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To cite this version:
P. Nozières. Kondo effect for spin 1/2 impurity a minimal effort scaling approach. Journal de Physique, 1978, 39 (10), pp.1117-1124. <10.1051/jphys:019780039010011700>. <jpa-00208852>

HAL Id: jpa-00208852
https://hal.archives-ouvertes.fr/jpa-00208852
Submitted on 1 Jan 1978
KONDO EFFECT FOR SPIN 1/2 IMPURITY
A MINIMAL EFFORT SCALING APPROACH

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(Reçu le 6 juin 1978, accepté le 22 juin 1978)

Résumen. — La mécanique d’une procédure de renormalisation, appliquée au cas simple d’une impureté Kondo de spin 1/2, est décortiquée en détail. Certaines ambiguïtés sur la nature des points fixes, en particulier près des points fixes, sont levées. On montre que des considérations simples de symétrie et d’invariance déterminent le comportement à basse température. En particulier, la diffusion potentielle a pour seul effet de translater en bloc les déphasages.

Abstract. — The mechanism of a poor man’s scaling as applied to the simplest spin 1/2 Kondo impurity is analysed in detail. Ambiguities concerning the nature of the effective Hamiltonian are clarified, especially near fixed points. Maximal use is made of symmetry and invariance arguments in determining the low temperature properties. Especially, potential scattering is shown to result in a uniform translation of phase shifts.

We consider the simplest Kondo system: a spin 1/2 impurity (spin $S$) is antiferromagnetically coupled to a free electron gas, via the usual Hamiltonian

$$[V_{kk'} \delta_{\sigma\sigma'} + J_{kk'} S_s s_{\sigma\sigma'}] c_{k\sigma} c_{k'\sigma'}.$$ 

(1)

The scalar and exchange matrix elements, $V_{kk'}$ and $J_{kk'}$, are cut off at a distance $\pm D$ from the Fermi level (chosen as the origin of energy).

Our present understanding of such a system relies on the idea of scaling originally introduced by Anderson [1]: as the temperature $T$ is reduced, one can eliminate an increasing number of states from the electron band, at the expense of correcting the effective Hamiltonian that couples the magnetic impurity to those electrons. When the scaling reaches a characteristic Kondo-temperature $T_K$, an initially small antiferromagnetic $J_{kk'}$ becomes large, thereby quenching the impurity spin into a local singlet. The validity of this simple picture was demonstrated in the numerical solution of Wilson [2], which fully describes the cross over from weak to strong coupling in the region $T \sim T_K$. Well below $T_K$, one recovers a simple physical problem which can be treated by an appropriate modification of the usual Landau theory of Fermi liquids [3].

In a sense, the problem may be considered as solved and indeed the basic ideas are by now well understood. Yet a number of points remain somewhat ambiguous: for instance the nature of the scaling procedure, which is often mis-represented or the extent to which a numerical solution is required in order to connect the regions $T \gg T_K$ and $T \ll T_K$. These questions may appear somewhat academic when a numerical solution exists. They become vital, however, if one wants to generalize the theory to realistic models, where orbital effects, crystal field anisotropies, spin orbit coupling introduce so many parameters that a numerical solution is impossible. Then, one needs a clear understanding of the scaling method, in order to fully exploit symmetry considerations, etc... (such a generalization is now within reach [4, 5]).

In the present note, we wish to clarify some of these points. We shall survey the mechanism of the scaling process, and the way in which the search for fixed points must be approached. The basic ideas are most easily laid out in the simple case of pure potential scattering ($J_{kk'} = 0$). In the real Kondo effect, we show how universality and symmetry allow us to by pass a numerical treatment of the cross over region. We discuss the effect of potential scattering $V_{kk'}$ on the Kondo cross over, a feature which was ignored in Wilson’s original calculation. We thus corroborate recent calculation by Lloyd et al. [6], who extended Wilson’s numerical approach to the asymmetric case $V_{kk'} \neq 0$. None of this is really new: the purpose...
of the paper is mostly pedagogical, as a stepping stone to more complicated problems.

