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COUPLED FIELDS IN ONE DIMENSION : CROSS-OVER FROM A CONTINUOUS SYMMETRY TO A DISCRETE ONE

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Abstract. — We study a pedagogical example of a one dimensional Landau Ginzburg functional with a symmetry-breaking perturbation which causes a cross-over behaviour from a continuous symmetry \( n = 2 \) to a discrete one. The correlation length is shown to be governed by the lowest energy defects, which are solitons in the weak anisotropy limit. As the symmetry breaking term is introduced in the Hamiltonian the Goldstone mode transforms into solitons, the energy of which is proportional to the square root of the anisotropy and the characteristic localization length is inversely proportional to this square root. Application to various physical cases is discussed: dilute magnet near the percolation threshold, three dimensional array of weakly coupled chains and structural transitions.
The existence of anisotropy terms which break from the start the rotation symmetry of the \( n = 2 \) Hamiltonian prevents the existence of a zero energy collective mode at \( 0 \) K. The order of magnitude of the symmetry breaking term in the Hamiltonian should be intimately related to the order of magnitude of the cross-over temperature.

We have treated the problem as a coupled field problem; starting from the Landau-Ginzburg functional for the partition function of two fields \( M_1 \) and \( M_2 \) we introduce a cubic anisotropy term \( \lambda M_1^2 M_2^2 \), so that for \( \lambda = 0 \) one has a scalar field problem, and for a particular value of \( \lambda \) a rotationally invariant complex field problem. Quantum effects are not discussed in this work.

This study has a bearing on various types of physical systems: linear conductors where the order parameter is a complex field [4, 5], critical properties of dilute ferromagnets near the percolation threshold [6-9] domain wall motion and thermodynamics in magnetic materials and ferroelectrics [10]. As for the dilute ferromagnet near the percolation threshold, one dimensional spin correlation functions govern its thermodynamic properties because of the peculiar topology of the infinite cluster near the percolation threshold [6, 9]. As regards structural phase transitions, the system we study represents a very simplified model for Rochelle salts, KDP, or systems such as \( \text{NaH}_3(\text{SeO}_3)_2\text{NH}_4\text{HSO}_4 \), which have two dipoles or more per paraelectric unit cell [11]. Domain wall motion for strong anisotropy is described by kink states [10, 12], and by solitons [12] for weak anisotropy. Solitons appear as solutions of the Sine-Gordon equation which has been studied some decades ago in connection with Bloch wall theory in magnetism [13], among others — our work connects with the latter and yields the thermodynamics of the problem as emphasized in reference [10]. Current interest in solitons stems from field theory [12, 20].

In one space dimension, it is useful to transform the partition function

\[
Z = \int DM_1 DM_2 \exp - \mathcal{K}(M_1, M_2)
\]

by using the transfer matrix method [16, 10, 4].

In the thermodynamic limit the free energy per unit length is equal to the ground state energy \( E_0 \) of a transfer Hamiltonian

\[
\mathcal{K} = -\frac{1}{2m} \nabla^2 M + \frac{r_0}{2} (M_1^2 + M_2^2) + \frac{u}{2} (M_1^4 + M_2^4) + v (M_1^4 + M_2^4)
\]

where

\[
\nabla^2 = \frac{\partial^2}{\partial M_1^2} + \frac{\partial^2}{\partial M_2^2}
\]

and \( m = 1/T^2 \). Equation (2) is the Hamiltonian for two uncoupled anharmonic oscillators when \( u = 0 \) for a 2-D cylindrically symmetric anharmonic oscillator when \( v = 0 \).

The order parameter correlation function can be

2. Phase transition in 1-D with cubic anisotropy. —

We consider 1-D systems defined by Hamiltonians of the type

\[
\mathcal{K} = \int \left[ \frac{1}{2} r_0 (M_1^2 + M_2^2) + \frac{1}{2} \left\{ (\nabla M_1)^2 + (\nabla M_2)^2 \right\} + u(M_1^2 + M_2^2) + v(M_1^4 + M_2^4) \right] dx
\]

when \( u = 0 \), this Hamiltonian describes two uncoupled fields with \( n = 1 \). When \( v = 0 \), it describes a complex field \( n = 2 \). When \( u \) is small, we have two weakly coupled scalar fields. When \( v \) is small, we have a small cubic anisotropy in a complex \( n = 2 \) field problem.

We are interested in situations when \( r_0 < 0 \) and \( u + v > 0 \).

