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On the dynamics of fission as a dissipative process

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Résumé. — Nous avons étudié la dynamique du processus de fission de la région du point selle à la configuration de scission. Pour cela nous résolvons une équation de type Fokker-Planck pour une fonction de Wigner multidimensionnelle. Cette équation prend en compte à la fois les effets dissipatifs mais aussi les effets quantiques. L'influence des conditions initiales, du choix des coefficients de friction et de diffusion, sur les valeurs finales des variances des distributions en énergie et en masse des produits, a également été étudiée.

Abstract. — We study the fission dynamics from the region of the saddle point to the scission configuration. For this we solve an equation of Fokker-Planck type for a multi-dimensional Wigner function. This equation accounts for dissipative as well as quantal features. We study the influence of the initial condition as well as friction and diffusion coefficients on the final values for the variances of energy and mass asymmetry.

1. Introduction. — One of the most challenging theoretical problems in present nuclear physics is the proper treatment of nonlinear collective dynamics. For such motion the overall coupling between collective and intrinsic degrees of freedom is expected to lead to features not encountered when studying small vibrations. This coupling may lead to an irreversible transfer of energy from the few collective degrees to the many residual nucleonic degrees of freedom. This dissipation will be accompanied by a simultaneous creation of statistical fluctuations of the collective variables.

There do exist two prominent examples of this kind of collective motion and for which detailed experimental studies have been performed. The first example is almost as old as nuclear physics; it is the fission process at all energy ranges. However over the last decade more effort, both theoretically and experimentally, has been put into the second example of collisions of heavy ions at small bombarding energies. High energy fission differs from the other processes mentioned in one respect. A large amount of the intrinsic excitation (heat) seen at the end of the process is not produced by the collective motion (leading to fission) itself. It is produced by the particle which excited the mother nucleus in the first stage of the process. This implies that in these cases the collective motion starts when the nucleus is already heated.

However, all these examples have in common one prerequisite for the introduction of statistical concepts like dissipation. The collective motion is slow compared to the motion of the nucleons. The ultimate reason for this is the large collective inertia being typically two orders of magnitude larger than the nucleonic mass.

The formal means to account for dissipation and the simultaneous creation of statistical fluctuations is to describe the collective dynamics by transport equations, like the Fokker-Planck equation. Such a description has been widely used (and to a great extent justified) over the last few years for the case of deeply inelastic heavy ion collisions. Such equations can doubtlessly be expected to work in the case of fission also. As a matter of fact, the first time a Fokker-Planck equation was applied in nuclear physics at all was in the case of fission. H. Kramers [1] used such an equation to obtain the probability for the escape over a barrier as far back as 1940. Neglecting quantum penetration the vehicle to overcome the barrier is the statistical fluctuation. Kramers showed that the escape probability can be expressed by the Bohr-Wheeler formula, derived in 1939 on the basis of a statistical counting of the transition states. This
equivalence holds true only if the transport coefficient appearing in the dynamical treatment with the Fokker-Planck equation meet certain conditions.

In this paper we will not try to calculate the escape probability. Such a calculation is an interesting problem on its own, because one must be able to treat the interplay between purely statistical and purely quantal effects. Instead, we shall start our dynamical treatment from behind the barrier and follow the time evolution of the system on its way down to the scission point. In this sense, our philosophy is similar to the one of reference [2] and also to those of the earlier treatments [3, 4] of fission dynamics.

Our approach differs from the one in reference [2] not only because we shall be treating a multi-dimensional system but because we will study different effects and will also reach different conclusions. The approaches of references [3] and [4] differ from ours, and the one of reference [2], in that no account of fluctuating forces was made in these studies.

2. Theoretical model. — 2.1 Equations of motion.

The Fokker-Planck equation (FPE) which we are going to use for the description of the collective dynamics is very similar to the one which was derived in reference [5] and finally used in several applications to deeply inelastic collisions; (for a thorough description see reference [6]). We shall finally use this equation for a multi-dimensional system. Before coming to this point however, we would like to illustrate the essential features of the FPE by means of its one-dimensional analog

$$\frac{\partial d(Q, P, t)}{\partial t} = - \{ d(Q, P, t), \mathcal{H}_\text{coll} \} + \gamma \langle Q \rangle \frac{\partial}{\partial P} \left( \frac{d(Q, P, t)}{m \langle Q \rangle} \right) + D \langle Q \rangle \frac{\partial^2}{\partial P^2} d(Q, P, t) \tag{1}$$

The first term on the right represents the non-dissipative dynamics (with the curly brackets defining the usual Poisson brackets). For $\gamma = D = 0$ (1) reduces to the Liouville equation, restricted to the locally harmonic approximation. This means that $\mathcal{H}_\text{coll}$ is obtained by expanding

$$\mathcal{H}_\text{coll} = \frac{P^2}{2 m(\alpha)} + \mathcal{U}(Q) \tag{2}$$

around the classical trajectory, defined by

$$Q_e(t) = \langle Q \rangle_t, \tag{3}$$
$$P_e(t) = \langle P \rangle_t, \tag{4}$$

to second order in $(Q - Q_e(t))$ and $(P - P_e(t))$.

