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H.A. Rose, P.L. Sulem

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FULLY DEVELOPED TURBULENCE AND STATISTICAL MECHANICS

H. A. ROSE (*) and P. L. SULEM

Centre National de la Recherche Scientifique,
Observatoire de Nice, B.P. 252, 06007 Nice Cedex, France

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Résumé. — Cet article passe en revue quelques progrès récents en théorie statistique de la turbulence développée. L'accent est mis sur les analogies mais aussi les différences avec la mécanique statistique hamiltonienne, en particulier les phénomènes critiques.

La méthode des équations spectrales qui joue un peu le rôle d'une théorie du champ moyen est discutée en détail. Elle est présentée comme une reformulation de la théorie de Kolmogorov de 1941, permettant d'étudier l'énergétique de la turbulence (spectres en loi de puissance, cascades d'énergie directes et inverses, dissipation d'énergie dans la limite de viscosité nulle...). En outre cette méthode éclaire de façon intéressante les résultats tant démontrés que conjecturés sur les équations de Navier-Stokes et d'Euler que l'on passe en revue en termes plus accessibles que dans la littérature mathématique.

Il existe de fortes indications expérimentales (intermittence) que la théorie de Kolmogorov de 1941 n'est en fait qu'une première approximation. Certains des efforts actuels pour prendre en compte des grandeurs statistiques au-delà du second ordre, au moyen de techniques formelles inspirées de la théorie quantique des champs ou des phénomènes critiques, sont aussi discutés.

Abstract. — This paper gives a self contained review of some recent progress of the statistical theory of fully developed turbulence. The emphasis is on both analogies and differences with Hamiltonian statistical mechanics, in particular critical phenomena.

The method of spectral equations, which plays to a certain extent the role of a mean field theory, is discussed in detail. It is here viewed as a reformulation of the Kolmogorov 1941 theory leading to quantitative insight into the energetics of turbulence (power-law spectra, direct and inverse energy cascades, energy dissipation in the limit of zero viscosity, etc.). In addition, it sheds light on the proven and conjectured properties of the Navier-Stokes and Euler equations which are reviewed in terms more accessible than those of the mathematical literature.

There are strong experimental indications (intermittency) that the Kolmogorov 1941 theory is only approximate. Some of the current efforts to handle higher than second order statistics by formal methods inspired from quantum field theory or critical phenomena are also discussed.

Table of contents

(*) Present address : Los Alamos Scientific Laboratory, division T.II, Los Alamos, New Mexico 87545, USA.

1. **Introduction.** — The word *turbulence* is used to describe a wide family of diverse phenomena, even when restricted to systems adequately described by the Navier-Stokes (NS) equation for incompressible fluids [1]

$$\left.\begin{array}{ll} \dfrac{\partial u}{\partial t}+(u.\nabla)u=-\nabla p+\nu\nabla^{2}u+f & (a) \\ \nabla.u=0\,. & (b) \\ (+\text{ boundary and initial conditions}). & \end{array}\right\} \quad (1.1)$$

f denotes a possible external force per unit mass and ν the kinematic viscosity. The pressure force per unit mass $-\nabla p$ which is necessary to maintain incompressibility, can be expressed as a quadratic functional of the velocity by taking the divergence of eq. (1.1.a) and solving the resulting Poisson equation. For zero viscosity, the NS equation is called the Euler equation. It is easily checked that for three-dimensional flows, the non-linear terms conserve both the energy

$$\mathcal{E}(t)=\frac{1}{2}\int|u(\mathbf{r})|^{2}\,\mathrm{d}^{(3)}r \qquad (1.2)$$

and the helicity [2, 3, 4]

$$\mathcal{H}(t)=\frac{1}{2}\int u(\mathbf{r}).\omega(\mathbf{r})\,\mathrm{d}^{(3)}r \qquad (1.3)$$

where

$$\omega=\operatorname{curl}u \qquad (1.4)$$

is called the vorticity.

As an illustration of the different regimes described by the NS equation, consider in figure 1 the photograph (taken from Prandlt [5]) of the flow past a cylindrical body. From the diameter L of the body and the (uniform) velocity V of the fluid far upstream, one constructs the Reynolds number $R = LV/\nu$ which appears as the ratio of the non-linear to the dissipative terms in the NS equation. If R is much less than one, the flow is time-independent and laminar. As R grows, several flow regimes are encountered :

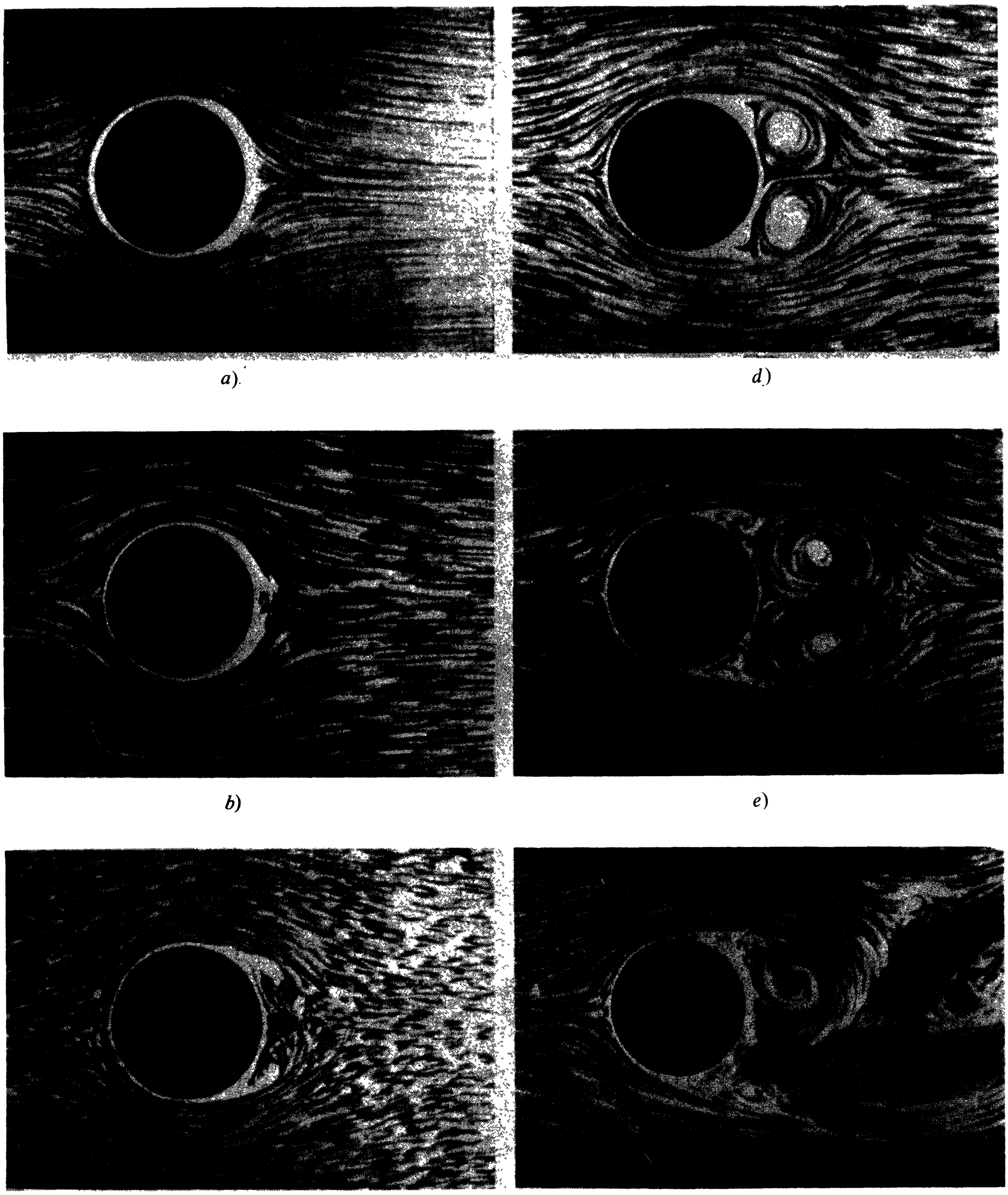
i) The laminar flow which was stable for $R < 1$, becomes unstable and is replaced by another laminar flow which has two counter-rotating vortices just downstream of the cylinder.

ii) The above laminar flow becomes unstable and is replaced by a time-dependent flow in which vortices are shed, in a quasi-periodic manner, and carried downstream.

iii) The vortex structure becomes less and less distinct.

iv) In the limit of infinite Reynolds number, a highly chaotic flow develops in regions of the fluid downstream of the body.

Let us stress the strong difference between the second regime referred to as *transition to turbulence* and the fourth one called *fully developed turbulence*. The former has recently attracted much attention for example in the context of the Rayleigh-Bénard convection and Couette flows where the fluid is driven respectively by thermally produced gravitational forces and centrifugal forces ; see Martin [6] and McLaughlin and Martin [7] for reviews. This regime is chaotic in the sense that the time-correlation-function, obtained by temporal averaging, tends to zero for large time separations. However, the presence of distinct spatial structures (the rolls in the Rayleigh-Bénard convection and the Taylor cells in the Couette flows) which are



a) d)

b) e)

c) f)

FIG. 1. — Photographs of different flow regimes past a cylinder (taken from Prandlt [5]).

characterized by just a few length scales means that the chaos is *temporal and not spatial.* Models which exhibit such behavior have been obtained from severe truncations of a modal representation of the primitive equations. The Lorenz [8] model for example is derived from the Rayleigh-Bénard system by retaining only three modes. This model has proved to be highly useful as a simple example of a bifurcating system. When the model's Rayleigh number (the ratio of the temperature induced buoyancy force to the viscous force) is increased, there is a first bifurcation from a purely conductive (zero velocity) solution to another stationary solution which corresponds to the usual convective rolls; after the second bifurcation, there is numerical evidence that the phase-space trajectory is ergodic on a *strange attractor* which locally appears as an infinite-sheeted surface. Similar models are discussed in Hénon and Pomeau [9]. It is possible to find mathematical models which develop strange attractors after four bifurcations. Their mathematical aspects are developed in Ruelle and Takens [10] and Ruelle [11], and their relation to higher order truncations of the Rayleigh-Bénard system are discussed in McLaughin and Martin [7]. We shall not dwell here on this very interesting aspect of *transitional turbulence*; see Clever and Busse [12], Joseph and Sattinger [13], Joseph [14, 15], Sattinger [16, 17], Iooss [18, 19], Marsden and McCraken [20], Normand, Pomeau and Velarde [21].

In contrast, *fully developed turbulence* (which we shall in this paper simply refer to as *turbulence*) is characterized by a *spatio-temporal* chaos : there is a hierarchy of flow structures (eddies) whose length scales extend from l_0, characteristic of boundary and/or initial conditions, to the dissipative scale $l_{\text{diss}} \ll l_0$, where viscous and inertial forces become comparable.

The *chaotic* aspect of fully developed turbulence can be characterized in several ways. In a given flow-realization the trajectory of each fluid particle is extremely intricate, leading to strong mixing of the flow and to drastically modified transport properties compared to those in the laminar state. This modification usually amounts to an enhancement of the transport coefficients (one then talks of turbulent or eddy transport coefficients). Even when the small scales of such a flow cannot be resolved (e.g. in distant stars), these changes in the transport properties may have observable consequences. Another characterization is found in the *instability* of a given flow-realization : a small amount of noise in initial conditions will be amplified and attain a significant level independent of its initial value [1]. This naturally leads to the use of a statistical description of fully developed turbulence which is in this paper our main concern. Note that there has been recently a renewed interest in the properties of individual flow-realizations in connection for example with large scale semi-coherent structures (Laufer [24]).

Let us now outline the content of the paper. In chapter 2, we introduce some of the fundamental concepts of fully developed turbulence. In chapter 3, we compare turbulence to systems in thermal equilibrium. Existence of energy transfer is motivated by a model whose properties are similar to those of a thermodynamic system in its equilibrium properties but ressemble turbulence when it is far from equilibrium. In chapter 4, we present a restatement of the classical Kolmogorov (1941) [25] phenomenological theory (in short K41), which contains many important concepts and has strong implications concerning the eventual nature of a mathematical theory. Chapter 5 is devoted to the results which can be rigorously obtained from the Euler and Navier-Stokes equations. The K41 theory also sets the framework for the method of spectral equations presented in chapter 6, a possible starting point for the reader mostly interested in closure. Within this approximate framework it is possible to answer a number of mathematical questions which are still open in the context of the primitive Euler and NS equations. The most important defect of the K41 theory is that it does not take into account intermittency (the very sparse spatial distribution of small eddies). Intermittency is one of the most challenging question in the theory of turbulence; it is considered in chapter 7, mostly on a phenomenological basis. In chapter 8, we compare turbulence with critical phenomena and also discuss the possible applications to turbulence of the calculational techniques recently developed in this field.

Finally, let us make some bibliographic comments. Review papers in somewhat the same spirit as the present one but with less emphasis on the relation to modern statistical mechanics and mathematics include Orszag [26] and Kraichnan [27, 28]. An extensive discussion of Kraichnan's DIA and related theories is given in Leslie [29]. A review of scaling behaviour at high Reynolds numbers is given by Nelkin [30]. Reviews concerned with the prediction of turbulence in realistic (usually inhomogeneous) situations include Tennekes and Lumley [31], Bradshaw [1, 22], Craya [32], Fernholz [33], Johnston [34], Bradshaw and Woods [35], Reynolds [23, 36], Reynolds and Cebeci [37], Launder [38], André [39]. A comprehensive survey of the physics of turbulence is given in Monin and Yaglom [40]. For more detailed reviews of the mathematics of the Navier-Stokes and Euler equations, we refer the reader to Lions [41], Ladyzenskaya [42, 43], Ebin and Marsden [44], Marsden, Ebin and Fischer [45], Bardos [46] and Bardos and Benachour [47]. Let us also mention the old but still illuminating paper by von Neumann [48].

2. **Fundamental concepts.** — 2.1 INHOMOGENEOUS TURBULENCE AND EDDY VISCOSITY.

— It is often the case for practical applications that a knowledge of only the simplest flow statistics is required. As an example, to calculate the mean turbulent drag on an object, all that is needed is the mean velocity $\langle u \rangle$. However, the nonlinear terms in the NS equation do not permit the derivation of an equation for $\langle u \rangle$ alone. If the velocity field is decomposed into a mean and a fluctuating part

$$u = \langle u \rangle + u' , \tag{2.1}$$

one can derive an equation of motion for $\langle u(x) \rangle$ in which terms of the form $\langle u'(x) u'(x) \rangle$ occur. Again,

[1] This also applies to noise in the boundary conditions. Although this is apparent experimentally, the construction of models for mean quantities which take this effect into account is difficult [22, 23].

because of the inertial terms, an attempt to derive an equation of motion for $\langle u'(x) u'(x) \rangle$, the so called Reynolds stress tensor, necessarily introduces terms of the form $\langle u'(x) u'(x) u'(x) \rangle$, etc... The traditional phenomenological approach to this *closure problem* is associated with the name of Prandtl [49]. It is motivated by the kinetic theory of gases and consists in the simplest case in approximating the effects of fluctuations on the transport of mean momentum (velocity) by enhancing the values of the kinematic viscosity by an *eddy viscosity* ν_E, which is defined by

$$\langle u_i'(x) u_j'(x) \rangle = -\frac{1}{2} \nu_E \left[\frac{\partial}{\partial x_j} \langle u_i(x) \rangle + \frac{\partial}{\partial x_i} \langle u_j(x) \rangle \right]. \tag{2.2}$$

The eddy viscosity is often approximated by

$$\nu_E \sim \langle | u | \rangle l_m$$

where the *mixing length* l_m is usually taken to be the integral scale of the velocity fluctuations (see Bradshaw [22], Mellor and Herring [50], Tennekes and Lumley [31], Reynolds [36], Craya [32] for reviews). These models are strictly valid only when the fluctuations have a characteristic length scale much smaller than that of the mean velocity. In the kinetic theory of gases, one usually has such a separation between, say, the mean free path and hydrodynamic scales. In inhomogeneous turbulence, in contrast, there is never a clear separation between the length scales of the fluctuations and those of the mean field.

More recent models work explicitly with the Reynolds stress tensor as well as the mean velocity ([2]) (Hanjalic and Launder [53], Launder [54], Launder, Reece and Rodi [55], Donaldson [56], Lumley and Khajeh-Nouri [57], Wyngaard and Coté [58], Wingaard, Coté and Rao [59], André [39]). As mentioned before, this introduces triple correlations whose significance is connected with the *energy transfer*. The conservation of energy by the non-linear terms of the NS equation implies a transfer of energy from one location to another (spatial transfer) and from one scale of motion to another (spectral transfer).

2.2 Homogeneous isotropic turbulence and the dynamics of the fluctuations.

— A quantitative description of the spectral transfer requires the explicit introduction of two-point-correlations. Their systematic study in the inhomogeneous geometries characteristic of real flows is difficult. Some attempts have nevertheless been made. Herring [60, 61] performed calculations of thermal convection with methods which are in some sense intermediate between the above one-point phenomenological models and the more quantitative two-point closure methods described in chapter 6. Kraichnan [62, 63], Herring [64] and Leslie [65, 29] have suggested that the latter can be use to deduce the former. We shall here restrict our study to *homogeneous* turbulence, a flow whose statistical properties are invariant under spatial translations : the mean velocity is uniform and hence may be eliminated by a Galilean transformation. Turbulence then reduces to the dynamics of fluctuations. A further simplification consists in assuming that the turbulence is *isotropic* (i.e. statistically invariant under rotations) and *non-helical* (i.e. statistically invariant by mirror symmetries). For a flow which possesses these symmetries, the two-point correlation tensor

$$U_{ij}(\mathbf{r}, t) = \langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \mathbf{r}, t) \rangle \tag{2.3}$$

is characterized by its trace [66, 67]

$$U(r, t) = \sum_{i=1}^{3} U_{ii}(r) \tag{2.4}$$

where $r = | \mathbf{r} |$. Since our conceptual understanding of turbulence involves the classification of eddies according to their size, it is often convenient to work with the Fourier transform of $U(r, t)$, denoted by $\hat{U}(k, t)$. In terms of $\hat{U}$, the kinetic energy per unit mass is

$$\mathcal{E}(t) = \frac{1}{2} \langle u^2(\mathbf{r}, t) \rangle =$$

$$= \frac{1}{2} \int \hat{U}(k, t) \, d^{(d)}k = \int_0^\infty E(k, t) \, dk \tag{2.5}$$

where d is the dimension of space and

$$E(k, t) \sim k^{d-1} \hat{U}(k, t) \tag{2.6}$$

is called the *energy spectrum*; it characterizes the energy distribution among the different scales of motion.

The experimental result on three-dimensional turbulence which has been the focus of theoretical interest, is the existence of a range of eddy-sizes, $\{ l ; l_0 \gg l \gg l_{\mathrm{diss}} \}$, called the *inertial range*, where turbulent kinetic energy production and dissipation are negligible (see Fig. 5 in ref. [68]) and where the spectrum exhibits a scaling behaviour [69]

$$E(k, t) \propto k^{-m}, \tag{2.7}$$

with m close to 5/3.

The spectrum with $m = 5/3$ is the celebrated Kolmogorov (1941) spectrum. It is shown in Appen-

([2]) Though such models succeed in describing a large number of situations, they are unsuitable for the cases where there are significant spatial variations of the integral scale, for example in some experiments on spatially decaying turbulence [51]. Recent investigations suggest that, in addition to the mean velocity and the Reynolds stress tensor, the vector potential of the velocity field must be explicitely considered; it contains in effect important information on the largest eddies [52].

dix 1 (cf. also Batchelor [66]) that if $1 < m < 3$, the correlation function $U(r)$ is locally determined in Fourier space in the sense that it is determined by $\hat{U}(k)$ for k in the neighbourhood of r^{-1}. This implies a scaling law for the *structure function*

$$U(0) - U(r) \propto r^{m-1} . \tag{2.8}$$

The huge Reynolds numbers required to produce an extended inertial range are experimentally accessible in geophysical flows such as the planetary boundary layer and tidal channels. Of course, the large scale eddies reflect the inhomogeneities characteristic of the forces and boundaries conditions which produce them. Nevertheless, since the characteristic times decrease geometrically with eddy size (see chapter 4), a large number of generations is produced during the life-time of the large anisotropic eddies, and the idealization of homogeneous isotropic turbulence is approximately realized in the small scales. We refer the reader to Van Atta [70] and Comte-Bellot [71] for a review of the measurement methods.

With the latest generation of computers which can handle of the order of a million words in central memory, direct numerical simulations of the NS equation with periodic boundary conditions and random initial conditions are now feasible with Reynolds numbers up to a few hundred in three-dimensions [72, 26] and up to about thousand in two-dimensions [73, 74, 75, 76]. Only in the two-dimensional case, are the Reynolds numbers sufficiently high for the spectrum of the solution to display an inertial range [77].

2.3 VORTICITY DYNAMICS AND ENERGY TRANSFER IN THREE DIMENSIONS. — The vorticity dynamics (which will be seen in section 7.1 to be also relevant in understanding intermittency) gives a configuration-space interpretation of the energy transfer to small-scale motions in three-dimensional turbulence. In effect, in the absence of viscosity the vorticity $\omega = \text{curl}\, u$ satisfies ($\mathrm{D}/\mathrm{D}t$ is the Lagrangian derivative)

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} \equiv \frac{\partial \omega}{\partial t} + (u.\nabla)\,\omega = (\omega.\nabla)\,u . \tag{2.9}$$

This reveals an intimate connection between the energy transfer and the distortion by velocity gradients of a small line element $\mathbf{dl}$ being carried by the flow : its evolution is governed by eq. (2.9) where ω has been replaced by $\mathbf{dl}$. The question of the stretching of a line element by a prescribed random velocity field has been studied by Batchelor [78], Cocke [79, 80] and Orszag [81]. The main result is that a line element which is initially statistically independent of the velocity field is, in the mean, stretched, but the exact law of stretching is not known. Such results cannot be used for vortex lines, the integral curves of the vorticity field because velocities cannot be assigned independently of vorticities. However, the following very simple argument taken from Bardos and Frisch [82] suggests that in the non linear case there may be a catastrophic growth of the vorticity : the vorticity is just the anti-symmetric part of the velocity gradient ∇u ; if we tentatively identify ω and ∇u and discard vector and tensor indices, we obtain that the Lagrangian rate of change of the vorticity is something like the squared vorticity, which implies that ω blows up at the time (³) $t_* = |\sup \omega_0|^{-1}$.

Growth of the mean squared vorticity can be demonstrated rigorously for short times with Gaussian initial conditions. (Ref. [26], p. 306). There is numerical evidence that this growth persists at least for some time (Ref. [26], p. 274). Mean square vorticity $\Omega = \langle \omega^2 \rangle$ is usually called *enstrophy*. It is related to the energy spectrum by

$$\Omega = \int_0^\infty k^2\, E(k)\, \mathrm{d}k .$$

Growth of enstrophy together with conservation of energy requires an energy transfer to high wavenumbers.

2.4 THE SPECIAL CASE OF TWO-DIMENSIONAL TURBULENCE. — Large scale flows in the earth atmosphere and oceans are known to be mostly two-dimensional but of course not exactly so [85, 86, 87]. In two dimensions, the Euler equations has the property of conserving in addition to the energy the vorticity of each fluid element as it follows the velocity field (⁴). This is immediately seen from the vorticity equation (2.9) whose right hand side vanishes identically in two dimensions. For example, if the flow is confined to the (x, y)-plane, then $\omega = (0, 0, \omega)$, while $\nabla = (\partial/\partial x, \partial/\partial y, 0)$ and $(\omega.\nabla) = 0$. A special consequence is the conservation of enstrophy. This excludes the possibility of significant energy transfer towards the small scales [90]. It is generally believed that the energy cascades to large scales (inverse cascade) and that enstrophy cascades to small scales ([91], see also Section 6.3).

Spectra derived from large scale atmospheric motion [92, 93, 94, 95] and from turbulent flows of mercury constrained by a strong magnetic field [96, 97] exhibit a k^{-3} range. This has been interpreted as evidence of enstrophy cascades, but the existence of physical effects related to the earth's rotation in the former and Joule dissipation in the latter which

(³) Note that this argument applies neither to two-dimensional flows (cf. Section 2.4) nor to axisymmetric flows (cf. the spherical vortex of Hill [83, 84] which is a steady axisymmetric solution).

(⁴) This property does not survive in MHD, which makes the two-dimensional MHD turbulence dynamics very different from the non-magnetic ones [88, 89].

compete with the transfer makes their relevance to the turbulent solutions of the two-dimensional NS equation doubtful (Leith 1975 private communication, cf. [98]). A k^{-3} energy spectrum has also be observed for turbulence in a steady stratified fluid, i.e. a fluid in a gravitational field with the density of the fluid decreasing upwards, assuming the Boussinesq approximation [99, 100].

2.5 Detailed conservation properties of the Euler equation. — After spatial Fourier transform, the Euler equation of an unbounded fluid reads

$$\begin{aligned} &\frac{\partial}{\partial t}\hat{u}_i(\mathbf{k}, t) = -\frac{i}{2}\sum_{j,m} P_{ijm}(\mathbf{k}) \int_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \hat{u}_j(\mathbf{p}, t)\, \hat{u}_m(\mathbf{q}, t)\, \mathrm{d}^{(d)}p \qquad (a) \\ &\mathbf{k}.\hat{u}(\mathbf{k}, t) = 0 \qquad (b) \end{aligned} \tag{2.10}$$

with

$$P_{ijm}(\mathbf{k}) = k_m P_{ij}(\mathbf{k}) + k_j P_{im}(\mathbf{k}) \tag{2.11}$$

where

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2 \tag{2.12}$$

is the projection operator on the plane perpendicular to $\mathbf{k}$.