Similar coupled field problems in 1-D were treated in reference [14]. In the latter reference, cubic anisotropy and quadratic anisotropy [11] are simultaneously present, and no special emphasis is put on the cross-over from \( n = 1 \) to \( n = 2 \) behaviour. In reference [4] as well as in many other works on quasi 1-D systems [5], a weak 3-D coupling is introduced among an infinite crystalline array of 1-D chains. The renormalization group has been used to study cubic anisotropy in 4-\( \varepsilon \) dimensions [15]. In one dimension, the renormalization group has been applied to a discrete spin model in the limit of infinite cubic anisotropy [29].
expressed in terms of the excited states $|q\rangle$ of $\mathcal{J}$ and the corresponding eigenvalues $E_q$:

$$\langle M_i(x) M_j(0) \rangle = \sum_q \langle 0 | M_i | q \rangle \langle q | M_j | 0 \rangle \exp(-\beta(x) [E_q - E_0]).$$  \hspace{1cm} (3)

When $v$ is small it is practical to transform $\mathcal{J}$ to polar coordinates:

$$M_1 = \rho \cos \varphi, \quad M_2 = \rho \sin \varphi$$

so that

$$\mathcal{J} = -\frac{1}{2m} \frac{\partial^2}{\partial M^2} + \frac{r_0}{2} \rho^2 +$$

$$(u + v) \rho^4 - \frac{v}{2} \rho^4 \sin^2 2\varphi$$

with

$$\frac{\partial^2}{\partial M^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2}.$$  \hspace{1cm} (3)

We transform again the eigenvalue equation into

$$\left[ -\left(\frac{2}{\alpha}\right)^2 \left[ \frac{\partial^2}{\partial z^2} + \frac{1}{z} \frac{\partial}{\partial z} + \frac{1}{z^2} \frac{\partial^2}{\partial \varphi^2} \right] - 2z^2 + z^4 + Gz^4 \sin^2 2\varphi \right] \psi_q = E_q \psi_q$$  \hspace{1cm} \(\text{with} \quad z = \frac{\rho}{\rho_0} \quad \text{and} \quad \rho_0^2 = \frac{1}{4} \frac{|r_0|}{u + v} \)

$$V_0 = \frac{1}{16} \frac{r_0^2}{u + v}$$

is minimum of the oscillator potential, at $\rho = \rho_0$.

$$\alpha = \frac{\sqrt{2} |r_0|^{3/2}}{4(u + v) \sqrt{T}} = \frac{T}{T}$$

and

$$G = \frac{v}{2(u + v)}.$$  \hspace{1cm}

For $v = 0$ and $\alpha \gg 1$ (low temperatures) the function $\psi_q \equiv R_m(z) \exp il\varphi$ ($l = 0, \pm 1$), because of the cylindrical symmetry of the potential [4]. $R_m(z)$, obeys the radial equation

$$\left[ -\left(\frac{2}{\alpha}\right)^2 \left[ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} + \frac{l^2}{z^2} \right] - 2z^2 + z^4 \right] \times$$

$$\times R_m(z) = \frac{E_m}{V_0} R_m(z).$$

The functions $R_m$ are sharply peaked about the minimum of the potential at $z = 1$. One finds the spectrum of a rigid rotator

$$E_{01} - E_{00} = V_0 \left(\frac{2}{\alpha}\right)^2 l^2 \sim V_0 \frac{T^2}{T^2} l^2$$

with

$$E_{00} = V_0 \left(1 + \frac{4}{\alpha}\right).$$

The behaviour of the chain at low temperature is determined by the lowest $l$ series $E_{0l}$. The correlation length is given by

$$\xi(T) = \frac{T}{E_{0l} - E_{00}} = \frac{T}{V_0 T}.$$

When $G \neq 0$, $G \ll 1$ and $\alpha \gg \frac{2}{v}$ (weak anisotropy at low temperatures), one may solve for an approximate wave function

$$\psi(z, \varphi) = Z(z) \psi(\varphi)$$

$Z(z)$ obeys the equation

$$\left[ 1 - \frac{2}{\alpha} \left[ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} + \frac{E_\varphi}{z^2} V_0 \right] - 2z^2 + z^4 \right] \times$$

$$\times Z(z) = \frac{E_{n\varphi}}{V_0} Z(z)$$

$\psi(\varphi)$ satisfy

$$\left[ 1 = \frac{2}{\alpha} \left[ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} + \frac{G z^4 \sin^2 2\varphi}{z^2} \right] \right] \psi = \frac{E_\varphi}{V_0} \psi$$  \hspace{1cm} \(\text{with} \quad z = 1 \quad \text{and} \quad |G| = \frac{v}{2(u + v)} \ll \frac{v}{2u} \ll 1 .$$

