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NON-LINEARITIES AND FLUCTUATIONS AT THE THRESHOLD OF A HYDRODYNAMIC INSTABILITY IN NEMATIC LIQUID CRYSTALS

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Abstract. — In nematics, the simple shear flow may become unstable when the director is perpendicular to the shear plane. Here we study the non-linear regime above the threshold of the Homogeneous Instability [1]. We use a perturbative approach derived from exact equations within the framework of the Frank-Ericksen-Leslie continuous theory of nematics: the distortion grows as the square root of the distance to the threshold, in quantitative agreement with experiments on MBBA. Thermal fluctuation effects are also analyzed. They turn out to be important only in a vanishingly narrow vicinity of the threshold.

Introduction. — Nematic liquid crystals are ordered fluids made of elongated molecules spontaneously aligned but without any macroscopic ordering in positions. In a non-uniform flow, the local orientation is coupled with the velocity gradients and peculiarities of this coupling lead to flow instabilities which are quite unique and have no counterpart in isotropic liquids [1, 2]. Recently much attention has been paid to the simplest case which is achieved in a shear flow when the orientation is perpendicular to the shear plane [1-7]. Linearized theories developed so far [3-6] have given a satisfactory understanding of the instability characteristics at threshold. However, many experimental results concern the non-linear regime above the threshold [2, 8]. The main purpose of this paper is to give a first quantitative account of this non-linear regime. This will be the content of section 3 but, we shall first give a general preview of the paper (§ 2) recalling instability mechanisms and discussing briefly the special features of the non-linearities involved.

An outline presentation of the Landau approach to hydrodynamic bifurcations [9] and the analogy with phase transitions [12] which derives from it, will also be given in section 2 and will serve as an introduction to the qualitative analysis of fluctuation effects developed in section 4. The essential results of the linearized theory are summarized in two appendices. The first one is devoted to the standard normal mode analysis [13], the second one to a novel variational determination of the instability threshold.

1. General survey. — Let us begin with a short description of the anisotropic flow properties of nematics [14] and point out the instability mechanism which stems from them. The mean orientation being labeled by a unit vector \( \mathbf{n} \) called the director, one can distinguish three flow geometries according to the position of \( \mathbf{n} \) relative to the shear plane [15] (Fig. 1).

i) In the two first geometries, the director \( \mathbf{n} \) lies in the plane of the shear which exerts a viscous torque on the molecules. This torque tends to make the orientation rotate and must be balanced by elastic torques and/or by field induced torques when external fields are applied. This balance adds two novel equations to the usual set of hydrodynamic equations.

ii) On the contrary when \( \mathbf{n} \) is perpendicular to the shear plane (geometry (3)) no viscous torque is exerted on the molecules for symmetry reasons. Then the...
FIG. 1. — The three Miezowicz geometries. In geometries (1) and (2) the flow exerts a non-vanishing viscous torque on the molecules; on the contrary, in geometry (3) the viscous torque is zero. $\alpha_2$ is negative for rod shaped molecules but $\alpha_3$ may be positive or negative; figure 1a corresponds to $\alpha_3 < 0$. If the orientation is kept fixed in each of these positions, one can measure three different effective viscosities $\eta_1, \eta_2$ and $\eta_3$:

$$\eta_1 = (\alpha_4 + \alpha_3 + \alpha_2)/2; \eta_2 = (\alpha_4 + \alpha_3 - \alpha_2)/2; \eta_3 = \alpha_2/2.$$ 

where the $\alpha_i$'s are Leslie's viscosity coefficients.

nematic looks rather like an isotropic liquid. However the direction perpendicular to the plane of the shear is special and any orientation fluctuation away from this direction reveals the anisotropic properties specific of nematics. In particular, such a fluctuation involves viscous torques and instabilities may emerge from a constructive coupling between these torque components [1, 3] as sketched in figure 2. Let the unperturbed orientation direction define the x-direction and let the flow be parallel to the y-axis with $v_y = v_y(z) = s \cdot z$ where $s$ is the shearing rate. Then the effect of an orientation fluctuation $n_z$ is described in figure 2a: a viscous torque $\Gamma_z = -\alpha_2 \cdot s \cdot n_z$ takes place. For $n_z$ positive, this torque is positive since $\alpha_2$ is negative in general. Then, it tends to create a fluctuation $n_y$ also positive. Now, this fluctuation induces a viscous torque $\Gamma_y = -\alpha_3 \cdot s \cdot n_y$, which may be positive or negative according to the sign of $\alpha_3$. When $\alpha_3$ is negative, this torque tends to reinforce the distortion $n_z$ and the coupling is destabilizing (Fig. 3c). The instability takes place when the mechanism is strong enough to overcome the stabilizing effect of elasticity and external fields. This occurs when the shearing rate $s$ is greater than a certain threshold values $s_\alpha$. The basic mechanism just described does not depend explicitly on the spatial variation of the orientation fluctuation in the plane of the flow. Thus it can lead to different instability modes according to the wave-length of the distortion which corresponds to the lowest critical value. The simplest case is a distortion which is uniform in the plane of the flow (i.e. of infinite wavelength). This homogeneous instability [1] follows directly from the mechanism described in figure 2 and thus can take place only when $\alpha_3$ is negative. A second instability mode is possible which corresponds to a distortion periodic in the direction of the unperturbed orientation [2]. It is associated with a secondary flow in form of rolls aligned parallel to the flow direction; a detailed understanding of the instability mechanism must treat the coupling between orientation and flow in a more complete way as above. However, taking into account back-flow effects leads to a mere renormalization of viscosity coefficients $\alpha_2$ and $\alpha_3$ [3] leaving the basic mechanism unchanged (which explains that the roll instability can exist even when $\alpha_3$ is positive if the corresponding renormalized value remains negative). The theoretical description of the roll instability is much less simple than that of the homogeneous instability which involves only a dependence on the $z$-coordinate. So in this first paper on non-linear effects we shall restrict ourselves to this case, leaving the case of rolls for subsequent investigations.

Now let us point out the origin of non-linearities in nematics and more especially in the present problem. First, let us stress the fact that the Frank [16] Ericksen [17] Leslie [18] description of nematics rests on linear constitutive equations (i.e. linear relations between conjugated stresses and strains). Indeed, on the one hand, the Frank orientational elastic energy [16] is a quadratic combination of curvature strains described by $\mathcal{V}_\kappa$, which corresponds to a linear relation between curvature stresses and strains. On the other hand, the Leslie viscous stress tensor [18] is a linear function of the strain rate (symmetric part
of $\mathbf{Vv}$) and of the rotation rate of the director relative to the fluid measured by $N = \frac{dn}{dt} - 1/2 (\text{curl} \ v) \times n$. Since we are interested in wavelengths large on the microscopic scale and rates very small when compared to microscopic equivalents it does not seem relevant to introduce non-linearities in the constitutive equations [19]. The intrinsic linearity of the description appears quite clearly when the orientation is nearly uniform in space. Then the remaining non-linearities are

i) the usual convective term $\mathbf{v}. \mathbf{V}(\ldots)$ already present in isotropic liquids and

ii) another term, specific of nematics, namely the Ericksen stress-tensor [17] $\sigma^E$ with components

$$
\sigma^E_{ij} = \sum_k \left[ \frac{\partial F}{\partial (\mathbf{n}_i, \mathbf{n}_k)} \right] \partial_j n_k
$$

obtained by differentiating the Frank elastic energy relative to curvature strains. Now it must be noticed that for a uniform distortion depending only on the $z$-coordinate, which is the case here, these two non-linearities do not contribute. Indeed, the derivative $\partial / \partial z$ only remains and from the continuity equation one gets $v_z \equiv 0$ so that $\mathbf{v}. \mathbf{V}(\ldots) = v_z \partial / \partial z (\ldots) \equiv 0$. Moreover the only non-vanishing component of $\sigma^E$ is $\sigma^E_{zz}$ which only enters the determination of the hydrostatic pressure once all other variables are known. In the present problem, non-linearities have then a different origin: they arise from the fact that at a finite distance above the threshold the orientation cannot be considered as nearly uniform; this situation is quite general in nematics and gives them their strongly non-linear character. Indeed, the constitutive equations are linear but the exact value of the coefficients entering the relation between stresses and strains depends explicitly on the orientation direction (see for example the form of the Leslie viscous stress tensor [18, 14]). As long as deviations from the uniform orientation were infinitesimal one could choose a laboratory frame linked to the unperturbed orientation; when distortions are finite, one has to change from a local frame linked to $n$ to a laboratory frame independent of $n$. In terms of the director components, this adds strong non-linearities. Now since the problem is in fact to determine the director field $n(r, t)$ one has to cope with a strongly non-linear problem.