The global conservation of energy implies a detailed conservation in which any triad

$$\{\, \hat{u}(\pm\,\mathbf{k}), \quad \hat{u}(\pm\,\mathbf{p}), \quad \hat{u}(\pm\,\mathbf{q}); \quad \mathbf{k}+\mathbf{p}+\mathbf{q}=0 \,\}$$

exchanges energy among its members conservatively. This is a direct consequence of the energy being a quadratic invariant [91], but can also be seen more directly. Indeed, starting from eq. (2.10), let us evaluate

$$\frac{\partial}{\partial t} \mid \hat{u}(\mathbf{k}) \mid^2 = \int S(\mathbf{k} \mid \mathbf{p}, \mathbf{q})\, \mathrm{d}^{(d)}p\, \mathrm{d}^{(d)}q \tag{2.13}$$

where

$$S(\mathbf{k} \mid \mathbf{p}, \mathbf{q}) = -\operatorname{Im}\{\, P_{ijm}(\mathbf{k})\, \hat{u}_j(\mathbf{p})\, \hat{u}_m(\mathbf{q})\, \hat{u}_i(\mathbf{k}) \,\}\, \delta^{(d)}(\mathbf{k}+\mathbf{p}+\mathbf{q})\,. \tag{2.14}$$

Use of the definition of P_{ijm} and the transversality of u's give

$$S(\mathbf{k} \mid \mathbf{p}, \mathbf{q}) = -\operatorname{Im}\{\, (\mathbf{k}.\hat{u}(\mathbf{q}))\, (\hat{u}(\mathbf{p}).\hat{u}(\mathbf{k})) + (\mathbf{k}.\hat{u}(\mathbf{p}))\, (\hat{u}(\mathbf{q}).\hat{u}(\mathbf{k})) \,\}\, \delta^{(d)}(\mathbf{k}+\mathbf{p}+\mathbf{q})\,. \tag{2.15}$$

Detailed conservation means that

$$S(\mathbf{k} \mid \mathbf{p}, \mathbf{q}) + S(\mathbf{p} \mid \mathbf{q}, \mathbf{k}) + S(\mathbf{q} \mid \mathbf{k}, \mathbf{p}) = 0\,. \tag{2.16}$$

The terms of this sum are cyclic permutations of

$$(\hat{u}(\mathbf{k}).\hat{u}(\mathbf{p}))\, ((\mathbf{k}+\mathbf{p}).\hat{u}(\mathbf{q}))\, \delta^{(d)}(\mathbf{k}+\mathbf{p}+\mathbf{q}) \tag{2.17}$$

which vanishes because of the transversality of $\hat{u}(\mathbf{q})$.

Similar calculations lead to detailed helicity and enstrophy conservations in three and two dimensions respectively.

3. **Relationship to equilibrium statistical mechanics.** — We begin this chapter by demonstrating a fundamental difference between turbulence and thermal equilibrium and then attempt to bridge the gap by using a model-system with the following features : it is similar to a thermodynamic system in its equilibrium properties but far from equilibrium, it resembles turbulence.

3.1 Energy transfer and the lack of a fluctuation-dissipation theorem in turbulence. — In an equilibrium system, fluctuations tend to restore thermodynamic equilibrium; in contrast three-dimensional turbulence is concerned with the fluctuations of energy transport from large to small scales of the motion. One might think that the non-equilibrating effect of the energy transfer could be compensated by driving the fluid with external forces in such a way that a statistically stable stationary state is maintained : the energy lost to the small scales being balanced by the external injection of energy at large scales. However, the fluctuations about this stationary state are different from the fluctuations about thermal equilibrium. Consider a classical interacting gas in thermal equilibrium. On the average, the energy density is spatially uniform. If at a given instant of time, a fluctuation is observed whereby the energy density in a given region is lowered under its mean value, then, because energy is conserved and because we presume the interaction between the molecules of the gas to be short-ranged, a compensating augmentation of energy must be found in the neighbourhood of this region. As the fluctuation relaxes, energy which was transferred out of the given region returns. There is thus a direct correlation between the original fluctuation and the subsequent relaxation process. The fluctuation-dissipation theorem is a compact mathematical representation of this effect [101, 102]. Now, consider a fluctuation in a turbulent fluid whereby, the energy in (inertial range) eddies of a certain size in a given spatial region is transferred to eddies of a comparable size in a neighbouring spatial region. If the turbulence is in a stationary state, then there will be a transfer of energy back into the given region. However, because the transfer of energy to smaller eddies occurs on a time scale comparable to that for spatial transfer, the energy that was initially transferred out is no longer available : it has been pushed to the smallest eddies and dissipated by viscosity. Thus, the energy which drives the local energy spectrum back towards its mean values is not closely correlated with that which was initially lost. Instead, it comes from the energy cascade generated by the new eddies which are being injected to maintain stationarity. This absence of correlation between fluctuations and relaxation is reflected in the non existence of a fluctuation-dissipation theorem for turbulence. This difference can be illustrated by considering an abstract space in which a point represents a probability distribution function. In this space, the evolution of the gas is self-corrective, whereas turbulence amplifies deviations from an initial truncated absolute equilibrium state with each step of the cascade (Kraichnan [28], Monin and Yaglom [40] chap. 8).

3.2 ABSOLUTE EQUILIBRIUM IN THREE DIMENSIONS. — The *absolute equilibrium model*, introduced independently by Burgers [103], Hopf [104], Lee [90] and Kraichnan [105] has the virtues of being closely related to the usual thermodynamic systems in its equilibrium properties, yet resembling turbulence when it is far from equilibrium. It is obtained when the discrete Fourier transform of the Euler equation (e.g. assuming cyclic boundary conditions) is truncated by retaining only those wavenumbers which fall in an interval $(k_{\min}, k_{\max})$, $k_{\min}$ being sometimes taken to be zero. The dynamics of the remaining finite number of modes

$$\left\{ \hat{u}(\mathbf{k}) \mid \mathbf{k} = \frac{2}{L}(n_x, n_y, n_z) \right\}, \tag{3.1}$$

where n_x, n_z, n are positive or negative integers such that $k_{\min} \leqslant k \leqslant k_{\max}$, is then governed by an equation similar to (2.10) where the integral is replaced by a discrete summation and $\hat{u}(\mathbf{k})$, $\hat{u}(\mathbf{p})$, $\hat{u}(\mathbf{q})$ are restricted by (3.1). The $\hat{u}$'s are further constrained by the transversality (2.10*b*), and by the reality of $u(\mathbf{x})$. These constraints allow us to associate with each $\mathbf{k}$ in the half-space $k_x \geqslant 0$, four independent variables which correspond to the real and imaginary parts of the components of $\hat{u}$ in a plane perpendicular to $\mathbf{k}$. Let $\{ y_a ;\ 1 \leqslant a \leqslant N \}$ be the set of independent variables. In the phase space of the y's, a representative point moves according to

$$\frac{\mathrm{d}y_a}{\mathrm{d}t} = \sum_{b,c=1}^{N} A_{abc}\, y_b\, y_c\,, \tag{3.2}$$

where $A_{abc} = A_{acb}$ are constant coefficients. Excluding excitation at $k = 0$ (uniform advection of the flow), a condition preserved by (2.10), we can assume that A_{abc} vanishes if two indices are equal. The detailed energy conservation by the original Euler equation yields

$$A_{abc} + A_{bca} + A_{cab} = 0\,. \tag{3.3}$$

This implies the global energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{a=1}^{N} |\, y_a \,|^2 = 0 \tag{3.4}$$

and the incompressibility of the phase space flow of an ensemble of points :

$$\sum_a \frac{\partial}{\partial y_a}\left(\frac{\mathrm{d}y_a}{\mathrm{d}t}\right) = 0\,. \tag{3.5}$$

In other words, the volume of a given region in the phase space of the y's is preserved as it evolves according to (3.2). Assuming that the flow in the phase space is mixing, we can carry over the usual analysis which uses first the microcanonical ensemble and then the canonical ensemble to describe the equilibrium statistical mechanics of (3.2). It has the probability distribution (for zero mean helicity)

$$P(y) \propto \exp\left\{ -\beta \sum_a y_a^2/2 \right\} \tag{3.6}$$

where β plays the role of an inverse *temperature*. Evolution towards this equilibrium was observed in the computer simulations of Orszag and Patterson [72].

The distribution (3.6) has two outstanding features :

(i) It is Gaussian in the variables y.

(ii) $\langle y_a^2 \rangle$ is independent of a, which is equivalent to an equipartition of energy among the Fourier modes $u(k)$.

Equipartition implies that $\hat{U}(k)$ is independent of k, and therefore $E(k) \propto k^2$. The energy per unit mass in the system at equilibrium is, assuming $k_{\min} = 0$,

$$\mathcal{E}(t) \propto \int_0^{k_{\max}} k^2 \,\mathrm{d}k \propto k_{\max}^3\,, \tag{3.7}$$

which diverges if the limit of $k_{\max} \to \infty$ is taken. This is analogous to the ultraviolet catastrophy of classical blackbody radiation. Since we would hope to regain the full Euler equation in this limit, we conclude that if the initial velocity field has a finite but non-zero energy per unit mass, a state of statistical equilibrium will never be attained as the velocity field evolves in time according to the Euler equation. This inability to reach equilibrium for $k_{\max} = \infty$, is one of the characteristics of turbulence. The implications of this statement are illustrated by a series of Gedanken Experimente which lead up to the limit $k_{\max} = \infty$. For example, begin with a distribution of energy which is in equilibrium, confined by $k_{\min} = 0$ and $k_{\max} = k_1$, then remove the constraint at $k = k_1$, and let the system evolve in time according to (3.2) until it has again approximately reached equilibrium at a time $t = T$, with a newly imposed cutoff at $k = k_2 > k_1$. If

$$E(k, t = 0) = \begin{cases} A_1 k^2 & k \leqslant k_1 \\ 0 & k > k_1 \end{cases} \quad (a)$$

and (3.8)

$$E(k, t = T) = \begin{cases} A_2 k^2 & k \leqslant k_2 \\ 0 & k > k_2 \end{cases} \quad (b)$$

then conservation of energy implies that (see Fig. 2)

$$\frac{A_1}{A_2} = \left(\frac{k_2}{k_1}\right)^3 > 1 . \tag{3.9}$$

This shows a tendency to transfer energy up to the highest wavenumbers available in Fourier space or in other words a tendency for large eddies to generate smaller eddies. Alternatively, one may see that regions of high velocity gradient (high shear) are created. This follows from comparison of the enstrophy at time $t = 0$ and $t = T$.

$$\frac{\langle \omega^2(T) \rangle}{\langle \omega^2(0) \rangle} = \frac{\int_0^{k_2} k^2 E(k)\, \mathrm{d}k}{\int_0^{k_1} k^2 E(k)\, \mathrm{d}k} = \frac{A_2}{A_1}\left(\frac{k_2}{k_1}\right)^5 = \left(\frac{k_2}{k_1}\right)^2 > 1 . \tag{3.10}$$

Therefore, for three-dimensional turbulence we may interpret the transfer of energy to small scales and the associated build-up of velocity gradients, as a consequence of the Euler equation seeking to attain an inaccessible state of statistical equilibrium.

More generally, if helicity is non-zero, the exponent in eq. (3.6) should be a linear combination of energy and helicity which is also conserved by the truncated Euler equation. In contrast with the enstrophy invariant in two dimensions (see section 3.3), the helicity invariant does not lead to absolute equilibria spectra which are peaked at small wavenumbers [106]; this is no longer true in MHD ([5]) [107].

([5]) The reader interested in helical MHD turbulence is referred to Léorat, Frisch and Pouquet [108], Pouquet, Frisch and Léorat [109], Léorat [110], Moffatt [111], Pouquet and Patterson [112].

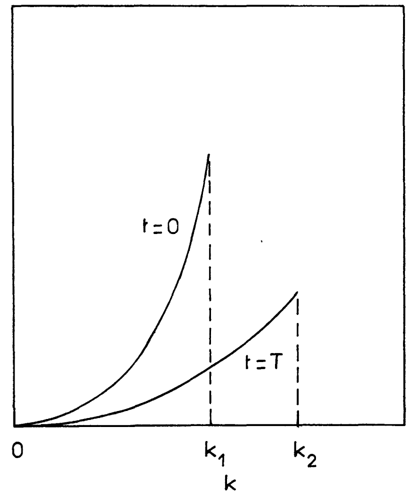


FIG. 2. — Evolution from an equilibrium truncated at k_1 to an equilibrium truncated at $k_2 > k_1$.

The effect of a non-zero mean helicity on energy transfer has been analysed on the basis of spectral equations [113] which indicate that the inertial range is unchanged except for an inhibition of the rate of energy transfer. This inhibition is suggested by considering the Euler equation in the form

$$\frac{\partial u}{\partial t} + u \times \omega = -\nabla\left(p + \frac{u^2}{2}\right) . \tag{3.11}$$

A non-zero helicity $\langle u . \omega \rangle$ implies a directional correlation between u and ω which tends to reduce $u \times \omega$, and hence the energy transfer (Patterson and Malkus, private communication cited in ref. [107]). In addition to energy and helicity, there exist the Kelvin circulation invariants which do not seem to have any counterparts in the truncated equations.

3.3 ABSOLUTE EQUILIBRA IN TWO DIMENSIONS. — The existence of an additional positive-definite quadratic invariant of the motion, the enstrophy

$$\Omega = \sum_a k_a^2 y_a^2 \tag{3.12}$$

profoundly changes the form of the absolute equilibria spectra ([6]). In place of the distribution given by (3.14), we have [91, 115]

$$P(y) \propto \exp \left\{ - \sum_a (\alpha + \beta k^2) \mid y_a \mid^2 \right\} \tag{3.13}$$

which yields

$$\langle \mid y_a \mid^2 \rangle \propto \frac{1}{\alpha + \beta k^2} \tag{3.14}$$

([6]) Hald [114] shows that for special truncations having only a very small number of modes (up to 8), there may be other invariant quantities, quadratic or cubic. For such systems, the flow on the intersection of the surfaces corresponding to the invariants is not mixing.

where α and β are parameters which together determine the mean energy $\mathcal{E}$ and mean enstrophy Ω; see Fox and Orszag [116] and Basdevant and Sadourny [117] for a computer simulation of the approach toward this equilibrium. The case $\beta = 0$ gives energy equipartition, $\alpha = 0$ gives enstrophy equipartition, while the possibility of negative β implies the existence of spectra whose energy peaks at small wavenumbers. The implication for the nonequilibrium behaviour displayed during the approach to equilibrium is complex because there are three different regimes of behaviour depending upon the values of $k_0 = (\Omega/\mathcal{E})^{1/2}$ relative to the truncation wavenumbers $k_{\min}$ and $k_{\max}$. We refer the reader to Kraichnan [115] who shows that an initial state in which energy and enstrophy are concentrated at k_0 with

$$k_{\min} \ll k_0 \ll k_{\max}$$

leads to an inverse (to small wavenumbers) transfer of energy and a direct (to large wavenumbers) transfer of enstrophy.

Extension of this treatment to two layer flows with or without topography and rotation has been made by Salmon, Holloway and Hendershott [118]. They found that the system behaves essentially like a single flow in the large scales and like two uncorrelated flows in the small scales.

In the context of the two-dimensional Euler equation, it is also possible to consider the statistical mechanics of a system of discrete point-vortices [119-123]. The relationship between this approach and absolute equilibrium has been investigated by Kraichnan [115]. It must also be mentioned that the numerical simulations of Boccara, Conte and Sarma [124] indicate that such systems of discrete vortices do not necessarily evolve towards statistical equilibrium.

4. **Phenomenology following Kolmogorov 1941.** —

4.1 K41 PHENOMENOLOGY FOR THREE-DIMENSIONAL TURBULENCE. — By the Kolmogorov 1941 theory (in short K41), we mean the general class of arguments developed by Kolmogorov [25], Obukhov [125] and others (see ref. [126] for review) which has led in particular to the 5/3 law for the energy spectrum.

Consider a stationary homogeneous isotropic turbulence where energy is injected with a forcing spectrum F(k) peaked about a wavenumber $k_0(= 1/l_0)$, as in figure 3. The energetics of eddies with wavenumbers smaller than a given wavenumber k are determined by an equilibrium between the injection, the losses due to viscous dissipation and the energy flux $\Pi(k)$ to higher wavenumbers (to smaller eddies):

$$0 \equiv \frac{\partial}{\partial t}\int_0^k E(p)\,\mathrm{d}p = \int_0^k F(p)\,\mathrm{d}p - \\ - 2\nu\int_0^k p^2\,E(p)\,\mathrm{d}p - \Pi(k) \quad (4.1)$$

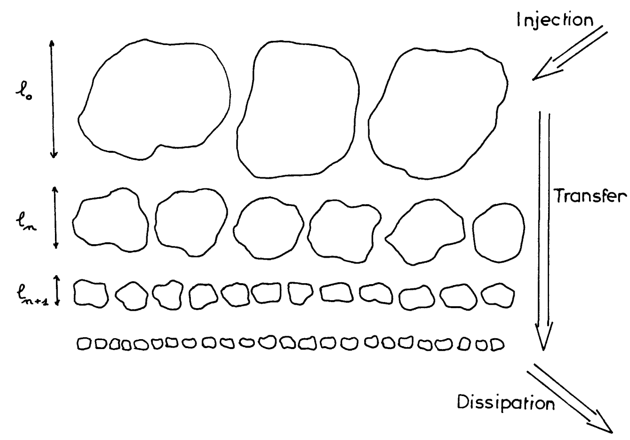


FIG. 3. — The energy cascade according to the 1941 Kolmogorov theory: at each step the eddies are space-filling.

Π depends upon the inertial terms in the Navier-Stokes equation, and is expressible in terms of the triple velocity correlation $\langle uuu \rangle$ (cf. appendix 3). We suppose that the flow will evolve into a quasi-stationary self-similar hierarchy of eddies whose wavenumbers form a geometric progression, say

$$k_n = 2^n\,k_0\,, \qquad n = 0, 1, 2, \dots. \quad (4.2)$$

The energy, E_n, carried by eddies of size $l_n(\sim 1/k_n)$ is related to the energy spectrum by

$$E_n = \int_{k_n}^{k_{n+1}} E(k)\,\mathrm{d}k \quad (4.3)$$

There are three time scales implied by the Navier-Stokes equation for an eddy of size $1/k$ in the hierarchy. In the inertial range, the viscous time scale,

$$\tau_{\mathrm{visc}}(k) \sim 1/\nu k^2 \quad (4.4)$$

is the largest of the three and may be ignored. The inertial terms determine two time scales:

(i) The eddy-turnover-time,

$$\tau_n = l_n/v_n \quad (4.5)$$

where

$$v_n = \langle\,|\,u(x) - u(x + l_n)\,|^2\,\rangle^{1/2} \quad (4.6)$$

is a typical velocity difference accross an eddy of size l_n. It is the time required for the eddy to be distorted and, in this distortion process, generate smaller eddies. Therefore, τ_n is associated with energy transfer.

(ii) The sweeping time,

$$\tau_{\mathrm{sweep}} = l_n/v_0 \quad (4.7)$$

is the time required for the eddy of size l_n to be simply advected without appreciable distortion past a point of observation fixed relative to the largest, most energetic eddies. It is therefore irrelevant to energy transfer.

The tendency of eddies to generate smaller eddies of comparable size makes the word *cascade* appropriate for the description of energy transfer (Fig. 3).

$\Pi(k_n)$ measures the rate at which energy is being transferred out of the interval $k_n \leqslant k \leqslant 2\,k_n$ into the interval $2\,k_n \leqslant k \leqslant 4\,k_n$. This rate is estimated by the amount of energy in $k_n \leqslant k \leqslant 2\,k_n$ divided by the eddy-turnover-time

$$\Pi(k_n) \sim E_n/\tau_n\,. \qquad (4.8)$$

If the flow is not intermittent (see chapter 7) and if the spectrum is local (see Appendix 1), then

$$E_n(t) \sim v_n^2(t)\,. \qquad (4.9)$$

Now, assuming the existence of a wavenumber range, the *inertial range*, where injection is absent and dissipation negligible (because $\tau_{\text{visc}}(k_n) \gg \tau_n$), then energy conservation implies that $\Pi(k_n)$ is a constant in this range. This constant is usually symbolized by ε, which gives, using (4.9)

$$v_n^3/l_n \sim \varepsilon\,. \qquad (4.10)$$

This leads to

$$E_n \sim \varepsilon^{2/3}\,l_n^{2/3}\,, \qquad (4.12)$$

which, by localness assumption, is equivalent to

$$E(k) = C\varepsilon^{2/3}\,k^{-5/3} \qquad (4.13)$$

This is the Kolmogorov spectrum. C is a universal constant, generally called the Kolmogorov constant. Note that such a spectrum, if not terminated by viscosity, implies an infinite enstrophy

$$\langle\,\omega^2\,\rangle = \int_0^\infty k^2\,E(k)\,\mathrm{d}k\,.$$

To estimate the extent of the inertial range, we use the condition that the viscous decay time becomes comparable to the eddy-turnover-time. Let this occur at a wavenumber k_{diss}. Then

$$\frac{1}{\nu(k_{\text{diss}})^2} \sim \frac{1}{\varepsilon^{1/3}(k_{\text{diss}})^{2/3}}$$

gives the Kolmogorov dissipation wavenumber ([7])

$$k_{\text{diss}} \sim (\varepsilon/\nu^3)^{1/4}\,. \qquad (4.15)$$

When ε is evaluated for $n = 0$, we obtain

$$\varepsilon \sim v_0^3/l_0\,, \qquad (4.16)$$

which is clearly independent of the viscosity. For a given value of viscosity, k_{diss} simply adjusts itself to make eq. (4.16) true.

Using eq. (4.1), the mean square velocity gradient becomes

$$\langle\,|\,\nabla u\,|^2\,\rangle \sim \langle\,\omega^2\,\rangle = \int_0^\infty p^2\,E(p)\,\mathrm{d}p \sim \varepsilon^{2/3}\int_0^{k_{\text{diss}}} p^{1/3}\,\mathrm{d}p \sim \varepsilon/\nu\,. \qquad (4.17)$$

We take this to mean that for any $\nu > 0$, the velocity field is differentiable. This justifies the usual manipulations to prove that the inertial terms in the Navier-Stokes equation conserve energy. It follows from eqs. (4.1) and (4.17) that the energy balance reads

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty E(k)\,\mathrm{d}k = -\,\varepsilon + \int_0^\infty F(k)\,\mathrm{d}k\,. \qquad (4.18)$$

Thus, ε, which was originally defined as an energy transfer rate, is also equal to the energy dissipation and energy injection rates. These considerations lead to the picture of energy being injected into the flow at small wavenumbers, and being transferred to higher wavenumbers where it is eventually removed by viscosity (see Fig. 3).

4.2 Extension of K41 phenomenology to two dimensions. — Consider the unforced inviscid initial value problem with an initial energy spectrum peaked at k_0. The conservation of enstrophy

$$\Omega = \langle\,\omega^2\,\rangle = \int_0^\infty k^2\,E(k)\,\mathrm{d}k \qquad (4.19)$$

prevents the establishment of a direct cascade of energy to high wavenumbers since such a cascade would increase the enstrophy. But an enstrophy cascade has been conjectured [91, 128, 129] : the (constant) rate of enstrophy transfer is given by

$$\eta = \Omega_n/\tau_n \qquad (4.20)$$

where

$$\Omega_n = \int_{k_n}^{k_{n+1}} k^2\,E(k)\,\mathrm{d}k \sim k_n^3\,E(k_n)\,. \qquad (4.21)$$

If we take for τ the same (local) expression that was used in three dimensions, i.e.

$$\tau_n = l_n/E_n^{1/2} = (k_n^3\,E(k_n))^{-1/2}\,,$$

we obtain

$$E(k) \sim \eta^{2/3}\,k^{-3}\,, \qquad (4.22)$$

([7]) This analysis can be extended to the case where the dissipative term in the NS equation, $\nu\,\Delta u$ is changed into $-\,\nu(-\,\Delta)^\alpha\,u$ with a dissipativity $\alpha \geqslant 0$. $(-\,\Delta)^\alpha$ is defined as the Fourier transform of multiplication by $|\,k\,|^{2\alpha}$. If $\alpha > 1/3$, a dissipation range is obtained at high wavenumbers; if $\alpha < 1/3$ and the viscosity is too small, the viscous time is everywhere longer than the transfer time and this modified NS equation is expected to be as singular as the Euler equation [127].

which invalidates the locality assumption

$$(E(k) \sim k^{-m}; m < 3) \quad \text{(see Appendix 1)}.$$

A more refined analysis has been made by Kraichnan [130] : if one uses for the eddy-turnover time τ the *non-local* expression (see Appendix 1)

$$\tau(k) = \left[\int_0^k p^2 E(p)\, \mathrm{d}p \right]^{-1/2}, \tag{4.23}$$

then eq. (4.20) becomes

$$\eta \sim k^3 E(k) \left[\int_0^k p^2 E(p)\, \mathrm{d}p \right]^{1/2} \tag{4.24}$$

which gives a log-corrected spectrum

$$E(k) \sim \eta^{2/3} k^{-3} \left(\ln \frac{k}{k_0} \right)^{-1/3}. \tag{4.25}$$

Note that with (4.22), all octaves below a given wavenumber k contribute the same amount to the strain acting on eddies on size close to $1/k$, whereas with (4.25) we have gained a certain amount of locality (not very much actually). Nevertheless the dynamics of the enstrophy cascade remains determined essentially by the quasi-uniform straining action of the largest eddies. This is a linear problem related to the somewhat simpler question of the quasi-uniform straining of a passive scalar [131].