1. The scaling procedure. — As far as the thermodynamics is concerned, we need only keep track of the states with energy ± T around the Fermi level: the wings of the electron band may therefore be eliminated, by renormalizing the Hamiltonian. More precisely, consider a new cut off D' < D. Any perturbation diagram may be broken into effective vertices, whose internal lines all lie in the eliminated range (D', D). The effective vertices acts as black boxes, connected by external lines that lie in the remaining range (−D', + D'). The physical problem is thus replaced by an equivalent one, with a smaller cut off D', the original Vkk, Jkk being replaced by the effective vertices. However, one pays a price: the effective vertices are now energy dependent (the corresponding time retardation accounts for the time spent in the k-states that are eliminated). Moreover, the scaling generates higher couplings, in which the impurity spin simultaneously interacts with several conduction electrons (the simplest such coupling is an interaction energy of two opposite spin conduction electrons on the impurity site). As a result, the scaling procedure is not just a change of V and J[7].

As the cut off D → 0, the effective Hamiltonian is supposed to evolve toward a fixed point. Usually, the fixed point Hamiltonian is something quite complicated (retarded, etc...), yet it is uniquely defined for a given scaling procedure. The fixed point itself determines the T = 0 properties; the low temperature behaviour depend on the approach to the fixed point.

As such, the scaling process is defined for any system, whether it displays logarithmic behaviour or not. As an illustration, consider pure potential scattering (Jkk = 0). For simplicity, we assume a separable Vkk, given by:

\[ V_{kk'}(z) = \begin{cases} V & \text{if } |\epsilon_k| < D' \\ 0 & \text{otherwise} \end{cases} \]

(this simplification is in no way essential, but it makes the equations shorter). In this way, we deal only with s-wave scattering. The scattering t-matrix takes the simple form:

\[ t(z) = \frac{V}{1 - V' \sum_{D'} \frac{1}{z - \epsilon_k}} \]

(the index D stands for the range (−D', + D')). The phase shift δ(\epsilon) is just the argument of t on the real axis, given by:

\[ \tan \delta(\epsilon) = \frac{\pi V \rho(\epsilon)}{1 - V' \sum_{D'} \frac{1}{z - \epsilon_k}} \]

ρ(\epsilon) is the density of states (not necessarily constant); P stands for the principal part.

Scaling the cut off from D to D', one rewrites the t-matrix (2) in the equivalent form

\[ t(z) = \frac{V'}{1 - V' \sum_{D'} \frac{1}{z - \epsilon_k}} \]

where the new interaction V' is given by

\[ V' = \frac{V}{1 - V \sum_{D-D'} \frac{1}{z - \epsilon_k}}. \]

As expected, V' is z-dependent: the effective interaction is retarded. Since the phase shift is a physical quantity, it must be unchanged in the scaling — at least in the range (−D', + D') covered by both descriptions. Indeed it is: scaling simply chops a slice of the δ(\epsilon) curve (Fig. 1). Note that retardation is essential: an instantaneous V' would necessarily yield δ = 0 or π at the new edges ± D', in contradiction to reality.

In order to compare the old and new potentials, we dilate the z-scale, bringing D' back to D. The new phase shift δ(\epsilon) is the dash-dot curve of figure 1(1). As scaling proceeds, δ(\epsilon) is flatter and flatter; when D' goes to 0, the new phase shift is constant in the range (−D', + D), equal to its Fermi surface value δ₀. That limit represents the fixed point of the scaling process.

We thus find a line of fixed points, parametrized by δ₀. The fixed point Hamiltonian is obtained by letting D' → 0 in (5). It may be cast in the form

\[ V^*(z) = \frac{\tan \delta_0}{\pi \rho_0} \frac{1}{1 - \frac{z}{D'}} \left(1 - \frac{\tan \delta_0}{\pi} \log \frac{1 - \frac{z}{D'}}{1 + \frac{z}{D'}}\right) \]

(1) Similarly, the density of states becomes ρ(\epsilon)(D'/D): it is flatter and flatter as D' decreases: in the end, ρ is a constant, equal to its Fermi level value ρ₀.
Once again, $V^*$ is retarded (which allows a constant phase shift in $(-D', +D')$). It is moreover clearly scale independent, depending as it does on the ratio $z/D'$. One point, however, is important: while $\delta_0$ is physically meaningfully, the shape of $V^*(z)$ is specific of our sharp cut off procedure. Different scaling techniques would yield the same $\delta_0$, but a different functional form (2) of $V^*(z)$.