The aim of this paper is to study the effect of non-linearities discussed above close to the threshold of the homogeneous instability which, from an experimental point of view, is known to be linear and stationary, that is to say: slightly above the threshold the distortion grows continuously from zero and without time oscillations. Taking for granted the linear character of the bifurcation, one can determine the threshold value within the framework of a linearized theory [3, 5] and show that the instability must be stationary [3]. The result of the normal mode analysis [13] is recalled in appendix A. In appendix B we develop a variational approach which in addition to an approximate threshold determination also derives the stationary character of the instability. Now the fact that the bifurcation is linear remains to be proved. This is the first goal of the non-linear analysis. Energy methods [20] give global stability criteria i.e. stability criteria not restricted to infinitesimal disturbances but, on the contrary, independent of the amplitude of the disturbances (or of the energy they contain). The instability is proved to be linear only when a global stability criterion coincides with the criterion obtained from the linearized theory. To our knowledge, no such result has been derived for the geometry we are interested in (1). Then we are left with the less ambitious Landau-Hopf approach of bifurcations [9, 20-22] which is restricted to finite but small disturbances. Let us adapt the Landau theory to the case of a stationary instability and write a phenomenological motion equation for the amplitude $A$ of the distorted state:

$$
\frac{dA}{dt} = \sigma A - \gamma A^3 \quad (1.1)
$$

$\sigma$ is the evolution rate as calculated by the linearized theory and the cubic term originates from non-linearities. No term in $A^2$ is present if the distortion is invariant through the symmetry operation corresponding to the change $A \rightarrow -A$. Let $p$ be the parameter which controls the instability and $p_e$ the threshold value. $\sigma$ is negative below the threshold ($p < p_e$, damping of infinitesimal disturbances) and positive above it. Close to the linear threshold, Landau simply assumes [9]:

$$
\sigma = \alpha (p - p_e)
$$

with $\alpha > 0$. Now the linear character of the instability is associated with the sign of $\gamma$. Indeed if $\gamma$ is positive, the stationary solution of equation (1.1) reads:

$$
A = \sqrt{\sigma / \gamma} \propto \sqrt{p - p_e} \quad (1.2)
$$

going continuously to zero when $p$ tends towards $p_e$ from above. Moreover the bifurcated state is stable (Fig. 3a). On the contrary when $\gamma$ is negative, the unperturbed flow is unstable for finite amplitude disturbances below the threshold and the non-trivial stationary solution of (1.1) is in fact unstable. Higher order non-linear terms have to be taken into account to explain the possible saturation of the distorted state amplitude (Fig. 3b). The problem is then to put the Landau theory on quantitative grounds in order to

(1) The stability criterion derived by A. Chauve (C. R. Acad. Sci. Paris 285A (1977) 73) does not seem to be truly relevant for nematics since $n$ is assumed to be clamped once for all in the flow which ignores a large part of the dynamics of nematics and in particular the mechanisms responsible for the Pieranski-Guyon instabilities [1, 2].
determine $\gamma$. Such a program has already been achieved for example in the case of isotropic liquids for the Rayleigh-Bénard instability of a fluid layer heated from below [22]. To our knowledge, only the case of electrohydrodynamic instabilities [23] has been examined in nematics (2). Moreover the study has been performed either using an approximate model [24, 25] or on the basis of exact equations suitably simplified [26, 27]. In both cases only qualitative agreement could be obtained. Here on the contrary we look for quantitative predictions and we must turn to a perturbative approach based on exact equations. Of course the price paid will be a validity restricted to a narrow vicinity of the threshold but we shall be able to show that at least for the well known nematic compound MBBA, the instability is linear. Moreover, coefficient $\gamma$ of the Landau-Hopf equation (1.1) will turn out to be in good agreement with the value deduced from experiments [2].

Now, result (1.2) is strongly reminiscent of the mean field theory of second order phase transitions. This analogy has received much attention recently both from theoretical and experimental points of view [12, 28]. It is in fact implied by the strong parallel between the Landau-Hopf bifurcation theory sketched above and the Landau-Khalatnikov theory of time-dependent critical phenomena close to a phase transition [29]. Indeed equation (1.1) has the same form as the motion equation for the order parameter:

$$\frac{d\eta}{dt} = -A \frac{\partial F}{\partial \eta}$$

(1.3)

where $A$ is a kinetic coefficient and $F$ the free energy which in the Landau theory close to a second order phase transition takes the form [30]:

$$F = F_0(T_0) + \alpha(T - T_c) \frac{\eta^2}{2} + \gamma \frac{\eta^4}{4}$$

(1.4)

with $\alpha$ and $\gamma$ positive. The Landau theory of phase transitions is known to lead to the mean-field behaviour which is incorrect due to an incomplete account of fluctuation effects close to the transition point. This suggested examination of the effect of fluctuations close to the instability threshold [11, 12]. The main step is the derivation of a generalized thermodynamic potential [31]; the procedure is roughly the following:

i) Following Landau and Lifshitz [11] hydrodynamic equations are transformed into stochastic equations containing a fluctuating force, the intensity of which is given by the Fluctuation-Dissipation theorem.

ii) Linearized hydrodynamic equations are transformed into linear Langevin equations for the amplitude of normal-modes, which allows for a classical treatment of fluctuations far enough from the threshold [32].

iii) Close to the threshold, the non-linear coupling between fluctuations cannot be ignored, so non-linear terms are introduced in the Langevin equations. Restricting oneself to the amplitude of the most unstable mode, one simply gets:

$$\frac{dA}{dt} = \alpha(p - p_c)A - \gamma A^3 + f$$

(1.5)

where $f$ is the noise term.

iv) Then one derives a Fokker-Planck equation [33] for the probability density $\Psi$ of having a particular amplitude $A$ and one looks for the stationary solution which is supposed to describe time-independent phenomena close to the instability threshold.

v) This stationary solution may be written as:

$$\Psi(A) = C \exp \left\{ -\Psi(A)/kT \right\}$$

(1.6)

where $\Psi$ has exactly the same structure as the free energy (1.4). Identification of $\Psi$ with a generalized thermodynamic potential is complete if one realizes that in order to incorporate fluctuation effects, the modern theory of phase transitions [34] replaces $F(\eta)$ by the average over the canonical ensemble:

$$\Psi(\eta) = C \exp \left\{ -F(\eta)/kT \right\}$$

(1.7)

In what concerns the value of the order-parameter, the Landau (or mean-field) theory is equivalent to determining the most probable value instead of working with the probability density (1.7). As to the fluctuations, it neglects non-linearities, which comes to a Gaussian approximation. This leads to the so-called classical exponents. As far as equilibrium phenomena are concerned, a non-classical (or critical) behaviour with modified exponents occurs inside a region roughly given by a Ginzburg Criterion [35, 34] where the non-linear coupling between fluctuations cannot be neglected. Following the analogy, one can derive a kind of Ginzburg Criterion giving the extent of the critical region where the Landau-Hopf theory fails to predict the actual steady behaviour

$$(A \propto (p - p_c)^\beta \text{ with } \beta \neq 0.5).$$

In fact fluctuation effects are so weak that the critical domain is vanishingly small: its relative width is given by the ratio of the thermal energy $kT$ to a typical distortion energy in a volume of the order of $h^3$ where $h$ is the thickness of the cell. This ratio is always very small ($10^{-6}$ at best) so that the classical Landau theory will be valid, even close enough to the threshold. A similar situation holds for the Rayleigh-Bénard instability in isotropic liquids [22, 36] and this seems to be a quite general feature of hydrodynamics instabilities.

---

(2) Let us also mention the Freedericks transition, an orientational instability for which the effect of non-linearities and fluctuations have been thoroughly studied close to the transition point [50].
2. Non-linear analysis. — 2.1 Non-linear hydrodynamic equations. — Consider the plane shear flow experiment described in figure 4. Two parallel plates are moving, the upper one at a velocity \( V \) relative to the lower one. Let \( h \) be the spacing between the plates, take the origin of coordinates \( O \) at midway in between, axis \( Oy \) in the flow direction and axis \( Oz \) normal to the plates. The director \( n \) used to label the nematic orientation will be defined in spherical coordinates

\[
\begin{align*}
n_x &= \cos \theta \cos \phi \\
n_y &= \cos \theta \sin \phi \\
n_z &= \sin \theta
\end{align*}
\]

Fig. 4. — When the distortion is no longer infinitesimal \( n \) must be defined in spherical coordinates. In addition to the usual set of hydrodynamic equations, nematodynamics adds two evolution equations for the director: equations (2.3a, b) corresponds to the cancellation of the total torque exerted on \( n \) in the referential \((x', y', z')\) linked to \( n \).