The angular equation has qualitatively different solutions at high temperature

$$\frac{2}{\alpha} \approx 2 \frac{T}{T} \gg \left(\frac{v}{u}\right)^{1/2}$$

and at low temperatures, i.e. : $\frac{2}{\alpha} \ll \left(\frac{v}{u}\right)^{1/2}$. In the former case the anisotropy term can be treated as a perturbation and the solution is $\psi(\varphi) \sim \exp[-i\varphi] + 0(\varphi)$. In the latter case, the $G \sin^2 2\varphi$ term produces angular potential wells, the lowest energy level corresponding to a tunnelling regime between the 4 potential wells at

$$2\varphi = n\pi \quad \text{when} \quad v < 0 \quad (G > 0)$$

or

$$2\varphi = (2n + 1) \frac{\pi}{2} \quad \text{when} \quad v > 0 \quad (G < 0).$$

Thus, in the low temperature regime we can find approximate solutions for the wave equation (4) which are linear combinations of harmonic oscillators in each potential well

$$\psi = \sum_n c_n \Phi_n.$$
The lowest eigenstate is
\[ \psi_0 = \frac{1}{2} (\Phi_0 + \Phi_1 + \Phi_2 + \Phi_3). \]
The first excited eigenstate is doubly degenerate
\[ \psi_{1,1'} = \frac{1}{\sqrt{2}} (\Phi_0 - \Phi_2); \quad \frac{1}{\sqrt{2}} (\Phi_1 - \Phi_3). \]
The next higher energy level corresponds to
\[ \psi_2 = \frac{1}{2} (\Phi_0 - \Phi_1 + \Phi_2 - \Phi_3). \]
The ground state energy is then found to be
\[ E_{\psi_0} = V_0 \left( 4 \sqrt{\frac{|G|}{\alpha}} - \frac{5}{2} G \exp - \frac{\alpha \sqrt{|G|}}{16} \right); \]
the exponential factor comes from the tunnelling of the wave function across the potential energy barriers.

The free energy of the system in the thermodynamic limit is then
\[ F = E_0 + E_{\psi_0} = V_0 \left[ -1 + \frac{4}{\alpha} (1 + \sqrt{G}) - \frac{5}{2} G \exp - \frac{\alpha \sqrt{|G|} \pi^2}{16} \right]. \] (5)

\[ \langle M_1(x) M_2(0) \rangle = \sum_n \langle \Phi_n | M_1 | \Phi_0 \rangle \langle \Phi_0 | M_2 | \Phi_n \rangle \exp - (\beta x/(E_n - E_0)) \]
\[ = \langle 1 | M_1 | 0 \rangle \langle 0 | 0 \rangle \langle 0 | M_2 | 0 \rangle \langle 0 | 1 \rangle = 0, \]
and to first order in \( u/v \), this yields
\[ \zeta(T) \approx \zeta(T) \left( 1 - \frac{3}{2} \frac{u \alpha}{v} \right) \] (7)
valid for \( \frac{3}{2} \frac{u \alpha}{v} \ll 1. \)

As expected the correlation length decreases at constant \( T \) with increasing positive (repulsive) coupling and increases with negative (attractive) coupling.

Comments — cubic fixed point in 1-D systems. — The picture which emerges from the study above is the following: for small anisotropy, the system behaves as a 1-D system with \( n = 2 \) above \( T_{c.o.} \sim \sqrt{|G|} \); the correlation function obeys a power law \( \zeta(T) \sim 1/T \). For \( T < T_{c.o.} \), the correlation function obeys an exponential law
\[ \zeta(T) \sim \exp \left[ \frac{2 \sqrt{|G|} \pi^2}{16} \right] \exp \left[ \frac{T}{\sqrt{|G|} \pi^2} \right]. \]
Although this behaviour is qualitatively similar to the behaviour of the correlation function for \( n = 1 \) which is governed by an Ising fixed point, it differs in an important aspect: instead of reflecting the radial tunnelling of the quantum particle of the transfer matrix treatment between the two wells of a \( n = 1 \) anharmonic oscillator, it arises because of the angular tunnelling through smaller potential wells of depth \( \sim G \), whence the smaller increase of \( \xi(T) \) with temperature.