As indicated in equation (1) the friction and the diffusion coefficient, $\gamma$ and $D$, are also to be calculated, not at $Q$ but instead at the mean value $Q_e(t)$. As a consequence, a solution $d(Q, P, t)$ of equation (1) can easily be obtained from the first and second moments in $Q$ and $P$:

$$Q_e(t) = \int dQ dP Q d(t), \tag{5}$$
$$P_e(t) = \int dQ dP P d(t), \tag{6}$$
$$\omega(t) = \frac{1}{2} \int dQ dP (P - P_e(t))^2 d(t), \tag{7}$$
$$\chi(t) = \frac{1}{2} \int dQ dP (Q - Q_e(t))^2 d(t), \tag{8}$$
$$\psi(t) = \frac{1}{2} \int dQ dP (Q - Q_e(t)) (P - P_e(t)) d(t). \tag{9}$$

Suppose the initial distribution

$$d_0 = d(Q, P, t) \tag{10}$$

is chosen to be a Gaussian defined by the mean values $Q_0$ and $P_0$ as well as the width $\omega_0$, $\chi_0$ and $\psi_0$. In this case the function $d$ will remain a Gaussian for all $t$:

$$d(t) = \frac{1}{\sqrt{\Delta}} \exp \left\{ - \frac{1}{4 \Delta} (\chi(P - P_e))^2 - 2 \psi(P - P_e)(Q - Q_e) + \omega(Q - Q_e)^2 \right\} \tag{11}.$$

Here the determinant $\Delta$ is defined as:

$$\Delta = \omega \chi - \psi^2 \tag{12}$$

All that we have to know are the values of the moments (5) to (9) calculated at any arbitrary time $t$. But these values can be obtained by solving a set of coupled ordinary differential equations. These equations for the moments can easily be derived from the Fokker-Planck equation (1). We will not repeat that here but instead refer to reference [6]. There the full equations are written down for a multi-dimensional system (cf. eqs. (11)-(13)); unfortunately with a factor $1/2$ missing in the terms proportional to the second derivatives of the inertial coefficients.

The present calculation differs from the previous one in two respects; a formal and a further difference concerning the interpretation of the solution. Both these differences have to do with the inclusion of quantal effects. The equations in reference [6] were meant to be written for the strictly classical limit. In dealing with effects of statistical mechanics, the use of the classical limit necessarily implies an assumption about the magnitudes of the collective frequency and the temperature of the heat bath. It is only for $|\hbar \Omega_{col}| \ll T \tag{13}$

(here and in the following we put Boltzmann's constant equal to unity) that the diffusion and friction coefficients are related by the Einstein relation:

$$D = \gamma T \tag{14}.$$
This was demonstrated in reference [5] and again recently in references [7-9].

In the case of condition (13) being violated, the Einstein relation has to be generalized to the form

\[ D = \gamma T^* . \]  

(15)

In this formula \( D \) is still proportional to \( \gamma \), a fact which shows the intimate connection between friction and statistical fluctuations. Within the linear response approach this proportionality between \( D \) and \( \gamma \) can be proved from rather general considerations (see Ref. [10]). On the other hand, our equation of motion (1) is assumed to describe the nonlinear dynamics within a locally harmonic approximation. This latter approximation can be seen to be consistent with the linear response approach. This justifies the relation (15) which we want to use in this paper (1).

Our general philosophy will be to treat \( T^* \) as a phenomenological parameter. However, since we know its microscopic expression we shall be able to specify \( T^* \) further in some cases, or particular limits. Indeed, according to the microscopic derivation of (1), within the linear response approach \( D \) is given [7-9] by

\[ D(\Omega) = \int_0^\infty dt \, \psi(t) \cos \Omega t = \] 

\[ = \coth \frac{\Omega}{2} T \int_0^\infty dt \, i \chi''(t) \sin \Omega t . \]  

(16)

Here, \( \psi(t) \) and \( \chi''(t) \) are the relevant microscopic correlation and response functions for the one body operator which couples the collective coordinate to the nucleonic degrees of freedom. The frequency \( \Omega \) is the one which describes the unperturbed collective motion. In equation (16) we assumed \( \Omega \) to be real. The second equality then is nothing else but the fluctuation dissipation theorem (see Ref. [5]). With the friction coefficient defined as (see Ref. [7])

\[ \gamma(\Omega) = \frac{i}{\Omega} \int_0^\infty dt \, \chi''(t) \sin \Omega t . \]  