One last comment: the fixed point describes properties at $T = 0$. The low temperature properties, on the other hand, are contained in the approach to the fixed point. Thus, for any finite $D'$, we should write $V'$ as

$$V'(z) = V^* \left( \frac{z}{D'} \right) + D' f \left( \frac{z}{D'} \right) \ldots$$  \hspace{1cm} (7)

The kernel $f$ is the leading correction to $V^*$ at low temperature.

2. The Kondo cross over. — When exchange coupling is restored, the scaling process becomes far more complicated. The one electron scattering amplitudes are of course renormalized, becoming retarded quantities $V'(z)$ and $J'(z)$, but in addition, scaling generates higher couplings. Their physical origin is clear: the history of the impurity spin provides a memory mechanism, whereby an electron can indirectly influence another electron (or several of them) that will pass by the impurity at a later time. The effective Hamiltonian is thus very complicated.

Moreover, a new feature appears: scaling generates logarithmic terms, proportional to $\log (D/D')$, which diverge when $D' \rightarrow 0$: these terms are the gist of the Kondo effect; their slow divergence as $T \rightarrow 0$ leads to the Kondo cross over.

In the pure potential scattering case, the only energy scale of the problem was the original cut off $D$. The fixed point [6] was thus reached as soon as $D'$ was a small fraction of $D$; put another way, the transient approach to $V^*$ was achieved over a range of temperature of the order of $D$. In contrast, Kondo scaling involves two characteristic energy scales: the original cut off $D$, and the Kondo temperature $T_k \sim D \exp[-1/J]$, at which logarithmic corrections become sizeable. As a result, Kondo scaling will proceed in two steps.

(i) In a first transient regime, $D' \sim D$, all the corrections $\sim z/D$ progressively die out. When $D' \ll D$, we are left with an effective Hamiltonian that depends only on $z/D'$, except for logarithmic terms $\sim \log (D/D')$ that diverge when $D' \rightarrow 0$. That first step ends up with an effective retarded and complicated Hamiltonian

$$\mathcal{H}^* \left( \frac{z}{D'}, \log \frac{D}{D'} \right).$$  \hspace{1cm} (8)

The logarithms in (8) are the only place where the original cut off $D$ survives.

(ii) Once the transients $\sim z/D$ are dead, the $\log (D/D')$ terms keep growing slowly. Hence another cross over, this one logarithmic, that covers many decades around $D' = T_k$. Eventually, a final fixed point will be reached at $T = 0$.

The mechanism of this cross over is described in detail in the beautiful review article of Wilson [2]. Here we repeat some of his arguments in order to lay the ground for subsequent discussion.

a) The original Hamiltonian depends on many parameters: shape of the density of states, of $V_{\nu \nu}$ and $J_{\nu \nu}$, etc..., in that sense, the first transient step is non universal (i.e. model dependent). In contrast, the intermediate Hamiltonian (8) only depends on two parameters, namely the two vertices that diverge when $D' \rightarrow 0$. These two relevant charges are here the values of $V'(z)$ and $J'(z)$ for $z = 0$, which we abbreviate as $V'$ and $J'$; they are enough to parametrize $\mathcal{H}^*$: once they are known, everything else is completely determined. All the multielectron couplings, as well as the energy dependence of every vertex (retardation effects) are forced by $V'$ and $J'$. Let us emphasize that these higher couplings are large: when we say they are irrelevant we only mean that they are triggered by $V'$ and $J'$. Put another way, they are not additional independent variables: the intermediate Hamiltonian is a two parameter family, yet a complicated one (3) (in the pure potential case, $\mathcal{H}^*$ depended on only one parameter, $\delta_0$, and it was much simpler).