We shall be concerned with a strong anchoring situation in planar geometry with \( n \) initially perpendicular to the shear plane \((yOz)\). This means that \( n \) is kept rigidly fixed parallel to \( Ox \) at the plates:

\[
\theta \left( z = \pm \frac{d}{2} \right) = \varphi \left( z = \pm \frac{d}{2} \right) = 0.
\]

In the unperturbed configuration, this alignment prevails everywhere in the bulk of the sample. As to the velocity boundary conditions, we assume the usual non-slip condition at the plates:

\[
\begin{align*}
v_x \left( z = \pm \frac{d}{2} \right) &= v_z \left( z = \pm \frac{d}{2} \right) = 0 \quad (2.1b) \\
v_y \left( z = \pm \frac{d}{2} \right) &= \pm \frac{V}{2}. \quad (2.1c)
\end{align*}
\]

In absence of distortion, the nematic behaves like an isotropic liquid and we get the usual linear velocity profile:

\[
v_y(z) = sz
\]

where \( s = V/d \) is the shearing rate. The homogeneous instability [1] is uniform in the \( xOy \) plane so that the velocity and director components depend only on the \( z \)-coordinate. Then the incompressibility condition reads:

\[
\frac{\partial v_z}{\partial z} = 0
\]

which according to (2.1b) leads to \( v_z(z) \equiv 0 \). In the following we shall denote \( u \) and \( v \) the two remaining velocity components \( v_x \) and \( v_y \), \( w \) anyone of the quantities \( \theta, \varphi, u \) or \( v \) and

\[
\frac{dw}{dt} = \frac{\partial w}{\partial t} = \dot{w} \quad \frac{\partial w}{\partial z} = w'.
\]

Within the framework of the Ericksen-Leslie hydrodynamic theory [14], Leslie has derived the whole set of non-linear equations governing the most general \( z \)-only dependent flow. He gets [5]:

2.1.1 Torque equations.

\[
\begin{align*}
\{ - \gamma_1 \dot{\varphi} \cos \theta + G_\theta(\theta) \varphi'' - 2 H(\theta) \theta' \varphi' + x_2 \sin \theta (u' \sin \varphi - v' \cos \varphi) \} &= \Gamma_x' = 0 \quad (2.3a) \\
- \{ - \gamma_1 \dot{\varphi} + G_\varphi(\varphi) \theta'' + \cos \theta [H(\theta) \varphi'^2 + (K_1 - K_2) \sin \theta \varphi'^2] \\
- L_1(\theta) (u' \cos \varphi + v' \sin \varphi) \} &= \Gamma' = 0 \quad (2.3b)
\end{align*}
\]

with

\[
\begin{align*}
G_\theta(\theta) &= \cos \theta (K_2 \cos^2 \theta + K_3 \sin^2 \theta) \\
G_\varphi(\varphi) &= K_1 \cos^2 \theta + K_3 \sin^2 \theta \\
H(\theta) &= [(2 K_2 - K_3) \cos^2 \theta + K_3 \sin^2 \theta] \sin \theta \\
L_1(\theta) &= x_3 \cos^2 \theta - x_2 \sin^2 \theta
\end{align*}
\]

These equations express the dynamical equilibrium of the director. It may be checked that they vanish identically when \( \varphi = \theta = 0 \) everywhere (undistorted configuration) and in presence of a distortion they correspond to the cancellation of the total torque (elastic + viscous) exerted on the molecules in a local referential linked to \( n(x', y', z') \) with \( x' \) along \( n \) and \( y' \) parallel to the \((x, y)\) plane.

2.1.2 Force equations (equations for the linear momentum).

\[
\begin{align*}
\rho \ddot{u} &= \frac{\partial}{\partial z} \{ [M(\theta) + N(\theta) \cos^2 \varphi] u' + N(\theta) \sin \varphi \cos \varphi' + K(\theta) \cos \varphi \dot{\varphi} + L_2(\theta) \sin \varphi \dot{\theta} \} \quad (2.3c) \\
\rho \ddot{v} &= \frac{\partial}{\partial z} \{ N(\theta) \sin \varphi \cos \varphi' + [M(\theta) + N(\theta) \sin^2 \varphi] v' + K(\theta) \sin \varphi \dot{\varphi} + L_2(\theta) \cos \varphi \dot{\theta} \} \quad (2.3d)
\end{align*}
\]
with
\[ L_2(\theta) = \alpha_2 \sin \varphi \cos \varphi \]
\[ M(\theta) = \frac{\alpha_4}{2} + \frac{\alpha_5 - \alpha_2}{2} \sin^2 \theta \]
\[ N(\theta) = \left[ \frac{\alpha_3 + \alpha_6}{2} + \alpha_1 \sin^2 \theta \right] \cos^2 \theta. \]

In all these formulas, \( K_1, K_2, K_3 \) are the Frank elastic constants [16], \( \alpha_1 \) ... \( \alpha_6 \) are the Leslie viscosity coefficients [18] linked by an Onsager relation [37]: \( \alpha_6 - \alpha_3 = \alpha_2 + \alpha_3 \). Finally \( \gamma_1 = \alpha_3 - \alpha_2 \) is the orientational viscosity.

2.2 Perturbation expansion. — As long as \( \mathbf{n} \) is parallel to \( \mathbf{x} \), \( v \) is given by (2.2) but when \( \mathbf{n} \) deviates, the shear flow exerts a viscous torque on the director and the destabilizing mechanism sketched in figure 2 develops when \( \alpha_3 \) is negative. The instability takes place when the stabilizing effect of elasticity is no longer able to maintain the orientation imposed by the anchoring conditions at the plates. The instability threshold is given by the linearized theory [3, 5] which corresponds to a first order expansion of equation (2.3). Due to the complicated form of these equations it does not seem possible to get an analytical solution above the threshold [27, 28] and we must look for a solution slightly above the threshold under the form of an expansion in terms of a small parameter [38, 22]

\[ w(z) = w_0(z) + \varepsilon w_1(z) + \varepsilon^2 w_2(z) + \ldots \] (2.4a)

where \( w_0 \) is the unperturbed solution:

\[ v_0 = \pm \frac{d}{2} \]
\[ \theta_0 = \varphi_0 = u_0 = 0. \]

Obviously the \( w_n \) have to fulfil the boundary conditions:

\[ w_n \left( \pm \frac{d}{2} \right) = 0 \quad n = 1, 2, \ldots \] (2.4b)

Usually in an experiment, the shearing rate \( s \) is kept fixed and the distortion amplitude (related to \( \varepsilon \)) is measured, this corresponds to a relation:

\[ \varepsilon = f(s). \] (2.5)

Here we will consider (2.5) as an implicit equation giving \( s \) as a function of \( \varepsilon \):

\[ s = g(\varepsilon). \]

Now, this function must be expanded in powers of \( \varepsilon \) in the vicinity of the threshold \( s_0 \) so that we set:

\[ s = s_0 + \varepsilon s_1 + \varepsilon^2 s_2 + \ldots. \] (2.6)

In (2.4a) and (2.6) \( w_n \) and \( s_n \) are to be determined. In order to solve the non-linear problem we substitute series (2.4a) and (2.6) in equations (2.3) and separate the different orders in \( \varepsilon \). At this stage it is useful to rescale all lengths in units of \( h \) (\( z \rightarrow zh, u \rightarrow uh, v \rightarrow vh \)). To first order one gets:

\[ \varphi''_n + \tau s_c \theta_1 = 0 \] (2.7a)
\[ \theta''_n + \tau (s_c \varphi_1 + u'_1) = 0 \] (2.7b)
\[ u'_1 + e_1 s_c \varphi'_1 = 0 \] (2.7c)
\[ v'_1 = 0 \] (2.7d)

with \( \tau = |\alpha_2| h^2/K_2 \), \( \tau' = |\alpha_2| h^2/K_1 \),

\[ e_1 = \frac{\alpha_3 + \alpha_6}{\alpha_3 + \alpha_4 + \alpha_6} = \frac{\eta_1 - \eta_3}{\eta_1} < 0. \]

From (2.7d) and the boundary condition \( v_1(\pm \frac{1}{2}) = 0 \) one gets \( v_1(z) \equiv 0 \) everywhere. Then the second order system reduces to:

\[ \varphi''_n + \tau s_c \theta_1 = - \tau s_1 \theta_1 \] (2.8a)
\[ \theta''_n + \tau (s_c \varphi_1 + u'_2) = - \tau s_1 \varphi_1 \] (2.8b)
\[ u'_2 + e_1 s_c \varphi'_2 = - e_1 s_1 \varphi'_1 \] (2.8c)
\[ v'_2 = d_0 + d_1 s_c \theta_1^2 + d_2 s_c (s_c \varphi_1 + u'_1) \] (2.8d)

with

\[ d_1 = - \frac{\alpha_3 - \alpha_2}{\alpha_4} = \frac{\eta_3 - \eta_2}{\eta_3} < 0 \]

and

\[ d_2 = - \frac{\alpha_3 + \alpha_6}{\alpha_4} = \frac{\eta_3 - \eta_1}{\eta_3} > 0. \]

