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HOLE THEORY IN NONORTHOGONAL BASIS

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Résumé. — En seconde quantification pour bases non-orthogonales nous avons développé la notion de trous. Nous présentons une application de la méthode au modèle nucléaire à N centres.

Abstract. — The concept of hole is developed in second quantization for nonorthogonal basis. Application is made to the N-centre model.

1. Introduction. — In a previous paper [1] we have shown how the second quantization for a nonorthogonal basis proposed by Moshinsky and Seligman [2] can help for calculations in the N-centre model. We propose here to develop the formalism further by introducing the notion of holes in order to describe more easily the 4N−1 and 4N−2 nuclei obtained by removing one or two nucleons from the corresponding 4N nucleus.

For that purpose we present first the second quantization method in a more general fashion in order to introduce the concept of occupied and empty states that will be used later on in the definition of holes. The next section is devoted to the hole theory, based on the definition of a new vacuum and the corresponding creation and annihilation operators.

In the last section we point out the advantages of our method by applying it to the 10Be case.

2. Second quantization. — 2.1 Closure property. — Let us consider the Hilbert space $\mathcal{H}$ of single-particle states and some set of vectors $|\kappa\rangle$ which is a nonorthogonal basis of $\mathcal{H}$. Here $\kappa$ stands for all the quantum numbers required to specify completely the vectors (for instance $\rho$ for the spatial part, $\sigma$ for the spin part and $\tau$ for the isospin part).

By assumption the set of vectors $|\kappa\rangle$ is complete so that it is possible to expand any vector $|\psi\rangle$ of $\mathcal{H}$ in terms of these vectors,

$$|\psi\rangle = \sum_{\kappa} c_{\kappa} |\kappa\rangle. \quad (1)$$

The coefficients $c_{\kappa}$ of the expansion (1) are determined uniquely by the relation

$$a_{\kappa} = \sum_{\lambda} R_{\kappa\lambda}^{-1}(\lambda | \psi) \quad (2)$$

where $R$ is the (infinite) matrix of the overlaps $\langle \kappa | \lambda \rangle$ of the basis vectors.

By introducing eq. (2) into eq. (1) we obtain

$$|\psi\rangle = \sum_{\kappa} |\kappa\rangle R_{\kappa\lambda}^{-1}(\lambda | \psi)$$

and the closure relation

$$\sum_{\kappa} |\kappa\rangle R_{\kappa\lambda}^{-1}(\lambda | = I. \quad (3)$$

In the case of an orthogonal basis,

$$\langle \kappa | \lambda \rangle = \delta_{\kappa\lambda},$$

so that

$$R_{\kappa\lambda}^{-1} = \delta_{\kappa\lambda},$$

and eq. (3) becomes the usual expression

$$\sum_{\kappa} |\kappa\rangle \langle \kappa | = I.$$

2.2 Definition of a new basis. — In practice we always have to deal with a finite number of basis vectors. These vectors will be denoted by a small letter $|\bar{k}\rangle$. The other vectors of the basis will be denoted by a capital letter $|K\rangle$. The complete space $\mathcal{H}$ is therefore the union of two subspaces, $\mathcal{H}_{m}$ with finite dimension $N'$ containing the vectors $|\bar{k}\rangle$, and $\mathcal{H}_{m}$ generated by the vectors $|K\rangle$.

We note that the left hand side of eq. (3) mixes both kinds of vectors. Moreover the terms connected with the vectors $|\bar{k}\rangle$ of $\mathcal{H}_{m}$, i.e.

$$\sum_{\bar{K}} |\bar{k}\rangle R_{\bar{k}\lambda}^{-1}(l | \\bar{k}),$$

require the knowledge of the whole basis because of...
This makes the closure relation (3) difficult to handle. In order to get rid of this difficulty let us define new vectors as follows:

\[ |k \rangle = \sum_l B_{lk}^{-1} |l \rangle \]  

(4)

and

\[ |K \rangle = |K\rangle - \sum_k B_{kl}^{-1} (l \mid K) \]

(5)

where \( B \) is the \( n \times n \) matrix of the overlaps \( (k \mid l) \) of the basis vectors of \( \mathcal{C}_m \).