This transfer of enstrophy to high wavenumbers is necessarily accompanied by some energy transfer; this would increase the enstrophy if a larger quantity of energy were not simultaneously transferred to small wavenumbers [132, 133]. To see more clearly the inverse energy transfer, let energy be injected in a narrow wavenumber-band near k_0 at a rate ε. This is necessarily accompanied by enstrophy injection at a rate $\eta \approx \varepsilon k_0^2$. The energy cannot be significantly transferred to large wavenumbers; instead, it is transferred to small wavenumbers at the rate ε. We then obtain an inverse energy cascade where the K41 analysis gives as usual $E(k) \sim \varepsilon^{2/3} k^{-5/3}$. It follows from the divergence of the integral of $k^{-5/3}$ at $k = 0$, that the development of such a range will never reach zero wavenumbers at a finite time. The conservation of energy implies that at time t the inverse cascade reaches up to wavenumber $k_{\min}(t)$ given by ($\mathcal{E}(0)$ is the initial energy)

$$\varepsilon t + \mathcal{E}(0) \sim \int_{k_{\min}(t)}^{k_0} \varepsilon^{2/3} k^{-5/3}\, \mathrm{d}k. \tag{4.26}$$

For large t we obtain

$$k_{\min}(t) \sim \varepsilon^{-1/3} t^{-3/2}. \tag{4.27}$$

5. Rigorous results for time-dependent Euler and Navier-Stokes equations.

— 5.1 PHENOMENOLOGY AND MATHEMATICS. — Turbulence phenomenology (K41 and almost any extension taking intermittency into account, see chapter 7) has strong implications concerning the nature of a mathematical theory. For the NS equation, it predicts global regularity (i.e. smoothness for all times). This regularity is in fact expected to hold even if the viscous term $-\nu \Delta u$ is changed into $\nu(-\Delta)^\alpha u$ with a *dissipativity* α larger than a critical value close to 1/3 (see section 4.1 footnote 5). For the Euler equation, global regularity is predicted in two dimensions. In three dimensions, in contrast, a vortex stretching argument suggests a catastrophic growth of vorticity (see section 7.1). Some direct numerical evidence of singularities has been found for the Taylor-Green vortex [26], a flow whose initial velocity field is a simple product of sines and cosines. There are also a few known exact solutions of the Euler equation which display singularities at a finite time, but these solutions are badly behaved at large distances. An example of such a solution, communicated to us by S. Childress and E. Spiegel is

$$u(x, y, z, t) = \left(\frac{y+z}{t-t_0}, \frac{z+x}{t-t_0}, \frac{x+y}{t-t_0} \right).$$

The pressure is then

$$p(x, y, z, t) = -\frac{x^2 + y^2 + z^2}{t - t_0}.$$

We shall examine in this chapter, which among these conjectures have a mathematical support :

i) The global regularity of the two dimensional Euler equation in a bounded domain has been known for a long time [134, 135, 136]. This is mainly a consequence of the conservation of vorticity. Global regularity in an unbounded domain is still unproven when no constraint is prescribed on the decrease of the solution at infinity.

ii) On the three-dimensional Euler equation, regularity of the solution during a finite time has been proved for a broad class of initial conditions ([8]) (see refs. [138, 44, 45, 139, 140, 82, 141, 142, 143, 144]), but the actual production of a singularity remains one of the most challenging mathematical problems of turbulence theory.

iii) The global regularity of the three-dimensional Navier-Stokes equation is still an open problem. Some regularity results are nevertheless known to hold for

— all times and large viscosities (typically, initial Reynolds numbers $R_0 < 1$),

— short times which, in absence of boundaries,

([8]) The first regularity result on three dimensional Euler equation goes back to Lichtenstein [137] where it is assumed that the initial vorticity is confined to a bounded domain and where persistence of regularity is shown only up to a time such that the fractional change in the vorticity is small compared to unity.

depend on the initial data but not on the viscosity (this is mainly a consequence of (ii)).

— all times and all (positive) viscosity for *modified dissipativity* $\alpha \geqslant 5/4$ [41].

iv) The global regularity of the two-dimensional Navier-Stokes equation for arbitrary viscosity in a domain without boundary results from the global regularity of the Euler equation. Proofs of global regularity in presence of boundaries are given in Ladyzhenskaya [42] and Lions [41]. The case $\nu \to 0$ is still open because of the boundary layers problems.

Extension of the results (ii), (iii) and (iv) to the MHD equations is found in ref. [145]. It is interesting to note that there are some indications that the two-dimensional MHD equations at zero viscosity and magnetic diffusivity produce a singularity after a finite time because of the non-conservation of vorticity [88, 89].

In this chapter, assuming the existence of solutions of the Euler and NS equations, we shall present in a rather simplified way the essence of the proof of the regularity results. Specifically, we shall remain at the level of the so called *a priori estimates.*

5.2 REGULARITY OF IDEAL FLOWS. — We shall follow the derivation given by Bardos and Frisch [82, 141] and Frisch and Bardos [146]. Another method based on differential geometry is presented by Ebin and Marsden [44] : It uses the fact that the solution of the Euler equation is a geodesic flow on an infinite dimensional manifold [147]. For simplicity, we shall restrict ourselves to flows with cyclic boundary conditions, though the results are also true for flows in bounded domains with slippery boundary conditions or in the entire space (with some restrictions on the initial conditions in two dimensions). It will be shown that :

— The velocity gradient is bounded as long as the vorticity is Hölder continuous, i.e. that the Hölder norm [148]

$$|\omega|_\alpha = \sup_x |\omega(x)| + \sup_{x,y} \frac{|\omega(x) - \omega(y)|}{|x - y|^\alpha} \quad (5.1)$$

with $0 < \alpha < 1$ is bounded.

— The vorticity is Hölder continuous during a finite time in three dimensions and indefinitely in two dimensions.

— If the initial velocity is n times ($n \geqslant 2$) continuously differentiable, it remains so as long as the velocity gradient is bounded.

Bound for the velocity gradient : From the definition of the vorticity, it follows that

$$\Delta u = - \operatorname{curl} \omega , \quad (5.2)$$

hence ($d = 2$ or 3),

$$u_i(x) = - \int g(x, z)\, \varepsilon_{ijk} \frac{\partial}{\partial z_j} \omega_k(z)\, d^{(d)} z \quad (5.3)$$

where ε_{ijk} is the alternating tensor and g denotes the Green's function of the Poisson equation which satisfies [149] :

$$|g(x, y)| \leqslant \begin{cases} \dfrac{1}{(d-2)\, S_d} \dfrac{1}{|x - y|^{d-2}} & d \neq 2 \\ \dfrac{1}{2\pi} \left| \ln \dfrac{1}{|x - y|} \right| & d = 2 \end{cases} \quad (5.4)$$

where S_d is the area of the unit sphere in d-dimensions [(5.4) becomes an equality when working in the entire space]. Moreover

$$|Dg(x, y)| \leqslant \frac{c_d}{|x - y|^{d-1}} \quad (5.5)$$

$$|D^2 g(x, y)| \leqslant \frac{c'_d}{|x - y|^d} \quad (5.6)$$

where D denotes any first order spatial derivative and c and c' are constants.

The boundedness of ω does not imply that of ∇u ; however, the latter is insured if we take ω Hölder-continuous. Indeed,

$$\frac{\partial u}{\partial x_l}(x) = - \int \frac{\partial}{\partial x_l} g(x, z)\, \varepsilon_{ijk} \frac{\partial}{\partial z_j} [\omega_k(z) - \omega_k(x)]\, d^{(d)}z \quad (5.7)$$

(integrating by parts)

$$= \int \frac{\partial}{\partial x_l} \frac{\partial}{\partial z_j} g(x, z)\, \varepsilon_{ijk} \frac{[\omega_k(z) - \omega_k(x)]}{|z - x|^\alpha} |z - x|^\alpha\, d^{(d)}z \quad (5.8)$$

and then

$$|\nabla u(x)| \leqslant c_d \sup_{x,x'} \frac{|\omega(x) - \omega(x')|}{|x - x'|^\alpha} \int \frac{|z - x|^\alpha}{|z - x|^d}\, d^{(d)}z . \quad (5.9)$$

It follows that, with a constant K_1 which depends on the size of the periodic domain, we have

$$\sup_x |\nabla u(x)| \leqslant \frac{K_1}{\alpha} \sup_{x,x'} \frac{|\omega(x) - \omega(x')|}{|x - x'|^{\alpha}} \leqslant \frac{K_1}{\alpha} |\omega|_{\alpha} . \tag{5.10}$$

A more refined analysis leads to the stronger result (Ladyzenskaya and Uralceva [150], Chap. III, § 2, lemma 2.2).

$$|\nabla u|_{\alpha} \leqslant K_2 |\omega|_{\alpha} . \tag{5.11}$$

5.2.1 *Finite time regularity in three dimensions.* — It is convenient to work with Lagrangian coordinates, i.e. in a frame following the stream-lines of the flow. More precisely, for a smooth enough flow, there exists a unique function

$$(a, t) \to X(a, t) \tag{5.12}$$

which will, in short, be denoted by $X(t)$, such that

$$\frac{dX}{dt}(a, t) = u(X(t), t) ; \qquad X(a, 0) = a . \tag{5.13}$$

The vorticity eq. (2.9) then reads

$$\frac{d}{dt} \omega(X(t), t) = f(X(t), t) \tag{5.14}$$

with

$$f(X(t), t) = \omega(X(t), t).\nabla u(X(t), t) \tag{5.15}$$

It follows from eq. (5.14) that

$$\frac{d}{dt} \sup_x |\omega(x, t)| \leqslant \sup_x |f(x, t)| \tag{5.16}$$

Furthermore,

$$\sup_{x_1,x_2} \frac{|\omega(x_1, t) - \omega(x_2, t)|}{|x_1 - x_2|^{\alpha}} = \sup_{a_1,a_2} \frac{|\omega(X_1(t), t) - \omega(X_2(t), t)|}{|X_1(t) - X_2(t)|^{\alpha}} \tag{5.17}$$

where $X_1(t)$ and $X_2(t)$ are the solutions of (5.13) corresponding to the initial conditions a_1 and a_2 respectively. Denoting

$$\rho(t) = |X_1(t) - X_2(t)| , \tag{5.18}$$

we get (with $\rho'(t) \equiv d\rho/dt$)

$$\frac{d}{dt} \sup_{a_1,a_2} \frac{|\omega(X_1(t), t) - \omega(X_2(t), t)|}{\rho(t)^{\alpha}} \leqslant \sup_{a_1,a_2} \frac{d}{dt} \frac{|\omega(X_1(t), t) - \omega(X_2(t), t)|}{\rho(t)^{\alpha}} \leqslant$$
$$\leqslant \sup_{a_1,a_2} \frac{|f(X_1(t), t) - f(X_2(t), t)|}{\rho(t)^{\alpha}} + \alpha \frac{|\omega(X_1(t), t) - \omega(X_2(t), t)|}{\rho(t)^{\alpha+1}} \rho'(t) . \tag{5.19}$$

Since

$$\rho'(t) \leqslant |u(X_1(t), t) - u(X_2(t), t)| \leqslant \sup_x |\nabla u(x, t)| \rho(t) , \tag{5.20}$$

we obtain upon adding eqs. (5.14) and (5.19)

$$\frac{d}{dt} |\omega|_{\alpha} \leqslant |f|_{\alpha} + \alpha \sup_x |\nabla u| |\omega|_{\alpha} . \tag{5.21}$$

Noticing that

$$|f|_{\alpha} = |\omega.\nabla u|_{\alpha} \leqslant |\omega|_{\alpha} |\nabla u|_{\alpha} \tag{5.22}$$

and using (5.11), we finally obtain

$$\frac{d}{dt} |\omega|_{\alpha} \leqslant K_3 |\omega|_{\alpha}^2 \tag{5.23}$$

which implies

$$|\omega(t)|_{\alpha} \leqslant \frac{|\omega(0)|_{\alpha}}{1 - K_3 |\omega(0)|_{\alpha} t} . \tag{5.24}$$

Thus, if the vorticity is initially bounded in Hölder norm, it remains so at least up to $T_* \sim 1/|\omega_0|_{\alpha}$.

Bound for higher order derivatives of the vorticity. Taking successive derivatives of the vorticity equation and using (5.11) to bound the gradient of the velocity derivatives by the same order vorticity derivatives, we prove by induction that *initial boundedness of any order velocity derivatives persists at least up to T_*.*

Remark : The regularity results can be extended from periodic domains to bounded or unlimited three-dimensional flows. In the latter case the time up to which regularity is insured varies like [141]

$$T^* \sim [|u_0|_\alpha + |\omega_0|_\alpha]^{-1}.$$

5.2.2 *Global regularity in two dimensions.* — To show that if the vorticity is initially Hölder-continuous, it remains so indefinitely, we make use of vorticity conservation and of an estimate on temporal evolution of pair separation; to obtain the later, we first use an

Estimate on velocity increments (true for $d = 2$ and $d = 3$) :

$$|u(x,t) - u(y,t)| \leqslant$$
$$\leqslant C \sup_x |\omega(x,t)| \, |x-y| \ln \frac{eL}{|x-y|} \quad (5.25)$$

were C is a constant, L the diameter of the fundamental periodic domain and $e = \exp 1$.

To prove eq. (5.25), we start from eq. (5.3) which leads to

$$|u(x,t) - u(y,t)| \leqslant$$
$$\leqslant \sup_x |\omega(x,t)| \int \left| \frac{\partial}{\partial z} g(x,z) - \frac{\partial}{\partial z} g(y,z) \right| d^{(d)}z \quad (5.26)$$

We then use in (5.26) the lemma 1.4 of Kato (9) [151]

$$\int \left| \frac{\partial}{\partial z} g(x,z) - \frac{\partial}{\partial z} g(y,z) \right| d^{(d)}z \leqslant$$
$$\leqslant C |x-y| \ln \frac{eL}{|x-y|}, \quad (5.27)$$

to obtain eq. (5.25).

Evolution of pair-separation (true in $d = 2$ and $d = 3$) : Let $X(t)$ and $Y(t)$ be the trajectory of two points advected by the flow. Let their separation

$$\rho(t) = |X(t) - Y(t)|$$

be initially ρ_0; then at any later time it is bounded from above and below by

$$\left(\frac{\rho_0}{eL}\right)^{\exp[C\int_0^t \sup_x|\omega(x,t)|d\tau]} \leqslant \frac{\rho(t)}{eL}$$
$$\leqslant \left(\frac{\rho_0}{eL}\right)^{\exp[-C\int_0^t \sup|\omega(x,\tau)|d\tau]}. \quad (5.28)$$

The proof consists in writting, from (5.25)

$$\left|\frac{d}{dt}\rho(t)\right| \leqslant |u(X(t),t) - u(Y(t),t)| \leqslant$$
$$\leqslant C \sup_x |\omega(x,t)| \, \rho(t) \ln\left(\frac{eL}{\rho(t)}\right) \quad (5.29)$$

and integrating forward and backward, to obtain (5.28).

An initially Hölder-continuous vorticity remains so with a smaller α. From vorticity conservation, we have

$$|\omega_0|_\alpha = \sup_x |\omega(x,t)| +$$
$$+ \sup_{X(0),Y(0)} \frac{|\omega(X(t).t) - \omega(Y(t).t)|}{\rho_0^\alpha} \quad (5.30)$$

with $\rho_0 = |X(0) - Y(0)|$. By (5.28),

$$\rho_0 \leqslant eL\left(\frac{\rho(t)}{eL}\right)^{\exp[-C\sup_x|\omega_0(x)|t]}, \quad (5.31)$$

and finally

$$|\omega_0|_\alpha \geqslant C(L) \, |\omega(t)|_{\alpha(t)} \quad (5.32)$$

with

$$\alpha(t) = \alpha \, e^{-C\sup|\omega_0|t}. \quad (5.33)$$

Bound for the velocity gradient. Since the persistence of Hölder-continuity for the vorticity is insured with an exponent given by (5.33), the bound for the vorticity gradient given by (5.10) becomes

$$\sup_x |\nabla u(x,t)| \leqslant \frac{K}{\alpha(t)} |\omega(t)|_{\alpha(t)} \leqslant$$
$$\leqslant K' |\omega_0|_\alpha \exp\left[Ct \sup_x |\omega_0(x)|\right]. \quad (5.34)$$

Bound for the vorticity gradient. Taking any first order space derivative of the vorticity equation, we get

$$\frac{\partial}{\partial t} D\omega + (u.\nabla) D\omega + (Du.\nabla)\omega = 0. \quad (5.35)$$

Since the two first terms represent the derivative in Lagrangian coordinates, we obtain for the supremum of the vorticity gradient

$$\frac{d}{dt} \sup_x |\nabla\omega| \leqslant \sup_x |\nabla u(x,t)| \sup_x |\nabla\omega(x,t)|. \quad (5.36)$$

Hence, by (5.34)

$$\sup_x |\nabla\omega(x,t)| \leqslant \sup_x |\nabla\omega_0(x)| \times$$
$$\times \exp\left\{\int_0^t C |\omega_0|_\alpha \exp\left[K' \sup_x |\omega_0(x)| \tau\right] d\tau\right\}. \quad (5.37)$$

(9) There is a misprint in ref. [151] lemma 1.4, p. 191. $\chi(s) = 1$ for $s \geqslant 1$ must be read instead of $\chi(s) = 0$.

Bound for higher order derivatives of vorticity. Similar results involving exponentials of exponentials of exponentials... etc. are obtained by induction for higher order derivatives of the vorticity and therefore of the velocity. *This proves that the solution of the two-dimensional Euler equation with C^∞ initial conditions remains so for all times.*

5.2.3 *Analyticity.* — A stronger result than regularity has been established on the two and three-dimensional Euler equations [152, 47], namely that for analytic initial conditions, the solution remains analytic both in space and time as long as the velocity gradient remains bounded. The essence of the proof of spatial (resp. temporal) analyticity consists in extending the *a priori* estimate (5.23) and (5.32) to a suitably chosen complex domain of spatial (resp. spatial and temporal) arguments. The first result on finite time analyticity of the three-dimensional Euler equation has actually been obtained as a special case of a general theorem due to Baouendi and Goulaouic [153], which does not however give an explicit lower bound for the time of analyticity.

5.3 REGULARITY OF VISCOUS THREE-DIMENSIONAL FLOWS. — The regularity of the Euler equation (global in two dimensions and during a finite time in three dimensions) is easily extended to the Navier-Stokes equation, at least in the absence of boundaries, because of the regularizing properties of the Laplacian. In addition, supplementary results have been obtained on the NS equation. First, global existence of a weak solution has been proved for arbitrary positive viscosity (Leray [154], see also Lions [41]), whereas at zero viscosity the existence is known only during the regularity time. But the global regularity of the NS equation for small viscosities remains an open problem. For simplicity, we shall restrict ourselves in this section, to flows extending in the entire $\mathbb{R}^3$-space. Those readers interested to the NS equation in bounded domains (with rigid boundary conditions) are referred to Lions [41].

For the NS equation it is useful to have a framework in which energy and its dissipation are simply related to the various norms. This is provided by a generalization of the L^2 space, the Sobolev space $H^s(R^3)$ with the norms

$$\| f \|_{H^s}^2 = \| f \|_{L^2}^2 + \sum_{i=1}^{3} \| D_i^s f \|_{L^2}^2 \qquad (5.38)$$

wherein

$$D_i = \frac{\partial}{\partial x_i}.$$

An equivalent norm is

$$\| f \|_{H^s}^2 = \| f \|_{L^2}^2 + \| \, | k |^s \hat{f} \, \|_{L^2}^2 \qquad (5.38')$$

or

$$\| f \|_{H^s}^2 = \| (1 + k^2)^{s/2} \hat{f} \|_{L^2}^2 \qquad (5.39)$$

where $\hat{f}$ denotes the Fourier transform of f and the L^2-norm $|\cdot|_{L^2}$ is defined by

$$\| f \|_{L^2}^2 = \int | f(x) |^2 \, d^{(d)}x \sim \int | \hat{f}(k) |^2 \, d^{(d)}k \,. \qquad (5.40)$$

The use of Sobolev norms, defined as integrals over the domain, is limited to finite energy flows. Notice that the definition in Fourier space makes sense even for non integer values of s.

5.3.1 *Finite time regularity for arbitrary viscosity.* — The essence of the method consists in obtaining estimates for the Sobolev norms of the solution of the NS equation, by writing a closed inequality for the smallest positive-order norm. If, as in Kato [139], we restrict ourselves to integer orders, this can be done for $\| u \|_{H^3}^2$. Let $b(u, v, w)$ denote the L^2-scalar product of $(u . \nabla) v$ with w,

$$b(u, v, w) \equiv ((u . \nabla) v, w) \equiv \int (u . \nabla) v . w \, d^{(d)}x \,. \qquad (5.41)$$

For divergenceless fields which are smooth enough and vanish at infinity,

$$b(u, v, w) = - b(u, w, v) \qquad (5.42)$$

and the L^2-scalar product of such a field with a gradient (e.g. the pressure force) vanishes. The energy equation follows

$$\frac{1}{2} \frac{\partial}{\partial t} \| u \|_{L^2}^2 = - \nu \| \nabla u \|_{L^2}^2 \,. \qquad (5.43)$$

Now, taking three spatial derivatives of the NS equation, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| D^3 u \|_{L^2}^2 + \nu \| D^3 \nabla u \|_{L^2} &= \\ &= - b(D^3 u, u, D^3 u) - 3 \, b(D^3 u, Du, D^3 u) \\ &\quad - 3 \, b(Du, D^2 u, D^3 u) \,. \end{aligned} \qquad (5.44)$$

We then use the Schwarz inequality and the following inequalities true for scalar functions defined on $\mathbb{R}^3$ [155]

$$\| fg \|_{L^2} \leqslant \begin{cases} C \| f \|_{H^1} \| g \|_{H^1} \\ C' \| f \|_{L^2} \| g \|_{H^2} \end{cases} . \qquad (5.45)$$

to obtain

$$\left. \begin{array}{l} | b(D^3 u, u, D^3 u) | \\ | b(D^2 u, Du, D^2 u) | \\ | b(Du, Du, D^3 u) | \end{array} \right\} \leqslant C_1 \| u \|_{H^3}^3 \,. \qquad (5.46)$$

It follows from eqs. (5.44) and (5.46) that

$$\frac{1}{2} \frac{d}{dt} \| u \|_{H^3}^2 + \nu \| \nabla u \|_{H^3}^2 \leqslant C_2 \| u(t) \|_{H^3}^3 \qquad (5.47)$$

$\nu \| Du \|^2_{H^3}$ being non negative, this implies

$$\| u(t) \|_{H^3} \leqslant \frac{\| u_0 \|_{H^3}}{1 - C_2 \| u_0 \|_{H^3} t}. \quad (5.48)$$

Higher order Sobolev norms ($s > 3$) satisfy

$$\frac{d}{dt} \| u(t) \|^2_{H^s} \leqslant C_3 \| u(t) \|_{H^3} \| u(t) \|^2_{H^s}. \quad (5.49)$$

This implies that *if a solution of the three-dimensional NS equation, with arbitrary viscosity* $\nu \geqslant 0$, *initially belongs to* $H^s(R^3)$ *with* $s \geqslant 3$, *it remains so at least up to a time* $T_* \sim 1/\| u_0 \|_{H^3}$ [139].

Remark : An improvement of this result corresponding to the replacement of 3 by $5/2 + \varepsilon$ ($\varepsilon > 0$) is presented in Section 5.3.3.