b) In the case $J \equiv 0$, the log terms disappear from (8): $\mathcal{H}^*$ is a real fixed point, and nothing else happens. In the Kondo case, they instead remain. They can be however embedded in the two independent charges $V'$ and $J'$, which consequently keep varying slowly. Hence an indirect $D'$-dependence of the effective Hamiltonian

$$\mathcal{H}^* \left( \frac{z}{D'}, V', J' \right).$$  \hspace{1cm} (9)

The crucial point is now the following: a change in $t = \log (D/D')$ may be absorbed in a redefinition

(3) As a result, one cannot map the universal $\mathcal{H}^*$ over any simple case with specific values of $V$ and $J$ that would happen to be soluble, at least approximately. Scaling to the Toulouse limit [8] is thus an approximation, a good one, though, as it does not distort the physics [9].
of $V'$ and $J'$. Put another way, $\mathcal{K}^*$ is invariant along a
certain curve $V'(t), J'(t)$ in the three dimensional
space $(V', J', t)$. It is this invariance which makes the
theory renormalizable in the field theoretical sense (*).
(Note once more that behind $V', J'$, a very complex
Hamiltonian is hidden !)

c) The universal trajectory $V'(t), J'(t)$ goes to a
stable fixed point when $t \to \infty$ (i.e., $D' \to 0$). One may
continue it backwards in the region $t \to -\infty$: it then
originates from another, unstable, fixed point of the
universal renormalization transformation.

In the spin 1/2 Kondo system under consideration,
the only fixed points known to date are

- $J' = 0, V'$ arbitrary
  - Stable for ferromagnetic coupling ($J' < 0$)
  - Unstable for antiferromagnetic coupling ($J' > 0$)

- $J' = \infty, V'$ arbitrary
  - Stable when $J' > 0$
  - Unstable when $J' < 0$.

In the antiferromagnetic case, the universal
trajectories are therefore a one parameter family, ori-
ginating from an arbitrary point of the $J' = 0$ axis,
and ending up somewhere in the $J' = \infty$ limit. The
situation is sketched on figure 2.

d) In reality, scaling begins with bare values $V, J,$
that correspond to some point $P_1$ in figure 2. The first
transient part of scaling corresponds to the dotted
line. By the time $D'$ has reached, say 0.1, the repre-
sentative point returns toward the universal curve

![Figure 2](image)  
**Fig. 2.** — The full line is a sketch of a universal scaling trajectory in
the $(V, J)$ plane, parametrized by the scaling variable $t = \log(D/D')$.  
A real physical problem would start at some point $P_1$. The initial
transient steps of scaling lead to the region around $P_2$, where one
joins the universal curve.

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(*4) Here, we scale keeping the original $D$ constant, and letting $D'$
and $z$ go to zero. Standard renormalization theory instead lets $z$
untouched, while $D \to \infty$. The transients are those terms $\sim z/D$
that vanish in such a limit. The terms $\sim \log(z/D)$ are instead
embedded in a redefinition of bare couplings. The two points of view
are clearly equivalent: since $\mathcal{K}^*$ depends only on $D/D'$, it does not
matter whether $D' \to 0$ or $D \to \infty$!

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(*5) Such a picture, which clearly separates the transient (dotted
curve) from the subsequent universal cross over, is somewhat
misleading, as it ignores higher couplings. The universal trajectory
is actually drawn in a multidimensional surface, any coupling being
a well defined function of $V'$ and $J'$. In contrast, the dotted transient
does not belong to that manifold, a statement that it is non universal.
impurity spin averages to zero during a collision; it is seen as non magnetic.

Although the impurity degree of freedom disappears from the problem, the effective Hamiltonian does not reduce to pure potential scattering. The multielectron terms that built up during scaling are still there; in the limit $T \ll T_k$, they appear as local interactions between several electrons near the impurity. In a real strong coupling case, the physical origin of that interaction is clear: one electron polarizes the singlet, and this polarization is felt by another electron. When strong coupling arises from the Kondo crossover, such a simple picture is somewhat too crude (the effective Hamiltonian is complicated), yet the basic idea is still valid: the low temperature regime corresponds to locally interacting electrons in the presence of a structureless impurity.