The basis transformation (4) which was proposed by Moshinsky and Seligman [2], possesses the property that

\[ \langle k \mid l \rangle = \delta_{kl} . \]  

(6a)

The vectors \( |K\rangle \) have been defined in eq. (5) in such a way that they are orthogonal to the vectors \( |k\rangle \),

\[ \langle K \mid k \rangle = 0 . \]  

(6b)

Definitions (4) and (5) prevent the mixing of the two kinds of vectors in the closure relation written in this new basis. Indeed we obtain the relation

\[ \sum_{kl} |k \rangle \langle k \mid l \rangle \langle l \mid k \rangle + \sum_{KL} |K \rangle \langle K \mid L \rangle \langle L \mid K \rangle = I \]  

(7)

which is particularly interesting because when applied on the subspace \( \mathcal{C}_m \), the second term vanishes due to eq. (6b). In \( \mathcal{C}_m \) the closure relation may thus be written as

\[ \sum_{kl} |k \rangle \langle k \mid l \rangle \langle l \mid k \rangle = I , \]

or according to eq. (4)

\[ \sum_k |k \rangle \langle k \mid = I , \]

\[ \sum_k |k \rangle \langle k \mid = I , \]

or

\[ \sum_{kl} B_{kl}^{-1} (l \mid k) = I . \]

2.3 SECOND QUANTIZATION OPERATORS. — Let us associate with each vector \( |k\rangle \) of the nonorthogonal basis a creation operator \( a_k^+ \) and an annihilation operator \( a_k = (a_k^+)^+ \). These operators satisfy the anticommutation relations

\[ \{ a^+, a^+ \} = 0 = \{ a_k^+, a_k^+ \} , \]

\[ \{ a_k, a_k^+ \} = (k \mid \lambda) . \]  

(8)

We can also define operators \( \tilde{a}_k^+ \) and \( \tilde{a}_k = (\tilde{a}_k^+)^+ \) associated with the new basis vectors \( |K\rangle \), and satisfying anticommutation relations similar to eqs. (8) except that the round brackets are replaced by angular ones.

Let us consider now \( n \)-body operators and let us look for their second quantized form in either nonorthogonal basis \( |k\rangle \) or \( |K\rangle \). As starting point we know their second quantized form in the position representation in terms of the operators \( \psi^+(q) \) \( \psi(q) \) that create (annihilate) one particle in an eigenstate of the position operator \( r \), the spin component \( S_z \) and the isospin component \( t_3 \). For a one-body operator \( T = \sum_i t_i \), for instance, we have

\[ T = \int dq_1 \ dq_2 \ \psi^+(q_1) (q_1 \mid l \mid q_2) \psi(q_2) . \]  

(9)

To write the operator in the nonorthogonal basis, we have to express \( \psi^+(q) \) in terms of the operators \( a_k^+ \) or \( \tilde{a}_k^+ \). To this end let us apply the closure relation (7) on the vector \( |q\rangle \). We obtain

\[ |q\rangle = \sum_{kl} |k \rangle \langle k \mid l \rangle \langle l \mid q \rangle + \sum_{KL} |K \rangle \langle K \mid L \rangle \langle L \mid q \rangle \]

so that

\[ \psi^+(q) = \sum_{kl} \tilde{a}_k^+ (k \mid l \rangle \langle l \mid q \rangle + \sum_{KL} \tilde{a}_{KL}^+ R_{KL}^{-1} (L \mid q \rangle , \]  

and

\[ \psi(q) = \sum_{KL} (q \mid l \rangle \langle l \mid k \rangle \tilde{a}_k + \sum_{kl} (q \mid L \rangle \langle L \mid k \rangle R_{kl} \tilde{a}_k \]

by using the definition \( \psi(q) = (\psi^+(q))^+ \).