\( d_0 \) is obtained from the boundary condition \( v_2(\pm \frac{1}{2}) = 0 \):

\[ \frac{d_0}{2} + \int_0^{1/2} \mathrm{d}z [d_1 s_c \theta_1^2 + d_2 s_c (s_c \varphi_1 + u'_1)] = 0 \] (2.8e)

Third order equations are much less simple:

\[ \varphi''_n + \tau s_c \theta_1 = \left[ \frac{3}{2} - \frac{K_1}{K_2} \right] \theta_1^2 \varphi_1 + 2 \left( 2 - \frac{K_3}{K_2} \right) \theta_1 \varphi_1' + \varphi_1'' + \tau \left\{ \theta_1 ((\varphi_1^2 + \theta_1^2/3) s_c/2 + \varphi_1 u'_1) - [\theta_2 s_1 + \theta_1 (s_2 + v'_2)] \right\} \] (2.9a)
\[ \theta_3' + \tau'(s_c \varphi_3 + u_3') = \left( 1 - \frac{K_3}{\bar{K}} \right) \left( \theta_1^2 \right) + \frac{K_3 - 2 K_2}{\bar{K}} \theta_1 \varphi_1^2 + \]
\[ + \tau' \left\{ \varphi_1^2(u_1' + \varphi_1 s_c/2)/3 - [\varphi_2 s_1 + \varphi_1 (s_2 + v_2')] + \tau' \theta_1^2(u_1' + \varphi_1 s_c) \right\} \]
\[ u_3' + e_1 s_c \varphi_3' = e_1 \left\{ \varphi_1^2(2 \varphi_1(u_1' + \varphi_1 s_c) + u_1' \varphi_1) - \varphi_2 s_1 + \varphi_1 (s_2 + v_2') + \varphi_1 v_2' \right\} + \]
\[ + \theta_1 e_2(2 \theta_1' u_1' + \theta_1 u_1') + e_3 s_c(\theta_1 \varphi_1' + 2 \theta_1' \varphi_1) \] (2.9b)

\[ u_1 = 2(a_2 + a_3 - a_1)/(a_3 + a_6 + a_4), \]
\[ e_1 = (a_3 + a_6 - 2 a_1)/(a_3 + a_6 + a_4). \]

Since we shall stop the expansion at 3rd order, the equation giving \( v_3 \) is useless and we shall not reproduce it here. The problem is to solve systems (2.7), (2.8) and (2.9) one after the other. They may be written as:

\[ \mathcal{L}(s_c) \mathcal{U}_1 = 0 \] (2.10a)
\[ \mathcal{L}(s_c) \mathcal{U}_n = \mathcal{F}_n \text{ for } n > 1 \] (2.10b)

where \( \mathcal{L}(s_c) \) is a linear differential operator and \( \mathcal{U}_n \) a vector with components \( \theta_1, \varphi_1, \) and \( u_1 \). The first order problem is homogeneous. It will have no trivial solutions only when \( s_c \) takes special values, the smallest corresponding to the threshold. Since \( s_c \) is kept equal to this value, \( \mathcal{L}(s_c) \) has no longer an inverse and the inhomogeneous higher order problem will have no solutions unless certain existence conditions are fulfilled [39]. These existence conditions will serve to determine the value of the unknown parameters \( s_n \) in the expansion (2.6).

### 2.3 Algebraic Properties of the Linearized System

The first order problem is linear and homogeneous. Its solution [3, 5] obtained by a normal mode analysis [13] is given in appendix A. Here let us rather turn to its algebraic properties [39]. Under the form (2.7) the differential system is not formally self-adjoint. However performing the change

\[ u = -\frac{e_1}{\tau'} s_c \tilde{u} = \gamma \tilde{u} \]

and using matricial notation we may rewrite (2.7) as:

\[
\mathcal{L}(s_c) \mathcal{U}_1 = \begin{bmatrix}
\tau s_c & \frac{d^2}{dz^2} & 0 \\
\frac{d^2}{dz^2} & \tau' s_c & \gamma' \frac{d}{dz} \\
0 & -\gamma' \frac{d}{dz} & \gamma \frac{d^2}{dz^2}
\end{bmatrix} \begin{bmatrix}
\theta_1 \\
\varphi_1 \\
\tilde{u}_1
\end{bmatrix} = 0
\]

where \( \mathcal{L}(s_c) \) is now formally self-adjoint. In the space of vectors \( \mathcal{U} \) which fulfill the boundary conditions:

\[ \mathcal{U}(\pm \frac{1}{2}) = 0, \text{ i.e. } \theta(\pm \frac{1}{2}) = \varphi(\pm \frac{1}{2}) = \tilde{u}(\pm \frac{1}{2}) = 0, \]

we define the scalar product:

\[ \langle \mathcal{U}_1, \mathcal{U}_1 \rangle = \int_{-1/2}^{1/2} dz (\theta' \varphi + \phi' \varphi_1 + \tilde{u} \tilde{u}_1) \]

where the star denotes complex conjugation. Then by a sequence of integration by parts, making use of boundary conditions (2.11) one checks that the differential problem is self-adjoint in the usual algebraic sense:

\[ \langle \mathcal{U}_1, \mathcal{F}_n \rangle = \langle \mathcal{F}_n, \mathcal{U}_1 \rangle = \langle \mathcal{U}_1, \mathcal{U}_1 \rangle^* \] (2.13)

This property which allows the derivation of a variational solution for the threshold problem (see appendix B) will be used here to obtain the existence conditions required for the inhomogeneous higher order problem to have a solution. Indeed, \( \mathcal{U}_1 \) being the solution of the linearized problem consider the scalar product \( \langle \mathcal{U}_1, \mathcal{F}_n \rangle \) where \( \mathcal{F}_n \) is the r.h.s. of the nth-order system (2.10b). Using (2.13), one gets:

\[ \langle \mathcal{U}_1, \mathcal{F}_n \rangle = \langle \mathcal{U}_1, \mathcal{L}(s_c) \mathcal{U}_n \rangle = \langle \mathcal{U}_n, \mathcal{L}(s_c) \mathcal{U}_1 \rangle^* = 0 \] (2.14a)

i.e. \( \mathcal{F}_n \) must be orthogonal to the first order solution \( \mathcal{U}_1 \). This necessary condition is also sufficient [39] and leads to the determination of \( s_{n-1} \) in the expansion for \( s - s_c \) once \( (s_{n-2}, \mathcal{U}_{n-1}) \) are known, the condition being fulfilled, one then looks for the solution \( \mathcal{U}_n \). This solution is not unique and to determine it completely one must add a supplementary condition for example

\[ \langle \mathcal{U}_1, \mathcal{U}_n \rangle = 0. \] (2.14b)

### 2.4 Explicit Solution

Let us first consider the second order problem. The orthogonality condition (2.14a) reads:

\[ \langle \mathcal{U}_1, \mathcal{F}_2 \rangle = 0 = -s_1 \times \int_{-1/2}^{1/2} dz (\theta_1^2 + \tau' \varphi_1^2 + e_1 \tilde{u}_1 \varphi_1) = -s_1 I \]

which leads to \( s_1 = 0 \) if \( I \neq 0 \). Using (2.7) and a sequence of integrations by parts one can simplify \( I \) as:

\[ I = 2 \tau \int_{-1/2}^{1/2} dz \theta_1^2 \]

(2.15)
which is different from zero since the first order problem has a non-trivial solution \( \mathcal{U}_1 \neq 0 \) for the threshold value \( s_c \). Then \( s_1 \) is equal to zero. This fact could be inferred from a symmetry property of equations (2.3). Indeed, if \( s_1 \) is different from zero, restricting oneself to the lowest significant order one gets

\[
s = s_c = \varepsilon s_1 , \quad \mathcal{U} = \varepsilon \mathcal{U}_1 \quad \text{and} \quad v = s x
\]

and two solutions with opposite amplitude \( \varepsilon \) and \( -\varepsilon \) are not strictly equivalent in contradiction with the invariance property of equations (2.3) under the change

\[
(\theta, \varphi, u, v) \rightarrow (-\theta, -\varphi, -u, v)
\]

that can be checked easily.