Equation (15) becomes a direct consequence of equation (16). So for real frequencies the effective temperature \( T^* \) will be given by

\[ T^*(\Omega) = \frac{1}{2} \bar{h} \Omega \coth \frac{\bar{h} \Omega}{2} . \]  

(17)

In the fission dynamics the evolving system will pass over regions in the collective phase space where the unperturbed local frequency will be purely imaginary, i.e. \( \Omega = i\Omega \). Then the fluctuation-dissipation theorem can be used only if a continuation from the real axis into the complex plane is possible up to \( \Omega = \Omega \). This question depends crucially on the analytic behaviour of the Fourier-transformed response and correlation functions, i.e. on the analytic behaviour of \( D(\Omega) \) and \( \gamma(\Omega) \). A detailed study will be possible only when calculating \( \psi(t) \) and \( \chi''(t) \) microscopically.

All these statements hold as well for the multidimensional case as long as the friction tensor is diagonal. This we will assume later on in this paper. For nondiagonal friction and diffusion tensors new features would arise which shall be discussed in a forthcoming publication.

So far we have discussed the appearance of a \( T^* \) in the Einstein relation as one of the differences to the equations of reference [6]. The second difference concerns the interpretation of \( d \). In contrast to the previous case where \( d(Q, P, t) \) represented the distribution function in a classical phase space we want to understand \( d \) as the Wigner transform of a quantal density operator \( \bar{d} \). This \( \bar{d} \) satisfies a quantal master equation of the type discussed in reference [5] (see Eq. (33)) or, more specifically, reference [7] (see Eq. (12)). The Wigner transform of this master equation turns out to be identical to equation (1). When solving this equation we then have to be sure of not violating the uncertainty relation. This implies restrictions on the possible choices of \( \omega_0, \chi_0 \) and \( \psi_\alpha \) of the form

\[ \omega_0 \chi_0 - \psi_\alpha \geq \frac{\hbar^2}{16} . \]  

(18)

2.2 CHOICE OF THE COLLECTIVE VARIABLES. — To define the collective coordinates we adopt the parametrization of the nuclear shape as proposed in reference [11]. The elongation and the neck of the nucleus are described by the parameters \( c \) and \( h \) respectively. Furthermore, a parameter \( \alpha \) characterizes the left right asymmetry of the (volume or) mass.

Trajectory calculations showed that the path to fission lies close to the surface \( h = 0 \). To obtain the full solution of the Fokker-Planck equation (1) we therefore, restricted the dynamics to the two-dimensional space given by \( h = 0 \). One should note that up to the region of the second barrier the path \( \alpha = h = 0 \) is approximately equal to the path of minimal liquid drop energy.

(1) These functions are expected to show a relaxational behaviour in the sense that they decay like \( \exp(-t/\tau) \) for large times. In this case \( D(\Omega) \) and \( \gamma(\Omega) \) can be analytically continued to complex frequencies up to \( | \text{Im} \Omega | < 1/\tau \). It is interesting to note that \( D(\Omega) \) and \( \gamma(\Omega) \) turn out to be analytic all over the complex plane when they are calculated within the random matrix model of reference [17]. This is true because in this case \( \psi(t) \) and \( \chi''(t) \) decay like \( \exp(-t^2/\tau^2) \) (see Ref. [10]).
The elongation of the nucleus is equal to $2 c R_o$, where $R_o$ is the radius of a sphere having the same volume as the deformed nucleus that we are considering here. Similarly, the mass ratio of the fission fragments can be expressed by $\alpha$ and $c$ approximately as (correct to first order in $h$ and $\alpha$)

$$\frac{A_1}{A_2} = \frac{1 + \frac{3}{8} c^3 \alpha}{1 - \frac{3}{8} c^3 \alpha}.$$  

(19)

In this two-dimensional manifold we now have to specify the potential as well as the tensors of the inertial, frictional and diffusional coefficients. The potential energy surface is shown in figure 1 for $h=0$.

![Potential energy surface of $^{238}U$ versus the elongation parameter $c$ and the left-right asymmetry deformation parameter $\alpha$. The solid lines correspond to layers of constant energies. Distances between them are 2 MeV. The dashed lines denote the half layers. The left part of the figure corresponds to the region of the saddle point (cross); the right one to the scission configuration. The potential energy is obtained as the sum of the liquid drop energy and the shell correction term of Myers-Swiatecki (details in the text).](image1)

Fig. 1. — The potential energy surface of $^{238}U$ versus the elongation parameter $c$ and the left-right asymmetry deformation parameter $\alpha$. The solid lines correspond to layers of constant energies. Distances between them are 2 MeV. The dashed lines denote the half layers. The left part of the figure corresponds to the region of the saddle point (cross); the right one to the scission configuration. The potential energy is obtained as the sum of the liquid drop energy and the shell correction term of Myers-Swiatecki (details in the text).