It is now easy to replace \( \psi^+(q) \) and \( \psi(q) \) in eq. (9) by the expressions given in eqs. (10) in order to express the one-body operator \( T \) in the nonorthogonal basis \( |K\rangle \).

2.4 MOSHINSKY-SELIGMAN FORMULATION. — Let us assume that the states describing a system of \( A \) particles are built from the individual states \( |k\rangle \) of \( \mathcal{C}_m \) only. Such states are expressed in second quantization as homogeneous polynomials of degree \( A \) in the operators \( a_k^+ \) acting on the particle vacuum \( |0\rangle \), \( P(a_k^+) |0\rangle \).

Using the anticommutation relation

\[ \{ a^+, a^+ \} = 0 \]

deduced from eqs. (5) and (8), we obtain

\[ \tilde{a}_k P(a_k^+) \mid 0 \rangle = 0 . \]

Consequently, when we evaluate matrix elements of \( n \)-body operators, the first part of \( \psi(q) \) as given by eqs. (10) is the only term that contributes. The expression of \( \psi(q) \) may therefore be truncated.

We can demonstrate a similar property of \( \psi^+(q) \). In the following we shall therefore only use the truncated expressions:

\[ \psi^+(q) = \sum_{kl} \tilde{a}_k^+ (k \mid l \rangle \langle l \mid q \rangle \]

and

\[ \psi(q) = \sum_{kl} (q \mid l \rangle \langle l \mid k \rangle \tilde{a}_k , \]  

(11)
so that we can get back the Moshinsky-Seligman formulation [2].

By putting eqs. (11) in eq. (9) we obtain the following expression for a one-body operator:

\[ T = \sum_{\mathcal{H}} a_+^k \langle k | t | l \rangle \mathbf{d}^l, \]

where

\[ \langle k | t | l \rangle = \sum_{k'} B_{kk'}^{-1}(k' | t | l). \]

For a two-body operator we proceed in the same way and obtain

\[ V = \frac{1}{2} \sum a_+^k a_+^{k'} \langle k_1 k_2 | v | l_1 l_2 \rangle \mathbf{d}^{l_1} \mathbf{d}^{l_2}, \]

where

\[ \langle k_1 k_2 | v | l_1 l_2 \rangle = \sum_{k'_{12}} B_{k1k2}^{-1} B_{k'_{12}}^{-1}(k'_1 k'_2 | v | l_1 l_2). \]

It is this formulation we used in the description of the two-centre model [1, 3].

3. Holes and particles. — 3.1 New vacuum. —

Let us consider the state obtained by filling up all the individual states of \( \mathcal{E}_m \). Such a state, which is unique on account of the exclusion principle, is given by

\[ | \Psi_o \rangle = \prod_{k \in \mathcal{E}_m} a_+^k \langle 0 |. \]

Since \( | \Psi_o \rangle \) contains all the states of \( \mathcal{E}_m \) and only these, it is easy to show that

\[ \mathbf{d}_k^+ | \Psi_o \rangle = 0 \quad \text{for all } k; \]
\[ \mathbf{d}_K^+ | \Psi_o \rangle = 0 \quad \text{for all } K. \]

Let us consider new annihilation operators \( b^k \), defined by

\[ b^k = \mathbf{d}_k^+ \]
\[ b^k = \mathbf{d}_k^+ \]

and such that

\[ b^k | \Psi_o \rangle = 0. \]

The corresponding creation operators \( b^+_k \) are given by \( b^+_k = (b^k)^+ \) i.e.

\[ b^+_k = \mathbf{d}_k^+ \]
\[ b^+_k = \mathbf{d}_k^+ \]

The state \( | \Psi_o \rangle \) behaves like a vacuum state for these new creation and annihilation operators. As \( b^+_k \) corresponds to a particle annihilation operator, it can be viewed as a hole creation operator [8, 9].