5.3.2 *Global regularity for « large » viscosity.* — This result was first obtained by Leray [154], a shorter derivation can be based on (5.47). Using (5.38′), we rewrite (5.47)

$$\frac{1}{2} \frac{d}{dt} \| u(t) \|^2_{H^3} + \nu \| u(t) \|^2_{H^4} \leqslant$$
$$\leqslant C_2 \| u(t) \|^3_{H^3} + \nu \| u(t) \|^2_{L^2} ; \quad (5.50)$$

since

$$\| u(t) \|^2_{H^4} \geqslant \| u(t) \|^2_{H^3} \quad \text{and} \quad \| u(t) \|^2_{L^2} \leqslant \| u_0 \|^2_{L^2} , \quad (5.51)$$

we also have

$$\frac{1}{2} \frac{d}{dt} \| u(t) \|^2_{H^3} \leqslant$$
$$\leqslant \| u(t) \|^2_{H^3} [C_2 \| u(t) \|_{H^3} - \nu] + \nu \| u_0 \|^2_{L^2} . \quad (5.52)$$

Denoting $y(t) = \| u(t) \|^2_{H^3}$, eq. (5.52) reads

$$\frac{dy}{dt} \leqslant H(y) \equiv C_2 y^{3/2} - \nu y + \nu \| u_0 \|^2_{L^2} . \quad (5.53)$$

If

$$\nu > \nu_0 = \frac{C_2 \| u_0 \|^3_{H^3}}{\| u_0 \|^2_{H^3} - \| u_0 \|^2_{L^2}}, \quad (5.54)$$

$H(y(0))$ is negative and $y(t)$ first decreases down to the first zero α of $H(y)$ (cf. Fig. 4). Then, y cannot increase above α because in such a situation y' would be positive whereas $H(y)$ would be negative. It follows that $y(t) < y(0)$, or

$$\| u(t) \|_{H^3} \leqslant \| u_0 \|_{H^3} . \quad (5.55)$$

We conclude that *for sufficiently large viscosity* (eq. 5.54), *or equivalently sufficient small initial Reynolds number, the solution of the NS equation remains regular for all times.*

5.3.3 *Global regularity for increased « dissipativity ».* — We are interested in this section in

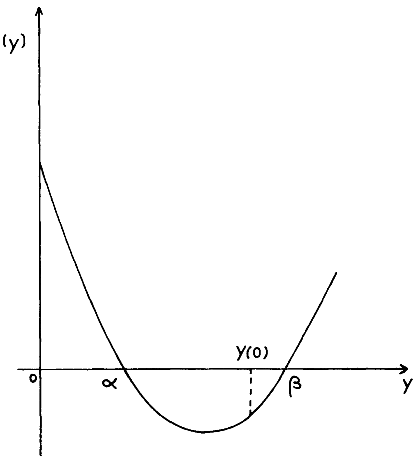


FIG. 4. — Global regularity for large viscosity : graph of $H(y)$ defined in eq. (5.53).

the NS equation in which the viscous term $\nu \Delta u$ is replaced by $-\nu(-\Delta)^\alpha u$ where the *dissipativity* α is a non-negative real number. In Fourier space, this term reads $- | k |^{2\alpha} \hat{u}$. This problem was first considered by Ladyzhenskaya (1963) who proved global regularity for $\alpha = 2$ and arbitrary viscosity. Lions (1969), using Sobolev space interpolation techniques, proved the same result for $\alpha \geqslant 5/4$. We shall give here a more direct proof (valid only for $\alpha > 5/4$) which is based on a generalization of the Kato estimate (3.47) where H^3 is replaced by H^s with $s > 5/2$. This requires

A *non-integral order Leibnitz formula* : For $s > 1 + \frac{d}{2}$ and $\frac{d}{2} < \gamma \leqslant s - 1$ (d = space dimension), we have (proof given in Temam [143])

$$\| D^s_i(fg) - f D^s_i g \|_{L^2} \leqslant C(\gamma, s, d) \times$$
$$\times \{ \| f \|_{H^s} \| g \|_{H^s} + \| f \|_{H^{\gamma+1}} \| g \|_{H^{s-1}} \} . \quad (5.54)$$

Now, following Bardos *et al.* [127] and Temam [143], we deduce from the Euler equation that for $s \geqslant 0$,

$$\frac{1}{2} \frac{d}{dt} \| D^s_i u \|^2_{L^2} = (D^s_i(u.\nabla u), D^s_i u)$$
$$= (D^s_i(u.\nabla u) - u D^s_i \nabla u, D^s_i u) +$$
$$+ (u D^s_i \nabla u, D^s_i u) . \quad (5.55)$$

Notice that

$$(u D^s_i \nabla u, D^s_i u) = b(u, D^s_i u, D^s_i u) = 0 . \quad (5.56)$$

Then, using the Schwarz inequality and the above Leibnitz formula, we obtain

$$| (D^s_i(u.\nabla u), D^s_i u) | \leqslant C(\gamma, s) \times$$
$$\times \{ \| u \|_{H^s} \| \nabla u \|_{H^\gamma} + \| u \|_{H^{\gamma+1}} \| \nabla u \|_{H^{s-1}} \} \| D^s_i u \|_{L^2} . \quad (5.57)$$

Finally we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| u \|_{\mathrm{H}^s}^2 \leqslant C(\gamma, s) \| u \|_{\mathrm{H}^s}^2 \| u \|_{\mathrm{H}^{\gamma+1}} \quad (5.58)$$

with $s \leqslant 1 + \gamma$ and $\gamma > 3/2$.

In particular, for $s = \gamma + 1 > 5/2$, this reads

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| u \|_{\mathrm{H}^s}^2 \leqslant C_s \| u \|_{\mathrm{H}^s}^3 . \quad (5.59)$$

It follows that *a solution of the Euler or NS equation which initially belongs to* H^s *with* $s > 5/2$, *remains so at least up to a time* $T_* \sim 1/\| u_0 \|_{\mathrm{H}^s}$.

In the presence of a dissipative term $-\nu(-\Delta)^\alpha u$, eq. (5.59) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| u \|_{\mathrm{H}^s}^2 + \nu \| u \|_{\mathrm{H}^{s+\alpha}}^2 \leqslant C_s \| u \|_{\mathrm{H}^s}^3 + \nu \| u \|_{\mathrm{L}^2}^2 . \quad (5.60)$$

To estimate the dissipative term, we need the *lemma* : if $0 \leqslant s_1 \leqslant s_2 \leqslant s_3$,

$$\| u \|_{\mathrm{H}^{s_2}}^{s_3 - s_1} \leqslant \| u \|_{\mathrm{H}^{s_1}}^{s_3 - s_2} \| u \|_{\mathrm{H}^{s_3}}^{s_2 - s_1} . \quad (5.61)$$

The proof of the lemma is readily obtained by noticing that if

$$p = \frac{s_3 - s_1}{s_3 - s_2} \quad \text{and} \quad q = \frac{s_3 - s_1}{s_2 - s_1}$$

then

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad \frac{s_1}{p} + \frac{s_3}{q} = s_2 .$$

Eq. (5.61) is then a direct consequence of the Hölder inequality

$$\left| \int fg \, \mathrm{d}^{(d)}x \right| \leqslant \left| \int f^p \, \mathrm{d}^{(d)}x \right|^{1/p} \left| \int g^q \, \mathrm{d}^{(d)}x \right|^{1/q} \quad (5.62)$$

applied to

$$\| u \|_{\mathrm{H}^{s_2}}^2 = \int [(1 + k^2)^{s_1/p} \, | \hat{u}(\mathbf{k}) |^{2/p}] \times [(1 + k^2)^{s_3/q} \, | \hat{u}(\mathbf{k}) |^{2/q}] \, \mathrm{d}^{(d)}k . \quad (5.63)$$

Taking $s_1 = 0$, $s_2 = s$, $s_3 = s + \alpha$ in eq. (5.63), we obtain

$$\| u \|_{\mathrm{H}^{s+\alpha}}^2 \geqslant \frac{\| u \|_{\mathrm{H}^s}^{2\frac{(s+\alpha)}{s}}}{\| u \|_{\mathrm{L}^2}^{2\alpha/s}} \geqslant \frac{\| u \|_{\mathrm{H}^s}^{2\frac{(s+\alpha)}{s}}}{\| u_0 \|_{\mathrm{L}^2}^{2\alpha/s}} . \quad (5.64)$$

Eq. (5.60) finally reads

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| u \|_{\mathrm{H}^s}^2 \leqslant \| u \|_{\mathrm{H}^s}^3 \left[C_s - \frac{\nu}{\| u_0 \|_{\mathrm{L}^2}^{2\alpha/s}} \| u \|_{\mathrm{H}^s}^{\frac{2\alpha - s}{s}} \right] + \nu \| u_0 \|_{\mathrm{L}^2}^2 \quad (5.65)$$

with $s > 5/2$.

Denoting $y(t) = \| u(t) \|_{\mathrm{H}^s}^2$, eq. (5.65) reads

$$\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}t} \leqslant h(y) \equiv y^{3/2}\left(C_s - \frac{\nu}{\| u_0 \|_{\mathrm{L}^2}^{2\alpha/s}} y^{\frac{2\alpha - s}{2s}} \right) + \nu \| u_0 \|_{\mathrm{L}^2}^2 . \quad (5.66)$$

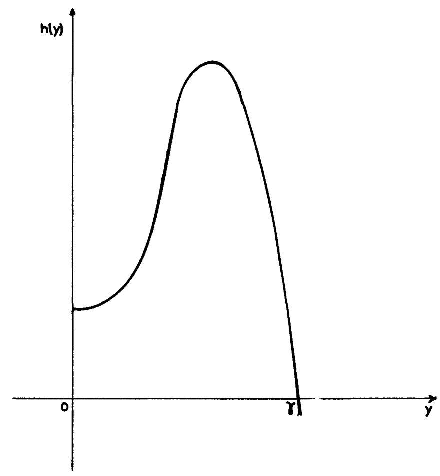


FIG. 5. — Global regularity for increased *dissipativity* : graph of $h(y)$ defined in eq. (5.66).

If $\alpha > s/2$, the aspect of $h(y)$ is represented in figure 5. If the unique zero of h, namely γ, is less than $y(0)$, then $h(y(0)) < 0$ and $y(t)$ first decreases down to γ. Then, y cannot increase above γ because this would simultaneously make $y' > 0$ and $h(y) < 0$. For the same reason if $y(0) \leqslant \gamma$, y remains so indefinitely. If follows that

$$y(t) \leqslant \sup (y(0), \gamma) . \quad (5.67)$$

Consequently, *for arbitrary positive viscosity, a solution of the NS equation with a dissipativity* $\alpha > 5/4$, *which initially belongs to* H^s *with* $s > 5/2$, *remains so indefinitely* [128].

Remark 1. — *Bounds* for statistical quantities have been obtained rigorously by Howard [156, 157] and Busse [158] for flows in domains bounded in at least one direction. For unbounded homogeneous isotropic turbulence, it appears difficult to derive exact bounds from the NS equation or from the equivalent hierarchy of moment-equations. The main reason is that high-order moments are not bounded from *above* by a suitable combination of lower-order moments, but only from *below*. For unbounded flows of finite energy where one can use integrals instead of averages, an upper bound on the energy flux in the inertial range has been obtained by Sulem and Frisch [159] ; see also ref. [88].

Remark 2. — *The random NS equation* with a forcing term which is a temporal white noise, has been investigated by Bensoussan and Temam [160], but the results are somewhat weaker than in the deterministic case.

Remark 3. — *Possible singularities of the NS equation* : Global regularity of the NS equation remains unproven and actually controversal. Scheffer [161, 162] has shown that if the solution of the NS equation has singularities, then the Hausdorff dimension (see Kahane [163]) of the set of singular times is at most 1/2, and the Hausdorff dimension of the spatial support of the singularities at a singular time is at most 1. This does not of course imply the actual existence of singularities in the viscous case (conjectured by Leray [154]). The direct numerical simulations at Reynolds numbers up to about a hundred times larger than required by the regularity theorem 5.3.2, give evidence against such singularities. So do the mathematically rigorous results derived from spectral equations (cf. section 6.2). Hopefully, Scheffer's results will be useful in proving that the set of singularities is empty.

6. **Spectral equations.** — 6.1 THE SPECTRAL EQUATIONS AS SEMI-HEURISTIC CLOSURES. — The experimental results on fully developed turbulence indicate that the solutions to the NS equation exhibit a self similar range [164], and, as we have just seen, this is related to the possible appearance of singularities in the limit of zero viscosity. As guidelines for a possible theory, let us list somes fundamental structural properties of the NS equation which must be preserved [165].

i) Invariance under space translations, rotations, reflections. Conservation of energy, helicity in three dimensions, and enstrophy in two dimensions by the inertial terms.

ii) Existence of absolute equilibrium solutions for the inviscid and unforced truncated equations.

iii) Positivity of the energy spectrum.

iv) Invariance of energy transfer under random Galilean transformations.

The importance of (i) is clear ; (ii) also is relevant because there is always some tendency to locally achieve absolute equilibrium (Léorat [110], section 2.7) ; see also the argument in chapter 3 relating absolute equilibrium and transfer ; (iii) is related to the fact that we are interested in the evolution of the spectrum. Should the spectrum become negative, spurious instabilities might develop [166] ; (iv) is one of the consequences of the locality of energy transfer in Fourier space. A random Galilean transformation is induced on a given statistical ensemble of initial conditions by adding to the velocity field of each realization a spatially uniform, time independent isotropic random vector $\mathbf{V}_0$, with zero mean and variance $\langle V_0^2 \rangle$. Since energy transfer is a consequence of eddy distortion, a uniform velocity field such as $\mathbf{V}_0$ has no effect on it, and hence it is also invariant after ensemble averaging over $\mathbf{V}_0$. The relevance of (iv) for a theory of fully developed turbulence follows from the presence of an energy concentration at low wavenumbers. Most of the effect of the large, strongly energetic eddies on the small inertial range eddies is simple uniform advection. An approximate theory which fails to satisfy criteria (iv) will overestimate this effect and predict incorrect inertial range behaviour.

There are two frameworks in which approximations to turbulence can be constructed, the Eulerian and the Lagrangian. The Eulerian representation is the one used in eq. (1.1). The Lagrangian representation is essentially characterized by using as one of the independent variables, the position of each fluid element at a given reference time. This means using a coordinate system which follows the motion of the fluid. In both frameworks, it is possible to make a very close analogy between the algebraic structure of the statistical theory of turbulence (and more generally, classical non-Hamiltonian systems) and that of quantum field theory [104, 167-178].

Eulerian perturbation theory : The simplest renormalized approximation in the Eulerian framework is the Direct Interaction Approximation (DIA) of Kraichnan [179]. This leads to a set of closed integro-differential equations for the two-time correlation and the infinitesimal response (Green's) function (cf. Ref. [29] for details). For the inertial range of homogeneous turbulence, this theory has the severe defect of being non-invariant under a random Galilean transformation [180]. Consider a given realization of a Galilean transformation, with $\mathbf{V}_0$ the uniform velocity field. This modifies the solution to the NS equation as follows

$$\hat{u}(\mathbf{k}, t) \to \mathrm{e}^{-i\mathbf{k}.\mathbf{V}_0 t}\, \hat{u}(\mathbf{k}, 0) \quad \text{for} \quad \mathbf{k} \neq 0 \,. \quad (6.1)$$

If $\mathbf{V}_0$ is chosen to be Gaussian and isotropic, the effect on the covariance

$$U_{ij}(\mathbf{k} \mid t, t') = \langle u_i(\mathbf{k}, t)\, u_j(-\mathbf{k}, t') \rangle \quad (6.2)$$

and the Green's function

$$G(\mathbf{k} \mid t, t') = \delta \langle u(\mathbf{k}, t) \rangle / \delta \langle f(\mathbf{k}, t') \rangle \quad (6.3)$$

is

$$\begin{pmatrix} U(k \mid t, t') \\ G(k \mid t, t') \end{pmatrix} \to \exp[-\tfrac{1}{6} k^2 \langle V_0^2 \rangle (t-t')^2] \begin{pmatrix} U(k \mid t, t') \\ G(k \mid t, t') \end{pmatrix}. \quad (6.4)$$

For homogeneous turbulence, equal-time correlation functions, and therefore energy transfer are invariant under a Galilean transformation. The Eulerian DIA, or any approximation which works directly with two-time covariances, and Green's functions will have difficulty in faithfully representing this invariance because there is no way for the exponential factors in (6.4) to simply drop out of an energy transfer calculation. As a consequence, they predict an inertial energy spectrum $E(k) \sim (\varepsilon V_0)^{1/2} k^{-3/2}$ where V_0 is the r.m.s. velocity [179].

Lagrangian perturbation theory : A description of the traditional Lagrangian representation is found in Tennekes and Lumley [31], and the relation between Eulerian and Lagrangian correlation functions is investigated in Weinstock [181]. We will restrict ourselves to the particular representation introduced by Kraichnan [182, 183, 184], which uses a generalized velocity field $u(\mathbf{x}, t \mid s)$, defined as the velocity measured at time s in a fluid element which passes through $\mathbf{x}$ at time t. Kraichnan has constructed two heuristic approximations within this framework, the *Lagrangian History Direct Interaction* (LHDI) and *Abridged Lagrangian History Direct Interaction* (ALHDI) approximations, which are random Galilean invariant and yield K41 spectra. Kraichnan [178] has shown that these approximations can be obtained as the lowest order of a systematic renormalized perturbation theory. In addition to being Galilean invariant, the Lagrangian representation has the advantage of capturing more of the physics than the Eulerian one at a given order of perturbation theory. For example, Kraichnan [185] has used the Lagrangian DIA to calculate the effects of helicity on a passive scalar and on magnetic field. As shown by Kraichnan [186, 187], such effects are missed by the Eulerian DIA and require higher order approximations, the so-called vertex approximations. Note also that the Lagrangian DIA and the Eulerian vertex renormalized approximations conserve energy and have absolute equilibria as solutions. However, it is still unknown whether they do or do not maintain the positivity of the energy spectrum.

In view of the above difficulties, a class of phenomenologically-inspired Eulerian closures working only with the energy spectrum has been developed. Such methods, referred to as *spectral equations* have been the object of both analytic and numerical studies at huge Reynolds numbers. There are several ways to establish spectral equations. Let us begin with the relationship to the formal calculation of short-time behaviour of the NS equation for Gaussian initial conditions. For this purpose, an abstract representation of the equation of motion which only makes explicit the quadratic nonlinearity is sufficient,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = uu \,. \tag{6.5}$$

We have dropped the viscous term which poses no closure problem.

Taking zero-mean Gaussian initial conditions, $u(0)$, and writing (we ignore numerical factors)

$$u(t) = u(0) + t\,\frac{\mathrm{d}u(0)}{\mathrm{d}t} + t^2\,\frac{\mathrm{d}^2u(0)}{\mathrm{d}t^2} + \mathrm{O}(t^3) \tag{6.6}$$

we obtain, after expressing fourth order moments as sum of products of second order moments

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\,u(t)\rangle = t\,\langle u(0)\,u(0)\rangle\,\langle u(0)\,u(0)\rangle + \mathrm{O}(t^2)\,. \tag{6.7}$$

Reverting the expression (6.6) to lowest order and inserting in (6.7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\,u(t)\rangle = t\,\langle u(t)\,u(t)\rangle\,\langle u(t)\,u(t)\rangle + \mathrm{O}(t^2)\,. \tag{6.8}$$

The non-autonomous character in this equation (appearance of a factor t in the r.h.s.) is due to the special choice of initial conditions (zero triple correlations) which singles out the initial time.

For the explicit calculation, we write the NS equation in Fourier space (eq. (2.10)), take into account homogeneity, isotropy and mirror symmetry, and obtain (cf. appendix 2 and also ref. [26] section 4.4 where the three-dimensional case is treated in the context of the quasi-normal approximation) :

$$\begin{aligned}\frac{\partial}{\partial t}E(k, t) = C_d \iint_{\Delta_k} t\,\frac{k^{4-d}}{pq}(1 - x^2)^{\frac{d-3}{2}} \times \\ \times\,[a_{kpq}^{(d)}\,k^{d-1}\,E(p, t)\,E(q, t) \\ - b_{kpq}^{(d)}\,p^{d-1}\,E(q, t)\,E(k, t)]\,\mathrm{d}p\,\mathrm{d}q + \mathrm{O}(t^2)\end{aligned} \tag{6.9}$$

d is the dimension of the physical space. The integration in the (p, q) plane extends over the domain Δ_k such that k, p, q can form the sides of a triangle (see Appendix 2). The coefficient $b_{kpq}^{(d)}$ is given by

$$b_{kpq}^{(d)} = \frac{1}{2}\,\frac{p}{k}\,[(d - 3)\,z + (d - 1)\,xy + 2\,z^3]\,, \tag{6.10}$$

where x, y, z are the cosines of the interior angles of the (k, p, q) triangle, and

$$a_{kpq}^{(d)} = \tfrac{1}{2}[b_{kpq}^{(d)} + b_{kqp}^{(d)}]\,. \tag{6.11}$$

The numerical coefficient C_d is related to the surface

$$S_d = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}$$

of the unit sphere in d dimensions by

$$C_d = \frac{4\,S_{d-1}}{(d - 1)^2\,S_d}\,. \tag{6.12}$$

In particular

$$C_2 = 4/\pi\,, \qquad C_3 = 1/2\,. \tag{6.13}$$

Higher order terms in the expansion (6.9) can be obtained similarly. But what do we know about the convergence properties of such an expansion ? *A priori*, there is no reason to believe that the formal Taylor series has more than zero radius of convergence [165, 188, 189]. Indeed, individual realization of the Euler equation in any dimension $d > 2$ are likely to blow up at a finite time which, by the Gaussian assumption can be arbitrarily close to $t = 0$. Fournier and Frisch [190] have investigated this question on Burgers' equation [191] which is known to produce singularities at a finite time. They have shown that the formal Taylor series in powers of t of the energy spectrum has, for fixed wavenumber, an infinite radius of convergence. There are also strong indications that the formal solution differs from the true ($\nu \to 0$) solution by a non-analytic function with an identically vanishing Taylor series, something like $\exp(-1/t^2)$. For the NS equation, the convergence properties of the formal expansion are unknown. Still, we shall assume that homogeneous isotropic turbulence can be defined for arbitrary (non integral) spatial dimensions by analytically continuing the formal expansion, term by term, as functions of dimension. In integral ($d \geqslant 2$) dimensions, the energy spectrum is by definition non-negative because it is realized as the mean square of the Fourier component of the velocity. A realizability problem can occur only by making some approximations. This is not so any more in non-integral dimensions since the analytic continuation of a positive function need not be positive. Realizability for $d > 2$ is still an open problem. At least it can be shown that for $d < 2$, if realizability holds at $t = 0$, it may be lost for arbitrary small positive times (Frisch, Lesieur and Sulem [192], Fournier and Frisch [193] where a proof is given in Appendix A).

The Taylor expansion (6.9) allows us to calculate the energy spectrum for short times ([10]). We have already noticed that (6.9) is non-autonomous. For sufficiently long times, the memory that the triple correlations were initialy zero should be lost and an autonomous equation seems more appropriate. Such an equation can be obtained from (6.9) by various heuristic modifications which we shall now outline.

Direct numerical simulations of the NS equation indicate that triple correlations saturate in a time of the order of the eddy turnover-time [203]. This suggests replacing the factor t by some eddy-turnover time θ (for times large compared to θ). If we also want to make contact with the K41 theory which is local in wavenumber space, we should use an expression θ_{kpq} involving only the local eddy turnover times $\tau(k)$, $\tau(p)$, $\tau(q)$ (see Appendix 1) of the interacting triad (k, p, q). Finally, in order not to loose energy conservation, it is convenient to assume complete symmetry of θ_{kpq} in k, p, q ([11]). We are then led to the following equation for the energy spectrum (viscosity reinserted)

$$\frac{\partial}{\partial t} E(k,t) + 2\nu k^2 E(k,t) = T(k,t) \equiv C_d \iint_{\Delta_k} \theta_{kpq}(t) \times \frac{k^{4-d}}{pq}(1-x^2)^{\frac{d-3}{2}} \left[a_{kpq}^{(d)} k^{d-1} E(p,t) E(q,t) - b_{kpq}^{(d)} p^{d-1} E(q,t) E(k,t)\right] \mathrm{d}p\, \mathrm{d}q \,. \tag{6.14}$$

A simple choice for θ_{kpq} with all the above requirements and which reduces to t for small times is

$$\theta_{kpq}(t) = \frac{1 - \exp[-\mu_{kpq}(t)\, t]}{\mu_{kpq}(t)} \tag{6.15}$$

with

$$\mu_{kpq}(t) = \mu_k(t) + \mu_p(t) + \mu_q(t) \tag{6.16}$$

$$\mu_k(t) = \nu k^2 + \lambda_d \left[\int_0^k p^2 E(p,t)\, \mathrm{d}p\right]^{1/2}. \tag{6.17}$$

The μ_k's are called eddy-damping rates (note that the viscous rate νk^2 is negligible except in the dissipation range). λ_d is a free numerical parameter of the approximation which can be adjusted to give a constant in front of the $k^{-5/3}$ law in agreement with the experimental data [113, 130].

An equation of the form (6.14) with a different (non-local) choice of θ_{kpq} has been introduced for the first time by Kraichnan [204]. Orszag [205, 206] and Orszag and Kruskal [207] have used

$$\theta_{kpq} = (\mu_{kpq})^{-1} \tag{6.18a}$$

$$\mu_{kpq} = \mu_k + \mu_p + \mu_q \tag{6.18b}$$

$$\mu_k = \nu k^2 + \lambda (k^3 E(k))^{1/2} \tag{6.18c}$$

equivalent to the above choice when one assumes stationarity and a power law spectrum shallower than k^{-3}. Expression (6.15) with (6.18b, c) which is also appropriate for short times has been proposed by Leith [208]. We shall refer to all these equations as the Eddy-Damped Quasi-Normal Markovian (EDQNM) approximation.

From the method of construction it follows that the absolute equilibria spectra are time independent solutions to the truncated, inviscid, EDQNM. This is a consequence of the Gaussianity and time independence of the absolute equilibria solutions to the truncated Euler equation, and the exactness of the inviscid EDQNM for short time. Since the choice of μ_k in (6.18) is not affected by adding to $E(p)$ a δ-function at $p = 0$; the energy transfer function is invariant under a random Galilean transformation, and hence the EDQNM satisfies the criteria for a spectral equation which were listed at the begining of this section. The positivity of the spectrum is guaranteed by that of a_{kpq}. In Appendix 2, it is proved that $a_{kpq} \geqslant 0$ if and only if $d \geqslant 2$.

Eq. (6.14) describes not only the EDQNM but several other spectral equations each determined by the choice of θ_{kpq}.

([10]) If t is replaced by $\int_0^t \mathrm{d}t'$, we essentially recover the Quasi-Normal Approximation [194, 195, 196] which has the well-known defect of yielding negative energy spectra [166, 197, 198]. A modification of the Quasi-Normal approximation, which is suitable for studying inhomogeneous turbulence, is the *clipping* approximation [199]; it insures positive energy spectra by enforcing certain inequalities between double and triple velocity correlations; cf. also refs. [200-202].

([11]) Actually, symmetry in k and p suffices [203].