In figure 2 we present the potential $V$ (solid line) along the path to fission in the $(c, \alpha)$ plane (dashed line). At the same time we show the absolute values of the stiffness coefficients

$$C_{ij} = \frac{\partial^2 V}{\partial Q_i \partial Q_j}$$

with $Q_i = c$ and $\alpha$, as calculated along the same path. To obtain the potential $V$ we approximated the sum of the Myers-Swiatecki shell correction and the liquid drop energy by a polynomial of fourth order in $c$ and $\alpha$; adding the Myers-Swiatecki shell correction [12] to the liquid drop energy. The shell correction is composed of two parts, one describing the shell effects of the mother nucleus and a second for the fission fragments. We shall assume the same deformation dependence of the shell correction as in paper [12]. The shell correction of the fission fragments is multiplied by some function $F$ of a Fermi type which depends on the proximity distance ($R_{prox}$) between the fragments:

$$F(R_{prox}) = 1 \left( 1 + \exp \left( - \frac{R_{prox}}{0.2} - 1 \right) \right).$$  

(20)

For configurations in the precission stage we use negative values for $R_{prox}$. More precisely we define $R_{prox}$ to be

$$R_{prox} = \pm \sqrt{\left| \alpha^2 - 4 AB \right|}$$  

(21)

with $A$ and $B$ as defined as in reference [11]:

$$A = 1/c^3 - B/5$$

and

$$B = 2 h + (c - 1)/2.$$  

The function $F$ is chosen in order to ensure that the potential energy surface is similar to the one obtainable in more advanced microscopic calculations when the shell effects produce asymmetric fission valleys whose depth increases with the elongation of the nucleus. All the parameters of the liquid drop and shell correction are the same as in reference [12]. So our approximation of the potential energy is similar to that of Hasse [4].

The mass tensor we assume to be constant and diagonal: $m_{ij} = \delta_{ij} m_o$. For $m_o$ we choose the value

$$\frac{1}{h^2} m_o = \frac{1}{8} A^{5/3} \text{ (MeV}^{-1}),$$  

(22)

which corresponds roughly to the average microscopic value [13] around the saddle point.

For the friction tensor we assume a diagonal form too viz.,

$$\frac{1}{h} \gamma_{ij} = \delta_{ij} \gamma_0 (1 - F).$$  

(23)
In contrast to the inertia we shall however allow $\gamma_{ij}$ to be coordinate dependent. This dependence will ensure that $\gamma_{ij}$ is practically constant for small deformations and vanishes rapidly around, or shortly after, the scission point. The constant $\gamma_0$ is chosen to be equal to:

$$\gamma_0 = nm_0.$$  

(24)

Later we shall discuss computations for which $n$ was chosen to be 0, 1, 2 or 3 units of MeV/$h$ (see below).

Finally, we need to specify the diffusion tensor. In most of the examples we used for $D_{ij}$ the classical expression as given by the Einstein relation (14). Furthermore, we have made calculations where $T^*$ was put equal to a constant different from $T$. We will come back to this point later.

2.3 Computation of observable quantities. — In principle, we should be able to calculate the distribution of any observable which can be expressed by our collective degrees of freedom $Q_i$ and their conjugate momenta $P_i$. In practice this might require following the evolution of the distribution function $d$ for $t \to \infty$. This is true because the shape of the function will change after scission. Indeed, in the applications to heavy ion collisions [5, 6], convergence was found only after the relative distance between the fragments had reached values of the order of a few hundred fm. In the present calculation we want to avoid complications which involve the dynamics after scission. The main reason is the lack of a suitable continuation of the shape coordinates beyond the scission point. These complications can be avoided if we are able to pick out quantities which remain constant after scission. But this will be the case for the total energy $E$ (and any function $f(E)$) as well as for the mass asymmetry. The total energy is a constant of the motion for nondissipative dynamics and after scission friction must be assumed to vanish. Similarly, any proper description for mass asymmetry must lead to a freezing of this degree behind the scission point. For these reasons we will concentrate on calculating the mean values and the variances of the total energy and the mass asymmetry. The final values for these quantities which we are going to compare with experiment are those reached around the scission configuration. As scission configuration we define those values of $c$, $h$, $\alpha$ for which $R_{prox}$ given by equation (21) is identically zero.