The operators \( b^+_k \) and \( b^k \) are still fermion operators as can be shown by looking at the definitions (14) and (15) and the anticommutation relations (8):

\[ \{ b^k, b^+_l \} = 0 = \{ b^+_k, b^+_l \}, \]

and

\[ \{ b^k, b^+_l \} = B^{-1}_{lk} \]
\[ \{ b^k, b^+_l \} = 0 \]
\[ \{ b^k, b^+_l \} = 0 \]
\[ \{ b^k, b^+_l \} = (K | L) - \sum_{l} (K | k) B_{lk}^{-1} (l | L). \]

In the following, we shall be interested in the anti-commutators \{ \( b^k, b^l \), \{ \( b^k, b^+_l \) \} and \{ \( b^+_k, b^+_l \) \} only.

3.2 Group theory and state classification. —

In order to describe a system of \( A \) particles \((A < N)\) we have to consider homogeneous polynomials of degree \( A \) in the particle creation operators \( a_+^k \) acting on the vacuum \( | 0 \rangle \). However, we can work equally well with homogeneous polynomials of degree \((N - A)\) in the hole creation operators \( b^+_k \), acting now on the new vacuum \( | \Psi_o \rangle \).

Let us sketch the classification of the polynomials in the particle operators before studying the case of hole operators.

Let \( GL(N') \) be the group of general linear transformations in the space \( \mathcal{E}_m \). Any transformation \( G \) of \( GL(N') \) transforms the individual states \( k \rangle \) of \( \mathcal{E}_m \) into

\[ G | k \rangle = \sum_{l} | l \rangle G_{lk}. \]

where \( \| G_{lk} \| \) is its matrix representation in \( \mathcal{E}_m \).

Relation (17) induces the following transformation law for the creation operators \( a^+_k \):

\[ G a^+_k G^{-1} = \sum_{l} a^+_l G_{lk}. \]

If we limit ourselves to infinitesimal transformations we obtain the generators of \( GL(N') \),

\[ C^{\alpha}_{k} = a^+_k \mathbf{d}^k. \]

The generators \( C^\alpha_k \) commute among themselves, and we may thus require that the polynomial \( P(a^+_k) \) satisfies the equations

\[ C^\alpha_k P(a^+_k) | 0 \rangle = w_k P(a^+_k) | 0 \rangle \]

where \((w_1, w_2, ..., w_N)\) is its weight. We find that the generators \( C^\alpha_k \), \( k < l \), raise the weight whereas the generators \( C^\alpha_k \), \( k > l \), lower the weight.

For a system of \( A \) particles, all the states belong to the I.R. \([1^A]\) of \( GL(N') \) because we deal with fermions.

In nuclear applications the index \( k \) of the individual wave functions stands for an index \( \rho \) characterizing their spatial part and an index \( s \equiv \langle \rho, \tau \rangle \) specifying their spin-isospin part. The dimension of \( \mathcal{E}_m \) is then \( N = 4 \ N \), if \( N \) denotes the number of values taken by the index \( \rho \). The generators of \( GL(N') \) are now written as \( C_{\rho s}^\alpha \). By contracting on the indices \( \rho \) or \( s \) we get the generators of two subgroups of \( GL(4 N) \),
namely $GL(N)$ for the spatial part and $U(4)$ for the spin-isospin part. They are given in

$$C_p^\rho = \sum_s C_p^{\rho s}$$

and

$$C_s^\rho = \sum_p C_p^{\rho s}$$

for $GL(N)$ and $U(4)$ respectively.