— The Test Field Model (TFM), first introduced by Kraichnan [209, 210] as a heuristic modification of the DIA to restore random Galilean invariance : the triad relaxation time θ_{kpq}, is calculated from an auxiliary problem, the advection of a *Test Field* by the turbulent flow [211]. This model is less ad hoc than the EDQNM but more difficult to handle both analytically and numerically. The EDQNM can be derived as a simplified version of this more refined model [212].

— The Markovian Random Coupling Model (MRCM) [213] is obtained by taking θ_{kpq} to be just a constant number. This is, from the view-point of the K41 theory, less realistic than the EDQNM, but has a simple dynamical model underlying, closely related to the NS equation ; furthermore the MRCM is simple enough to form the subject of rigorous mathematical studies (ref. [214], and next section).

It must be mentioned that the EDQNM can be obtained in other ways.

1) From the DIA (Kraichnan 1964*c*) : if the two-time covariance and the Green's function are assumed of the form

$$\begin{aligned} U(k \mid t, t') &= \mathrm{e}^{-(t-t')\mu_k}\, U(k \mid t, t) \quad (a) \\ G(k \mid t, t') &= \mathrm{e}^{-(t-t')\mu_k} \quad (b) \end{aligned} \qquad (6.20)$$

the EDQNM equation is recovered in the equation for $U(k \mid t, t)$.

2) As a closure of the hierarchy of moment equations : (*a*) the equation of evolution of triple correlation involves in the r.h.s., a fourth order moment which is written as a sum of products of second order moments (as in the Quasi-Normal approximation) plus a fourth order cumulant $\langle uuuu \rangle_c$. Symbolically, γ denoting the bare vertex $-\frac{i}{2} P_{ijl}(\mathbf{k})$, and dropping the viscous term, one has

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle uu \rangle = \gamma \langle uuu \rangle \qquad (a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle uuu \rangle = \gamma \sum_{3\ \text{terms}} \langle uu \rangle \langle uu \rangle + \gamma \langle uuuu \rangle_c\,. \qquad (b) \qquad (6.21)$$

(*b*) Following Orszag, [26, 205, 206] one takes into account the statistical irreversibility by relating $\langle uuuu \rangle_c$ *linearly* to the third order moment (or cumulant) $\langle uuu \rangle$ through an *eddy relaxation* operator μ determined phenomenologically as above, or obtained more systematically on the basis of the full hierarchy [215], (also Frisch, private communication 1977). One then obtains the closed equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle uu \rangle = \gamma \langle uuu \rangle \qquad (a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle uuu \rangle = \sum \gamma \langle uu \rangle \langle uu \rangle - \mu \langle uuu \rangle\,. \qquad (b) \qquad (6.22)$$

(*c*) Eq. (6.22*b*) is solved with zero initial third order moments :

$$\langle u(t)\, u(t)\, u(t) \rangle = \int_0^t \exp\left[- \int_s^t \mu(\tau)\, \mathrm{d}\tau \right] \times \sum \gamma \langle u(s)\, u(s) \rangle \langle u(s)\, u(s) \rangle\, \mathrm{d}s\,. \qquad (6.23)$$

Finally, the markovianization consists in neglecting memory effects and writing

$$\langle u(t)\, u(t)\, u(t) \rangle = \sum \theta(t)\, \gamma \langle u(t)\, u(t) \rangle \langle u(t)\, u(t) \rangle \qquad (6.24)$$

with

$$\theta(t) = \int_0^t \exp\left[- \int_s^t \mu(\tau)\, \mathrm{d}\tau \right] \mathrm{d}s \approx \frac{1 - \exp(-\mu t)}{\mu}\,. \qquad (6.25)$$

3) *Stochastic Models.* — they are obtained by coupling many replicas of the NS equation characterized by an index $\alpha = 1, \ldots, N$ to each other with random coupling coefficients. Symbolically they read ([12])

$$\frac{\mathrm{d}u^{(\alpha)}}{\mathrm{d}t} = \frac{1}{N} \sum_{\beta,\gamma=1}^{N} \varphi_{\alpha\beta\gamma}(t)\, u^{(\beta)}\, u^{(\gamma)} - \nu u^{(\alpha)} + f^{(\alpha)} \qquad (6.26)$$

$\alpha = 1, 2, \ldots, N$.

$\varphi_{\alpha\beta\gamma}(t)$ is a Gaussian white noise process with zero mean and covariance $\delta(t - t')\, \theta(t)$. Furthermore, the various $\varphi_{\alpha\beta\gamma}$'s are identically distributed and independent with the restriction that they are completely symmetric in α, β and γ to insure energy conservation. One then proves that the spectrum constructed from

$$U(k, t) = \lim_{N \to \infty} \frac{1}{N} \sum_{\alpha=1}^{N} \langle u^{\alpha}(t)\, u^{\alpha}(t) \rangle \qquad (6.27)$$

exactly satisfies the EDQNM equation.

If the $\varphi_{\alpha\beta\gamma}$'s are chosen to be time-independent, one obtains the DIA [216, 217, 218].

6.2 Results in three dimensions. — The method of construction of the EDQNM equation only guarantees that the Kolmogorov spectrum is a stationnary solution in the limit of zero viscosity. It is of interest to see if the time dependent solutions evolve toward the K41 spectrum for arbitrary initial conditions. At Reynolds numbers $R_0 \sim 100$, the integration of the spectral equations such as EDQNM, DIA or TFM is in good agreement with the direct numerical simulation and the experimental results [203]. One of the very attractive features of the EDQNM equations is that, contary to the primitive NS equation, they can be integrated numerically at very high Reynolds numbers. The reason is that the spectrum $E(k)$, being an averaged quantity, has a very gentle variation

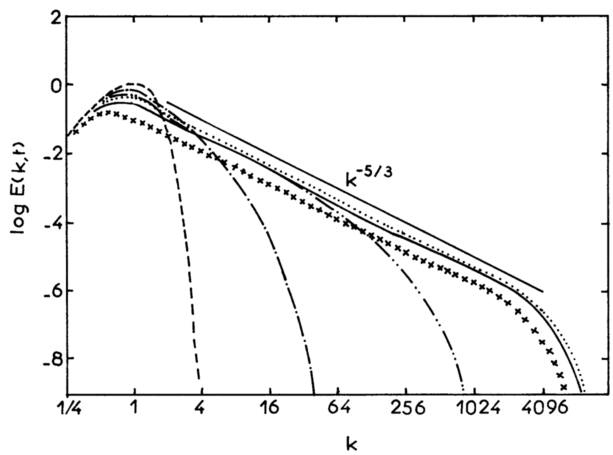
([12]) In the case of inhomogeneous turbulence, only the fluctuation-fluctuation terms are randomly coupled.

with k which can be adequately represented by a few points per octave. Typically, one takes $k_L \propto 2^{L/F}$ $L = 1, ..., LS$ with $F = 4$ or 6. The total number of points (LS) depends on the ratio of maximum wavenumber $k_{\max}$ to minimum wavenumber $k_{\min}$ one wants to achieve; this in turn depends on the Reynolds number because $k_{\max}$ must be large enough to allow dissipation to remove energy at high wavenumbers before truncation effects become important. The reader interested in the numerics will find details in Kraichnan [219], Leith [208], Herring and Kraichnan [203], Pouquet, Lesieur, André and Basdevant [220]. In this way, (large scale) Reynolds numbers R_0 up to 10^6 have been achieved [113]. It is then easily checked that a true asymptotic regime is attained at high wavenumbers : when the Reynolds number is changed, say from 10^4 to 10^6, only the dissipation wavenumber varies in accordance with the Kolmogorov law

$$k_{\text{diss}} \propto R_0^{3/4} .$$

Figure 6*a* extracted from André and Lesieur [113] represents the temporal evolution of an energy spectrum according to the EDQNM. The initial spectrum is $E(k, 0) \sim k^4 \exp - 2(k/k_0)^2$ with the large scale Reynolds number ([13]) $R_0 = \langle V_0^2 \rangle^{1/2}/\nu k_0 \approx 10^5$ and there is no external forcing. k is measured in units of the energy wavenumber, k_0, t in units of the large-eddy turnover time $1/k_0 \langle V_0^2 \rangle^{1/2}$ and $E(k, t)$ in units of $\langle V_0^2 \rangle/k_0$. It is seen that with increasing time, there is a transfer of energy towards large wavenumbers. For $t > t_* \approx 5$, a zone where $E(k) \sim k^{-5/3}$ is established, limited at the upper end by a *dissipation zone* where the spectrum has a rapid decrease and which is pushed further and further away when $R \to \infty$ [179, 205]. In this limit, the $k^{-5/3}$ spectrum reaches to infinity (cf. Penel [221], Bardos *et al.* [127] for a mathe-

([13]) Many papers make use of the Reynolds number R_λ constructed from the Taylor micro-scale which is more easily accessible to experiment. For fully developed turbulence $R_0 \sim R_\lambda^2$. This relation is not true in absence of an inertial range.



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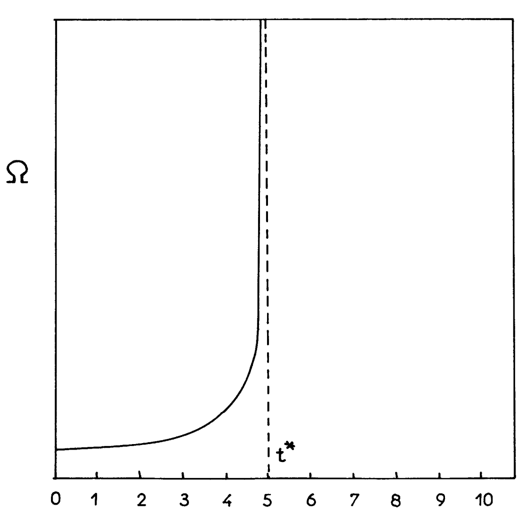

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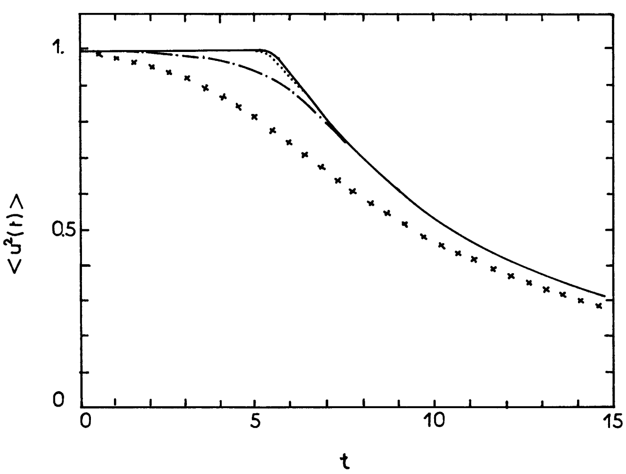
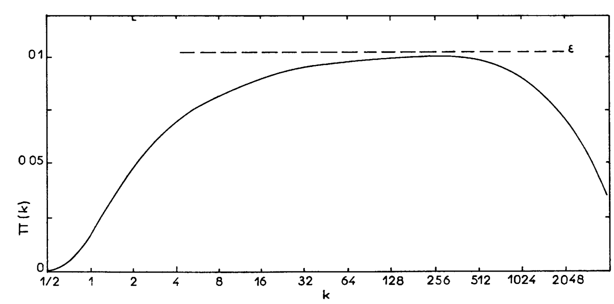


(*c*)



(*d*)

FIGS. 6. — Evolution of high Reynolds number three-dimensional turbulence (taken from André and Lesieur [113]) : *a*) Temporal evolution of energy spectrum $E(k, t)$; no helicity, initial spectrum $E(k, 0) \sim k^4 \exp(-2k^P)$, Reynolds number $R_0 = 524\,000$. - - - $t = 0$, ·—·—· $t = 3$, —··— $t = 5$, ... $t = 6$, —— $t = 8$, × × × × $t = 15$. *b*) Evolution of the enstrophy in the limit of infinite Reynolds number. *c*) Temporal evolution of the total energy $\langle u^2(t) \rangle$; same initial conditions as in figure 3a. × × × $R_0 = 128$; —·— $R_0 = 813$; ..... $R_0 = 32\,800$; —— $R_0 = 524\,000$. *d*) Variation of the energy flux $\Pi(k, t)$ as a function of k for $t = 8$; same conditions as in figure 6*a*. - - - energy dissipation rate.

matical demonstration of a similar result on a simplified model, the MRCM applied to Burgers' equation), and the enstrophy diverges. Figure 6*b* shows the evolution of the enstrophy in the limit $R_0 \to \infty$: the enstrophy becomes infinite at a finite time.

These results obtained numerically with the EDQNM can be demonstrated on the MRCM (where the triad relaxation time is chosen to be constant). One establishes in this case the following equations [222, 214] (cf. also appendix 4) :

$$\frac{\mathrm{d}}{\mathrm{d}t} \mid E(t) \mid_0 + 2 \nu \mid E(t) \mid_1 = 0 \qquad (6.28)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mid E(t) \mid_1 + 2 \nu \mid E(t) \mid_2 = \frac{\theta_0}{3} \mid E(t) \mid_1^2 \qquad (6.29)$$

and for $s > -2$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mid E(t) \mid_s \leqslant C_s \mid E(t) \mid_1 \mid E(t) \mid_s \qquad (6.30)$$

where

$$\mid E(t) \mid_s = \int_0^\infty k^{2s} E(k, t) \, \mathrm{d}k \qquad (6.31)$$

denote the moments of the energy spectrum which play a role similar to that of the Sobolev norms in the analysis of the primitive NS equation. $\mid E \mid_0$ is the energy and $\mid E \mid_1$ the enstrophy. It follows from (6.29) that at zero viscosity the enstrophy is given by

$$\mid E(t) \mid_1 = \frac{3 \mid E(0) \mid_1}{3 - \theta_0 \, t \mid E(0) \mid_1} \qquad (6.32)$$

and therefore becomes infinite at the instant

$$t_* = 3/\theta_0 \mid E(0) \mid_1 .$$

In contrast to what happens with the primitive Euler equation, the initial analyticity of the covariance (in configuration space) may be immediately lost on the inviscid spectral equations [127]. This probably reflects the fact that individual realizations of the three-dimensional Euler equation are likely to blow up at a finite time which for Gaussian initial conditions can be arbitrarily close to $t = 0$.

For positive viscosity, in contrast, the enstrophy remains uniformly bounded in time. The Schwarz inequality leads in effect to

$$\mid E(t) \mid_2 \geqslant \mid E(t) \mid_1^2 / \mid E(t) \mid_0 \qquad (6.33)$$

and (6.29) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \mid E(t) \mid_1 \leqslant \mid E(t) \mid_1 \left\{ \frac{\theta_0}{3} \mid E(t) \mid_1 - \nu \mid E(t) \mid_1 \mid E(t) \mid_0^{-1} \right\} . \qquad (6.34)$$

From the energy eq. (6.28), one obtains

$$\frac{1}{\mid E(t) \mid_1} \frac{\mathrm{d}}{\mathrm{d}t} \mid E(t) \mid_1 \leqslant - \frac{1}{3} \frac{\theta_0}{\nu} \frac{\mathrm{d}}{\mathrm{d}t} \mid E(t) \mid_0 + \frac{1}{\mid E(t) \mid_0} \frac{\mathrm{d}}{\mathrm{d}t} \mid E(t) \mid_0 \qquad (6.35)$$

or again

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \left\{ \frac{\mid E(t) \mid_1}{\mid E(t) \mid_0} \right\} \leqslant - \frac{\theta_0}{3 \nu} \frac{\mathrm{d}}{\mathrm{d}t} \mid E(t) \mid_0 , \qquad (6.36)$$

thus

$$\mid E(t) \mid_1 \leqslant \mid E(0) \mid_1 \exp \left\{ \frac{\theta_0}{3 \nu} \mid E(0) \mid_0 \right\} . \qquad (6.37)$$

Eq. (6.30) then clearly shows that for positive viscosity, all the $\mid E(t) \mid_s$ $s \geqslant 0$, remain bounded, which signifies that the energy spectrum has a rapid fall-off at large wavenumbers. In this case, one speaks of *global regularity*, in opposition to the case $\nu = 0$, where there appears a *singularity* at a finite time. Notice that both of these questions are still open on the original equations (cf. chapter 5).

Let us indicate that in the case where $\theta_{kpq}(t)$ is not a constant but is given by the EDQNM expression (6.16), it has not yet been possible to demonstrate analytically that for $\nu = 0$, the enstrophy becomes infinite at a finite time, although this is shown very clearly by the numerical results : for the moment one only has the following inequality [113], which is not better than the estimate obtained from the original NS equation (see chapter 5)

$$\frac{\mathrm{d}}{\mathrm{d}t} \mid E(t) \mid_1 + 2 \nu \mid E(t) \mid_2 \leqslant C \mid E(t) \mid_1^{3/2} .$$

Nevertheless, and in contrast with the corresponding estimate on the primitive NS equation, this estimate insures global regularity for $\nu > 0$.

Since the enstrophy becomes infinite for $t \geqslant t_*$ when $\nu \to 0$, it is interesting to consider the limit of the dissipation of energy, $\nu \mid E(t) \mid_1$ (cf. 6.28). Figure 6*c* shows in the case of the EDQNM, the temporal evolution of the energy $\mid E(t) \mid_0$ for smaller and smaller values of the viscosity : it appears that when $\nu \to 0$, the energy is only conserved during the time of regularity, t_*. After this, an infinitesimal viscosity suffices to produce a finite dissipation of energy. This is called *energy catastrophe* in Brissaud *et al.* [223].

This infinite Reynolds number dissipation makes the energy tend to zero when $t \to \infty$ (cf. Foias and Penel [224] for a rigorous proof on the Burgers MRCM equation). For large times, the entire spectrum evolves to a universal form with all the time-dependence contained in scale factors (Ref. [225]; see [226] for experimental results).

Figure 6*d*, extracted from André and Lesieur [113] shows that in the $k^{-5/3}$ spectral range, the flux of energy (or energy transfer rate) through each wavenumber k

$$\Pi(k) = - \int_0^k T(p, t) \, \mathrm{d}p \qquad (6.38)$$

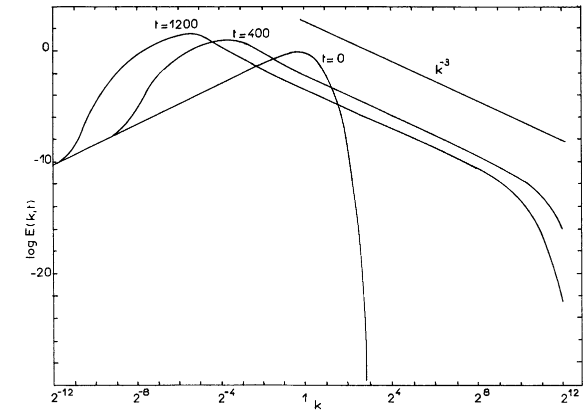
is constant. In addition, it can be shown (cf. Appendix 2) that a $-5/3$ spectrum, extending from k_0 to infinity, assures the constancy of $\Pi(k)$.

One example of practical application of the second order spectral equations is found in the predictability problem [208, 227, 228, 229]. Consider a naturally occurring flow whose initial velocity field can be experimentally resolved only for length scales larger than a certain given scale, the smaller scales of motion being completely unresolved. As the (supposedly turbulent) flow evolves, will the cascade sweep out the initial uncertainty in the experimental data to even smaller scales, or, instead, will the instability of the flow allow the *errors* to contaminate the large scales and thus make the flow completely unpredictable ? This question is of interest in weather prediction where a finite world-wide grid of weather stations limits the initial condition resolution used in numerical forecasting [230]. The mathematical model introduced by Lorenz assumes two random solutions with the same statistics which are initially strongly correlated except in the smaller scales where the errors are confined. In the special case of isotropic, unforced, Navier-Stokes turbulence, both direct numerical simulations and the spectral equations of the form introduced in this chapter, indicate that errors will contaminate the entire spectrum, even if they were initially confined to wavenumbers strongly damped by viscosity (Herring, Riley, Patterson and Kraichnan [231]). There are indications that this happens only for Reynolds numbers exceeding a certain critical value [232]. Rigorous bounds on error growth can be obtained within the same framework as for the regularity of the Euler equation (Arnold [147] p. 343, Sulem [233]).

6.3 RESULTS IN TWO DIMENSIONS. — The numerical integration of the EDQNM equation in two dimensions is consistent with the phenomenological conjectures. It is the triadic description of the energy transfer which enables the EDQNM equation to correctly describe the simultaneous transfer of energy and enstrophy in opposite directions [91]. Figure 7*a*, extracted from Pouquet *et al.* [220], shows the evolution of the energy spectrum, without external forcing, for

$$E(k, 0) \sim k^3 \exp\{ -\tfrac{3}{2}(k/k_0)^2 \},$$

with a (large scale) Reynolds number $R_0 \approx 2.4 \times 10^7$. We see the establishment of a k^{-3} range (the logarith-



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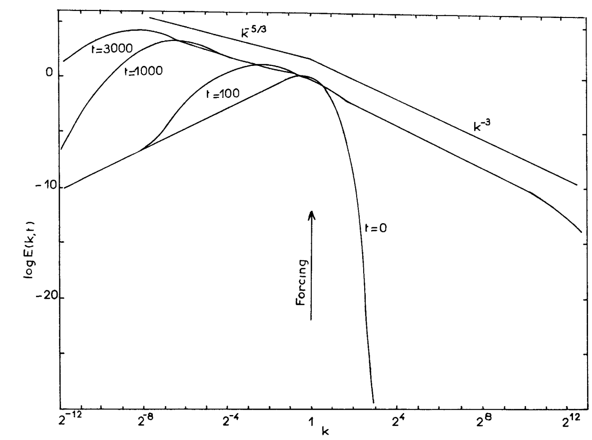

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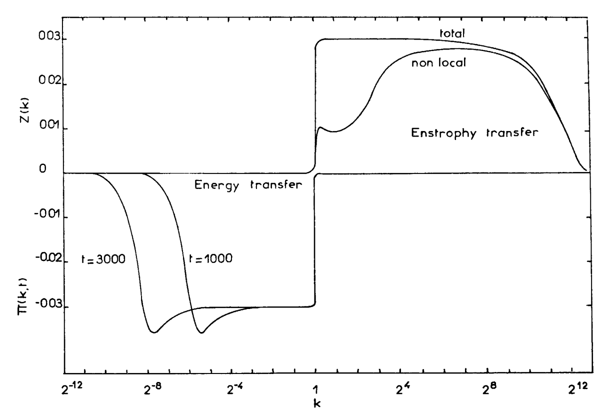


c)

FIGS. 7. — Evolution of high Reynolds number two-dimensional turbulence (taken from Pouquet, Lesieur, André and Basdevant [220]. *a*) Free evolution of energy spectrum without forcing. Initial spectrum $E(k, 0) \sim k^3 \exp(-\frac{3}{2}(k/k_0)^2)$. Reynolds number $R_0 = 2.4 \times 10^7$. *b*) Quasi-steady energy spectrum $E(k, t)$ for $t = 100$, 1 000 and 3 000 corresponding to an injection spectrum constant in a a half-octave band around $k_0 = 1$ with injection rates $\varepsilon = 0.03$ and $\eta = 0.03$. Reynolds number $R_0 = 2.4 \times 10^7$. *c*) Quasi-steady energy transfer rate $\Pi(k, t)$ and enstrophy transfer rates $Z(k, t)$ and $Z_{\mathrm{NL}}(k, t)$ for $t = 1\,000$ and 3 000. Same conditions as in figure 7*a*.

mic correction ([14]) is not discernable), and an inverse transfer of energy. Since the enstrophy is conserved, we know that the k^{-3} range cannot extend to infinity after a finite time. More precisely, the spectrum must be fast decreasing for large enough wavenumbers because all the moments $| E(t) |_s$ (defined in (6.31)) remain finite. For example,

$$\frac{\mathrm{d}}{\mathrm{d}t} | E(t) |_2 \leqslant t \, | E(0) |_1 \, | E(t) |_2 \qquad (6.39)$$

hence

$$| E(t) |_2 \leqslant | E(0) |_2 \exp \{ \tfrac{1}{2} | E(0) |_1 \, t^2 \} . \quad (6.40)$$

To observe an inverse cascade of energy, it is necessary to force the fluid externally. The EDQNM results are shown in figure 7*b*, where the spectrum behaves like k^{-3} for $k \gg k_0$ and $k^{-5/3}$ for $k \ll k_0$ (k_0 is where energy and enstrophy are injected by the external random force). Figure 7*c* represents the levels of energy and enstrophy fluxes Π and Z, whose constancy, in their respective wavenumber ranges, implies a direct cascade of enstrophy and an inverse cascade of energy (cf. Appendix 3).

It is of interest to analyze these cascade processes in terms of the detailed transfer between triads of wavenumbers, k, p, q (with $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$ by homogeneity). We have previously defined an energy spectrum as *local* if its values in the neighbourhood of $k \sim 1/l$ determines the velocity difference across an eddy of size l. With the aid of the EDQNM this notion of locality can be made quantitative. As in equation (6.38) for Π, the enstrophy flux Z can be written in the form

$$Z(k) = - \int_0^k p^2 \, T(p, t) \, \mathrm{d}p . \qquad (6.41)$$

Using the definition of T, the energy and enstrophy fluxes can be reexpressed as integrals over triangles characterized by a shape parameter, v, which specifies the ratio of the smallest leg of the triangle to the leg of intermediate size. Numerical calculations [130] then indicate that the contribution of triangles with small v are somewhat more important than in three dimensions and that most of the enstrophy flux comes from very elongated triads (cf. Appendix 3). Figure 3*c* shows the contribution $Z_{\mathrm{NL}}(k, t)$ to the enstrophy flux which comes from the non-local interactions such that the ratio of the smallest to the middle wavenumber of the interacting triads is less than 0.19.