As for the energy, we should like to add the following remarks. It is clear that only the total energy will be a constant of motion for vanishing friction. Therefore only the fluctuation $\Delta E^2$ will reach a constant value when the system approaches the scission configuration. This behaviour cannot be found for parts of the collective Hamiltonian like for example the kinetic energy $T_{kin}$. For nondissipative motion the time evolution $\langle (T_{kin} - \langle T_{kin} \rangle_t)^2 \rangle_t$ is governed by the following equation

$$d \frac{d}{dt} \langle (T_{kin} - \langle T_{kin} \rangle_t)^2 \rangle_t = \langle \{ (T_{kin} - \langle T_{kin} \rangle)^2, H \} \rangle.$$  

(25)

The Poisson bracket built of the kinetic energy $T_{kin}$ and the total Hamiltonian $H$ cannot be expected to vanish. So the right hand side of equation (25) will be different from zero provided the width in $T_{kin}$ is finite, which is the case after scission.

Since $(\Delta T_{kin})^2$ will clearly change after scission, no definite conclusion (with respect to comparisons of measured widths) can be drawn if $(\Delta T_{kin})^2$ is calculated only to the scission point. Such a calculation was used to show the importance of statistical fluctuations in reference [2]. Concerning the comparison with the measured widths $\Delta T_{rel}$ of the kinetic energy of relative motion $(T_{rel})$ we wish to point out that the Coulomb energy at scission cannot be neglected. One knows that the mean value of $T_{rel}$ (for $t \to \infty$) is mainly determined by $V_{\text{Coulomb}}$. Similarly, the Coulomb potential contributes largely to the width of the total energy. Indeed, due to the comparatively strong coordinate-dependence of $V_{\text{Coulomb}}$ even small fluctuations in the coordinates lead to a large $\Delta V_{\text{Coulomb}}$. We will later substantiate these statements by looking at the calculated values for the different contributions to the fluctuations.

On the other hand, it is clear that finally the measured kinetic energy will be related to one coordinate and its conjugate momentum namely to the kinetic and potential energy of the degree of relative motion. We assume that around the scission point this degree of freedom (the fission degree) decouples from that of the mass asymmetry. The energy of this other collective degree of freedom may finally show up in the excitation of the fragments, but not in their relative motion. Since both modes are decoupled, we will have for $t \gg t_{scission}$:

$$\Delta E^2 = (\Delta E_f)^2 + (\Delta E_{asy})^2$$  

(26)

with $E_f = $ energy in the fission mode and $E_{asy} = $ energy in the perpendicular mode (here assumed to be identical to mass asymmetry). Therefore, our calculation of $(\Delta E)^2$ before scission will then lead to an upper estimate of $(\Delta E_f)^2$. Numerical estimates show $\Delta E_f$ to be equal to about 95% of $\Delta E$, when calculated within the scission region.

Both the mass asymmetry as well as the total energy are nonlinear functions of our collective degrees and their conjugate momenta. In order to get simple expressions for the variance of these quantities we have used the assumption of $d(Q, P, t)$ as a narrow distribution. In this case the variance of some $O(x)$ can be approximately calculated as

$$\langle (O - \langle O \rangle_t)^2 \rangle_t = \sum_{i,j=1}^{2N} \left| \frac{\partial O}{\partial u} \right|_{u = \langle x_i \rangle} \left| \frac{\partial O}{\partial \nu} \right|_{\nu = \langle x_j \rangle} \times \langle (x_i - \langle x_i \rangle) (x_j - \langle x_j \rangle) \rangle_t.$$  

(27)
Here $N$ is the number of collective degrees and the $x_i$ are defined as

$$x_i = Q_i \quad \text{for} \quad i = 1, 2, ..., N$$

$$x_i = P_{i-N} \quad \text{for} \quad i = N + 1, N + 2, ..., 2N.$$  \hspace{1cm} (28)

The approximation (27) is, of course, consistent with our use of the Fokker-Planck equation (1). Its solution is defined uniquely by the first and second moments, even for the case of nonlinear motion.

3. Numerical results. — We shall now discuss the computations which we have performed for the model we have presented above. Our main concern is to study the influence of the various parameters (entering the dynamical equations) on the final values of the mass asymmetry and total energy, which in the following will be denoted by $A$ and $E$ respectively. For the most part we will be interested in the variances:

$$\sigma_1^2(t) = \langle (A - \bar{A})^2 \rangle_1,$$  \hspace{1cm} (29)

and

$$\sigma_2^2(t) = \langle (E - \bar{E})^2 \rangle_1.$$  \hspace{1cm} (30)

All computations were performed for the nucleus $^{238}\text{U}$.