Since \( s_1 = 0 \), the second order problem is also homogeneous and \( \mathcal{U}_2 \) is simply proportional to \( \mathcal{U}_1 \). In view of condition (2.14b), one gets

\[
\mathcal{U} = (\theta_2, \varphi_2, \tilde{u}_2) \equiv 0
\]

\((\varepsilon_2 \) remains given by equations (2.8d, e) as a function of \( \mathcal{U}_1 = (\theta_1, \varphi_1, \tilde{u}_1) \). Now the existence condition for the third order problem \( \langle \mathcal{U}_1 | \mathcal{F}_3 \rangle = 0 \) may be written as:

\[
s_2 = J / I
\]

where \( I \) is the positive integral (2.15) and \( J \) another integral, the expression of which is very complicated and cannot be reduced significantly. For \( \varepsilon \) small enough the solution of the non-linear problem then reads:

\[
s = s_c = s_2 \varepsilon^2 + O(\varepsilon^3) \quad (2.16)
\]

The physical relevance of this solution is closely related to the sign of \( s_2 \). Indeed, if \( s_2 \) is negative, solution (2.16) is defined for \( s < s_c \) but, as recalled in section 2, this solution is in fact unstable for \( \varepsilon \) small, the trivial solution \( \mathcal{U} = 0 \) being the stable one. Another stable solution can exist for \( \varepsilon \) large but the expansion is truncated at a too low order to account for it. On the contrary if \( s_2 \) is positive, solution (2.16) is defined for \( s > s_c \) and is the stable one. The instability is linear and result (2.16) is relevant as long as \( \varepsilon \) is small enough. Now integral \( J \) depends on the whole set of viscoelastic constants so that in absence of global stability criterion no general statement can be given about the sign of \( s_2 \) and one is left with a numerical application valid for a particular compound. This calculation has been performed for MBBA at 25°C which was used in experiments [2] and for which viscoelastic constants are rather well known [14]. For \( h = 200 \mu m \), which corresponds to experimental conditions, one gets:

\[
s_c = 0.101 \text{ s}^{-1} , \quad s_2 = 6.59 \text{ s}^{-1} \quad (2.16')
\]

Here \( s_2 \) is positive and remains so for reasonable variations of viscoelastic constants about the values given in reference [14]. This is in agreement with the linear character of the instability as it has been observed experimentally. Of course this does not prove that the bifurcation is linear for any other compound. This qualitative point being recognized, let us now consider the problem from a more quantitative point of view. In their experiments, Pieranski and Guyon [2] have measured the rotation angle \( \Phi \) of the conoscopic image (Fig. 5). This angle is related to the amount of twist deformation in the sample roughly given by the data of \( \phi(x) \). But the linearized theory allows to determine the distortion profiles (see appendix A and Ref. [3]). Moreover the variational method of appendix B shows that the approximate profile:

\[
\begin{align*}
\theta(z) &= \varepsilon_0 \cos \pi z \\
\phi(x) &= \varepsilon_0 \cos \pi x \\
\varepsilon_0 &= \frac{\sqrt{\tau \varepsilon}}{h}
\end{align*}
\]

is very close to the actual one (Ref. [3], Fig. 8a) and that \( \varepsilon_0 \) may be identified with \( \varepsilon \) (Eq. (4a)). Now for the present problem, it can be shown [40] that \( \Phi \) and \( \phi \) are related through

\[
tg 2 \Phi = \int_{1/2}^{1/2} dz \sin 2 \phi \int_{-1/2}^{1/2} dz \cos 2 \phi
\]

which leads to

\[
\Phi = \frac{2}{\pi} \varepsilon_0 = \frac{2}{\pi} \sqrt{\frac{\tau}{h} \varepsilon}
\]

![Fig. 5. — Experimental results (by courtesy of P. Pieranski and E. Guyon) \( s = V/h \) is the applied shearing rate. \( \Phi \) is the rotation angle of the conoscopic image linked to the amount of twist in the distortion. Clearly one has \( \Phi \propto \sqrt{s - s_c} \) for \( s = s_c \) small enough. Agreement between these experimental results and the theoretical ones is excellent.](image)
for $\varepsilon$ small enough. For quantitative comparison, result (2.16) is best visualized under the form:

$$\varepsilon_\varphi = \varepsilon_0 \sqrt{\frac{s - s_e}{s_e}}$$  \hspace{1cm} (2.18)

where

$$\varepsilon_0 = \sqrt{\frac{s_e}{s}}$$

is the twist distortion amplitude extrapolated for $s = 2 s_e$.

Contrary to $s_c$ and $s_t$ which depend on the thickness $h$ of the cell, $\varepsilon_0$ turns out to be an intrinsic quantity; moreover $\varepsilon_0$ can be expected to be of the order of $\pi/2$ since there is no other angle scale for twist distortions in the problem (see the note below). From experiments reported in figure 5 one gets:

$$s_e = 0.09 \text{ s}^{-1}, \quad \varepsilon_0 \sim 80^\circ$$

to be compared with the theoretical result (2.16):

$$s_e = 0.101 \text{ s}^{-1}, \quad \varepsilon_0 = 71^\circ.$$

The precision on the experimental value for $\varepsilon_0$ is difficult to estimate but the agreement is quite satisfactory if one takes into account the fact that certain viscoelastic constants are not well known even for MBBA (see Ref. [14a], p.66).

Note: extent of the square root behaviour. — Considering figure 5 one can see that the square-root behaviour is restricted to a relatively narrow neighbourhood of the threshold $s < 1.5 s_e$. This fact may be easily understood when looking at the aspect of the flow for $s$ much larger that $s_e$ as sketched in figure 6.

**Fig. 6.** — Aspect of the distortion well above the threshold. The two boundary layers of thickness $\xi$ are separated by a central layer where the director lies in the plane of the shear at an angle $\theta_L$ with the flow direction.

a) At the centre of the cell ($z \sim 0$) $\varphi$ and $\theta$ increase with $s$; $\varphi$ tends towards $\pi/2$ while $\theta$ tends towards the Leslie value $\theta_L$ [18] given by $\tan \theta_L = \sqrt{\alpha_3/\alpha_2}$. This limiting orientation $\theta = \theta_L, \varphi = \pi/2$ is expected to be stable at high shearing rates. Indeed Pikin [41] has shown that the flow was stable against fluctuations keeping the director in the $yOz$ plane. As to fluctuations out of this plane, one can develop an argument parallel to that of Pieranski, Guyon and Pikin [42]: isolating the viscous torque component in equation (2.3b) and assuming a small fluctuation about $\varphi = \pi/2$ one gets

$$\Gamma' \varepsilon' = - \alpha_2 v' \sin \theta \cos \varphi \sim - \alpha_2 s \theta_1 \left( \frac{\pi}{2} - \varphi \right)$$

which is positive when $\varphi < \pi/2$, negative in the opposite case but always tends to restore the initial orientation $\varphi = \pi/2$, so that the flow stabilizes the orientation.

b) Close to the plates ($z \sim \pm \xi$) we expect to find boundary layers [18] of thickness $\xi \sim (K/\alpha_2 s)^{1/2}$, where most of the distortion will be concentrated, mostly a twist deformation connecting the orientation $\varphi = 0$ at $z = \pm \xi$ with the orientation $\varphi = \pi/2$ in the centre (Splay adjustment is small since $\theta_L \sim 10^\circ$).

Apart from a possible instability of the boundary layer and/or an anchoring crack, we then expect the limiting flow to be stable. Notice that for $s = 2 s_e$ the extrapolated value $\varepsilon_0$ of the amplitude of twist deformation $\varepsilon_\varphi$ is already quite large, in the range $70^\circ$-$80^\circ$ of the order of the limiting value $\pi/2$. This fact explains that for $s \geq 1.5 s_e$ one already needs corrections to the square-root behaviour and that higher order terms cannot be neglected in the expansion.

3. Fluctuation effects. — From the analogy between phase transitions and bifurcations [12], we may expect pretransitional effects driven by thermodynamic fluctuations close to the instability threshold. Here we shall give order of magnitude estimates and only look for the extent of the domain where the classical Landau (or mean-field) theory breaks down, so we shall introduce fluctuations at a very qualitative level.

First as in section 2 let us write a phenomenological motion equation for the distortion amplitude measured by $\varepsilon_\varphi$ simply denoted as $\varepsilon$ in the following:

$$\gamma \frac{d\varepsilon}{dt} = \nu_s \left( \frac{s}{s_e} - 1 \right) \varepsilon - \frac{\varepsilon^3}{\varepsilon_0^2}$$  \hspace{1cm} (3.1)

which admits the stationary solution (2.18). $\gamma$ and $\nu_s$ can be calculated within the framework of the linearized theory. For simplicity let us consider the relaxation of an orientation distortion in absence of shear; it is governed by

$$\gamma_1 \frac{d\varepsilon}{dt} = - K \left( \frac{\pi}{h} \right)^2 \varepsilon$$  \hspace{1cm} (3.2)

where $\gamma_1$ is the orientational viscosity and $K$ a Frank elastic constant. Letting $s = 0$ in (3.1) and comparing to (3.2), we deduce:

$$\gamma \sim \gamma_1 \quad \text{and} \quad \nu_s \sim \frac{K\pi^2}{h^2}.$$. 