6.4 *d*-DIMENSIONAL TURBULENCE ($2 < d < 3$). — The above sections clearly point out a strong difference between turbulence in two and three dimensions, particularly in connection with the direction of the energy cascade : *ultraviolet* (to high k) in three dimensions and *infrared* (to small k) in two dimensions. This leads us to ask what happens for $2 < d < 3$. The question is easily settled within the framework of the EDQNM. First one checks that the enstrophy does not go over continuously into another conserved energy moment. However, a continuity argument indicates that at least for short times, the inverse transfer will still be favored for $d \approx 2$. Will this behaviour persist or will energy eventually leak through to higher wavenumbers ?

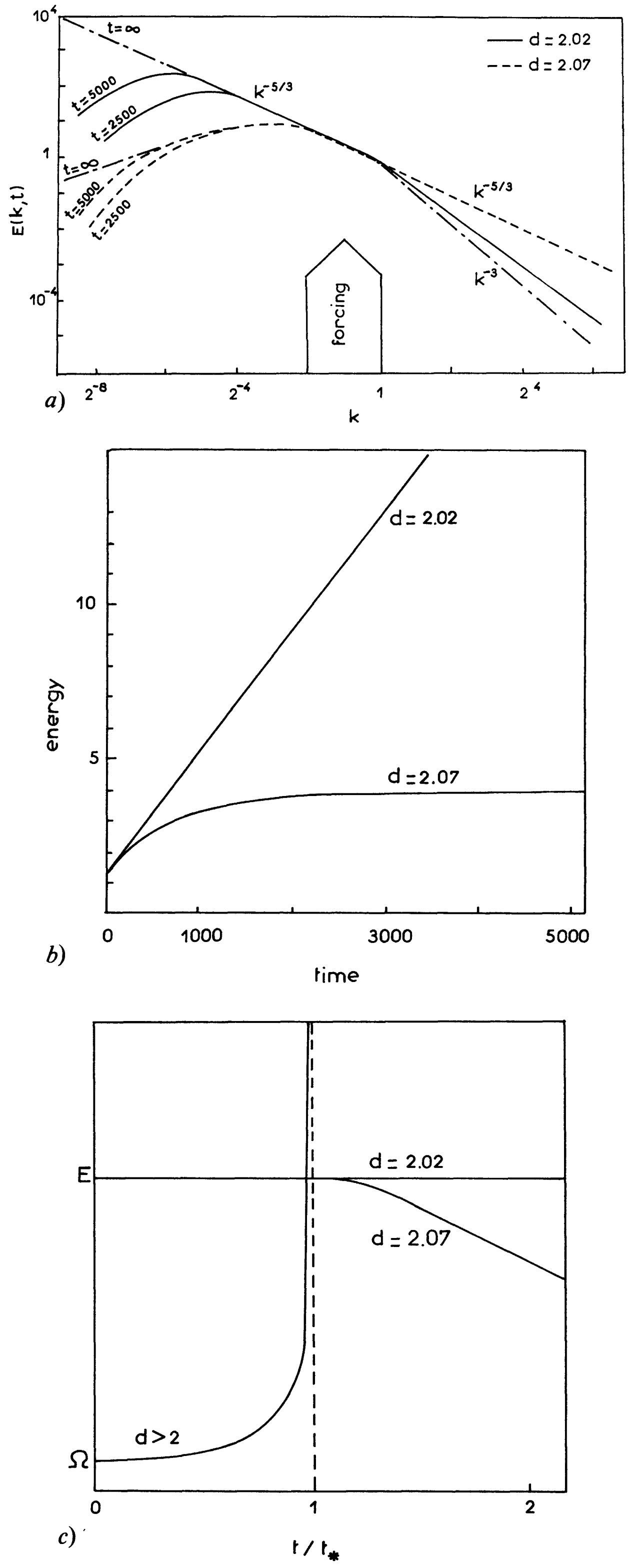
The following results have been obtained both analytically and numerically with Reynolds numbers $R_0 \approx 10^6$ [192, 193].

For any $d \geqslant 2$, there is an inertial-range solution with $E(k) \sim k^{-5/3}$, the energy cascade being in the infrared direction for $2 \leqslant d < d_c$ ($d_c \approx 2.03$) ([15]), and in the ultraviolet direction for $d > d_c$ ([16]). The behaviour at d_c appears to be singular (see also chapter 7) in the sense that the Kolmogorov constant becomes infinite when $d \to d_c$. When energy is injected in a narrow wavenumber-band near $k = 1$, one obtains for $d > d_c$ (e.g. $d = 2.07$), a stationary direct cascade and for $2 \leqslant d < d_c$ (e.g. $d = 2.02$) an inverse cascade such that the bottom of the $-5/3$ range moves to ever smaller wavenumbers (Fig. 8*a*). In the former case, total energy saturates and in the latter, it increases linearly at the injection rate (Fig. 8*b*). Since in the numerical integrations, the dimension differs from 2 by only a few percent, a naive continuity argument suggests a quasi-two-dimensional behaviour for about a hundred turnover-times at forcing wavenumbers (here of the order of one) : this is why the EDQNM equations were integrated up to $t = 5\,000$. For $2 \leqslant d < d_c$, in addition to the infrared $-5/3$ range, there appears an ultraviolet $-m(d)$ range with m varying from 3 to 5/3 (Fig. 8*a*). For $d = 2$, this is the usual enstrophy inertial-range but for $2 < d < d_c$, no conserved quantity cascades along this *fluxless inertial range*. The unforced equations with smooth initial data in the inviscid limit are also considered in Frisch *et al.* [192] : for any $d > 2$, the enstrophy becomes infinite at a finite time $t_*(d)$ proportional to $(d-2)^{-1}$ near $d = 2$ (cf. Appendix 4) ; for $2 < d < d_c$, this singularity is not accompanied by energy dissipation (Fig. 8*c*). This is consistent with the direction of the energy cascade.

([14]) A k^{-3} energy spectrum gives an infrared logarithmic divergence of the enstrophy transfer. Convergence requires a logarithmic correction [130]. Such a difficulty can be avoided by cutting off non-local interactions, that is by removing all non-linear interactions between triads kpq such that $\min(k, p, q)/\max(k, p, q) < a$ where a is a cut off parameter. One then has an exact k^{-3} enstrophy-inertial range.

([15]) More acurate numerical calculations indicate that the crossover dimension may be slightly higher [193].

([16]) Similar results were obtained by Bell and Nelkin [234, 235] using a simple phenomenological model introduced by Desnyansky and Novikov [236]. This model depends upon a parameter C whose value determines the direction of the energy transfer.



FIGS. 8. — Evolution of high Reynolds number d-dimensional turbulence $2 < d < 3$ (according to Frisch, Lesieur and Sulem [192] and Fournier and Frisch [193]) : *a*) Evolution of the energy spectrum below and above the crossover dimension d_c. For $d = 2.02$, there is an ultraviolet power-law-range and a $-5/3$ infrared energy-inertial range proceeding to ever smaller wavenumbers. For $d = 2.07$, there is an ultraviolet energy inertial range. *b*) Evolution of the energy with forcing below ($d = 2.02$) and above ($d = 2.07$) the crossover dimension d_c. *c*) Evolution of the energy E and enstrophy Ω without forcing in the limit of infinite Reynolds number. The singularity time t_* varies like $(d - 2)^{-1}$ near $d = 2$.

7. **Intermittency.** — 7.1 INTERMITTENCY AS A CONSEQUENCE OF VORTEX LINE STRETCHING. — Since the first experiments of Batchelor and Townsend [237], there has been strong evidence that the energy associated with small-scale structures is distributed very unevenly in space, being confined in a smaller and smaller fraction of the available space as the eddy-size decreases [238, 230] (see also Monin and Yaglom [40] and Craya [32] for reviews). This spottiness of the small scales is called intermittency ; vortex line stretching is believed to be the dynamical mechanism behind this phenomenon [240, 241], as suggested from the following plausible argument [242]. Consider the point M within a large scale structure which at the initial time has the largest vorticity amplitude $| \omega |$. This point is also likely to have a large velocity gradient $| \nabla u | \sim | \omega |$. The straining action of the velocity gradient on the vorticity may then be described by a crude form of the vorticity equation

$$\frac{\mathrm{D} | \omega |}{\mathrm{D}t} \sim | \omega |^2 \tag{7.1}$$

where $\mathrm{D} = \frac{\partial}{\partial t} + (u . \nabla)$ denotes the derivative following the flow. Hence, it is expected that the vorticity downstream of M will rise to very large values (probably infinite at zero viscosity) in a time of the order of the large eddy turnover time $t_0 \sim \sup_x | \omega_0(x) |^{-1}$. Associated with this local vorticity-increase is a local augmentation of vortex line stretching in the volume originally occupied by the vorticity excess. It is true that its volume must remain constant because of the incompressibility constraint (which has already been used in the derivation of the vorticity equation), however the self amplifying feature of vorticity will cause the shearing of the volume to be non uniform, with the strongest concentration of vorticity found in a small subvolume [241]. So, small-scale structures may be generated in a very localized fashion. Note however that it is crude to write $| \nabla u | \sim | \omega |$ since the velocity gradient at a point $\mathbf{x}$ is not related in a simple way to the vorticity at $\mathbf{x}$; it is given by a Poisson integral (cf. eq. (5.7)) with a fairly substantial local contribution, but also with some coupling to nearby points. This could smooth out the vorticity peak, but the smallest-scale structures will still have some tendency not to occur uniformly. The intermittency will by necessity also be temporal because of the sweeping of small structures by the large ones, but there probably also exists an intrinsic temporal intermittency [241, 243, 244].

7.2 A DYNAMICAL MODEL FOR INTERMITTENCY. — To take into account the fact that in the cascade, the small eddies become less and less space filling, let us now define the β-model [242]. In contrast with almost all the previous intermittency models [245, 246, 247, 248, 249, 250, 251] in which the key quantity is the energy dissipation, the β model works with dynamically relevant variables such as non-linear energy transfer [241]. It can then also be used to study the possible intermittency of the inverse cascade in two dimensions (see section 7.3).

It is assumed (as in K41) that at each step of the cascade process, any eddy of size $l_n = l_0 2^{-n}$, produces on the average N eddies of size l_{n+1} but contrary to K41, the assumption is made that these $N\, l_{n+1}$-eddies are concentrated in a fraction $\beta (0 \leqslant \beta \leqslant 1)$ of the volume occupied by the l_n-eddy. It follows that, if the largest eddies are space-filling, after n generations only a fraction

$$\beta_n = \beta^n \qquad (\beta = N/2^3 \leqslant 1) \tag{7.2}$$

of the space will be occupied by active fluid (see Fig. 9). Furthermore, it is assumed that the nth generation of eddies (in short, n-eddies) are positionally correlated with $(n + 1)$-eddies by embedding or attachment (for the sake of pictorial clarity this feature is not included in figure 16).

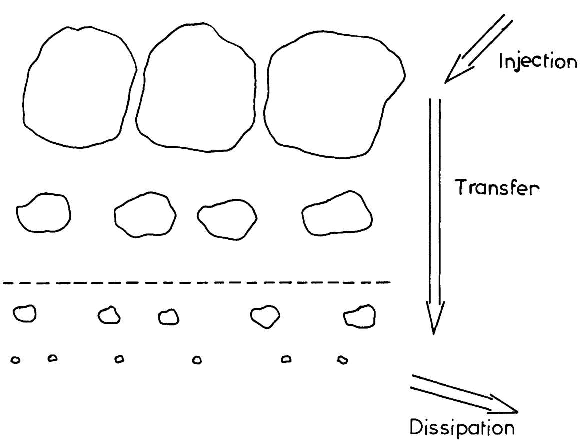


FIG. 9. — The energy cascade for intermittent turbulence : eddies become less and less space-filling. The reader must be warned that this picture is very schematic : actually, the successive eddies are imbedded within each other and the eventual product of the cascade, where dissipation takes place, should be thought of as some sort of highly convoluted sheets.

It is straightforward to work out the modification to K41 in the β-model. Let v_n now denote a typical velocity difference over a distance l_n in *an active region*. The kinetic energy per unit mass in scales $\sim l_n$ is then given by

$$E_n \sim \beta_n v_n^2 . \tag{7.3}$$

The characteristic dynamical time for transfer of energy from active n-eddies to smaller scales is still given by the turnover time $\tau_n = l_n/v_n$ as in K41 : the generation of $(n + 1)$-eddies arises from the internal dynamics of the n-eddies in which it is embedded. The rate of energy transfer from n-eddies to $(n + 1)$-eddies is expressed exactly as in K41, and as in K41 this quantity must be independent of n in the inertial range

$$\varepsilon_n \sim E_n/\tau_n \sim \beta_n v_n^3/l_n \sim \varepsilon . \tag{7.4}$$

Equations (7.2)-(7.4) are combined to obtain

$$v_n \sim \varepsilon^{1/3} l_n^{1/3} (l_n/l_0)^{-\frac{(3-D)}{3}} \tag{7.5}$$

$$\tau_n \sim \varepsilon^{-1/3} l_n^{2/3} (l_n/l_0)^{\frac{(3-D)}{3}} \tag{7.6}$$

$$E_n \sim \varepsilon^{2/3} l_n^{2/3} (l_n/l_0)^{\frac{(3-D)}{3}} \tag{7.7}$$

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} (kl_0)^{-\frac{(3-D)}{3}} \tag{7.8}$$

In equations (7.4)-(7.8), all the intermittency corrections have been expressed in terms of the *self-similarity dimension D*, a special case of Mandelbrot's [252] *fractal* dimension, related to the number of offsprings N by

$$N = 2^3 \beta \overset{\text{def.}}{=} 2^D . \tag{7.9}$$

The self-similarity dimension of a self-similar object is defined as follows : if one reduces the linear dimensions of the object by a factor 2, as in the cascade process, the number of offsprings needed to reconstruct the original object is 2^D. The usual dimension is recovered if the considered object is an interval, a square or a cube, but for more complicated self-similar objects [252], $N = 2^D$, where D need no longer take only integer values, is a natural interpolation.

Eq. (7.7) was first derived by Mandelbrot [251] using the Novikov-Steward [247] model for the spatial distribution of energy dissipation (see Monin and Yaglom [40] Vol. 1, p. 609). In this model, a cube of size l_0 of the order of the integral scale is split into N equal but smaller cubes of length $l_0 N^{-1/3}$. Then the entire dissipation is concentrated in m randomly chosen subcubes. The procedure is repeated until scales comparable to the dissipation length scale are obtained. This model is called *absolute curdling* or *fractal homogeneity* in Mandelbrot [251]. In this context, $D = \text{Log } N/\text{Log } m$ appears as the Hausdorff dimension (see e.g. ref. [163]) of the dissipative structures in the limit of zero viscosity [250].

A generalization of the absolute curdling is the *weighted curdling* (Yaglom [248], Mandelbrot [250, 251], already implicitly contained in Kolmogorov [245] and Obukhov [246]). The dissipative structures occupy all the available space but the density of dissipation in each subcell of a cell is obtained by multiplication of the dissipation in the cell by a random variable W of unit mean value. Absolute curdling is recovered when W has a Bernouilli distribution. Kolmogorov [245] and Obukhov [246] assume a log-normal W. Weighted curdling leads to a correction to the 5/3 law of the energy spectrum which is generally less than $(3 - D)/3$; D and W are then related by $3 - D = \langle W \log W \rangle$ [251].

Let us also mention that several authors have proposed non-hierarchical models of intermittency, where the small scale structures consist typically of extensive thin sheets or ribbons of vorticity [240, 253, 254, 255]. Kraichnan ([241] section 3) explains why such elementary structures are unlikely candidates. One of the main reason is their difficulty in producing an inertial range spectrum with an exponent close to 5/3.

The β-model can also be used to calculate higher order statistics for which the effect due to intermittency is more easily measurable [164, 256, 257, 258, 259, 260]. The β-model then produces the same results as the Novikov-Stewart model [247]. As an example, consider the dimensionless structure functions

$$a_p(l) = \frac{\langle \mid u(x) - u(x+l) \mid^p \rangle}{\langle \mid u(x) - u(x+l) \mid^2 \rangle^{p/2}} . \qquad (7.10)$$

Since the only external parameter which pays a role in K41 is ε, $a_p(l)$ should not depend on l in the inertial range. Instead, measured values increase dramatically with both p and l^{-1}. Using the β-model, these results can be understood at least in their qualitative features : one obtains in effect [242]

$$a_p(l) \propto \left(\frac{l}{l_0}\right)^{\frac{(3-D)(2-p)}{2}} . \qquad (7.11)$$

In contrast to the linear dependence on p in eq. (7.11), the Kolmogorov (1962) log-normal theory [245] predicts a quadratic dependence. Actually, there are indications that only linear-with-p deviations of the exponents from K41 are compatible with the infinite cumulant hierarchy of fully developed turbulence (Frisch 1977 private communication). Nelkin and Bell [261] have reached a similar conclusion with the assumption that there exists a single dissipation scale for all the structure functions.

The skewness

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^3 \right\rangle \Big/ \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle^{3/2}$$

and the flatness

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^4 \right\rangle \Big/ \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle^{2}$$

which measure the deviation of the small scales of the velocity field from Gaussian behaviour can also be calculated in the framework of the β-model, and are found to vary as positive powers of the Reynolds numbers if $D \neq 3$, in the limit of large Reynolds numbers. The correlation function of the dissipation $\varepsilon(r) = \nu\omega^2(r)$ is found to satisfy

$$\langle \varepsilon(r)\, \varepsilon(r+l) \rangle - \langle \varepsilon^2 \rangle \propto \langle \varepsilon^2 \rangle (l_0/l)^{\mu} \quad (7.12)$$

with $\mu = 3 - D$. Experimentally, a power law seems to work quite well and defines an exponent $\mu \approx 0.5$ [262, 263] (see Monin and Yaglom [40] vol.2, Chap. 25 for review). This leads to $D \approx 2.5$.

Since the Hausdorff dimension of the dissipative structures in the limit of zero viscosity satisfies $0 \leqslant D \leqslant 3$, the corrections to K41 cannot make the spectral exponent larger than 8/3. The same upper bound can be derived from the NS equation for finite energy turbulence (Sulem and Frisch ([17]) [159]). This exponent corresponds to a dissipation concentrated on a zero-dimensional set (e.g. isolated points). The true dimension is likely to be greater than two. Indeed, as noticed by Mandelbrot [251], the intersection of a D-dimensional set with a line has D-2 dimensions if $D \geqslant 2$, and is almost surely empty for $D < 2$. Since intermittency is easily seen in measurements made with a hot-wire anemometer, which is essentially a measurement along a line, it follows presumably that $D \geqslant 2$.

7.3 Intermittency in two dimensions. — The presence of highly non-local interactions makes the model unusable in the enstrophy cascade. Intermittency is present, but, as shown by Kraichnan [115], it probably does not change the energy spectrum.

The inverse cascade, in contrast, is local and intermittency corrections to the 5/3 law cannot be ruled out. The closest thing to the β-model would be a cascade of the kind shown in figure 10 which becomes less and less space-filling with increasing scale-size. Let us assume that after n octave-steps the fraction of space filled with active n-eddies is

$$\beta_n = 2^{-n(2-D)} = (l_n/l_0)^{D-2} . \qquad (7.13)$$

Repeating the calculation of section 7.2, we obtain

$$E(k) \sim \varepsilon^{2/3}\, k^{-5/3} (k l_0)^{\frac{(2-D)}{3}} . \qquad (7.14)$$

([17]) Pouquet [88] has noted that this reference contains a mistake in the bound on enstrophy flux in two dimensions. Eq. (8) of Sulem and Frisch [159] must read

$$\begin{aligned} \Pi_{\text{enst}}(k_L) \leqslant C^{st} \Bigg\{ & E_L^{1/2}\, \Omega_{L+1}^{1/2} \left(\sum_{n=0}^{L} k_n^2\, \Omega_n^{1/2} \right) + \\ & + k_L\, \Omega_L^{1/2}\, \Omega_{L+1}^{1/2} \left(\sum_{m=0}^{L-1} k_m\, E_m^{1/2} \right) \\ & + \left(\sum_{l=0}^{L} k^2\, \Omega_l^{1/2} \right) [E_{L+1}^{1/2}\, \Omega_{L+1}^{1/2} + E_{L+1}^{1/2}\, \Omega_{L+2}^{1/2} \\ & + \sum_{m=L+2}^{\infty} E_m^{1/2} (\Omega_{m-1}^{1/2} + \Omega_m^{1/2} + \Omega_{m+1}^{1/2})] \Bigg\} . \end{aligned}$$

It follows that if $E_l \propto C k_l^{-\alpha}$, we find, $\lim_{L \to \infty} \Pi_{\text{enst}}(k_L) = 0$, provided that $\alpha > 8/3$.

This mean that in the enstrophy range, the energy spectrum cannot be steeper than $k^{-11/3}$. If we insert $E(k) \propto k^{-11/3}$, we find that the dominant contribution comes from l and n close to L and small m, an indication that the enstrophy transfer is strongly non local.

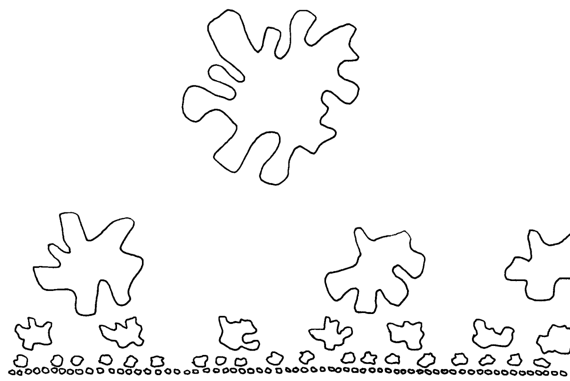


FIG. 10. — An intermittent inverse cascade.

Thus as first noticed by Kraichnan [115], intermittency corrections to the two-dimensional inverse cascade will, if they exist, *decrease* the 5/3 exponent.

We have seen that with the EDQNM, a crossover dimension, $d_c \approx 2.03$, is found where the direction of energy cascade reverses. Intermittency corrections to the inertial range exponent for the energy spectrum are thus expected to change sign upon crossing d_c. It is an open question whether the intermittency corrections vary continuously in the neighbourhood of d_c.

8. **Comparison with critical phenomena.** — 8.1 FORMAL ANALOGIES. — Detailed analogies have been sought between fully developed three-dimensional turbulence and critical phenomena. Both have asymptotically universal self-similar behaviour, the former in the limit of infinite wavenumber, the latter in the limit of zero wavenumber, and a suggestive table can be constructed [264-267]. We shall use the inverse Reynolds number R_0^{-1} and not viscosity to avoid confusion with the standard use of the exponent ν in critical phenomena (see Kadanoff *et al.* [268] for a review of notations and pre-renormalization group phenomenology).

Critical phenomena	Turbulence
limit $T - T_c \downarrow 0$ (T_c = critical temperature)	limit $R_0^{-1} \downarrow 0$
distance r	wavenumber k
non-universal small scale fluctuations (characteristic small scale ξ_0)	non-universal small wavenumber fluctuations (characteristic scale of the energetic eddies $L = 1/k_0$).
correlation length $\xi = \xi_0 \left(\dfrac{T - T_c}{T_c}\right)^{-\nu}$.	dissipation wavenumber $k_{\text{diss}} = k_0\, R_0^{+\nu}$
order parameter (magnetization) $M(\mathbf{r})$	Fourier transform of vorticity $\hat{\omega}(\mathbf{k})$
fluctuations of M have an infinite range in configuration space in the limit $T - T_c \downarrow 0$.	fluctuations of $\hat{\omega}$ have an infinite range in Fourier space in the limit $R_0^{-1} \downarrow 0$
spin-spin correlation function $g(r)$	Fourier transform of vorticity-vorticity correlation function $\langle \mid \hat{\omega}(\mathbf{k}) \mid^2 \rangle \sim E(k)$ in three dimensions
$\lim_{T \downarrow T_c} g(\mathbf{r})$ exists	$\lim_{R_0^{-1} \downarrow 0} E(k)$ exists
if H is an external field which couples to M, then the susceptibility $\left.\dfrac{\partial M}{\partial H}\right\rvert_{H=0} \sim \displaystyle\int_0^\infty r^2\, g(r)\, \mathrm{d}r \propto (T - T_c)^{-\gamma}$	total vorticity $\displaystyle\int_0^\infty k^2\, E(k)\, \mathrm{d}k \sim \varepsilon R_0$. This corresponds to a critical exponent $\gamma = 1$
$g(r) \sim r^{1+\eta} f\left(\dfrac{r}{\xi}\right)$. scaling relation $\gamma = (2 - \eta)\, \nu$	K41 implies that for $k \gg k_0$ $E(k) \sim \varepsilon^{2/3}\, k^{-5/3} f(k/k_{\text{diss}})$ and $k_{\text{diss}} = k_0\, R_0^{+3/4}$. This corresponds to critical exponents $\eta = 2/3$, $\nu = 3/4$ which satisfies the scaling relation $\gamma = (2 - \eta)\, \nu$. If intermittency is accounted for, then in the inertial range $E(k) \sim \varepsilon^{2/3}\, k^{-5/3} (k_0/k)^{\zeta}$. This corresponds to critical exponents $\eta = \dfrac{2}{3} + \zeta\,;\quad \nu = \left(\dfrac{4}{3} - \zeta\right)^{-1}\,;\quad \gamma = 1$

These analogies are not meant to be taken literally. It is the underlying concept of scaling which may be useful in understanding turbulence. In critical phenomena, scaling means the existence of various fields, the scaling fields, whose range of correlation is determined by a single length scale which diverges as $T \downarrow T_c$. The inertial range behaviour of the energy spectrum naturally leads to the identification of the velocity field (or equivalently any component of the velocity gradient, denoted $\psi(\mathbf{r})$) as a scaling field. Note that Nelkin [266] has conjectured that the local dissipation rate $\varepsilon(r) = \nu\psi^2(\mathbf{r})$ is another scaling field. This conjecture is based upon the experimental result that the dissipation spectrum appears to have the same viscous cut-off as the energy spectrum [257].