In order to fit this picture into our scaling description, we briefly discuss the nature of the effective Hamiltonian $\mathcal{H}(D')$ when $D' \ll T_k$. It involves only the local operator

$$\psi_a = \sum_{D'} c_{ka}$$

(any $k$-dependence has been washed out in the transient). A typical term describing the interaction of $n$ electrons involves a retarded product of $2n$ operators; in the time representation, this product appears as

$$\psi^*_a(t_1) ... \psi^*_a(t_n) \psi_a(t'_1) ... \psi_a(t'_n).$$

We note that, at this stage, the exclusion principle is of no avail: since the interaction is retarded, nothing prevents an electron of, say $\uparrow$ spin, from affecting another electron with the same spin which would reach the impurity at a later time, when the first one is gone.

One fact, however, is clear: since $\psi$ involves a momentum summation over a number of states $N' = 2 \rho D'$, the bigger $n$, the less relevant the corresponding term of $\mathcal{H}(D')$. A simple dimensional argument [3] shows that the $n$th order term is of order $D'^{-n}$ . We can thus identify the leading terms of $\mathcal{H}$ when $D' \to 0$.

(i) The fixed point, corresponding to the limit $D' \to 0$, involves only one electron scattering off the impurity. Returning to a frequency description, the corresponding Hamiltonian is

$$V^*(\frac{z}{D'}) \psi^*_a \psi_a$$

(10)

(the only form which is both scale invariant and spin conserving). (10) must lead to a constant phase shift $\delta_0$ in the range $(-D', +D')$. Hence $V^*$ is necessarily the expression (6) found for pure potential scattering. The only parameter is $\delta_0$, the Fermi surface phase shift.

(ii) The leading corrections to the fixed point are two-fold:

- One may have corrections to the one electron scattering

$$D' \int \frac{z}{D'} \psi^*_a \psi_a.$$ 

They correspond to the energy dependence of the phase shift $\delta(a)$ (and indeed $f$ is uniquely determined in such a way that the corresponding phase shift is linear in $a$).

- One may also have two particle scattering of the form

$$g \left( \frac{z}{D'} \right) \psi^*_a(\epsilon_1) ... \psi^*_a(\epsilon_n).$$

(11)

The central question is now: how many parameters do we need in order to characterize the interaction (11)? Can we devise a simple argument that we could transpose to more complicated situations?

In order to answer this question, we note that the retardation of the interaction (11) occurs over times $\sim \hbar/T_k$, which remain finite when $T \to 0$ (it corresponds to the memory of the spin). As soon as $T \ll T_k$, the electron collisions spread over a longer time $\tau_s \sim \hbar/T$: as a result, they simply ignore the existence of retardation. The effect of the interaction is the same as if it were instantaneous. But then, the exclusion principle comes in, which precludes the double occupation of the same orbital $\psi_a$: we conclude that the effective interaction only affects electrons of antiparallel spins. The argument may be cast in a more formal way by choosing in (11) a cut off $D'$ that is $\ll T_k$, but still much bigger than the temperature of interest $T$. The former condition ensures that only the two particle interaction (11) is important, the latter one implies that $z_s/D' \to 0$. In this limit, the energy dependence disappears, and the Hamiltonian behaves as if instantaneous. The number of independent parameters follows immediately from:

- symmetry considerations and conservation laws (here invariance under spin rotation),

- exclusion principle.

Such an approach will prove extremely powerful in more complicated situations [5].

