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QUANTUM ENERGY DISTRIBUTION FUNCTION OF HOT ELECTRONS IN CROSSED ELECTRIC AND MAGNETIC FIELDS

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Résumé. — A quelles conditions la notion de température électronique a-t-elle un sens pour des électrons couplés à un gaz de phonons et placés dans des champs électrique et magnétique croisés, tous les deux intenses ? La transformation de l'équation maîtresse de Pauli en une équation de Fokker-Planck montre que dans certains cas la fonction de distribution électronique est effectivement maxwellienne. Nous retrouvons ce résultat par des arguments qualitatifs fondés sur un modèle de marche au hasard.

Abstract. — We investigate the conditions under which the notion of electron temperature is meaningful for electrons interacting with phonons and moving in high crossed electric and magnetic fields. The transformation of the Pauli master equation into a Fokker-Planck equation shows that in some cases the electron distribution function is maxwellian. We obtain this result by qualitative arguments based on a random walk approach.

1. **Introduction.** — Many papers [1] have been devoted to the analysis of hot electron phenomena in the presence of a magnetic field. However, only in very few of them [2] has a quantum theoretical calculation of the stationary electron distribution function (E.D.F.), in the limit of high electric and magnetic fields, been carried out without introducing phenomenological considerations. In this paper we present such a calculation and show that the concept of electron temperature, very often used in the interpretation of experimental results on electrical conduction [3], is meaningful at least in some cases.

Section 2 is devoted to the formulation of the problem. Since we have a weak electron-phonon coupling and a strong magnetic field, we are able to use the Pauli master equation [4] and the expression for the electrical current known as the Titeïca formula [5]. We also discuss the influence of the boundary conditions on the solutions of the master equation.

In section 3 we study the case when kinetic energy transfers upon scattering are small compared to the effective electron temperature. We show that for a given Landau band, the master equation can be transformed into a Fokker-Planck type differential equation, the solution of which may be of a maxwellian type with an effective electron temperature. Complications due to transitions between different Landau bands are examined.

In section 4 we consider some assumptions from which an expression for the E.D.F. may be obtained. First of all, we neglect the Pauli exclusion principle and limit the study to the case in which only the first Landau level is occupied. For the phonons we use an Einstein model, appropriate for describing the optical phonons. We show that the E.D.F. reduces to a maxwellian distribution, characterized by an electron temperature different from the phonon temperature. This result is only valid in certain ranges of electric field and phonon temperature. The extension of this result to less restrictive cases is considered. Finally we show how the resulting behaviour may be understood as a *random walk* of the electrons in energy space under the joint action of phonon transitions and of the electrostatic potential energy changes.

2. **Background.** — Let us consider an electron gas without mutual interactions but colliding with phonons. The electrons move in uniform static electric and magnetic fields $\mathcal{E}$ and **B** applied parallel to the x axis and z axis respectively. Due to collision with phonons, an electric current with a non zero component along $\mathcal{E}$ appears allowing the electrons to receive a certain amount of energy from the electric field which is given back to the phonons during collisions. We assume that by interacting with an outside bath the phonons always remain in thermal equilibrium at temperature T.

With obvious notations the Hamiltonian of the electron phonon system can be written :

$$H = H_e + H_p + H_{ep} .$$

where H_e includes the effects of the electric field on the electron motion. (By including $\mathcal{E}$ from the outset, we depart from the standard linear response descriptions.) We neglect all the effects related to the spin of the electrons.

It is well known that the electron states and energy levels in crossed electric and magnetic fields $\mathcal{E}$ and **B** can be specified by the three quantum numbers : n which can assume any positive integral value and k_y and k_z which are multiples of $2\pi/L$, where L represents a characteristic dimension of the electron-enclosing box. We summarize these three indices by v. The eigenfunctions and associated eigenvalues of the Hamiltonian H_e can be expressed as [2] :

$$\Psi_v(\mathbf{r}) = \langle r \mid v \rangle = \text{const.}\ e^{ik_y y}\, e^{ik_z z}\, \Phi_n(x - X_v) , \quad (1)$$

$$E_v = \left(n + \frac{1}{2}\right)\hbar\omega_c + \frac{\hbar^2 k_z^2}{2m} + e\mathcal{E}X_v + \frac{1}{2}mv^2 , \quad (2)$$

where

$$X_v = -\left(\frac{\hbar k_y}{m\omega_c} + \frac{e\mathcal{E}}{m\omega_c^2}\right) , \quad (3)$$

$$v = \frac{\mathcal{E}}{B} , \qquad \omega_c = \frac{eB}{m} \quad (4)$$

and $-e$ is the electron charge.