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Now let us introduce fluctuations. In the case of the Rayleigh-Bénard instability of isotropic fluids this has been performed for the first time by Zaitsev and Shliomis [32] using the Landau-Lifshitz theory of hydrodynamic fluctuations [11]. Their work has been extended by Graham [31] who has taken non-linearities into account. Here we rather follow the phenomenological approach of Normand et al. [22] and simply complement equation (3.1) by a Langevin fluctuating term which describes the stochastic effect of degrees of freedom other than $\varepsilon$ just as in the theory of Brownian motion.

$$\frac{de}{dt} = \frac{K\varepsilon^2}{h^2} \left[ \left( \frac{s}{s_c} - 1 \right) \varepsilon - \frac{\varepsilon^3}{\varepsilon_0^2} \right] + f. \quad (3.3)$$

As usual the fluctuating term $f$ is assumed to be a stationary $\delta$-correlated random process with zero mean-value

$$\langle f(t) \rangle = 0 \quad \langle f(t) f(t + \tau) \rangle = F \delta(\tau). \quad (3.4)$$

(The bar denoting statistical average.) The intensity coefficient $F$ is given by the fluctuation-dissipation theorem which relates $F$ to the friction coefficient $\gamma$ and the thermal energy $kT$:

$$F = 2 kT \gamma / \Omega \quad (3.5)$$

(where $\Omega$ is the volume over which relevant fluctuations take place; as it will be justified further we must take here $\Omega = Sh$ where $S$ is the surface of the experimental cell).

Notice that (3.5) is implicitly assumed to be valid even far from equilibrium ($s \neq 0$) and close to the instability threshold, and that $\gamma$ is regular at threshold. This is the equivalent of the Van Hove assumption in the theory of critical phenomena [43]. Of course the direct calculation of $\gamma$ using the linearized theory cannot introduce critical singularities since it ignores fluctuation effects. The usual justification for the regular character of $\gamma$ is that it corresponds to friction processes at a microscopic scale which are not affected by the destabilizing mechanism which works on a length scale of the order of $h$ (see also Refs. [12, 36]).

Each term of the Langevin equation being evaluated one turns to the Fokker-Planck equation [33] for the probability $\mathcal{P}$ of having a particular value $\varepsilon$ at time $t$:

$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{\partial}{\partial \varepsilon} \left\{ \frac{K\varepsilon^2}{h^2} \left[ \left( \frac{s}{s_c} - 1 \right) \varepsilon - \frac{\varepsilon^3}{\varepsilon_0^2} \right] \mathcal{P} \right\} + \frac{F}{2 \gamma \varepsilon^2} \frac{\partial^2 \mathcal{P}}{\partial \varepsilon^2}. \quad (3.6)$$

The time independent equilibrium probability density is readily obtained:

$$\mathcal{P}(\varepsilon) \propto \exp \left\{ \frac{2 \gamma K\varepsilon^2}{Fh^2} \left[ \left( \frac{s}{s_c} - 1 \right) \varepsilon^2 - \frac{\varepsilon^4}{4 \varepsilon_0^2} \right] \right\}. \quad (3.7)$$

Recalling the relation (3.5) between $F$, $\gamma$ and $T$, we may write $\mathcal{P}$ under the form:

$$\mathcal{P}(\varepsilon) \propto \exp \left\{ - \left( \frac{\varepsilon}{kT} \right) \right\} \quad (3.8)$$

where

$$\mathcal{P}(\varepsilon) = \frac{K\varepsilon^2}{h^2} \Omega \left[ \left( 1 - \frac{s}{s_c} \right) \varepsilon^2 + \frac{\varepsilon^4}{4 \varepsilon_0^2} \right] \quad (3.9)$$

is the generalized thermodynamic potential [32] for the present problem. $K\varepsilon^2/h^2 \Omega$ is the amount of Frank elastic energy contained in a typical distortion (of the order of 1 rad.). Up to now we have assumed that the distortion was uniform over the whole surface $S$ of the experimental cell. But the analogy with phase transitions suggests that this is the case only close to the threshold and that we must include the effect of spatial inhomogeneities. This leads to a definition for a potential density $\psi$ such that

$$\psi = \int dxdy \psi(x, y)$$

and which takes into account the contribution of inhomogeneities to the potential through a term of the form

$$\xi_x^2 \frac{\partial^2 \psi}{\partial x^2} + \xi_y^2 \frac{\partial^2 \psi}{\partial y^2} \quad (3.10)$$

where $\xi_x$ and $\xi_y$ — the coherence lengths far from the threshold — must be of the order of $h$ for dimensional reasons. The true spatial dependence of the amplitude can be derived from the linearized theory (see Ref. [22] or [36] for the Rayleigh-Bénard case). Here we shall avoid the calculation, the form (3.10) must then be considered as a plausible guess rather than an exact result. For simplicity we assume moreover:

$$\xi_x^2 \sim \xi_y^2 \sim h^2/2.$$  

Then $\psi(x, y)$ reads:

$$\psi(x, y) = \frac{K\varepsilon^2}{h} \left[ \left( 1 - \frac{s}{s_c} \right) \varepsilon^2 + \frac{\varepsilon^4}{4 \varepsilon_0^2} + \frac{h^2}{2} (\nabla \psi)^2 \right]$$

which has exactly the form of the Landau free energy density for a continuous bidimensional (spatial dimension $d = 2$) magnetic system with an Ising-like ($n = 1$) order parameter [30, 34].

(1) The third spatial dimension always disappears from such problems. This is due to the fact that one is interested only in the unstable mode which is coherent over the whole thickness $h$ of the cell. Indeed the mathematical procedure which leads to eq. (3.3) involves the calculation of a scalar product which makes one spatial coordinate disappear through a quadrature and leaves a ($d = 2$)-problem.
Scaling i) \(x, y\) in units of \(h\) : \((x, y) = (\tilde{x}, \tilde{y}).h\); ii) \(\Psi\) in units of \(kT\) : \(\Psi = \Psi_k/kT\); and iii) \(\varepsilon\) in units of \(\varepsilon_r\) the amplitude of thermal fluctuations over a volume \(h^3\) : \(\varepsilon = \tilde{\varepsilon}_r, \varepsilon_r\) being given by the equipartition theorem : \(K(\pi/h)^2 \varepsilon_r^2 = kT/h^3\).

One gets :

\[\Psi = \int d\tilde{x} d\tilde{y} \tilde{\Psi}\]

with

\[\tilde{\Psi} = \left(1 - \frac{s}{s_e}\right)^{\frac{3}{2}} + \frac{1}{4} \left(\frac{\varepsilon_r^2}{\varepsilon_e^2}\right) \varepsilon^4 + \frac{1}{2} (\nabla \varepsilon)^2 \]

\[= \frac{1}{2} r_0 \varepsilon^2 + u_0 \varepsilon^4 + \frac{1}{2} (\nabla \varepsilon)^2.\]

In this dimensionless formulation, the intensity of fluctuations is given by the order of magnitude of the coupling constant

\[u_0 = \frac{1}{4} \frac{\varepsilon_r^2}{\varepsilon_e^2} \]

Here, at ordinary temperature \((T \approx 300 K)\), with the Frank elastic constant \(K\) of the order of \(10^{-6}\) CGS and \(h\) of the order of 100 \(\mu\)m, one gets :

\[\varepsilon_r^2 \sim 10^{-6} \text{ rad}\]

and

\[u_0 \sim 2 \times 10^{-7}.\]

Thus fluctuation effects will modify significantly the classical Landau behaviour only in a close vicinity of the threshold. The extent of the critical domain is given by a Ginzburg criterion [35] that we can derive from \(\Psi\) using a dimensional argument [34b].

The correlation length for Gaussian fluctuations is

\[\xi_G = 1/\sqrt{r_0}\]

and \(\tilde{\Psi}\) allows to define a second length

\[\xi_{nl} = 1/\sqrt{u_0} \gg 1\]

which is characteristic of non-linearities. Far from the threshold we have \(r_0 \gg u_0\) and \(\xi_G \ll \xi_{nl}\); we may neglect the non-linear coupling between fluctuations and the classical theory is satisfactory. It will fail when

\[\xi_G \sim \xi_{nl}\]

or equivalently :

\[r_0 \sim u_0 \Rightarrow \frac{s}{s_e} - \frac{s_e}{s} \sim \varepsilon_r^2/\varepsilon_e^2. \quad (3.11)\]