In the present case of a spin 1/2 impurity with no orbital effect, the low temperature behaviour may be described phenomenologically [3] by expanding the phase shift as a functional of the particle distribution $n_a(\epsilon)$

$$\delta'[\epsilon, n_a(\epsilon')] = \delta_0 + \alpha \epsilon + \varphi \delta n_\uparrow(\epsilon')$$

(12)

(where $\delta n = n - n_0$ is the departure from equilibrium). Only three independent parameters survive, $\delta_0$, $\alpha$ and $\varphi$, which determine all the thermodynamical quantities: specific heat, susceptibility, etc... The only remaining issue is how to match these few parameters to the high temperature properties.
4. Matching the low and high T regions. — Let us first consider the fixed point, i.e. the Fermi surface phase shift $\delta_0$. In the particular case of electron hole symmetry, its value follows at once from simple symmetry considerations. Such a symmetry holds when the following conditions are met:

- no potential scattering: $V_{kk'} = 0$;  
- density of states $\rho(\varepsilon_k)$ and exchange coupling symmetric with respect to the Fermi surface (for instance, both constant in the range $(-D, +D)$).

The Hamiltonian is then invariant under the transformation

$$c_{k\uparrow} \rightarrow i c_{k'\downarrow}, \quad c_{k\downarrow} \rightarrow -i c_{k'\uparrow}$$

(13)

(where $-k$ means here symmetric with respect to the Fermi level taken as $k = 0$). Since every line in each diagram is reversed, all the energies $z$ should also change sign (as in time reversal). Such a built-in symmetry will persist throughout the crossover, and the final fixed point Hamiltonian must also be invariant under (13). It follows that $V^*(z)$, given by (6), should be an odd function of $z$, which implies $\tan \delta_0 = \infty$. Electron hole symmetry therefore leads to $\delta_0 = \pm \pi/2$ (actually $-\pi/2$ as can be inferred from a continuity argument). This is the so-called unitarity limit, at which the scattering cross section is maximal. Of course, we do not need all that scaling machinery in order to find $\delta_0$: since electron hole symmetry implies a $\delta_0$ halfway between 0 and $-\pi$, it has to be $-\pi/2$! However, it is instructive to cast such an argument in our general language.

One can push the electron hole symmetry argument further. Let us consider the elastic scattering matrix, $T_{aa}$ for scattering of a single electron on the impurity with no spin flip (given by all diagrams with one incoming and one outgoing electron line). The argument of $1 - 2 i n \rho T_{aa}$ is the elastic phase shift for such a collision (see [3]). We thus define two elastic phase shifts $\delta_\parallel$ and $\delta_\perp$ respectively for parallel and antiparallel spins. Note that they are defined at any temperature; however, they are physically meaningful only when the spin is unquenched, i.e. when $T > T_K$. Electron hole symmetry implies $T_\parallel(z) = -T_\perp(-z)$. It follows that at the Fermi level $\delta_\parallel = -\delta_\perp$ at all temperatures.

When $T \to 0$, the impurity locks into a singlet: then the difference $(\delta_\parallel - \delta_\perp)$ must tend to $\pi$, a statement of a generalized unitarity limit. Indeed, the impurity traps one electron of opposite spin, which by Levinson’s theorem reduces $\delta_\parallel$ by an amount $\pi$. The two phase shifts are thus brought back to the same value—precisely the $\delta_0$ of our phenomenological description: thus

$$\delta_0 = \lim_{T \to 0} \delta_\parallel(z = 0).$$

(14)

(One can view the impurity either as a free spin with $\delta_\parallel - \delta_\perp \to \pi$, or as a locked spin with a single phase shift $\delta_\parallel$: it is a matter of taste.) The value of $\delta_0$ then follows from two arguments:

- unitarity limit $\delta_\parallel - \delta_\perp = \pi$;
- electron hole symmetry $\delta_\parallel = -\delta_\perp$.

Hence the result $\delta_0 = -\pi/2$.