Calling \( \xi(T) \) the Ising correlation length, we have \( \xi(T) = (\xi(T))^{\frac{1}{2}} \) with \( \sqrt{G} \ll 1 \). Furthermore the \( \text{Ln} \) of the correlation length scales with the cross-over temperature

\[
\text{Ln} \xi \sim + \frac{T}{T \sqrt{G}} = \frac{T_{c.o.}}{T}.
\]

For the Ising chain, the tunnelling behaviour represents the domain walls between up spin domains and down spin domains. The low \( T \) behaviour in the weak anisotropy limit reflects the existence of walls between domains of correlated \( M_1 \) and \( M_2 \) fields.

This is commented on further in section 3.3.

The transfer matrix formulation of the 1-D magnetic phase transition in the presence of a small anisotropy field allows a non-perturbative treatment of the latter: however small the initial values of the cubic anisotropy, at sufficiently low temperature (i.e., sufficiently near the critical temperature \( T = 0 \) K), the system behaves in a qualitatively different manner from the system with zero anisotropy. This is manifest through the construction of the lowest states of the system for \( T \ll T_{c.o.} \) through WKB semi-classical non-perturbative methods [12]. (Gaussian wave functions centred around the minima of the potential wells). This non-perturbative behaviour is reminiscent of the non-perturbative behaviour exhibited by the Renormalisation Group transformation [11] so that we can say that for \( T < T_{c.o.} \), the system we are studying is governed by the 1-D cubic fixed point. Near space dimension \( d = 4 \), there exists a critical value \( n_c(d) \) for \( n \) such that the cubic fixed point is the most stable one for \( n > n_c(d) \) [2]. The discussion above shows that for \( d = 1 \) and in the small anisotropy limit \( n_c(d = 1) < 2 \). We shall see below that this non-perturbative behaviour is also related in one-space dimension to the existence of non-perturbative solutions of the classical non-linear wave equations derived from equation (1), i.e., solitons.

In critical phenomena vocabulary, our results mean that in 1-D, the anisotropy coefficient is a relevant operator, but this statement loses in part its meaning for \( T < T_{c.o.} \), where the various thermodynamic quantities obey exponential laws, not power laws.

For small interchain coupling (or large anisotropy \( \mu/v \ll 1 \)) the concept of cross-over temperature has practically lost its meaning, the potential barriers between the various minima being of comparable height; the cross-over temperature is practically equal to the mean-field transition temperature of each Ising field.

To lowest order in \( \mu \), the cross correlation function \( \langle M_1(x) M_2(0) \rangle \) is zero, so that there is no correlation between the magnetic configuration in each chain. The coupling appears only through a perturbative modulation of the logarithm of the correlation function, each chain is only sensitive to the average square magnetization of the other.

3. Applications. — 3.1 COUPLED CHAINS IN THE PRESENCE OF CUBIC ANISOTROPY. — The type of problem studied above has its most direct application in one-dimensional magnets, the fields \( M_1 \) and \( M_2 \) representing then straight-forward magnetization field components. Experimental data on such systems can be found in reference [22]. An interesting linear magnet which seems to exhibit a cross-over from \( n = 3 \) to \( n = 2 \) behaviour is CsNiF\(_3\) [22], but we do not know of detailed experimental studies of a cross-over from \( n = 2 \) to \( n = 1 \).

Another one-dimensional physical system well described by Hamiltonian (1) is a one-dimensional conductor. The most prominent representatives of 1-D conductors are KCP(K\(_2\)Pt(CN)\(_x\)Br\(_{0.33}\)H\(_2\)O) TTF-TCNQ (tetrahydrofulvalinium tetracyano-quinodimethan) and SN\(_x\) [23, 24]. Those systems exhibit a weak interchain coupling and behave as 1-D systems above an ordering temperature \( T_{c} \) characteristic of 3-D interchain coupling [23]. The order parameter is a complex number describing the most general deformation of the chain of equidistant atoms [25], with its amplitude and its phase. When the wavelength of the deformation is twice the lattice spacing, however, the order parameter is real (\( n = 1 \)).

Although the expansion coefficients in the free energy (Eq. (1)) are discontinuous functions of the deformation wave vector [26], one may consider the anisotropy term studied in section 2 as some sort of phase-amplitude coupling near the commensurate-noncommensurate transition.