This condition is rather stringent due to the order of magnitude of \(u_0\). For \(r_0 \lesssim u_0\) we expect the critical behaviour of the two dimensional Ising model. However this is unlikely to be observed for two reasons: first, one has to achieve very precise and stable values of the parameter which controls the instability, and second, the slowing down of fluctuations is considerable. Indeed, in the classical regime, the time constant for an orientation fluctuation remains given by

\[\tau = \tau_0 \left(1 - \frac{s}{s_e}\right) \quad (3.12)\]

with \(\tau_0 \sim \gamma/K(\pi/h)^2 \sim 1\) to 10 s, so that \(\tau\) is very large close to the threshold \((\sim 10^6 s \sim 10 \text{ days} for (s - s_e)/s_e \sim 10^{-6}!)\). Even if critical corrections to (3.12) are required, this gives an order of magnitude. So, in the limit of a true bidimensional problem (cell of infinite lateral extent, see the discussion below) we expect the Landau theory to be valid except in a vanishingly small vicinity of the threshold. The situation is even worse than for superconductivity. In this latter case, the wide validity range of the Landau theory is associated with the large size of the coherence length \(\xi_0\) at 0 K relative to the atomic spacing \(a\). In fact, the preceding discussion may be repeated in terms of the comparison between a coherence length far from threshold \(\xi_0 \sim \xi_x\) or \(\xi_y \sim h\) and a microscopic length \(a\). Indeed we have

\[u_0 \sim \frac{\varepsilon_r^2}{\varepsilon_e^2} \sim \frac{kT}{Kx^2} \quad \text{with} \quad T_0 \sim 1 \text{ 000 K}\]

but the Frank elastic constant may be evaluated as the energy necessary to break the orientational ordering over a molecular length [14a]. This energy is typical of interactions between molecules and thus of the order of 2 kcal./mole or 1 000 K in temperature units so that :

\[K \left(\frac{\pi}{a}\right)^2 \sim \frac{kT_0}{a^3} \quad \text{with} \quad T_0 \sim 1 \text{ 000 K}\]

then, apart from a numerical factor of the order of unity we have

\[u_0 \sim \frac{a}{h}.\]

(Notice that a similar argument would hold for other convective instabilities in ordinary liquids.)

Now close to the threshold, the critical behaviour will correspond to the bidimensional case only if

\[\xi_G \sim \xi_{nl} \ll l/h\]

where \(l\) is the smallest lateral dimension. Indeed if this is not the case, when \(r_0\) is decreased, size effects become important before one enters the critical domain and a dimensionality reduction occurs : as soon as

\[\xi_G \sim l/h \quad \text{or} \quad r_0 \sim (h/l)^2 \gg u \quad (3.13)\]

one may neglect spatial inhomogeneities in the direction of the smallest lateral dimension (here the direction of the unperturbed orientation) and perform the
integration over the corresponding variable. One is then led to a undimensional \((d = 1)\) problem [44]. Further dimensionality reduction occurs when the distortion amplitude becomes uniform over the whole cell surface \((d = 0)\)-problem [45]). We shall not enter the details of this dimensionality reduction since it depends in a crucial way on (3.10) but simply consider the zero dimensional case for simplicity. Then the integration over the space variables is trivial and gives back the factor \(\Omega\) in the expression of the generalized thermodynamic potential (3.9) assumed at the beginning. Now in the \((d = 1)\)-case, and \textit{a fortiori} in the \((d = 0)\)-case, fluctuation effects are known to suppress the sharp transition and to lead to a smearing-out of the transition. The width of this smearing-out is easily obtained from an adaptation of the Ginzburg criterion to the \((d = 0)\)-case. Returning to (3.9), we define the amplitude of thermal fluctuations over the whole volume \(\Omega = S.h\) as:

\[
\mathcal{K}\left(\frac{\pi}{h}\right)^2 (S.h) \varepsilon_{T}^2 = kT
\]

so that

\[
\varepsilon_{T}^2 = \varepsilon_{T}^2(h^2/S) \ll \varepsilon_{T}^2.
\]

Below the threshold the probability distribution for \(\varepsilon\) is Gaussian with a dispersion

\[
\sigma^2 = \varepsilon_{T}^2/(1 - s/s_c)
\]

as long as non-linearities are negligible (term \(\varepsilon^2\)). The harmonic approximation breaks down when the \(\varepsilon^4\)-term in (3.9) is of the same order of magnitude as the \(\varepsilon^2\)-term for a typical fluctuation of size \(\sigma\):

\[
\left(1 - \frac{s}{s_c}\right) \sigma^2 \sim \frac{\sigma^4}{4 \varepsilon_{T}^2}
\]

this leads to:

\[
\left(1 - \frac{s}{s_c}\right) \leq \frac{1}{\sqrt{2}} \varepsilon_{T}/\varepsilon_0
\]

an analogous criterion can be derived above the threshold and the width of the smearing-out is roughly given by:

\[
1 - \frac{s}{s_c} \sim \frac{\varepsilon_{T}}{\varepsilon_0} \quad \text{or} \quad \frac{\varepsilon_{T}}{\varepsilon_0} \frac{h}{\sqrt{S}}
\]

(a more exact numerical approach has been given by Smith [46] referring explicitly to the Rayleigh-Bénard instability in isotropic liquids studied by Graham [31]).

Condition (3.15) is less stringent than (3.11) as long as \(h/\sqrt{S} \gg \varepsilon_{T}/\varepsilon_0\), which is precisely the condition to have dimensionality reduction (see (3.13)). Nevertheless the width of the smearing-out is very small due to the order of magnitude of \(\varepsilon_{T}\) and is unlikely to be observed for the same reasons as those given above. Indeed assuming \(h \sim 100 \mu m\) and \(S \sim 100 h^2\), one gets

\[
1 - \frac{s}{s_c} \sim 10^{-4}.
\]

Close to the threshold, we then expect the Landau theory to be valid. This does not mean that fluctuations have no effects. In fact they drive the instability and they are enhanced close to the threshold but non-linear coupling between fluctuations is negligible and the enhancement is \textit{classical}. In fact such pretransitional effects have already been observed in nematics in the case of electrohydrodynamic instabilities [25]. Thus, from the point of view of fluctuation observations, nematics remains more \textit{privileged} than isotropic fluids [47] since orientation fluctuations are much easier to observe by light scattering than velocity fluctuations [48] as to the steady behaviour, the critical domain does not appear to be drastically widened relative to what it is in isotropic liquids [22, 36]. Finally, one should notice that weak parasitic effects such as a lack of symmetry of the experimental set-up (*) would reintroduce a term \(A^2\) or \(\varepsilon^2\) in the Landau-Hopf equation (1.1) or (3.1) which would lead to a first-order like behaviour masking any critical effects.

4. Conclusion. — In this paper we have given a first quantitative account of non-linear effects above the threshold of a hydrodynamic instability in nematics. Theory has been developed within the framework of the Frank-Ericksen-Leslie continuous description. In order to take into account all the viscoelastic parameters we have performed a complete perturbation expansion [38] up to third order instead of using simplified models [25, 26]. As for the Rayleigh-Bénard instability in isotropic liquids, the \(\textit{simple} \ \text{shear} \ \text{flow}\) homogeneous instability turns out to be linear and the Landau result

\[
\varepsilon = \varepsilon_0 \sqrt{\frac{s}{s_c} - 1}
\]

is shown to hold. We have obtained an excellent agreement between experimental [2] and theoretical results. This leaves little doubt on the overall validity of the Ericksen-Leslie theory at least as long as shearing rate are small on a microscopic scale. Moreover we have shown that non-classical fluctuation effects occur only in a vanishingly small vicinity of the threshold and that the \textit{mean field} theory is expected to be valid.

Extensions of our calculation should concern the

(*) This was the case in the first Pieranski-Guyon experiments [1] where a weak magnetic field was applied in order to break the symmetry and to privilege one of the two equally possible distortions [49]. This led to a distortion amplitude measured by \(\Phi \propto (s - s_c)\) instead of \((s - s_c)^{1/2}\) close to the threshold.
Roll Instabilities. The theoretical analysis of weakly distorted regimes by perturbation methods is interesting in that it can lead to quantitative checks; for that reason it should be continued. However the study of strongly distorted flows should also be considered as a further step towards the approach of turbulence in nematics, but a lot remains to be done, in particular to include the effects of orientation defects. Finally for the study of non-linearities, shear flow instabilities in nematics, due to their intrinsic character and to their relative simplicity, may play a role comparable to that of the Rayleigh-Bénard instability in isotropic fluids.