We now return to the general case, with no electron hole symmetry (i.e., $V_{kk'} \neq 0$ and/or asymmetric bands). The unitarity limit still holds, but we need another condition for fixing $\delta_0$. This can be found by changing the basis in which we represent the conduction electron states. Up to now, we used free electron states, $|k\rangle$, corresponding to a free s-wave function $e^{ikr}/r$. Actually, we could as well use scattering states $|k\rangle$ corresponding to some arbitrary reference potential $V_r$. The energies $\varepsilon_k$ are unchanged, but the matrices $V_{kk'}$ and $J_{kk'}$ become

$$V = UVU^+, \quad J = UJV^+$$

(15)

in which $U = |\tilde{k}\rangle \langle k'|$ is the unitary transform that carries from the $|k\rangle$ to the $|\tilde{k}\rangle$ basis. Let $\delta_r$ be the reference phase shift due to $V_r$ (which depends on $\varepsilon_k$). Clearly, replacing $V, J$ by $\tilde{V}, \tilde{J}$ amounts to subtracting $\delta_r$ from all phase shifts. Put another way, changing the basis is equivalent to measuring the phase shifts from a translated origin.

Such a transformation does not affect the physics, but is provides enough lee-way to restore electron hole symmetry: hence an explicit solution. More precisely, let us consider a cut off $D'$ such that:

(i) $D' \ll D$: the transients are dead;
(ii) $D' \gg T_K$: a perturbation expansion in $J$ is still valid.

We know that the effective Hamiltonian depends on two independent parameters $V'(z = 0)$ and $J'(0)$. Under the change of basis, they become $\tilde{V}' = UVU^+ U'^+$ and $\tilde{J}' = UJV^+ U'^+$. If we choose the (arbitrary) reference potential $V_r$, such that $\tilde{V}'(z = 0) = 0$, we have restored electron hole symmetry: thereafter we can apply our earlier discussion. The new potentials $\tilde{V}', \tilde{J}'$ give a phase shift $\delta_r$ which goes to $\delta_0 = -\pi/2$ at $T = 0$. The phase shifts due to the old potentials $V, J$ are simply shifted by $\delta_r$, and consequently

$$\delta_0 = \delta_r - \pi/2.$$  

(16)
(In (16), \( \delta_1 \) is evaluated in the range \( -D', +D' \), i.e. near the Fermi level: it is a constant). The only thing we need is to find the \( V' \), that makes \( V'(0) = 0 \) once the transients are dead.

Actually, the above discussion improves our understanding of universality. Of the two parameters, \( V'(0) \) and \( J'(0) \), that characterize the universal cross over Hamiltonian, one can be absorbed in a redefinition of the basis states, which translates all phase shifts by a constant amount throughout the cross over. It follows that in the universal region

\[
\frac{1}{2} (\delta_{\parallel} + \delta_{\perp}) = \text{const.} .
\]

The \( V' \) parameter reduces to a trivial translation in \( \delta \)-space: only \( J' \) is relevant to the Kondo effect. In order to eliminate \( V' \) (and thus to restore electron hole symmetry), we choose \( \delta_1 \) as the constant of (17).

The relation (17) was first found numerically by Lloyd [6]. He also showed how a scattering state basis allows us to get rid of \( V \) (yet in leading order in \( J \) only). Here we modify his arguments in such a way as to respect universality.

With these remarks in mind, we can get \( \delta_1 \) directly. We need only calculate the Fermi surface phase shifts \( \delta_{\parallel} \) and \( \delta_{\perp}(\varepsilon = 0) \) by a plain perturbation expansion in \( J \), valid as long as \( T \gg T_k \) (we need not expand in \( V \)). In the non universal region \( T \sim D, \delta_{\parallel} \) and \( \delta_{\perp} \) are model dependent, and their sum need not be constant. In contrast, once we enter the universal region \( T \ll D \), the sum of the shifts stabilizes to a constant value \( 2 \delta_1 \) (the log terms being opposite). It is this \( \delta_1 \) which fixes the \( T = 0 \) phase shift through (16). For very small \( J \), the difference between \( \delta_{\parallel} \) and \( \delta_{\perp} \) is small, and consequently \( \delta_1 \) reduces to the phase shift \( \delta_1 \) due to potential scattering alone; the correction \( \delta_1 - \delta_1 = J^2 \).