As discussed in details in references [4] and [5], the behaviour of the 1-D correlation length has a bearing on the critical temperature of the three-dimensional array of weakly coupled chains; for 1-D scalar fields, the 3-D ordering temperature \( T_{c} \) is proportional to the inverse logarithm of the interchain coupling, while it is proportional to its square root for \( n = 2 \) one-dimensional fields. For \( n = 1 \), the exact result is

\[
(E_1 - E_0) \big|_{T = T_c} = 1.57 \left| J \right|_{T = T_c}
\]

where

\[
J_T = \sum_i - \lambda_i M_i M_j
\]

where \( \lambda_j M_i M_j \) is the interchain coupling [4]. For \( n = 2 \) there is no exact treatment. The molecular field approximation [4] gives

\[
(E_{o,1} - E_{o,0}) \big|_{T = T_c} = \left| J \right|
\]
where \(E_{o,1}\) and \(E_{o,o}\) are the two lowest levels of the rigid rotator.

There exists no exact result in the presence of an anisotropy term. However, it is clear that the qualitative results are correctly given by the general expression

\[
(E_1 - E_0) \big|_{T = T_c} = \delta \big| J \big|
\]

where \(E_1\) and \(E_0\) are the lowest energy levels of the perturbed anharmonic oscillator and \(\delta\) is a coefficient of the order of 1 (\(\delta = 1.57\) for \(v = 0\)).

As a consequence, in the presence of a weak cubic anisotropy, the dependence of \(T_c\) on \(\lambda\) is qualitatively different if \(T_c\) is larger or smaller than the 1-D crossover temperature. When \(T_c > T_{c,o}\), the two lowest levels are those of a rigid rotator, both equally perturbed to 1st order in \(v\), so that

\[
(E_{o,1} - E_{o,o}) \big|_{T = T_c} = \big| J \big|
\]

and

\[
T_c = \frac{\frac{1}{2} r_0}{u} \sqrt{\frac{\lambda}{2}}
\]

when

\[
T_c < T_{c,o}.
\]

The equation for \(T_c\) involves the tunnelling energy across the \(v \sin^2 2\varphi\) potential barriers:

\[
(E_1 - E_0) \big|_{T = T_c} \approx \big| J \big|
\]

so that

\[
T_c \sim \sqrt{G T} \left( \ln \left( \frac{\frac{1}{2} r_0}{u} \frac{1}{\sqrt{2} \lambda} \right) \right)^{-1}.
\]

For sufficiently large interchain coupling the 3-D ordering temperature is insensitive to the anisotropy field.

In the strong anisotropy limit, the dependence of \(T_c\) on \(\lambda\) differs only through numerical constants from the dependence found in the Ising case, since the 1-D correlation length is not qualitatively altered by a weak coupling between chains.

The equation for \(T_c\) is

\[
(E_1 - E_0) \big|_{T = T_c} = 1.57 \big| J \big|
\]

so that

\[
T_c \sim \left( \ln \left( \frac{1}{\frac{1}{2} r_0} \frac{1}{1.57 \lambda} \right) \right)^{-1} \left( 1 - \frac{3}{2} \frac{u}{v} \right)
\]

\(T_c\) is enhanced for \(u < 0\) (attractive intrachain coupling) and is decreased otherwise.

To the best of the author’s knowledge, no experimental data are available now which would allow a qualitative comparison with these results. However, the possibility of intrachain anisotropy in the evaluation of 3-D ordering temperature for quasi 1-D systems, should be considered.

3.2 Phase transitions in diluted systems near the percolation threshold. — Consider a magnetic system randomly diluted with non magnetic impurities, such as \(\text{Rb}_2\text{Mn}_{1-x}\text{Mg}_x\text{F}_4\). Let \(p\) be the probability that a given site is occupied with a magnetic ion, and \(1-p\) be the probability that it is occupied with a non magnetic ion. Recent theoretical and experimental work [6-9] on this kind of system has emphasized that near the percolation threshold \((p > p_c)\) [17], the lattice can be viewed as a collection of nodes connected by links that can be thought of as random zig-zag paths. This picture is originally due to de Gennes [18]. The zig-zag path length \(l\) between the nodes is proportional to the number of steps in the segment connecting two nodes. Lubensky [6] introduces an exponent \(\zeta\) to describe the divergence of \(l\) near \(p_c\) :

\[
l \sim | p - p_c | ^{-\zeta}
\]

and argues that

\[
\nu_p < \zeta < \nu_v / \nu_s
\]

where \(\nu_v\) and \(\nu_s\) are the correlation length exponent for the percolation problem and for the self avoiding walk problem, respectively.

Stauffer [8] and Stanley et al. [9] have proposed that the point \(p_c = p, T = 0\) be viewed as a multicritical point.

If the magnetic correlation length \(\xi_1(T)\) along a zig-zag path is small compared to \(l(p - p_c)\), the system behaves magnetically as a collection of non interacting self-avoiding one-dimensional chains. If on the contrary \(\xi_1(T) > l(p - p_c)\), 3-D correlations set in (2-D in the case of \(\text{Rb}_2\text{Mn}_{1-x}\text{Mg}_x\text{F}_4\), which is a 2-D system).

