 and 2.

Our best guess for the exponents is then:

$$
\gamma = 1.238 \pm 0.002 \\
v = 0.630 \pm 0.001 \\
\eta = 0.035 \pm 0.003
$$

Nickel [2] obtains:

$$
\gamma = 1.239 \pm 0.002 \\
v = 0.631 \pm 0.003
$$

while the R.G. values [3] are:

$$
\gamma = 1.241 \pm 0.002 \\
v = 0.630 \pm 0.001 \\
\eta = 0.031 \pm 0.004
$$

The consistency between all these results is quite impressive. The main difference between the B.C.C. series and the R.G. results comes from $\eta$ which is larger in the series results. We do not know if this fact is significant, but we notice that a similar situation did already arise in the comparison between the exact values of the spin 1/2 2D Ising model and the corresponding R.G. estimates.

3.5 The A.F. Singularity. — As explained in section 2, we can try to estimate the combinations $\gamma - \alpha$ and $2v - \alpha$ from the contributions to the series due to the A.F. singularity. The R.G. predictions are:

$$
\gamma - \alpha = 1.13 \\
2v - \alpha = 1.15
$$

The convergence is not good enough to allow us to distinguish very well the difference between $\gamma$ and $2v$.

In table X we show one of the series which has the best apparent convergence and which corresponds to the spin $\infty$ susceptibility series. One verifies only that the results seem consistent with the R.G. predictions.

4. Conclusions. — The longer series generated by Nickel for the B.C.C. lattice, both for the magnetic susceptibility and the correlation length, and for arbitrary spin, lead to new estimates of critical exponents which are in remarkable agreement with the R.G. predictions.

It seems that the former difference [17] between the H.T. series and R.G. estimates was mainly due to unreliable estimates of the amplitude of the leading confluent singularity. The studies based on the comparison between series corresponding to different values of the spin [11] were inconclusive because the series were too short. On the other hand from one series alone, it is extremely difficult, even at present, to make such an estimate. The effects of other singularities is just too strong.

As a consequence we find it now very hard to believe that there remains any serious problem with the violation of hyperscaling [18]. Indeed we see that the value of $v$ has changed by almost $10^{-2}$ between order 12 and 20 and we know that $v$ enters with a factor 3 into the relation:

$$
\alpha = 2 - 3v
$$

We think that more work should be done on the B.C.C. series using various forms of integral approximants or any other available methods to get a better feeling for the remaining uncertainties in the values of the exponents, and that the results should be confirmed by calculations on other lattices presumably with different spin distributions. However we do not believe that the general picture can change drastically any more.

Table X. — Values for the difference $\gamma - \alpha$ from the spin $\infty$ series. The R.G. prediction is: $\alpha - \gamma = 1.131 \pm 0.006$. 

<table>
<thead>
<tr>
<th>$n$</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma - \alpha$</td>
<td>1.250</td>
<td>1.251</td>
<td>1.222</td>
<td>1.213</td>
<td>1.181</td>
<td>1.172</td>
<td>1.156</td>
<td>1.151</td>
<td>1.144</td>
<td>1.142</td>
<td>1.139</td>
<td>1.137</td>
</tr>
</tbody>
</table>
Acknowledgments. — I thank Dr. B. G. Nickel for communicating me the coefficients of the high temperature expansions of the magnetic susceptibility and the correlation length on the B.C.C. lattice, prior to publication, and Dr. P. Moussa for a careful reading of the manuscript, and useful remarks.

References


[20] For a review see :