Besides translating phase shifts, the change of basis also modifies \( J' \) according to (15), a correction which reacts on \( T_k \). For small \( J \), a first order calculation is enough. Assuming for instance a constant density of states, a potential \( V \) yields a phase shift \( \delta = \arctan(\pi \rho V) \). The effective \( J \), describing the difference between \( \delta_{\parallel} \) and \( \delta_{\perp} \), is thus given by

\[
J = \frac{1}{2 \pi \rho} \left[ \arctan(\pi \rho(V + J)) - \arctan(\pi \rho(V - J)) \right] = J \cos^2 \delta_1
\]

a well known result due to Fischer [10]. For larger \( J \), that result is easily generalized.

Finally, we make one last comment on \( \delta_1 \). We discussed the case of a pure one electron scattering off the impurity. In reality, the electrons are also interacting between themselves, and the potentials \( V_{kk'} \) and \( \delta_{kk'} \) should be viewed as self consistent potentials, screened by the electron-electron interaction. As a result, they will depend on temperature, readjusting continuously to the changing electronic environment. The problem becomes far more complicated. The only handle we have is the condition of perfect screening, leading to a Friedel sum rule which determines \( \delta_0 \).

If we assume only s-wave scattering, electrical neutrality implies

\[
\frac{2 \delta_0}{\pi} + 1 = Z
\]

where \( Z \) is the impurity valency. In (19) the term 1 accounts for the electron locked into the singlet, the first term corresponding instead to the remaining deformation due to the locked impurity. In actual life, (19) is more realistic than the result (16) based on a constant \( V, J \) [11].

Let us finally turn to the approach to the fixed point, i.e. to the parameters \( \alpha \) and \( \varphi \) of (12), \( \alpha \) provides the energy scale of the problem: \( \alpha \sim 1/T_k \). It should be connected to the high temperature properties, i.e. ultimately to the original \( V \) and \( J \). Such a calculation is tantamount to fixing the number of decades in the cross over region: clearly that can only be done via a numerical analysis. It is precisely there that Wilson’s formulation is needed: it connects the low and high \( T \) approaches to \( T_k \) (see ref. [3]). Nothing simple therefore for the parameter \( \alpha \).

The ratio \( \varphi/\alpha \), however, follows from a simple poor man’s version of universality which proves to be extremely powerful. As long as \( T_k \ll D \), the Kondo singularity must be attached to the Fermi level: it is carried along if the latter moves. As a result, the phase shift must be invariant if the chemical potential \( \mu \) and the energy \( \varepsilon \) are shifted by the same amount (within corrections \( \sim T_k/D \) which are small and non universal). A glance at (12) shows that such a universality requirement implies

\[
\alpha + \rho \varphi = 0
\]

\( \varphi \) is thus fixed. An immediate consequence of (20) is that the total impurity susceptibility is exchange enhanced to twice the value it would have for free electrons: \( \chi T/C_J = 2 \), a result which was first found numerically by Wilson.

The above argument is typical of what one can do by using the universality concept without resorting to actual calculations. Our poor man’s universality is of course one aspect of the more general concept of Wilson, yet a self contained one! In more complicated problems [5], it will be even more useful.

5. Conclusion. — From the above discussion, the following philosophy of a scaling theory emerges:

(i) Identify the low temperature fixed point, usually by qualitative arguments.

(ii) In order to reduce the number of independent parameters below the cross over, exploit two kinds of arguments.
— Symmetry and invariance considerations (e.g. electron hole symmetry).
— Universality requirements (e.g. invariance under translation of the Fermi level).

One thus achieves an economy class version of renormalization. It should be realized, however, that not all questions are answered. For instance, while the total spin susceptibility $\chi_T$ is known, the local one $\chi_L$ (which controls the impurity Knight shift), is not simply expressed in terms of the parameters $\alpha$ and $\varphi$. Its value can certainly be obtained from a numerical calculation; it remains to be seen whether the answer follows from simple arguments ($^6$).

Acknowledgments. — The author wishes to thank Dr. P. Lloyd for an extremely useful correspondence. The present analysis of potential scattering was rooted in his work on the subject.

($^6$) Can one for instance apply the compensation theorem of Anderson, which states that the average conduction electron polarization is small, in which case $\chi_T \approx \chi_L$?

References