The transition temperature satisfies

\[
\xi_1(T_c(p)) = l \sim (p - p_c)^{-\zeta}.
\]

For a classical \(n\)-component spin-system (without anisotropy), one finds [6, 8]

\[
T_c(p) \sim \left\{ \begin{array}{ll} 
\frac{T}{\xi} \ln (p - p_c) & n = 1 \\
\frac{T}{(p - p_c)^{n/2}} & n = 2 
\end{array} \right.
\]

In the presence of spin anisotropy, the dependence of \(T_c(p)\) on \((p - p_c)\) will exhibit a cross-over from power law behaviour to logarithmic behaviour for weak anisotropy, i.e. using equation (5')

\[
T_c(p) \sim \left\{ \begin{array}{ll} 
\frac{T}{\xi} \ln (p - p_c) & T_c(p) > T_{c,o} \\
\sqrt{u} \frac{T}{\xi} \ln (p - p_c) & T_c(p) < T_{c,o} 
\end{array} \right.
\]

In the strong anisotropy limit (weakly coupled Ising chains), \(T_c(p)\) exhibits a logarithmic dependence on \((p - p_c)\), with first order corrections in the interchain coupling ; using equation (7) one has :

\[
T_c = \frac{T}{\xi} \left( 1 - \frac{3}{2} \frac{u}{v} \right).
\]
Recent experiments on Mn$_x$Zn$_{1-x}$Fe have stressed the role of spin anisotropy in the magnetic phase transition near the percolation threshold [7]. The cross-over from power law to exponential behaviour occurs around 6 K. The anisotropy field is not a cubic anisotropy in this case so that detailed comparison between theory and experiment is not in order.

### 3.3 Displacive Phase Transitions and Bloch Walls

Consider now $M_1(x)$ and $M_2(x)$ as ionic displacements with respect to some heavy ion or reference lattice. Calling $M_1(x)$ and $M_2(x)$ the velocities, we introduce a generalization of the model studied by Krumhansl and Schrieffer [10] for displacive phase transitions; in our case there are two coupled dipoles per unit cell:

$$\mathcal{E} = \int \frac{dx}{T} \left[ \frac{1}{2} \left( \frac{\dot{M}_1^2(x)}{m} + \frac{\dot{M}_2^2(x)}{m} \right) + A \left( M_1^2(x) + M_2^2(x) \right) 
+ u \left( M_1^2(x) + M_2^2(x) \right)^2 + v \left( M_1^4(x) + M_2^4(x) \right) 
+ \frac{mc_0^2}{2} \left( \frac{dM_1}{dx} + \frac{dM_2}{dx} \right)^2 \right]$$

where $l$ is the lattice spacing. $A$ and $v$ are potential coefficients, $c_0$ is the velocity of low amplitude sound waves if $A$, $u$ and $v$ were zero. We are interested in situations where $A$ is negative and $u + v$ is positive.

If $v$ is zero, as previously, (8) reduces to the Hamiltonian for one complex field ($n = 2$).

For $u$ and $v$ non zero, (8) can be a crude model Hamiltonian for displacive phase transitions in Rochelle salts, KDP, etc., which are characterized by more than one ionic displacement field (pseudo spin) per unit cell [11].

The classical partition function studied in section 2 follows from (8) as a functional integral in the field variables $M_i(x)$ and $P_i(x) = m\dot{M}_i(x)$.

The interest for Hamiltonians such as (8) has been revived recently in connection with the central peak observed in the neutron scattering spectrum of systems exhibiting displacive phase transitions. Various theoretical explanations exist for this central peak: non linear solitary waves [10], soft mode coupling to slowly relaxing degrees of freedom, impurities, or entropy fluctuations [19]. It is thus of interest to study models different from but comparable to those which have been put forward until now.

The equation of motion for the displacement field $M_1(x)$ with $u = 0$ is

$$\frac{d^2\eta}{ds^2} + \eta - \eta^3 = 0$$

with

$$M_{1,2} = f(x - v_0 t)$$
$$\eta = f(x - v_0 t)(|A|/v)^{1/2}$$
$$s = (x - v_0 t)/L_0$$
$$L_0^2 = m(c_0^2 - v_0^2)/|A| = \text{length squared}.$$
Equations (10) and (10') have solutions different from, but similar to the kink solution of equation (9).

$$\varphi(x, t) = \tan^{-1} \left( \exp \frac{x - v_0 t}{L_0} \right). \quad (11)$$

Like the kink solution (11) is a solitary wave but, unlike it is a soliton (the scattering with other solitons can at most lead to a time delay).