The states are classified according to the chains

$$GL(4N) \supset GL(N) \times U(4),$$

$$GL(N) \supset GL(N - 1) \supset \cdots \supset GL(1)$$

$$U(4) \supset SU_2(2) \times SU_2(2).$$

The same argument still holds when we deal with hole operators. Let us consider now the action of the group $GL(N)$ of general linear transformations on the hole operators rather than on the particle operators,

$$G b_k^+ G^{-1} = \sum_l b_l^+ G_{lk}.$$

The generators of $GL(N)$ can be written in the hole basis as

$$\overline{C}_k^\rho = b_k^+ b^l \delta_k^l$$

where

$$b^l = \sum_{l'} b^{l'}(l' | l).$$

The relation between these generators and those defined in eq. (18) is

$$\overline{C}_k = \delta_k^l - C_l^k,$$

as results from definition (15).

As for the particle operators, we may choose the generators $C_k^\rho$ to give the weight and the generators $\overline{C}_k^\rho$ for $k < l (k > l)$ to raise (lower) the weight.

We shall characterize the states written in terms of hole operators by the I.R. $[f_1, f_2, \ldots, f_N]$ of $GL(N)$, where $(f_1, f_2, \ldots, f_N)$ is the highest weight with respect to the generators $C_k^\rho$. We have seen that for a system of $A$ fermions all the states belong to the I.R. $[1, A]$ of $GL(N)$. We shall now show that the same vectors are characterized by the I.R. $[1, N - A]$ of $GL(N)$. We first notice that $[1, A]$ contains one vector of highest weight $(1^0 0^{N-A})$ and one vector of lowest weight $(0^N A 1^A)$. The latter is of highest weight with respect to the generators $\overline{C}_k^\rho$ as can be seen from

$$C_k^\rho = - \overline{C}_k^\rho, \quad \text{for} \quad k \neq l.$$

Moreover it corresponds to the weight $(1^N A 0^A)$ given by the generators $\overline{C}_k^\rho = I - C_k^\rho$.

We have thus proved that the states of particle symmetry $[1, A]$ are characterized by the hole symmetry $[1, N - A]$.

Let us go back to the nuclear case where index $k$ is $(\rho, s)$ and consider the subgroup $GL(N) \times U(4)$. The generators of $GL(N)$ and $U(4)$ are now given by

$$\overline{C}_k^\rho = \sum_s C_p^{\rho s},$$

and

$$\overline{C}_s^\rho = \sum_p C_p^{\rho s}$$

respectively.

It is interesting to know to what hole spatial symmetry $[h_1, \ldots, h_N]$ of $GL(N)$ belong the states of particle spatial symmetry $[h_1, \ldots, h_N]$. By the same procedure as we used for $GL(N)$ it can be shown that

$$\overline{C}_k^\rho = 4 \delta_k^\rho - C_k^\rho,$$

and that the particle spatial symmetry $[h_1, \ldots, h_N]$ is equivalent to the hole symmetry

$$[4 - h_N, \ldots, 4 - h_1].$$

For instance, for a system of $4N - 2$ particles there are two spatial symmetries and we have the equivalences

$$[4 \ldots 42] \equiv [2]$$

and

$$[4 \ldots 433] \equiv [1^2]$$

so that the states associated with $[4 \ldots 42]$ ($[4 \ldots 433]$) are symmetrical (antisymmetrical) with respect to the permutation of the spatial indices of the two hole creation operators.

3.3 n-BODY OPERATORS. — In order to evaluate matrix elements of one and two-body operators between states written in terms of hole creation operators, we have now to express those operators in the hole formalism.

For that purpose let us look at the operators (10) and take account of the definitions (14) and (15). We obtain

$$\psi^+(q) = \sum_k b^q(k | q) + \sum_{kL} b^q_k R_{kL}^{-1} \left< L | q \right>,$$