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Appendix A. — Solution of the linearized stationary problem. — The explicit solving of the homogeneous first order problem has already been given [3, 5]. Here, we shall only recall the main results of the normal mode analysis. Solutions can be classified according to their parity; let us begin with the simplest case of fluctuations $\theta$ and $\varphi$ which are odd functions of $z$. The critical shearing rate is given by

$$k = k^0(\lambda) = 2 \lambda \pi \quad (\lambda \text{ positive integer}) \quad (A.1)$$

where $k$ is defined by

$$k^2 = \frac{s^2 \tau \tau'(1 - e_4)}{1 - e_1}. \quad (A.2)$$

The corresponding solution reads

$$\varphi = \sin (2 \lambda \pi z) \quad (A.3a)$$

$$\theta = \frac{(2 \lambda \pi)^2}{2 \lambda \pi} \sin (2 \lambda \pi z) \quad (A.3b)$$

$$\dot{u} = -\frac{\tau}{2 \lambda \pi} \left( \cos (2 \lambda \pi z) - (-1)^\lambda \right). \quad (A.3c)$$

When $\theta$ and $\varphi$ are even functions of $z$, one gets

$$\theta = \cos (kz) - A \cosh (kz) \quad (A.4a)$$

with $A = \cos (k/2) / \cosh (k/2)$

$$\varphi = \frac{k^2}{1 - e_1} \frac{\tau}{s_\kappa} \left( \cosh (kz) + A \cosh (kz) + B \right) \quad (A.4b)$$

with $B = -2 \cos (k/2)$ and finally

$$\dot{u} = \frac{k^2}{e_1 (1 - e_1) s_\kappa} \left[ e_1 \left( \sin (kz) + A \sinh (kz) + Bz \right) \right]. \quad (A.4c)$$

the boundary condition $\dot{u}(\pm \frac{1}{2}) = 0$ which reads

$$k - e_1 (\tang (k/2) + \tanh (k/2)) = 0 \quad (A.5)$$

leads to the critical value for $k$ when $\theta$ and $\varphi$ are even; a numerical application for MBBA with $e_1 = -0.7479 [14a]$ gives

$$k^0(1) = 3.4942. \quad (A.6)$$

Higher values are $k^0(\lambda) \approx (2 + 1) \pi (\lambda \text{ integer greater than 1}).$ From the comparison between (A.1) (with $\lambda = 1$) and (A.6) one deduces that the threshold corresponds to the lowest even solution (A.4).

Appendix B. — Variational approach of the threshold problem. — Let us now consider a slightly more general problem allowing for time dependence of the fluctuations and including the effects of a magnetic field parallel to $Ox$. Due to the susceptibility anisotropy of the nematic phase, the field exerts a torque

$$\Gamma = \chi_\kappa (n \times H) \quad (\chi_\kappa = \chi_{||} - \chi_{\perp} > 0)$$
on the molecules. Here this torque tends to restore the perfect alignment along $Ox$ when it is perturbed: the magnetic field has a stabilizing effect and accordingly the value of the threshold increases.

Linearized equations including magnetic field effects [3, 5] now read

$$\tau_1 \dot{\varphi} = \varphi'' - m \varphi + \tau s_\kappa \theta \quad (B.1a)$$

$$\kappa \tau_1 \dot{\theta} = \theta'' - \kappa m \theta + \tau (s_\kappa \varphi + u') \quad (B.1b)$$

$$\tau_0 \dot{u} = u'' + e_1 s_\kappa \varphi' + \eta \theta' \quad (B.1c)$$

with

$$\tau_1 = \gamma_1 h^2 / K_1, \quad \kappa = K_2 / K_1, \quad m = \chi_\kappa H^2 h^2 / K_2$$

$$\tau_0 = \rho h^2 / (\chi_\kappa + \chi_\kappa + \chi_\kappa + \chi_\kappa) \quad \text{and} \quad \eta = \chi_\kappa (\chi_\kappa + \chi_\kappa + \chi_\kappa).$$

In the present problem, the inertial term $\tau_0 \dot{u}$ may be neglected since

$$\frac{\tau_0}{\tau_1} = \frac{\rho K_2}{\gamma_1 (\chi_\kappa + \chi_\kappa + \chi_\kappa)} \approx 10^{-6} \ll 1.$$

This corresponds to an adiabatic elimination of the most quickly fluctuating part of the velocity field when compared to the much slower variations of the director field. Now, time dependent solutions may be searched under the form

$$w(z, t) = \exp(\sigma t).w(z).$$

Performing the transformation

$$u \rightarrow \dot{u} = \gamma \dot{u} = -\frac{e_1 s_\kappa}{\tau'} \dot{u}$$

we may rewrite system (B.1) in matricial notation...
The differential operator is not formally self-adjoint due to the elasticity anisotropy \( (\kappa = K_2/K_1 \neq 1) \) and to the presence of the term \( \eta \sigma \frac{d}{dz} \) related to the peculiar structure of the viscous stress tensor. In order to recover a self-adjoint problem we shall simply neglect these two sources of difficulties but this may be justified [3]. Then, using boundary conditions

\[
\mathcal{U}(\pm \frac{1}{2}) = 0
\]

and the scalar product already defined by (2.12), we reobtain the self-adjointness property (2.13).

As an immediate consequence, one can derive the stationary character of the instability [23]. Let us consider the vector \( \mathcal{U} \) solution of \( \mathcal{L}(\sigma) \mathcal{U} = 0 \) and the scalar product \( \langle \mathcal{U}, \mathcal{U} \rangle \) which reads

\[
0 = \langle \mathcal{U}, \mathcal{L}(\sigma) \mathcal{U} \rangle = -\sigma \tau_1 \times \left[ \int_{-1/2}^{1/2} dz(\theta^* \varphi + \theta \varphi^*) \right] + \langle \mathcal{U}, \mathcal{L} \mathcal{U} \rangle
\]

where \( \mathcal{L} \) denotes the \( \sigma \)-independent part of \( \mathcal{L}(\sigma) \) and is also self-adjoint (see sect. 2). Since

\[
\langle \mathcal{U}, \mathcal{L} \mathcal{U} \rangle = \langle \mathcal{U}, \mathcal{L} \mathcal{U} \rangle^* ,
\]

\( \sigma \) must be real and the instability be stationary. Then the threshold corresponds to \( \text{Re} \{ \sigma \} = 0 \).

Now the condition

\[
\langle \mathcal{U}, \mathcal{L} \mathcal{U} \rangle = 0 \quad (B.2)
\]

gives us a variational formulation of the threshold problem [22]. We may restrict ourselves to only two independent functions \( \theta \) and \( \varphi \) considering \( \hat{u} \) as given by equation (B.1c) which simply reads

\[
\hat{u}'' = \tau' \varphi'. \quad (B.3)
\]

The final form of the scalar product (B.2) may be written as

\[
s_c = 2 \int_{-1/2}^{1/2} dz \left[ \tau^2 \theta^2 + \tau' \varphi^2 + \frac{e_1}{\tau} (\hat{u})^2 \right] \quad (B.4)
\]

and variations on \( \theta \) and \( \varphi \) lead back to equations (B.1a, b), when (B.3) is taken into account. In order to estimate the threshold, we must choose test-functions which fulfill the actual boundary conditions \( \theta(\pm \frac{1}{2}) = \varphi(\pm \frac{1}{2}) = 0 \). Here we simply take

\[
\theta = A \cos \pi z \quad (B.5a)
\]

\[
\varphi = B \cos \pi z \quad (B.5b)
\]

and deduce \( \hat{u} \) from (B.3)

\[
\hat{u}' = \frac{\tau' B}{\pi} (\pi \cos \pi z - 2) . \quad (B.5c)
\]

Then, replacing (B.5) in (B.4) we get

\[
s_c = 2 \frac{(\pi^2 + m) AB}{\tau A^2 + \tau'(1 - e'_1) B^2} \quad (B.6)
\]

with \( e'_1 = e_1 \left( 1 - \frac{8}{\pi^2} \right) \).

The threshold estimate is given by the extremum condition

\[
\frac{\partial s_c}{\partial A} = \frac{\partial s_c}{\partial B} = 0 ,
\]

which leads to

\[
\tau'(1 - e'_1) B^2 = \tau A^2
\]

so that

\[
s_c = \frac{\pi^2 + m}{\sqrt{\tau\tau'(1 - e'_1)}}
\]

or, recalling def. (A.2),

\[
k^2 = \frac{1 - e_1}{1 - e'_1} \left( \pi^2 + m \right) . \quad (B.7)
\]

Numerical application for MBBA in zero magnetic field \( (m = 0) \) gives

\[
k = 3.494 \ 6
\]

which is slightly higher than the exact result (A.6) as it should be. However, the small difference between the exact and approximate value is a strong indication of the general adequacy of the simplified distortion profile (B.5). This is no longer the case in presence of a high magnetic field. Here we get

\[
k^2 \sim m \sqrt{\frac{1 - e_1}{1 - e'_1}}
\]

instead of \( k^2 \sim m \) which is the exact result [6]. The
approximate value is much higher than the actual one (\(\sqrt{1.237}\) for MBBA) and (B.5) should be complemented to get a better estimate as one easily understands when considering figure 8 of reference [3] which displays the exact profiles.

We should notice here that \(\alpha_3\) was assumed to be negative. We have already shown [3] that this condition is necessary for the instability to take place. Here, when \(\alpha_3\) is positive we get

\[
\frac{2}{\int_{-1/2}^{1/2} dz \left( \theta m - \varphi' \right)}
\]

instead of (B.4) the denominator is no longer positive and definite. Looking for a solution of the form (B.5) we obtain \(A = B = 0\), which is an indication of the stability of the flow.

References

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[47] See also R. Graham in Ref. [12a], p. 268.
[48] Direct observation is out of reach of present techniques [22] however an indirect check can be envisaged using Brownian Motion.