To the best of the authors' knowledge the literature on Bloch wall theory does not mention the fact that equation (10') also exhibits another class of solutions, e.g. the doublet solution [12], which has a known equivalent in the scalar theory only for weak coupling [20] ($|u_0|/|A| \ll 1$). It reads

$$f'_{v_0} = 4 \tan^{-1} \left[ \frac{\sin v_0 t/L_0(1 + \delta_0^2/c_0^2)^{1/2}}{v_0 \cosh x/L_0(1 + \delta_0^2/c_0^2)^{1/2}} \right] \quad \text{where } L_0^0 = L_0^2 \text{ for } v_0 = 0.$$

This doublet, which represents a bound soliton-anti-soliton pair, vibrates with period

$$\tau = 2u \frac{L_0^0}{v_0} (1 + \delta_0^2/c_0^2)^{1/2}.$$

The vibration can be thought of as the modulation of the relative separation between the domain walls corresponding to the soliton and anti-soliton.

As of now there seems to exist no experimental observation of this doublet solution (also called a breathing solution). Perhaps this is because dissipative effects, such as friction [36], which are not taken into account in this simple model, reduce the life time of this object in a way which impedes observation. However, it may be worthwhile to devote some experimental effort to this study [21].

As is clear from section 1, the existence of soliton, anti-soliton, and doublet is restricted to low temperature $T \ll \sqrt{v/u}$. The energy of the soliton is found from standard treatments [13]

$$E_{\text{sol}} = 4 \sqrt{2} \frac{L_0^0 v_0}{1} \frac{v}{u} \frac{v}{u} \frac{\sqrt{m c_0^2}}{u} \frac{|A|^{3/2}}{u}$$

$$= \frac{1}{2} k_B T \left( \frac{v}{u} \right)^{1/2}, \text{ where } k_B \text{ is Boltzmann’s constant}$$

$$= \frac{1}{2} \sqrt{\frac{v}{u}} k_B T_{\text{e.o.}}.$$

Our previous result on the correlation length in the weak anisotropy limit thus fits nicely with the Krumhansl Schrieffer treatment on the Ising chain [10]; the exponential growth of the correlation length below $T_{\text{e.o.}} \sim \sqrt{\frac{v}{u} T}$ is related to the exponentially vanishing Boltzmann occupation probability of the soliton levels. From the transfer matrix treatment (Sect. 2) we find that the soliton matrix contribution to the free energy is

$$F_{\text{sol}} \sim V_0 \exp \left( - \frac{\pi^2 \sqrt{2} E_{\text{sol}}}{16} \right) \frac{k_B T}{|A|^2} \quad \text{(see Eq. (5)).}$$

This in turn is related to the breaking of the continuous $n = 2$ symmetry by the anisotropy term: for $v \to 0$, the soliton energy ($\sim \sqrt{v}$) goes continuously to zero, so that the $n = 2$ system has a zero energy mode down to zero temperature (Goldstone mode). In this picture the soliton finite energy can be viewed as the finite energy gained by the Goldstone mode as symmetry breaking terms are continuously introduced in the Hamiltonian. More precisely, since in the case of cubic anisotropy a domain wall is a 90° wall, two successive solitons of like chirality go over to a zero wave vector spin wave when the anisotropy goes to zero [34]. These localized defects go over to a delocalized collective mode as the inverse square root of the anisotropy.

In comparison with the kink states discussed in reference [10], the solitons (or domain walls) which appear in the weak anisotropy case are angular solitons, which do not involve a variation of the magnitude of the displacement field, but only of the phase [13]. The tunnelling of the phase is much easier, in the $v/u \ll 1$ limit, than the tunnelling of the amplitude because the angular potential barrier is much smaller ($\sim \frac{1}{4} \frac{v}{u} |A|^2$) than the radial one ($\sim \frac{1}{4} \frac{|A|^2}{u}$). As a well-known consequence, the wall thickness $L_0$ is larger by a factor $(v/2 u)^{1/2}$ in the phase case than in the amplitude one.

It is clear from all this discussion that the domain walls in ferromagnets and ferroelectrics are to be discussed on the same footing, once the appropriate pseudo-spin Hamiltonian has been set up for the ferroelectric material. Domain walls much larger than the lattice spacing are to be expected in ferroelectrics, since the ratio of the potential coefficients $v/u$ is small (strong intra-cell dipole-dipole coupling). Parts of this discussion can be found in reference [37], which deals with the thermodynamics of the sine-Gordon field, and in reference [38], which mentions the low temperature result equation (5) in connection with current carrying excitations in 1-D conductors.