and

$$\psi(q) = \sum_k (q | k) b^q_k + \sum_{kL} (q | L) R_{kL}^{-1} b^q_k.$$

In the case where we may truncate these expressions, we have

$$\psi^+(q) = \sum_k b^q(k | q)$$

and

$$\psi(q) = \sum_k (q | k) b^q_k.$$
The one-body operator (9) becomes then
\[ T = \sum_{kl} (k | t | l) b^+_k b^+_l. \]
It can be written in a normal form
\[ T = T_0 - \sum_{kl} (k | t | l) b^+_l b^+_k \]
where
\[ T_0 = \sum_{kl} B^{-1}_{kl} (l | t | k) \]
is nothing but the mean value of \( T \) in the new vacuum \( | \Psi_0 \rangle \).
For a two-body operator we find
\[ V = V_0 - \sum_{kl} v_{kl} b^+_k b^+_l + \frac{1}{2} \sum_{kl} (kk' | v | ll')_A b^+_k b^+_l b^+_l b^+_k \]
where
\[ v_{kl} = \sum_{k'l'} (kk' | v | ll')_A B^{-1}_{k'l'}, \]
\[ V_0 = \frac{1}{2} \sum_{kl} (kk' | v | ll')_A B^{-1}_{ik} B^{-1}_{lk} , \]
and
\[ (kk' | v | ll')_A = (kk' | v | ll') - (kk' | v | l' l) . \]
The matrix elements of \( T \) and \( V \) may be calculated either by a group theoretical method using the matrix elements of the generators of \( GL(N) \), or by a more direct method based on Wick's theorem. In this last method the only contraction which is different from zero is given by
\[ \overline{b^+_k b^+_l} = B^{-1}_{ik}. \]

4. Applications. — 4.1 4 \( N - 2 \) particle systems.
A direct application of the second quantization in nonorthogonal basis and the hole formalism can be found in the \( N \)-centre model [4]. The individual wave functions, composed of a spatial part \( | \rho \rangle \) and a spin-isospin part \( | s \rangle \), are nonorthogonal in their spatial parts only.
Thus, we have
\[ (\rho s | \rho' s') = (\rho | \rho') \delta_{ss'}. \]
To illustrate the method we shall deal here with the \( 4 N - 2 \) particle system. The vectors that are used to describe such a system are characterized by the two empty individual states, \( | k \rangle \) and \( | k' \rangle \). They are denoted by \( | kk' \rangle \) and are written in terms of two hole creation operators,
\[ | kk' \rangle = b^+_k b^+_k | \Psi_0 \rangle . \]
For a small number of holes, it has been established in ref. [3] that the use of Wick’s theorem is most efficient for the calculation of matrix elements. Proceeding this way, the overlaps are immediately given by
\[ (kk' | ll') = (\Psi_0 | \Psi_0) (B^{-1}_{ik} B^{-1}_{ik'} - B^{-1}_{ik} B^{-1}_{ik'}) . \]
For the matrix elements of one and two-body operators we obtain a more compact expression if we introduce
\[ \langle k | t | l \rangle = \sum B^{-1}_{ik} (k' | t | l') B^{-1}_{lk} , \]
\[ V_{kl} = \sum B^{-1}_{ik} v_{kl} B^{-1}_{lk} \]
(19)
4.2 Scalar Force. — When the individual wave functions are \( | \rho_\sigma \rangle ( \rho = 1 \ldots N ; s = (\sigma, \tau) ) \), we can develop the theory further. We have considered two cases: the nuclear force scalar with respect to 

\[ SU_\sigma(2) \times SU_\tau(2) \]

(Wigner, Majorana, Bartlett and Heisenberg mixing) and the Coulomb force. We give here the results for the scalar force.