Finally, let us mention the analogy which may exist between the inverse cascade in two-dimensional turbulence and a Hamiltonian system, initially out of thermal equilibrium, but with a total energy such that for large times it approaches a critical point. At the critical point, there are infinite fluctuations in the order parameter, such as the magnetization in a ferromagnet, which can be interpreted as the result of the *condensation* of spins into larger and larger locally aligned spin cluster. This appears to be analogous to the inverse cascade in two-dimensional turbulence [269]. Intermittency does not have a readily identifiable counterpart, but a relationship with the convolutions of the cluster boundaries has been envisaged [270].

8.2 UNIVERSALITY. — In order to compare what is usually meant by universality in critical phenomena and in turbulence, it is necessary to start with non-equilibrium critical phenomena. It has long been believed that the sufficiently small scales of fully developed turbulence are independent of initial conditions [25]. This, we plausibly take to correspond to the tacit assumption that there is a unique state of thermal equilibrium associated with each class of initial conditions for a Hamiltonian system. The values of the isolating integral of motion serve as class characteristics; in particular, the initial energy per unit volume may be chosen such that the state of thermal equilibrium attained is a critical point. In the modern theory of phase transitions, the feeling that certain properties of the system in the vicinity of the critical point, such as the critical exponents, are unsensitive to the detailed form of its Hamiltonian has been elevated to a principle which is called universality. The degree of universality is not generally quantifiable. However, given a specific Hamiltonian and one of its critical points, certain possible changes in the Hamiltonian are classified as irrelevant (relevant) if they lead to the same (different) critical point behaviour. Usually, changes which preserve the various symmetries of the Hamiltonian are irrelevant. Consider the class of lattice models for magnets which are partially characterized by the number of spatial dimensions d and the number of spin components n. It is known that changes in the lattice structure which are changes in symmetry are irrelevant (e.g. for $d = 2$, spin on triangular or square lattices have the same exponents). In contrast, changes in n or d, or the introduction of a preferred spatial direction in which for example the coupling is strongest are relevant. One is then led to compare the universality in critical phenomena with the sensitivity of turbulence to changes in the NS equation.

One of the essential symmetries to be preserved is the detailed energy conservation as expressed in (2.16). If in eq. (2.10a), the coupling coefficient $P_{ijl}(\mathbf{k})$ is multiplied by a totally symmetric function $\varphi(\mathbf{k}, \mathbf{p}, \mathbf{q})$, a modified NS equation is obtained which belongs to the same symmetry class as the original. Let us examine the qualitative significance of such a change in three dimensions. In the original NS equation, the dependence of the coupling coefficient $P_{ijl}(\mathbf{k})$ upon the sole wavevector $\mathbf{k}$ implies that in configuration space, the triadic interaction couples two eddies at the same point $\mathbf{r}'$ to produce a third eddy at point $\mathbf{r}$. Though the pressure contribution to the coupling appears to be *long ranged*, it varies in fact like r^{-4} because of the quadrupole character of the pressure source $\partial^2 u_i u_j / \partial x_i \partial x_j$. This is a rapidly decreasing function as compared to the velocity structure function in the inertial range

$$\langle v^2 \rangle - \langle v(\mathbf{r})\, v(0) \rangle \propto r^{m-1} \quad \text{with} \quad m \approx 5/3 .$$

A modification of the coefficient $P_{ijl}(\mathbf{k})$ by multiplication by a function $\varphi(\mathbf{k}, \mathbf{p}, \mathbf{q})$ would couple eddies at two distinct points $\mathbf{r}'$ and $\mathbf{r}''$ to produce an eddy at $\mathbf{r}$. In critical phenomena, it is known that interactions of infinite range must be added to a Hamiltonian before a change in critical exponents is produced. On the basis of a superficial analogy, it would seem that if the nonlinear terms are modified without introducing any long range interactions (in physical space), then the associated turbulence would be the same as in the original NS equation. Loosely following Kraichnan [28], we will now attempt to cast a serious doubt upon this conclusion. A change in the coupling coefficient will in general affect the locality of energy transfer in Fourier space, a measure of which is given by the energy transfer locality function $W(v)$ (see Appendix 3). It is likely that the locality degree of transfer determines the effective number of cascade steps needed to reach a given wave-number and hence the intermittency. If this conjecture is true, a modification of the coupling coefficient which leaves the scaling properties of turbulence invariant coincides with a modification which leaves W invariant. In this context, non local interactions (between scales of arbitrary separation) seems to be a prerequisite for the existence of intermittency. This is suggested by an analysis of the cumulant hierarchy (Frisch 1977 private communication).

8.3 IS THERE A MEAN FIELD THEORY IN TURBULENCE ? — In comparisons between fully developed turbulence and critical phenomena, K41 has been considered as the analog of the Landau mean field theory of phase transition, in the sense that they neglect statistical fluctuations of the energy flux and of the order parameter respectively [235, 243]. In critical phenomena, it is possible to carry out a systematic expansion of the various statistical quantities, the so-called loop expansion which gives the mean field theory at the lowest order [271]. In contrast, for turbulence, the only know systematic expansion analog to the loop expansion gives the Direct Interaction Approximation to lowest order [173], which, as noticed in section 6.1 is not compatible with K41.

8.4 IS THERE A CROSSOVER DIMENSION FOR VALIDITY OF K41 ? — By applying to the energy dissipation an argument parallel to the Ginsburg argument for critical phenomena, the existence of an 8/3 crossover dimension for intermittency correction to K41 has been proposed [6, 265, 267]. However, dissipation is not an inertial range quantity, and therefore the dimension 8/3 is of no particular interest. So far, there are no indications that intermittency corrections disappear in any dimension ([18]).

8.5 INFINITE-DIMENSIONAL TURBULENCE. — It has been suggested that simplifications occur as $d \to \infty$ and in particular that K41 could become exact (Migdal 1976, Siggia 1976 private communications). In the case of a passive scalar advected by a velocity field which is spatially smooth and white noise in time, Kraichnan [131] found that temporal fluctuations in the smallest scales become negligible as $d \to \infty$. More recently, the limit $d \to \infty$ has been investigated on the NS equation [273]. It is found that short-time expansions and the (Eulerian) DIA yield well defined limits if a rescaled time $\tilde{t} = t/\sqrt{d}$ or equivalently a rescaled energy spectrum $\tilde{E}(k) = E(k)/d$ is used. The latter is equivalent to assuming finite energy per component rather than a finite total energy. All the energy transfer comes then from triads which have one right angle. A simple understanding is obtained by noting that two independent unit vectors are almost surely orthogonal in infinite dimensions. This result has an important consequence : if initially the energy is confined to a wavenumber band $k_1 \leqslant k \leqslant k_2$, then it can never be transferred to wavenumbers less than k_1, since the interaction of two orthogonal wavevectors within that band necessarily results in a wavevector larger than k_1. The characteristic time for the dynamics of the energy containing scales (presumably the time for appearance of a singularity at zero viscosity) becomes independent of d in the limit $d \to \infty$, when the reference velocity from which this time is constructed is the r.m.s. velocity per component. The incompressibility constraint was found to still play a role as $d \to \infty$. Nevertheless, the pressure which is given by a Poisson equation with a source div $((u . \nabla) u)$ should only depend weakly on individual components of velocity and velocity gradient. Therefore, an anisotropy affecting a finite number of velocity components cannot be relaxed in a few large eddy turnover times as in three dimensions [274, 275]. When short-time expansions going beyong the second order, or renormalized perturbation theory going beyong the DIA, are considered, the corrections remain as important as the DIA terms as $d \to \infty$. At this level there is so far no evidence that K41 becomes exact as $d \to \infty$. But the fact that such corrections remain finite does not rule out exactness of K41 as $d \to \infty$ (Kraichnan 1976 private communication).

([18]) Grossmann and Schnedler [272] have made a Renormalization Group calculation suggesting a crossover at $d = 2$. Their starting point is however not the NS equation.

8.6 RENORMALIZATION GROUP METHODS AND APPLICATIONS. — The tool which has proved to be very useful for describing and calculating critical phenomena is the Renormalization Group (RG for short) as recently developed by Wilson [276]. See for reviews Wilson and Kogut [277], Wilson [278], Fischer [279], Ma [280, 281], Toulouse and Pfeuty [282]. In brief, it has two essential features :

a) Since (in a ferromagnet) spins tend to align, it is natural to consider in place of solitary spins, *block spins* [283] which are essentially a local average of the solitary spins and which are labelled by suitably rescaled values of the original position and time indices.

b) The self similarity of the large scales at a critical point imply that the sequence of Hamiltonians in terms of solitary spins, block spins, groupings of block spins, etc... converges to a *fixed point*. For (possibly non-Hamiltonian) dynamics, a sequence of equations of motion takes the place of the sequence of Hamiltonians [284, 285]. The critical thermodynamic properties are then determined by the approach to the fixed point.

Critical phenomena methods have been applied to the study of convective instabilities [286, 287, 288, 289], but their application to fully developped turbulence, proposed by Nelkin [264], has just begun. The first attempt to really implement the RG ideas in connection with fully developed turbulence is due to Forster. Nelson and Stephen (FNS) [290, 291]. This work is concerned with the infrared properties of a randomly stirred fluid and makes use of a very powerful method carried over from critical dynamics [284, 292, 293]. Its essence can be sketched as follows. Let the NS equation be written in Fourier space with wavenumbers extending from 0 to $k_{\max}$. Without any approximation, reference to velocity Fourier components $u^{>}$ with wavenumbers larger than $k_{\max}/b$ ($b > 1$) can be suppressed by the following procedure : using the NS equation, $u^>$ can, in principle, be expressed in terms of velocity Fourier components $u^<$ with wavenumbers less than k_{max}/b and of the stirring forces $f^>$ (and also the initial conditions $u^>(0)$ if needed); the expression for $u^>$ is then substituted in the equation of motion for $u^<$ wherever $u^>$ appears. In the resulting equation for $u^<$, variables are then rescaled to make it, as much as possible, look like the original equation (but unavoidably, new terms are also generated). This defines the RG transformation; when iterated indefinitely, the equation of motion may converge to a fixed form. In such a case, the scaling laws of the problem are easily extracted from the fixed equation and the rescaling factors. Since the RG transformation cannot be carried out explicitly, the real problem is to calculate perturbatively. It turns out that for the problems considered in FNS, there are crossover dimensions d_c above which the non-linear terms become essentially negligible as $k \to 0$, so that for d slightly *less* than d_c, one can calculate perturbatively (cf. the ε-expansion of Wilson and Fisher [294]). The most interesting result of FNS from the view-point of turbulence concerns their *Model* B. To give an indication of what is done, let us first use the K41 phenomenology. We write the equation of detailed balance of spectral energy density for stationary solutions

$$0 = \frac{\partial E(k)}{\partial t} = F(k) + T(k) - 2\, \nu k^2\, E(k) \quad (8.1)$$

($F(k)$ = forcing spectrum, $T(k)$ = transfer spectrum).

Inertial range solutions correspond to situation with negligible forcing and viscosity and identically vanishing transfer, whereas in FNS's Model B stationarity results (for $d < 4$) from competition between forcing and transfer with negligible viscosity. FNS assumed a particular power law forcing $F(k) \propto k^{d-1}$; following Fournier and Frisch [193], let us more generally assume

$$F(k) \propto k^{-r}\,; \quad (8.2)$$

let the energy spectrum be a prescribed power law

$$E(k) \propto k^{-m}\,. \quad (8.3)$$

Using the ideas of K41, and assuming locality, we can write

$$T(k) \sim k^{3/2}\, E(k)^{3/2}\,. \quad (8.4)$$

When we substitute (8.3) into (8.4) and equate the transfer to minus the forcing spectrum, we obtain

$$r = 3(m-1)/2\,. \quad (8.5)$$

This argument can be made more quantitative by using the EDQNM of chapter 6. Since the EDQNM is compatible with K41, the k-dependence of $T(k)$ automatically agrees with (8.4). Convergence of transfer and negativity put some constraints on r which depend on the dimension (see Fournier and Frisch [193], section V.C.). A noteworthy feature is that the DIA and the EDQNM which usually differ in their predictions (cf. the $k^{-3/2}$ and the $k^{-5/3}$ inertial ranges [180]) do agree for the solution (8.4) as long as $m < 1$, the reason being that the DIA has then no infrared divergence so that its non-Galilean invariance become irrelevant. The main interest of the FNS solution is that it agrees with K41 calculations although *it does not use closure*. Indeed, at a technical level, the FNS calculation is based on an $\varepsilon = (4 - d)$-expansion. They calculate an approximate recursion relation valid to order ε by a diagramatic perturbation method. Only second order diagrams contribute, so that the calculation is equivalent to using the lowest order *mass renormalized* equation, namely the DIA (see Martin *et al.* [173] for a field-theoretical view point of the DIA and higher order approximations). Nevertheless, as they noticed, this calculation is probably valid to all orders in ε. Fournier and Frisch [193] have shown that the actual crossover parameter is not the dimension but the spectral exponent m or equivalently the forcing exponent r : the relation (8.5) can be obtained by a RG calculation for $r = -3 + \varepsilon$ and arbitrary $d \geqslant 2$ (Fournier [295] ch. II). Other results obtained by FNS are relevant to large scale dynamics of stationary turbulence but correspond essentially to negligible non-linear terms (model A and C).

There is another infrared aspect of turbulence which can be investigated by RG methods, namely *subgrid scale modeling* [296]. The existence of a long range of quickly evolving small eddies, the inertial range, in fully developed turbulence implies both the impossibility of direct numerical calculation based on the NS equation because the number of degrees of freedom is too large, and the possibility of a numerical calculation which only explicitly refers to the large eddies because the properties of the small eddies are universal and need not be calculated explicitly. Existing subgrid-scale calculations are based either on phenomenological arguments [297-301] or on closures [302, 303, 304]. It may be shown that closure-based calculations are roughly equivalent to doing only one step in a RG iterative process. However, a distinctive characteristic of the RG applied to subgrid scale modeling is that the statistics of the scales to be eliminated are determined by the large scales which are the object of the calculation. This is in contrast with the application of the RG to critical phenomena where statistical properties are explicitly determined by the Gibbs ensemble.

The most challenging question for the application of RG ideas is fully developed three-dimensional turbulence which is clearly an ultraviolet problem. There has been an attempt, loosely inspired from the RG ideas to calculate the exponent of the dissipation correlation function for intermittent fully developed turbulence by numerical integration of a suitably chosen set of hierarchically distributed Fourier modes [244]. In trying to implement true RG ideas, there is no particular difficulty involved in *defining* the RG for an ultraviolet problem; it suffices to interchange $k_{\max}$ and $k_{\min}$, $u^<$ and $u^>$, b and $1/b$. The real problem is that, so far, we do not know how to carry out the perturbation calculation due to lack of a known small expansion parameter.

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Appendix 1 : Dependence of the Eddy-turnover-time on spectral shape when intermittency effects are ignored. — We assume the turbulence to be homogeneous and isotropic. For an arbitrary spatial dimension of space d, it then follows that

$$\langle \mid \Delta u(l_n, t) \mid^2 \rangle \equiv \langle \mid u(x + l_n, t) - u(x, t) \mid^2 \rangle = \int_0^\infty E(p, t) \times \left[1 - \int e^{ipl_n \cos\theta}\, d^{(d-1)}\,\Omega \right] \mathrm{d}p \quad \text{(A1.1)}$$

where $d^{(d-1)}\,\Omega$ is the differential solid angle which is normalized according to

$$\int d^{(d-1)}\,\Omega = 1$$

and θ is the azimuthal angle. The integral over p can be broken into three parts

$$0 \leqslant p < k_n/\sqrt{2}\,, \quad k_n/\sqrt{2} \leqslant p < \sqrt{2}\,k_n\,, \quad \sqrt{2}\,k_n \leqslant p < \infty\,,$$

where $k_n = 1/l_n$. We roughly identify the integral over the middle range with E_n. In the first and third integrals, we make the asymptotic replacements $(pl_n \to 0)$

$$1 - \int e^{ipl_n \cos\theta}\, d^{(d-1)}\,\Omega \approx c^2\, p^2\, l_n^2\,, \quad \text{(A1.2)}$$

[where c is a constant of order unity which will be suppressed] and $(pl_n \to \infty)$

$$1 - \int e^{ipl_n \cos\theta}\, d^{(d-1)}\,\Omega \approx 1\,. \quad \text{(A1.3)}$$

We obtain

$$\langle \mid \Delta u(l_n, t) \mid^2 \rangle \approx l_n^2 \int_0^{k_n/\sqrt{2}} p^2\, E(p)\,\mathrm{d}p + E_n + \int_{k_n\sqrt{2}}^\infty E(p)\,\mathrm{d}p\,. \quad \text{(A1.4)}$$

There are two possibilities. Either E_n dominates in (A1.4) in which case the spectrum is called local, or it does not and the spectrum is non-local. With the substitution $E(p) \propto p^{-m}$ (inertial range), we see that the first integral is well behaved if $m < 3$, and the second is well behaved for $m > 1$. Therefore, if $1 < m < 3$, the spectrum is local. The case $m \leqslant 1$ is observed only in connection with infrared problems while the limit $m = 3$ appears in the calculation of the enstrophy-transfer-range in two-dimensional turbulence. Let us here exclude $m \leqslant 1$, which then allows us to replace (A1.4) by a simple formula valid both in the local and non-local cases,

$$\langle \mid \Delta u(l_n, t) \mid^2 \rangle \approx l_n^2 \int_0^{k_n} p^2\, E(p, t)\,\mathrm{d}p\,. \quad \text{(A1.5)}$$

This implies that

$$\tau_n^{-1} = \sqrt{\langle \mid \Delta u(l_n, t) \mid^2 \rangle}/l_n \approx \sqrt{\int_0^{k_n} p^2\, E(p, t)\,\mathrm{d}p}\,. \quad \text{(A1.6)}$$

It is of some interest to give a physical interpretation of the non-local contributions to $\langle \mid \Delta u \mid^2 \rangle$ as they appear in (A1.4). Consider an eddy of size $1/p \gg l_n$. Its velocity field varies slightly over the length scale l_n and may be approximated by a simple shear, i.e. a flow having a spatially constant velocity gradient $\omega(p)$. The velocity difference across the eddy of size l_n is $l_n\,\omega(p)$ and the mean squared velocity difference is $l_n^2 \langle \omega^2(p) \rangle$. This allows a correspondence with the first integral in (A1.4) under the identification $\langle \omega^2(p) \rangle = \int p^2\, E(p)\,\mathrm{d}p$. Now consider an eddy of size $1/p \ll l_n$. Its velocity field varies over length scales much smaller than l_n and, in effect, causes the boundary of the l_n-eddy to execute a highly convoluted random walk. In the mean, this effect is diffusive, and as in the usual diffusion models which are based upon a fine grained random walk, the relevant parameter is the mean square velocity of the carrier field, which corresponds to the second integral of (A1.4).

Appendix 2 : The technical aspects of second order spectral equations. — In d spatial dimensions, the conditions of homogeneity, isotropy, reflection invariance and incompressibility allow us to write the velocity covariance in the form

$$\langle \hat{u}_l(\mathbf{k}, t)\, \hat{u}_m(\mathbf{p}, t) \rangle = \frac{U(k, t)}{d-1}\, P_{lm}(\mathbf{k})\, \delta^{(d)}(\mathbf{k} + \mathbf{p})\,. \quad \text{(A2.1)}$$

The Navier-Stokes equation

$$\left(\frac{\partial}{\partial t} + \nu k^2 \right) \hat{u}_i(\mathbf{k}, t) = -\frac{i}{2}\, P_{ijm}(\mathbf{k}) \int_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \hat{u}_j(\mathbf{p}, t)\, \hat{u}_m(\mathbf{q}, t)\, \mathrm{d}^{(d)}p \quad \text{(A2.2)}$$

together with the above definition of U, the short time expansion and the replacement of t by θ_{kpq} yield

$$\frac{\partial U}{\partial t}(k, t) + 2\,\nu k^2\, U(k, t) =$$
$$= \frac{4}{(d-1)} \int \mathrm{d}^{(d)} p k^2\, \theta_{kpq}(t) [a^{(d)}_{kpq}\, U(p, t)\, U(q, t) - - b^{(d)}_{kpq}\, U(q, t)\, U(k, t)] \quad \text{(A2.3)}$$

where

$$a^{(d)}_{kpq} = P_{lmn}(\mathbf{k})\, P_{mi}(\mathbf{p})\, P_{nj}(\mathbf{q})\, P_{lij}(\mathbf{k})/4\,k^2 \quad \text{(A2.4)}$$
$$b^{(d)}_{kpq} = P_{lmn}(\mathbf{k})\, P_{nli}(\mathbf{p})\, P_{mi}(\mathbf{q})/2\,k^2\,. \quad \text{(A2.5)}$$

Some properties of these coefficients which follow immediately from their definitions, are

$$2\,a^{(d)}_{kpq} = b^{(d)}_{kpq} + b^{(d)}_{kqp} \quad \text{(A2.6)}$$
$$2\,k^2\,a^{(2)}_{kpq} = p^2\, b^{(d)}_{kpq} + q^2\, b^{(d)}_{kqp} \quad \text{(A2.7)}$$
$$k^2\, b^{(d)}_{kpq} = p^2\, b^{(d)}_{pkq} \quad \text{(A2.8)}$$

In terms of x, y, z, the cosines of the interior angles α, β, γ of the (k, p, q) wavenumber triangle (Fig. 11),

$$b^{(d)}_{kpq} = \frac{p}{2\,k}[(d-3)\,z + (d-1)\,xy + 2\,z^3]\,. \quad \text{(A2.9)}$$

Using the identity

$$x + yz = \sin\beta\,\sin\gamma \quad \text{(A2.10)}$$

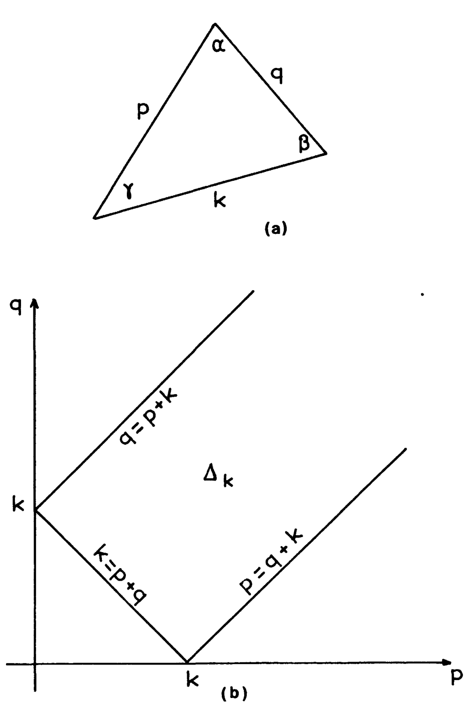

FIG. 11. — *a*) The (k, p, q) triangle. *b*) The Δ_k domain limited by the triangular inequalities $|\,p - q\,| \leqslant k \leqslant p + p$.

and eqs. (A2.6) and (A2.9), we obtain

$$4\,a^{(d)}_{kpq} = 2(d-2) + (3-d)(y^2+z^2) - 2\,yz\cos(\beta-\gamma)$$
$$= d(\sin^2\beta + \sin^2\gamma) - 4 + 3(\cos^2\beta + \cos^2\gamma)$$
$$- 2\cos\beta\cos\gamma\cos(\beta-\gamma)\,, \quad \text{(A2.11)}$$

from which we conclude that $a^{(d)}$ is an increasing function of d. For $d = 2$,

$$4\,a^{(2)}_{kpq} = y^2 + z^2 - 2\,yz\cos(\beta-\gamma) \geqslant 0 \quad \text{(A2.12)}$$

and consequently

for $d \geqslant 2$ $\quad a^{(d)}_{kpq} \geqslant 0$.
In contrast,
for $d < 2$ $\quad a^{(d)}_{kpp} = \frac{1}{2}(d-2)(1-y^2) < 0$.

The bUU term in (A2.3) can be absorbed into an integration factor :

$$\frac{d}{dt}\left[U(k, t)\exp\int_0^t V_k(t')\,\mathrm{d}t'\right] =$$
$$= \frac{4}{d-1}\left[\exp\int_0^t V_k(t')\,\mathrm{d}t'\right] \times$$
$$\times \int \mathrm{d}^{(d)} p\, k^2\, \theta^{(t)}_{kpq}\, a^{(d)}_{kpq}\, U(p, t)\, U(q, t)\,, \quad \text{(A2.15)}$$

where

$$V_k(t) = 2\,\nu k^2 + \frac{4}{d-1}\int \mathrm{d}^{(d)} p\, k^2\, \theta^{(t)}_{kpq}\, b_{kpq}\, U(q, t)\,. \quad \text{(A2.16)}$$

From this and the positivity and complete symmetry of $\theta_{kpq}(t)$, it is easily checked that a necessary and sufficient condition for the preservation in time of the property $\{\, U(k, t) \geqslant 0$ for all $k \,\}$ is $a^{(d)} \geqslant 0$ for all its allowed wavevector arguments, violated in dimensions $d < 2$.