Reference [13b] discusses the theory and application of the sine-Gordon Equation to such problems as magnetic-flux propagation in a large Josephson function, Bloch-wall motion, propagation of a crystal dislocation, propagation of ultra-short optical pulses, and field theory. All these problems are thus related.
to the classical phase transition problem in 1-D, with 

\[ n = 2 \] and a weak anisotropy term.

As mentioned in the introduction, the backward scattering model for the electron gas [27] leads to equations quite similar to equation (10) above. The essential difference with the cubic anisotropy problem in 1-D is that the argument of the sine in the sine-Gordon equation is \( \gamma \) with

\[ \gamma = 8 \pi \beta \cos \theta \]

where \( \psi \) and \( \psi \) depend on the electron gas parameters. This (generally) non integer value reflects the complexity of the coupling term between the two fields in the backward scattering model, which reduces to a cubic anisotropy term only for special values of \( \gamma \).

Our work can be viewed as a classical solution for the backward scattering model for a particular value \( \gamma = 4 \). However, a classical solution for this value is bound to be inaccurate, since quantum fluctuations renormalize the coupling constant from \( \gamma \) to \( \gamma^2(1 - \gamma^2/8 \pi) \). An exact classical solution for the correlation function [28] can be obtained in the limit \( \gamma \to 0 \). The limit, however, lacks the simplicity which allows the WKB solution described above.

As this paper was completed, we became aware that the thermodynamic aspect of our problem (Sect. 2) has been treated in part in reference [31]. This paper deals with the Peierls transition and the commensurability problem. The Landau-Ginzburg functional derived for a 1-D conductor contains a phase term proportional to \( \cos 4 \phi \) (Eqs (4) and (10)) when Umklapp processes are taken into account [32]. This term, which is thus analogous to a cubic anisotropy term, ensures the pinning of the phase (commensurate CDW) below the cross-over temperature, the expression of which in reference [31] is in agreement with our result. It is stated in reference [31] that an exponential behaviour of the correlation length below \( T_{c,0} \) is characteristic of an Ising model. As we have discussed in section 3, we believe it is significant to keep in mind the difference between a weak anisotropy behaviour (domain walls much larger than the interatomic distance) and an Ising behaviour (domain walls of the order of the interatomic distance). This difference corresponds essentially to that between an angular tunnelling and a radial one in the transfer matrix picture.

Spin correlations for a classical linear magnet \( n = 3 \) with single-site anisotropy have been studied with the transfer matrix technique and numerical methods [35]. For positive single site anisotropy, the system exhibits a cross-over from \( n = 3 \) to \( n = 2 \), so that the correlation length which diverges obeys a power law in both the high and low temperature regime. A cross-over from \( n = 3 \) to \( n = 1 \) occurs for negative single site anisotropy, but has only been briefly mentioned in reference [35]. Our work has the advantage of having analytical expressions in the case \( n = 2 \) to \( n = 1 \) and a clear interpretation in terms of the low lying excitations of the system.

4. Conclusion. — We have studied in detail various aspects and applications of a relatively simple and pedagogical example of a one dimensional system. The choice of the symmetry breaking term allows continuous passage from a continuous symmetry \( (n = 2) \) to a simple scalar theory with discrete symmetry \( (n = 1) \). The cross-over behaviour is shown to correspond to the difference in the nature of low energy excitations in systems which possess a Hamiltonian with continuous symmetry and those which do not. The delocalized zero energy collective mode (spin wave) of the \( n = 2 \) model evolves into a localized collective mode with finite energy (soliton) when a symmetry breaking term is introduced in the Hamiltonian. The characteristic length of localization is proportional to the inverse square root of the anisotropy while the energy is proportional to the square root of the anisotropy. As a result, the correlation length is governed by those lowest energy defects. It is activated, with an exponential behaviour, below the temperature which corresponds to the soliton energy; it obeys a power law, characteristic of a continuous symmetry, for temperatures larger than the soliton energy. It is clear that this simple behaviour is related to the 1-D topology, for which any cut separates the system in two disconnected pieces. The situation for the same type of Hamiltonian containing symmetry breaking terms has been recently investigated in 2-D [33]. As expected, the interrelation of the correlation length and of the defects (spin waves and vortices) in this system is more complex, the more so because it has a finite critical temperature.

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