First of all, let us construct two-hole states characterized by a given total spin and total isospin by coupling the individual spins and isospins:

\[
| \rho_1 \rho_2 \rangle S M S T M_T = (-)^{S-M_S} (-)^{T-M_T} \times \\
\sum (\frac{1}{2} \sigma_1 \frac{1}{2} \sigma_2) | S - M_S \rangle \\
\times (\frac{1}{2} \tau_1 \frac{1}{2} \tau_2) | T - M_T \rangle b_{\rho_1 \sigma_1 \tau_1}^+ b_{\rho_2 \sigma_2 \tau_2}^+ | \Psi_0 \rangle.
\]

(21)

The phases in the right hand side of eq. (21) come from the fact that, whereas \( a_{\rho\sigma} \) is the \((\rho, \sigma)\) component of a \((\frac{1}{2}, \frac{1}{2})\) tensor with respect to \( SU_\sigma(2) \times SU_\tau(2) \), for the annihilation operator the \((\rho, \sigma)\) component is given by

\[
(-)^{\frac{1}{2}-\rho} (-)^{\frac{1}{2}-\sigma} \hat{a}_{-\rho-\sigma} = (-)^{\frac{1}{2}-\rho} (-)^{\frac{1}{2}-\sigma} b_{\rho-\sigma}^+.
\]

From the symmetries of the Clebsch-Gordan coefficients, we verify immediately that

\[
| \rho_1 \rho_2 \rangle S M_S T M_T = (-)^{S+T} | \rho_2 \rho_1 \rangle S M_S T M_T,
\]

so that the symmetric states belonging to the I.R. [2] of \( GL(N) \) have \( S = 0, T = 0 \) or \( S = 1, T = 0 \) while the antisymmetric states belonging to [11] have \( S = 0, T = 0 \) or \( S = 1, T = 1 \), in agreement with the chain

\[ GL(4N) \Rightarrow GL(N) \times U(4). \]

Let us assume that we can write the nuclear force as a product

\[ v(i, j) = u(i, j) w(i, j) \]

of a spatial part \( u \) and a spin-isospin part \( w \) and that in addition \( w \) is scalar with respect to \( SU_\sigma(2) \times SU_\tau(2) \).

Under these conditions \( w \) appears in the matrix elements through its reduced matrix element

\[ w_{ST} = (ST \parallel w \parallel ST). \]

Consequently the quantities \( V_{\rho \rho' \rho'' \rho'''} \) defined in eq. (19) factorize into

\[ V_{\rho \rho' \rho'' \rho'''} = V_{\rho \rho'} \delta_{\rho \rho'} \delta_{\rho'' \rho'''}., \]

where

\[ V_{\rho \rho'} = \frac{1}{2} \left[ X_d \sum_{\rho \rho' \rho''} \langle \rho \rho' | u | \rho'' \rangle (\rho_2 | \rho_1) + X_e \sum_{\rho \rho' \rho''} \langle \rho \rho' | u | \rho_2 \rho' \rangle (\rho_2 | \rho_1) \right] \]

with

\[ X_d = \frac{1}{2} \sum_{ST} (2S + 1)(2T + 1) w_{ST}, \]

\[ X_e = \frac{1}{2} \sum_{ST} (-)^{S+T}(2S + 1)(2T + 1) w_{ST}., \]

and

\[ \langle \rho_1 \rho_2 | u | \rho_3 \rho_4 \rangle = \sum B_{\rho_2 \rho_3}^{-1} B_{\rho_3 \rho_4}^{-1} B_{\rho_4 \rho_1}^{-1} B_{\rho_1 \rho_2}^{-1}. \]

(22)

In writing eq. (22) we used the property

\[ B_{\rho \rho' \rho'' \rho'''}^{-1} = B_{\rho' \rho'' \rho'''}^{-1} \delta_{\rho \rho'} \delta_{\rho'' \rho'''}., \]