As mentioned in the introduction, we do not want to solve the problem of how the fissioning system decays from the first minimum into the continuum of the fragments relative motion. The picture we have in mind is that of a wavepacket impinging on the fission barrier, being partly reflected and partly transmitted. Our dynamical treatment shall be restricted to the transmitted part of the distribution. At $t = 0$, where our calculation starts, this part shall be called $d_0(Q_0, P, t = 0)$. It will be assumed to be of the Gaussian form discussed in subsection 2.1. Its centre in coordinate space $\langle Q_0 \rangle_0$ will always lie behind the barrier either at the exit point of the potential for low energy fission, or close to the top of the barrier. By exit point we mean more specifically that point on the line of steepest descent at which the potential energy is equal to the ground state energy. If not specified differently, the examples shown later will use the exit point as a starting position. In all cases the overall width of the distribution $d_0$ is chosen to be equal to, or larger than, the one for the ground state fluctuation in the first well. In subsection 3.2 we shall study somewhat more carefully the importance of this initial distribution on the final variances.

All computations presented in subsections 3.1 and 3.2 were performed using the Einstein relation $D_1 = \gamma_1 T$ for the diffusion coefficient. The temperature $T$ was chosen to be zero at $t = 0$. Its increase in time was calculated from the heat of the intrinsic degrees of freedom produced by the dissipation collective energy. As the relation of the excitation energy to the temperature we used the simple formula $E_{\text{int}} = ((A_1 + A_2)/10) T^2$. In other words, we used the same procedure as in the application of the Fokker-Planck equation to heavy ion collisions [6, 9]. As mentioned in section 2 the Einstein relation can be considered to be correct only in certain limiting cases. Therefore, in subsection 3.3 we shall discuss calculations for which the diffusion coefficients were treated as free parameters. In this connection we shall also study effects which arise when the temperature of the intrinsic system is different from zero right from the start. Such a situation is encountered in high energy fission.

One might be inclined to connect the situation in which we start at the exit point with temperature zero and a distribution chosen equal to the one for the ground state fluctuation with the realistic one encountered in low energy fission. However, we should like to emphasize strongly that this particular choice of the initial distribution $d_0$ can only be considered as a rough guess. It is neither clear that the correct $d$ is a Gaussian nor that for the low energy fission the overall width behind the barrier is the same as before the barrier. These problems can be attacked only in the context of a proper treatment of the barrier penetration, and consequently of a calculation of the decay probability.

The solution of our dynamical equations gives us the variances $\sigma_1^2$ and $\sigma_2^2$ as a function of the time. We find it more illustrative to present the results as functions of the elongation $c$ of the system. The translation is easily performed by means of the solution for the mean value $\langle c \rangle$, of this elongation. In figure 3 we show the time the system needs to move from $\langle c \rangle_0$ to some $\langle c \rangle_t > \langle c \rangle_0$. The calculation has been performed for different values of the ratio

$$\gamma_0/m_0 = n \text{(MeV/\hbar)}.$$

Fig. 3. — The time ($t$) in which the nucleus (starting from the exit point $\langle c, a(c) \rangle_0$ reaches a given point $\langle c, a(c) \rangle$, on its trajectory to fission. The end of the plot corresponds to the scission configuration. The calculation was performed for four sets of values of the friction parameters ($\gamma_0$).

3.1 Evolution of the variances along the fission path. — In figures 4 and 6 we present the variances $\sigma_1^2$ and $\sigma_2^2$ as a function of $c$. Additionally, in figure 5 we show a similar plot for the variance $\sigma_{\text{kin}}^2$ of the precission kinetic energy. Again all calculations were carried out for different values of
The variance of the total energy $U_2$ as a function of $c$, for a few values of the friction coefficients. The diffusion coefficients are chosen to be those given by the Einstein relation $D_{ij} = \gamma_{ij} T$. The initial temperature $T_0$ is taken to be zero.

As in the figure 4 but plotted for the variance $U_2^{\text{kin}}$ of the precission kinetic energy.

As in the figure 4 but plotted for the variance $\sigma_A^2$ of the mass of fission fragments. $\gamma_0/m_0$. Furthermore, the diffusion coefficient was approximated by the Einstein relation (see Eq. (14)).

The figures show the following features:

a) In general, the fluctuations increase strongly with $c$;

b) For the mass asymmetry the variance is almost independent of the friction coefficient. This implies that the growth of $\sigma_A^2$ is only due to the coupling of $A$ to the other modes by means of conservative forces (in this model the forces which result from the conservative potential).

c) However, figure 4 shows clearly that the fluctuation in $E$ is due only to the dissipative processes. The value of $\sigma_E$ is constant for $\gamma = 0$ and increases strongly with $\gamma$. This is in accordance with the general remarks in subsection 2.3.

d) Contrary to the case for the total energy, the variance of the precission kinetic energy decreases with $\gamma$. Its highest value is obtained for $\gamma = 0$. Furthermore, in this case we observe a very strong increase of the variance with $c$. This supports our reservations formulated in subsection 2.3 on the use of $\sigma_E^{\text{kin}}$ as a measure for the experimental variance of the kinetic energy of the fission fragments.