In discussing the conservation laws, it is traditional to work with the energy spectrum $E(k, t)$. Using the rotational symmetry of the problem, we perform the angular integrations in $d^{(d)} p$ and express the energy per unit mass

$$\frac{1}{2}\langle \mathbf{u}.\mathbf{u} \rangle = \frac{1}{2}\int U(k, t)\, d^{(d)} k \quad \text{(A2.16)}$$

in terms of the energy spectrum by

$$\frac{1}{2}\langle \mathbf{u}.\mathbf{u} \rangle = \int_0^\infty E(k, t)\,\mathrm{d}k\,. \quad \text{(A2.17)}$$

$E(k, t)$ is related to $U(k, t)$ by the use of the properties of the d-dimensional volume element

$$\mathrm{d}^{(d)} k = k^{d-1}\,\mathrm{d}k\, d^{(d-1)}\,\Omega \quad \text{(A2.18)}$$

where $d^{(d-1)}\,\Omega$ is the differential solid angle. When integrated over all the angular variables, it yields the area of the d-dimensional unit sphere

$$\int d^{(d-1)}\,\Omega = S_d = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}. \tag{A2.19}$$

Therefore

$$E(k, t) = \tfrac{1}{2}\, S_d\, k^{d-1}\, U(k, t). \tag{A2.20}$$

If γ is chosen to be the azimuthal angle, then the integrand in (A2.3) is independent of the other angular variables implicitly contained in $d^{(d-1)}\,\Omega$ and we may therefore make the replacement

$$\begin{aligned} d^{(d-1)}\,\Omega &\to S_{d-1}(\sin\gamma)^{d-2}\,\mathrm{d}\gamma & 0 \leqslant \gamma \leqslant 1 \\ &\to S_{d-1}(\sin\gamma)^{d-3}\,\mathrm{d}z & -1 \leqslant z \leqslant 1\,. \end{aligned} \tag{A2.21}$$

Using the law of cosines

$$q^2 = k^2 + p^2 - 2\,kpz\,, \tag{A2.22}$$

$$\mathrm{d}p\,\mathrm{d}z \to \left|\frac{\partial(p, z)}{\partial(p, q)}\right| \mathrm{d}p\,\mathrm{d}q = \frac{q}{kp}\,\mathrm{d}p\,\mathrm{d}q \tag{A2.23}$$

and the law of sines

$$\frac{k}{\sin\alpha} = \frac{q}{\sin\gamma} = \frac{p}{\sin\beta}\,, \tag{A2.24}$$

we are led to the replacement

$$d^{(d)}\,p \to S_{d-1}\left(\frac{qp}{k}\right)^{d-2} (\sin\alpha)^{d-3}\,\mathrm{d}p\,\mathrm{d}q\,. \tag{A2.25}$$

The variables of integration p and q are restricted to the region Δ_k limited by the triangular inequalities $|\,p - q\,| \leqslant k \leqslant p + q$, which is illustrated in figure 18. Eq. (A2.3) can now be rewritten as

$$\begin{aligned} &\frac{\partial E}{\partial t}(k, t) + 2\,\nu k^2\,E(k, t) = \\ &= C_d \iint_{\Delta_k} \theta_{kpq}(t)\left(\frac{\sin\alpha}{k}\right)^{d-3} \frac{k}{pq}\,[a^{(d)}_{kpq}\,k^{d-1}\,E(p, t)\,E(q, t) - \\ &\qquad - b^{(d)}_{kpq}\,p^{d-1}\,E(q, t)\,E(k, t)]\,\mathrm{d}p\,\mathrm{d}q \end{aligned} \tag{A2.26}$$

where

$$C_d = 4\,S_{d-1}/(d-1)^2\,S_d\,. \tag{A2.27}$$

In particular

$$C_2 = \frac{4}{\pi}\,, \qquad C_3 = \frac{1}{2}\,. \tag{A2.28}$$

To demonstrate the detailed conservation properties of (A2.26) when $\nu = 0$, its symmetries must be made explicit. By construction, θ_{kpq} is completely symmetric. The factor $\sin\alpha/k$ may be regarded as a completely symmetric function of k, p, q by the law of sines. The region of integration Δ_k is symmetric in p and q, and when considering the evolution of total energy, we shall have integrals of the form

$$\begin{aligned} \int_0^\infty \mathrm{d}k \iint_{\Delta_k} \mathrm{d}p\,\mathrm{d}q = \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}p \int_0^\infty \mathrm{d}q\, \times \\ \times\, Y(p+q-k)\; Y(k+q-p)\; Y(k+p-q)\,, \end{aligned} \tag{A2.29}$$

(Y is the unit step function) which are completely symmetric in k, p, q. Finally, let us make the integrand in (A2.27) symmetric in p and q by writing

$$\frac{\partial E}{\partial t}(k, t) = \iint_{\Delta_k} S(k \mid p, q)\,\mathrm{d}p\,\mathrm{d}q \tag{A2.30}$$

with

$$S(k \mid p, q) = S(k \mid q, p) \tag{A2.31}$$

and

$$\begin{aligned} S(k \mid p, q) &= \frac{C_d}{2}\,\theta_{kpq}(t)\left(\frac{\sin\alpha}{k}\right)^{d-3} \times \\ &\times \frac{k}{pq}\,[2\,a^{(d)}_{kpq}\,k^{d-1}\,E(p, t)\,E(q, t) \\ &- b^{(d)}_{kpq}\,E(q, t)\,E(k, t) - b^{(d)}_{kpq}\,E(p, t)\,E(k, t)]\,. \end{aligned} \tag{A2.32}$$

As shown in Section 3.1, a quadratic conserved quantity such as the energy, must be conserved in detail. In terms of $S(k \mid p, q)$ this is equivalent to

$$S(k \mid p, q) + S(p \mid q, k) + S(q \mid k, p) = 0\,. \tag{A2.33}$$

To see this explicitly, form the above sum for eq. (A2.32). It contains a series of terms quadratic in the energy spectrum whose coefficients vanish. For example, the coefficient of $E(p, t)\,E(q, t)$ in this series is proportional to

$$\frac{k^{d-2}}{pq}\,[2\,k^2\,a_{kpq} - p^2\,b_{kpq} - q^2\,b_{qkp}] \tag{A2.34}$$

which vanishes by virtue of eqs. (A2.6) and (A2.8). Similarly, the detailed conservation of enstrophy in two dimensions,

$$k^2\,S(k \mid p, q) + p^2\,S(p \mid q, k) + q^2\,S(q \mid k, p) = 0 \tag{A2.35}$$

follows from (A2.7) and (A2.8). More generally, the conservation of the energy moment

$$|\,E(t)\,|_s = \int_0^\infty k^{2s}\,E(k, t)\,\mathrm{d}k$$

is equivalent to the condition

$$k^{2s} S(k \mid p, q) + p^{2s} S(p \mid q, k) + q^{2s} S(q \mid k, p) = 0 \quad \text{(A2.36)}$$

from which it can be shown that the enstrophy does not go over continuously into another energy moment for $d \neq 2$ [193].

Appendix 3 : Evaluation of energy and enstrophy fluxes. — The constancy of the energy and enstrophy fluxes in the energy and enstrophy cascades respectively, together with the symmetry properties of the transfer function $S(k \mid p, q)$ implied by these cascades, can be used to quantitatively establish the notion of locality in the transfer processes. Though eqs. (A2.30), (A2.31), (A2.33) and (A2.35) were derived in the context of semi-phenomenological spectral equations, they are also true for the exact equations.

The energy flux through wavenumber k is ([19]) (viscosity and forcing set equal to zero)

$$\Pi(k) = -\frac{\partial}{\partial t}\int_0^k E(k')\,\mathrm{d}k' = = -\int_0^k \mathrm{d}k' \iint_{\Delta_{k'}} S(k' \mid p, q)\,\mathrm{d}p\,\mathrm{d}q\,. \quad \text{(A3.1)}$$

A sufficient set of conditions for $\Pi(k)$ to approach a constant positive value ε in the limit of large k (the direct cascade of energy) is given by

(i) The scaling law

$$S(ak' \mid ap, aq) = \frac{1}{a^3} S(k' \mid p, q) \quad \text{(A3.2)}$$

to be satisfied for $a \to \infty$.

(ii) The contributions from very elongated triads ($k \ll p \sim q$ or $p \ll k \sim q$ or $q \ll k \sim p$) should be vanishingly small, or in other terms the integral (A3.1) has neither infrared nor ultraviolet divergence when the inertial range expression is substituted in S; furthermore it should be positive. The condition for the inverse cascade are similar except that $a \to 0$ and the flux integral must be negative. A constant enstrophy flux

$$\eta = Z(k) = -\int_0^k \mathrm{d}k' \iint_{\Delta_{k'}} k'^2\, S(k' \mid p, q)\,\mathrm{d}p\,\mathrm{d}q \quad \text{(A3.3)}$$

requires, in place of (i), the condition ($a \to \infty$)

$$S(ak' \mid ap, aq) = \frac{1}{a^5} S(k' \mid p, q)\,. \quad \text{(A3.4)}$$

([19]) More precisely, to prevent possible infrared divergence, the energy flux must be defined as the limit for $k_{\min} \to 0$ of the integral (A3.1) with k', p, q restricted to exceed $k_{\min}$.

From the expression for $S(k \mid p, q)$ given in (A2.32), and that for θ_{kpq} given in (6.15-6.18) with $\nu = 0$, it is seen that the scaling law (A3.2) which is associated with an energy cascade is consistent with a Kolmogorov (1941) spectrum for $E(k)$. Similarly, a k^{-3} spectrum is consistent with the scaling law (A3.4) which is associated with the enstrophy cascade, but the logarithmic correction is required to insure the convergence of the integral in eq. (A3.3).

Before investigating the consequence of the above conditions in the context of the spectral equations, let us first put (A3.1) and (A3.3) into convenient forms, with the following sequence of manipulations which are based upon the general properties of $S(k \mid p, q)$. As noticed by Kraichnan [179], the triple integral in (A3.1) collects four types of interactions according to the values of k', p and q as compared to k (see Fig. 12)

$$\Pi(k) = \Pi_{\mathrm{I}}(k) + \Pi_{\mathrm{II}}(k) + \Pi_{\mathrm{III}}(k) + \Pi_{\mathrm{IV}}(k) \quad \text{(A3.5)}$$

$$\Pi_{\mathrm{I}}(k) \equiv -\int_0^k \mathrm{d}k' \int_0^k \mathrm{d}p \int_0^k \mathrm{d}q S(k' \mid p, q) \quad \text{(A3.6)}$$

$$\Pi_{\mathrm{II}}(k) \equiv -\int_0^k \mathrm{d}k' \int_k^\infty \mathrm{d}p \int_k^\infty \mathrm{d}q S(k' \mid p, q) \quad \text{(A3.7)}$$

$$\Pi_{\mathrm{III}}(k) \equiv -\int_0^k \mathrm{d}k' \int_k^\infty \mathrm{d}p \int_0^k \mathrm{d}q S(k' \mid p, q) \quad \text{(A3.8)}$$

$$\Pi_{\mathrm{IV}}(k) \equiv -\int_0^k \mathrm{d}k' \int_0^k \mathrm{d}p \int_k^\infty \mathrm{d}q S(k' \mid p, q)\,. \quad \text{(A3.9)}$$

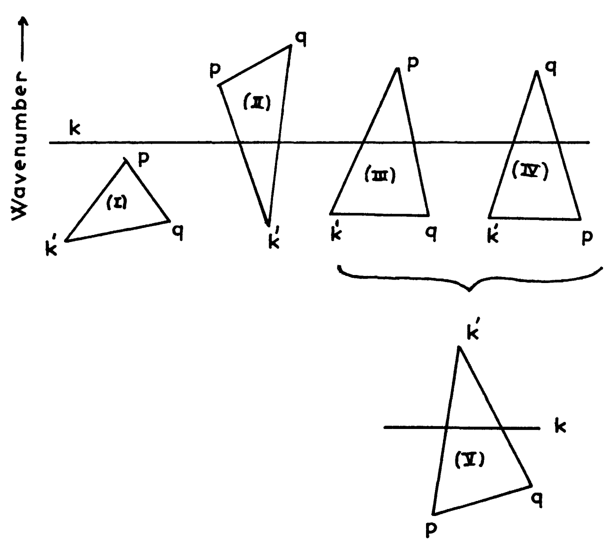

FIG. 12. — Classification of triad interactions involved in the energy flux $\Pi(k)$.

The above four integrals are also implicitly constrained by the condition that k', p and q form the sides of a triangle. Detailed energy conservation (A2.33) implies that $\Pi_{\mathrm{I}} = 0$. Eq. (A2.31) implies that $\Pi_{\mathrm{III}} = \Pi_{\mathrm{IV}}$

$$\begin{aligned}\Pi_{\rm III}(k)+\Pi_{\rm IV}(k) &= 2\,\Pi_{\rm IV}(k)\\ &= -2\int_0^k \mathrm{d}k' \int_0^k \mathrm{d}p \int_k^\infty \mathrm{d}q S(k' \mid p, q)\\ &= -\int_0^k \mathrm{d}k' \int_0^k \mathrm{d}p \int_k^\infty \mathrm{d}q[S(k' \mid p, q) + \\ &\qquad + S(p \mid k', q)]\\ &= \int_0^k \mathrm{d}k' \int_0^k \mathrm{d}p \int_k^\infty \mathrm{d}q S(q \mid k', p)\\ &= \int_k^\infty \mathrm{d}k' \int_0^k \mathrm{d}p \int_0^k \times\\ &\qquad \times\ \mathrm{d}q S(k' \mid p, q) \equiv \Pi_{\rm V}\,. \quad \text{(A3.10)}\end{aligned}$$

Thus (Kraichnan [179])

$$\Pi(k) = \int_k^\infty \mathrm{d}k' \int_0^k \mathrm{d}p \int_0^k \mathrm{d}q S(k' \mid p, q) - \\ - \int_0^k \mathrm{d}k' \int_k^\infty \mathrm{d}p \int_k^\infty \mathrm{d}q S(k' \mid p, q)\,. \quad \text{(A3.11)}$$

The first term on the r.h.s. is the total rate of gain in the range $k' > k$ due to triad interactions with $p, q < k$, while the second term is the total rate of loss in the range $k' < k$ due to triads with $p, q > k$. These two classes of triad interactions are mutually exclusive and exhaust the interactions which contribute to the net transfer across k. Similarly, in two dimensions, the mean rate of enstrophy transfer is

$$Z(k) = \int_k^\infty k'^2\,\mathrm{d}k' \int_0^k \mathrm{d}p \int_0^k \mathrm{d}q S(k' \mid p, q) - \\ - \int_0^k k'^2\,\mathrm{d}k' \int_k^\infty \mathrm{d}p \int_k^\infty \mathrm{d}q S(k' \mid p, q)\,. \quad \text{(A3.12)}$$

Still following Kraichnan [91, 179], we use the symmetries of the integrand and of the domains of integration in (A3.11) to make the replacements

$$\int_0^k \mathrm{d}p \int_0^k \mathrm{d}q \to 2\int_0^k \mathrm{d}p \int_0^p \mathrm{d}q$$

$$\int_k^\infty \mathrm{d}p \int_k^\infty \mathrm{d}q \to 2\int_k^\infty \mathrm{d}p \int_p^\infty \mathrm{d}q \to 2\int_k^\infty \mathrm{d}p \int_p^{p+k'} \mathrm{d}q \quad \text{(A3.12)}$$

where the last replacement is a consequence of the triangle constraint. Changing the variables of integration of the first integral in (A3.11) according to

$$u = k/p\,,\quad v = q/p\,,\quad w = k'/p \quad \text{(A3.13)}$$

and in the second according to

$$u = k/p\,,\quad v = k'/p\,,\quad w = q/p \quad \text{(A3.14)}$$

we obtain

$$\Pi(k) = 2\,k^3\left[\int_0^1 \mathrm{d}v \int_1^\infty \frac{\mathrm{d}u}{u^4} \int_u^\infty \mathrm{d}w S\left(\frac{k}{u}w \,\Big|\, \frac{k}{u}, \frac{kv}{u}\right) - \right. \\ \left. - \int_0^1 \frac{\mathrm{d}u}{u^4} \int_0^u \mathrm{d}v \int_1^{1+v} \mathrm{d}w S\left(\frac{kv}{u} \,\Big|\, \frac{k}{u}, \frac{kw}{u}\right)\right]. \quad \text{(A3.15)}$$

We now restrict our attention to the energy cascade (see Kraichnan [91] for the enstrophy cascade analysis) and use (A3.2) to remove the factor k/u from S,

$$\Pi(k) = 2\left[\int_1^\infty \frac{\mathrm{d}u}{u} \int_0^1 \mathrm{d}v \int_u^\infty \mathrm{d}w S(w \mid 1, v) - \right. \\ \left. - \int_0^1 \frac{\mathrm{d}u}{u} \int_0^u \mathrm{d}v \int_1^{1+u} \mathrm{d}w S(v \mid 1, w)\right]. \quad \text{(A3.16)}$$

In the first term we may replace

$$\int_1^\infty \mathrm{d}u \int_u^\infty \mathrm{d}w \quad \text{by} \quad \int_1^\infty \mathrm{d}w \int_1^w \mathrm{d}u \quad \text{(A3.17)}$$

and in the second term

$$\int_0^1 \mathrm{d}u \int_0^u \mathrm{d}v \quad \text{by} \quad \int_0^1 \mathrm{d}v \int_v^1 \mathrm{d}u\,. \quad \text{(A3.18)}$$

Note that in the first term w can never reach infinity because of the triangle constraint,

$$w = k'/p < (p+q)/p = 1+v\,. \quad \text{(A3.19)}$$

The integration over u can now be performed, yielding

$$\Pi(k) = \varepsilon = 2\int_0^1 \mathrm{d}v \int_1^{1+v} \times \\ \times\ \mathrm{d}w[\ln w\, S(w \mid 1, v) + \ln v\, S(v \mid 1, w)]\,. \quad \text{(A3.20)}$$

Since v is always less than one and w always greater than one, each choice of the pair (v, w) corresponds to a unique triangle shape, with v the ratio of the shortest to the middle leg. The energy transfer locality function

$$W(v) = \frac{2}{\varepsilon}\int_v^1 \mathrm{d}v' \int_1^{1+v'} \times \\ \times\ \mathrm{d}w[\ln w\, S(w \mid 1, v') + \ln v'\, S(v' \mid 1, w)] \quad \text{(A3.21)}$$

gives the fraction of energy transfer due to triangles whose smallest leg is larger than v times the middle leg.

In two dimensions, detailed energy and enstrophy conservation, eq. (A2.33) and (A2.35) lead to

$$\frac{S(p \mid q, k)}{S(q \mid k, p)} = \frac{q^2 - k^2}{k^2 - p^2} \quad \text{(A3.22)}$$

or

$$S(w \mid 1, v) = \frac{1 - v^2}{v^2 - w^2} S(1 \mid v, w) \quad (A3.23)$$

$$S(v \mid 1, w) = \frac{w^2 - 1}{v^2 - w^2} S(1 \mid v, w) \quad (A3.24)$$

and therefore

$$\Pi(k) = \varepsilon = 2 \int_0^1 \mathrm{d}v \int_0^{1+v} \mathrm{d}w [(1 - v^2) \ln w + (w^2 - 1) \ln v] \, S(1 \mid v, w) \quad (A3.25)$$

(Kraichnan [130], eqs. (2.5) and (2.9)). The energy locality function reads

$$W(v) = \frac{2}{\varepsilon} \int_v^1 \mathrm{d}v' \int_0^{1+v'} \mathrm{d}w [(1 - v'^2) \ln w + (w^2 - 1) \ln v'] \, S(1 \mid v', w) \quad (A3.25)$$

Figure 13 taken from Kraichnan [118] shows the function $W(v)$ computed in the context of the TFM in two and three dimensions. It illustrates that the energy transfer in two dimensions is less local than in three dimensions.

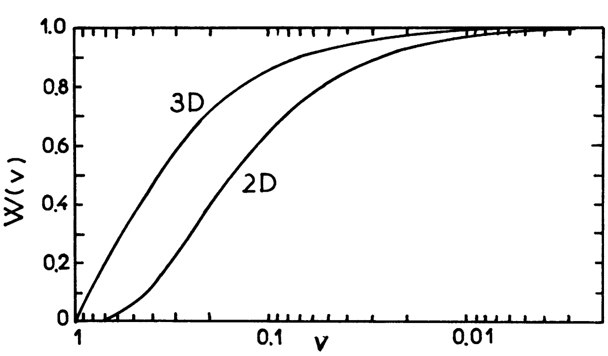

FIG. 13. — The transfer locality function $W(v)$ in three dimensions (3D) and in two dimensions (2D) (taken from Kraichnan [130]).

Through the use of eqs. (A3.20) and (A3.24), and the corresponding equation for the enstrophy flux, the various Kolmogorov constants, which are the dimensionless factors in the inertial-range expressions of $E(k)$, can be related to the adjustable constant λ_d which appears in the expression for θ_{kpq} [113, 130].

Appendix 4 : Appearence of a singularity at zero viscosity in dimensions $d > 2$ on the MRCM (communicated by M. Lesieur). — From eq. (6.14) with a constant triad relaxation time $\theta_{kpq}(t) = \theta_0$ (corresponding to the Markovian Random Coupling Model of Frisch *et al.* [213]), we derive a closed equation for the enstrophy

$$| E(t) |_1 = \int_0^\infty k^2 E(k, t) \, \mathrm{d}k .$$

$$\frac{\mathrm{d}}{\mathrm{d}t} | E(t) |_1 = \frac{C_d}{2} \theta_0 \iiint_\Delta \frac{k^2}{q} \left(\frac{\sin \alpha}{k} \right)^{d-3} \times [(d-3) z + (d-1) xy + 2 z^3] \times [k^{d-1} E(p) E(q) - p^{d-1} E(q) E(k)] \, \mathrm{d}p \, \mathrm{d}q \, \mathrm{d}k \quad (A4.1)$$

where Δ is the set of wavenumber-triads able to form a triangle. Exchanging k and p in the 2nd term on the r.h.s. and using the law of sines, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} | E(t) |_1 = \frac{C_d}{2} \tau_0 \int_0^\infty q^2 \, \mathrm{d}q \times \int_0^\infty k^2 \, \mathrm{d}k E(q) \, E(k) \, A^{(d)}(k, q) \quad (A4.2)$$

wherein

$$A^{(d)}(k, q) = \left(\frac{k}{q} \right)^2 \int_{|k-q|}^{k+q} (\sin \beta)^{d-3} \times [(d-3) z + (d-1) xy + 2 z^3] \left(\frac{p}{k} \right)^2 \left[\left(\frac{p}{k} \right)^2 - 1 \right] \frac{\mathrm{d}p}{q} . \quad (A4.3)$$

$A^{(d)}(k, q)$ is a homogeneous function of k and q of degree zero, and thus may be expressed as a function $\tilde{A}(q/k)$. Symmetrizing in q and k, eq. (A4.2) then reads

$$\frac{\mathrm{d}}{\mathrm{d}t} | E(t) |_1 = \frac{C_d}{4} \theta_0 \int_0^\infty \mathrm{d}q \times \int_0^\infty \mathrm{d}k E(q) \, E(k) \left[\tilde{A}\left(\frac{q}{k} \right) + \tilde{A}\left(\frac{k}{q} \right) \right] . \quad (A4.4)$$

Writing $q/k = \lambda$ and $p/k = \mu$, we have

$$\tilde{A}(\lambda) = \frac{2}{\lambda^2} \int_{|1-\lambda|}^{1+\lambda} (\sin \beta)^{d-3} (xy + z^3) \, \mu^2 (\mu^2 - 1) \, \mathrm{d}\mu + \frac{d-3}{\lambda^2} \int_{|1-\lambda|}^{1+\lambda} (\sin \beta)^{d-3} (xy + z^3) \, \mu^2 (\mu^2 - 1) \frac{\mathrm{d}\mu}{\lambda} . \quad (A4.5)$$

The trigonometric relation

$$xy + z = \frac{\sin^2 \beta}{\mu} \quad (A4.6)$$

and the sine and cosine laws lead to

$$\tilde{A}(\lambda) = \int_{-1}^{+1} (\sin \beta)^{d-1} \left[1 + \frac{1 - \lambda^2}{2 \lambda y - (\lambda^2 + 1)} \right] \mathrm{d}y + (d - 3) \int_{-1}^{+1} (\sin \beta)^{d-1} \, \mathrm{d}y . \quad (A4.7)$$

It follows that

$$\tilde{A}(\lambda) + \tilde{A}\left(\frac{1}{\lambda}\right) = 2(d-2)\int_{-1}^{+1}(\sin\beta)^{d-1}\,dy$$
$$= 2(d-2)\,\pi^{1/2}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)} \qquad \text{(A4.8)}$$

and then

$$\frac{d}{dt}\,|\,E(t)\,|_1 = \frac{C_d}{2}\,\theta_0(d-2)\,\pi^{1/2}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}\,|\,E(t)\,|_1^2$$
$$= \frac{2(d-2)}{d(d-1)}\,\theta_0\,|\,E(t)\,|_1^2\,. \qquad \text{(A4.9)}$$

Integrating, we find

$$|\,E(t)\,|_1 = \frac{|\,E(0)\,|_1}{1 - \frac{2(d-2)}{d(d-1)}\,\theta_0\,t\,|\,E(0)\,|_1}\,. \qquad \text{(A4.10)}$$

Hence, we obtain that for zero viscosity, the enstrophy becomes infinite at a finite time

$$t_* = \frac{d(d-1)}{2(d-2)\,|\,E(0)\,|_1\,\theta_0}$$

in any dimension $d > 2$.

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