When introducing these relations into eq. (20b), we get finally

\[
(\rho_1 \rho_2 S' M'_S T' M'_T | V | \rho_1 \rho_2 S M_S T M_T) = \delta_{S'S} \delta_{M'5M_5} \delta_{T'T} \delta_{M'5M_5}(\Psi_0 | \Psi_0) \times \\
\times \{ V_{\rho_1 \rho_2} B_{\rho_1 \rho_2}^{-1} B_{\rho_1 \rho_2}^{-1} - (-)^{S+T} B_{\rho_1 \rho_2}^{-1} B_{\rho_1 \rho_2}^{-1} \}
\]

\[ + (-)^{S+T} V_{\rho_2 \rho_1} B_{\rho_2 \rho_1}^{-1} - V_{\rho_2} B_{\rho_2}^{-1} + (-)^{S+T} V_{\rho_2} B_{\rho_2}^{-1} - V_{\rho_2} B_{\rho_2}^{-1} \]

\[ + \{ \langle \rho_1 \rho_2 | u | \rho_1 \rho_2 \rangle - (-)^{S+T} \langle \rho_1 \rho_2 | u | \rho_2 \rho_1 \rangle \} w_{ST}. \]
4.3 RESULTS FOR $^{10}$Be. — To illustrate the usefulness of the formalism developed in this paper, let us apply it to study the $^{10}$Be nucleus in a three-centre model. The geometry we adopt is an isosceles triangle whose parameters are defined as on figure 1. The individual wave-functions are given by

$$e^{-\frac{1}{2b^2}(r-r_i)^2} \chi_{\sigma \tau}$$

where $r_i$ are the coordinates of the centres and $b = \sqrt{\hbar/m\omega}$ is the oscillator parameter. We get 12 individual states if we take account of the spin-isospin $\chi_{\sigma \tau}$ functions. To describe $^{10}$Be we have thus at our disposal 66 Slater determinants or linear combinations of them. These 66 states can be separated into basis states of I.R. of $GL(3) \times U(4)$ as is shown in table I. In doing so we reduce the diagonalization of the hamiltonian matrix to that of a 6 x 6 and a 3 x 3 matrix associated with the spatial symmetries $[442]$ and $[433]$ respectively. We can reduce the dimension of the matrices further if we classify the states with respect to the symmetry group of the isosceles triangle $C_{2v}$ [6]. The reductions are given by

$[442] \rightarrow 4 A_1 \oplus 2 B_2$

and

$[433] \rightarrow A_1 \oplus 2 B_2$.

We are thus left finally with a 4 x 4 and a 2 x 2 matrix for the symmetry [442] and a 1 x 1 and a 2 x 2 matrix for the symmetry [433]. This is to be compared with the dimension 66 of our initial matrix.

The results we obtain with the $B_1$ force [7] by minimizing on the three parameters $b$, $d_1$ and $d_2$ are summarized in table II.

We would like to thank Professor M. Demeur, C. Quesne and G. Béart for useful discussions in the course of this work.

### Table I

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Spin (isospin $T = 1$)</th>
<th>$E$</th>
<th>$b$</th>
<th>$d_1$</th>
<th>$d_2$</th>
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<tr>
<td>$[442]$</td>
<td>$A_1$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$B_2$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$A_1$</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$B_2$</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
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<td>-</td>
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</tr>
<tr>
<td></td>
<td>$A_1$</td>
<td>-</td>
<td>-</td>
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<td>-</td>
</tr>
</tbody>
</table>

$[442] \rightarrow 4 A_1 \oplus 2 B_2$

and

$[433] \rightarrow A_1 \oplus 2 B_2$.

We are thus left finally with a 4 x 4 and a 2 x 2 matrix for the symmetry $[442]$ and a 1 x 1 and a 2 x 2 matrix for the symmetry $[433]$. This is to be compared with the dimension 66 of our initial matrix.

The results we obtain with the $B_1$ force [7] by minimizing on the three parameters $b$, $d_1$ and $d_2$ are summarized in table II.

We would like to thank Professor M. Demeur, C. Quesne and G. Béart for useful discussions in the course of this work.

### Table II

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Parity</th>
<th>$E$ (in MeV)</th>
<th>$b$ (in fm)</th>
<th>$d_1$ (in fm)</th>
<th>$d_2$ (in fm)</th>
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<td>1.64</td>
<td>2.07</td>
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<tr>
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<td>1.55</td>
<td>2.42</td>
<td>1.49</td>
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References