We conclude this subsection by noting that the large final value for $\sigma_A^2$ (see Fig. 6) may be a particular feature of our model, in that our inertias are independent of the coordinates. This may be inferred from figure 7. There we present a calculation in which the inertia in the $x$-direction was multiplied by a constant factor $n$. We observe that $\sigma_A^2$ decreases strongly with $n$, whereas $\sigma_E^2$ remains almost unchanged.

3.2 Influence of the Initial Distribution. — The previous results were obtained with initial widths $(\chi_{\alpha}(t = 0)$ and $\omega_{\alpha}(t = 0))$ of the distribution $d_0$ chosen to be similar to those for the ground state in the first well. This ground state was approximately described by the ground state of a multi-dimensional oscillator fitted to the potential and mass parameters of the first well. It is interesting to see in which way the final result for the variances is sensitive to the initial distribution. (For the $x$-degree this oscillator is more or less similar to the one (with possibly slight modifications) in the region of the exit point.) We have made two tests. Firstly, we multiplied the initial widths $\chi_{\alpha}(t = 0)$ and $\omega_{\alpha}(t = 0)$ by an overall scaling factor $a$:

$$\chi_{\alpha}(t = 0) = a \chi_{\alpha}^{\text{ground}}(t = 0),$$

and

$$\omega_{\alpha}(t = 0) = a \omega_{\alpha}^{\text{ground}}(t = 0).$$

Figure 7. — Change of the variances of total energy ($\sigma_E^2$) and mass ($\sigma_A^2$), calculated for the scission configuration, with the inertial mass parameter $m_\alpha$. The value of $m_\alpha$ used in figure 4 is indicated by an arrow. All other parameters are chosen as in figure 4.
in which the $\hbar \Omega_n$ are the approximate frequencies of the motion in the first well. The result of this calculation is presented in figure 8 in which is shown the final variances $\sigma_E^{\text{scission}}$ and $\sigma_A^{\text{scission}}$.

\[
\omega_{\mu}(t = 0) = a \omega_{\nu}^{H,0}(t = 0) = \left( \frac{\hbar}{\Omega} \right) \left( m_\mu \Omega_\mu \right) \delta_{\mu \nu},
\]

(32)

\[
\psi(t = 0) = 0
\]

(33)

By this transformation the overall broadness of the distribution is not changed because the determinant $\Delta$ built from all the variances $\chi$, $\omega$ and $\psi$ remains unchanged. (Because of the last equation the initial value of this determinant of course factorizes according to $\Delta = \| \chi_{\nu}(t = 0) \| \cdot \| \omega_{\mu}(t = 0) \|$. The result is shown in figure 8b. In both figures we observe a sizeable influence of the initial distribution. This influence is only slightly weaker for the case of figure 8b.

### 3.3 Influence of the Diffusion Coefficient

In section 2 we mentioned that for the fission case the proper relation between the diffusion and the friction coefficient can be deduced only after a suitable microscopic calculation of response and correlation function is performed. So far some calculations have been reported for fission recently (see Refs. [15] and [18]) but only with respect to the zero frequency limit and only for a few deformations. We have therefore chosen $T^*$ appearing in equation (15) as a free but constant parameter. In figure 9a and b we show $\sigma_E^2$ and $\sigma_A^2$, respectively, taken as a function of $c$ for two different values of $\gamma$ with $T^* = 0.3$. The initial distribution was chosen as in figures 4 and 6. A comparison with these

![Fig. 8.](image)

![Fig. 9.](image)
results (which were obtained by using the Einstein relation) shows great similarity, both in the functional dependence on \( c \) as well as for the magnitude. The fact that even the magnitude is almost unchanged may have the following reasons. The value \( T^* = 0.3 \) is about equal to the average value of the temperature between the starting and end point of our trajectory.

In table I we present the final variances (at the scission point) calculated for the case of \( \gamma_0 = m_0 \) and for several values of \( T^* \). We observe that these variances change comparatively little with \( T^* \). They change by only a factor of about two when \( T^* \) is altered by a factor of 30.

Table I. — Dependence \( \sigma_A^2 \) and \( \sigma_A^4 \) on \( T^* \).

<table>
<thead>
<tr>
<th>( T^* ) (MeV)</th>
<th>0.1</th>
<th>0.3</th>
<th>0.9</th>
<th>2.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>122.2</td>
<td>137.6</td>
<td>184.0</td>
<td>323.0</td>
</tr>
<tr>
<td>( E ) (MeV(^2))</td>
<td>34.4</td>
<td>38.0</td>
<td>48.6</td>
<td>80.8</td>
</tr>
</tbody>
</table>

Finally, we would like to come to the case of high energy fission, i.e. those fission processes which are initiated in a nuclear reaction. In these reactions the fissioning nucleus is highly excited, the typical temperatures reaching values of up to about three MeV. Using the picture of a compound nucleus we would then say that within our model the fissioning system already starts from a situation with finite (and even comparatively large) temperatures. This implies that here the Einstein relation can be used safely.

The picture of the compound nucleus has a further implication. It means that the initial width \( d_0 \) will be governed by this large temperature \( T_0 \). We may therefore take \( d_0 \) to be given by a Boltzmann distribution with temperature \( T_0 \). This distribution will have a much larger width than the one for the ground state fluctuations. In subsection 3.2 we have already seen that broad initial distributions will influence broad variances in the scission point. We therefore expect a large dependence on the temperature. This effect can be seen in table II where we present the final variances \( \sigma_A^2 \) and \( \sigma_A^4 \) as a function of \( T \). The calculation was made for \( \gamma_0 = m_0 \) and a change of \( T \) with \( c \) was neglected. For these examples the starting point was chosen close to the top of the barrier.

Table II. — Dependence of \( \sigma_A^2 \) and \( \sigma_A^4 \) on the initial temperature \( (T_0) \) of the fissioning nucleus.

<table>
<thead>
<tr>
<th>( T_0 ) (MeV)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>3494</td>
<td>18 066</td>
<td>35 540</td>
<td>50 220</td>
</tr>
<tr>
<td>( E ) (MeV(^2))</td>
<td>1 024</td>
<td>5 300</td>
<td>10 430</td>
<td>15 022</td>
</tr>
</tbody>
</table>

4. Summary. — We studied the dynamics of fission from the region of the saddle point to that of scission. The shape of the fissioning nucleus was parametrized by three collective degrees of freedom. The dynamical evolution of the system was studied by means of the Wigner function of these variables and their conjugate momenta. The equation of motion for this \( d(t) \) is of a Fokker-Planck type, in which all nonlinear forces are linearized locally. In contrast to the treatment of reference [16] this equation does not only account for quantal fluctuations but for features of non-equilibrium statistical mechanics as well.

The input for this equation involves the knowledge of conservative forces on one hand and on dissipative and conservative ones on the other. As for the first type of force, we based our input on experience with cranking model type calculations (cf. Ref. [14]). For the second type of force we used phenomenological ansätze which were partly justified by microscopic studies, namely as far as the generalized relationship between the friction and the diffusion coefficients is concerned.

The quantities we were aiming at were the variances at scission of the total energy \( \langle \sigma_A^4 \rangle \) and of the mass asymmetry \( \langle \sigma_A^2 \rangle \). These quantities can be expected to be constant after scission and may thus be compared to experiment. We argue that the same will not be true for the prescission kinetic energy. This makes us believe that the latter is not a good measure of the importance of the statistical fluctuations.

Within our model we observe a large increase of \( \sigma_A^2 \) and \( \sigma_A^4 \) on the way from saddle to scission. Furthermore, the final values of \( \sigma_A^2 \) and \( \sigma_A^4 \) depend strongly on the initial broadness of the distribution \( d \), whereas only a moderate influence of the diffusion coefficient was found. Both questions are somewhat related, because, for high energy fission both the initial distribution as well as the diffusion coefficient will be governed by the large temperature of the fissioning system.

Let us finally comment on a comparison with experimental results. We did not try to adjust parameters to obtain a fit to experiments. We may, however, note that for low energy fission the values for \( \sigma_A^2 \) and \( \sigma_A^4 \) are of the same order of magnitude as those for experiment. For high energy fission our model predicts much larger values (see above). This is probably due to deficiencies of our simple model. For instance, we did not take into account any temperature dependence of the friction, the inertia or the potential energy. As an immediate consequence, the mass distribution does not change from asymmetry to symmetry when the excitation energy increases. This could be improved upon by taking into account the well-known decrease of the shell correction energy with temperature. Besides the lack of a suitable temperature dependence, the assumed constancy of the mass parameters may be too crude. An increase of the inertia in the scission region will certainly decrease the fluctuations. We have made a test by changing the inertia in the \( \alpha \)-direction. Then, indeed, the final value of \( \sigma_A^4 \) was reduced.

It is clear that a more quantitative calculation can
be done only after microscopic inputs become available for the coefficients of the Fokker-Planck equation. We believe, however, that the features which we find and which were mentioned before are of somewhat more general nature. In particular, the strong dependence on the initial distribution certainly calls for a proper treatment of the problems of barrier penetration. It is here where, from a theoretical point of view, the interplay between quantal and dissipative phenomena is most stringent.

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