It is interesting to note that the projection of the classical electron motion on the xy plane is a cycloid ; the centers of its arches having a constant abscissa : $-\left(\frac{v_y^0}{\omega_c} + \frac{e\mathcal{E}}{m\omega_c^2}\right)$, where v_y^0 is the y component of the initial electron velocity. In other words $X_v = \langle v \mid x \mid v \rangle$ represents the mean abscissa of the electron in the state Ψ_v and $e\mathcal{E}X_v$ its potential energy V_v.

Consequently E_v can be split into two parts, one being V_v and the other the mean kinetic energy ε_v. When the electric field is zero, E_v reduces to ε_v.

The electron-phonon interaction will be characterized by the following matrix elements [3] :

$$\gamma_{vv'}(\mathbf{q}) = C(q)\, J_{nk_y,n'k'_y}(q_x)\, \delta_{k_y,k'_y+q_y}\, \delta_{k_z,k'_z+q_z} \quad (5)$$

with

$$J_{nk_y,n'k'_y}(q_x) = \int_{-\infty}^{+\infty} \Phi_n(x - X_v)\, e^{iq_x x}\, \Phi_{n'}(x - X_{v'})\, dx . \quad (6)$$

It can easily be shown that $| J_{nk_y,n'k'_y}(q_x) |^2$ is independant of the electric field. In (5) $C(q)$ specifies the nature and strength of the electron-phonon coupling which is assumed isotropic.

Thus an electron that passes from the state v to the state v' during a collision with a phonon of wave vector **q** sees its mean abscissa change by the amount :

$$X_v - X_{v'} = \frac{\hbar(k_y - k'_y)}{m\omega_c} = \frac{\hbar q_y}{m\omega_c} .$$

Such a shift of X is nothing but the quantum counterpart of the shift of the center when the electron is scattered from one circular orbit to another.

Now, the electron distribution function (E.D.F.) can be obtained, as well as the electric current **j**, by solving the equation of evolution of the density operator :

$$i\hbar\frac{\partial\rho}{\partial t} = [H, \rho] . \quad (7)$$

The E.D.F. is simply the diagonal element

$$\rho_v = \langle v \mid \rho \mid v \rangle$$

and the current is $\mathbf{j} = -e\ \mathrm{Tr}\ \rho\mathbf{v}$ where **v** is the electron velocity operator.

Within the framework of the Born approximation, when the electron-phonon interaction is weak and for times much larger than the duration of a given collision $\rho_v(t)$ obeys the master equation [4] :

$$\frac{\partial\rho_v}{\partial t} = \sum_{v'} (\rho_{v'}(t)\, W_{v'v} - \rho_v(t)\, W_{vv'}) . \quad (8)$$

In (8) we have neglected the Pauli exclusion principle. Similarly the current component parallel to E is given by the following expression (briefly demonstrated in appendix 1) :

$$j_{\parallel} = \sum_v \sum_{v'} \frac{X_v - X_{v'}}{2} (\rho_{v'}\, W_{v'v} - \rho_v\, W_{vv'}) . \quad (9)$$

In these two last equations, $W_{vv'}$ is the probability per unit time for an electron to be scattered from state v to v' and is given by the Fermi golden rule :

$$W_{vv'} = \frac{2\pi}{\hbar} \sum_q | \gamma_{vv'}(q) |^2 \{ N(\omega_q)\, \delta(E_v - E_{v'} + \hbar\omega_q) + [1 + N(\omega_q)]\, \delta(E_v - E_{v'} - \hbar\omega_q) \} \quad (10)$$

$N(\omega_q) = (e^{\beta\hbar\omega_q} - 1)^{-1}$ is the distribution function of the phonons assumed in thermal equilibrium at temperature $T = (k\beta)^{-1}$.

We note that (8) contains no driving term : the electric field is included in the definition of the basis states v. In this respect, our picture is somewhat unusual : *the effect of* $\mathcal{E}$ *appears only in the transition probability* $W_{vv'}$. Such a description is only possible because we have a large magnetic field, which allows a steady situation in the absence of collisions.

The physical information is entirely contained in $W_{vv'}$, which has the following properties :

(i) $W_{vv'}$ depends only on the difference $X_{v'} - X_v$ and not on X_v and $X_{v'}$ separately ; this property is a consequence of the homogeneity of the medium in which electrons move.

(ii) $W_{vv'}$ satisfies the principle of detailed balance :

$$\frac{W_{vv'}}{W_{v'v}} = \exp \beta(E_v - E_{v'}) \tag{11}$$

(11) relies on thermal equilibrium of *the phonons*, irrespective of whether the states v are current carrying or not. (The term in $\delta(E_v - E_{v'} + \hbar\omega_q)$ corresponds to absorption of a phonon while $\delta(E_v - E_{v'} - \hbar\omega_q)$ corresponds to emission.)

(iii) The electric field occurs only in the δ function associated to the energy conservation during the collision ; moreover $\mathcal{E}$ appears in these terms only through the difference $e\mathcal{E}(X_v - X_{v'})$ of the mean potential energy of the electron in states v and v'.

We can immediately remark that if we replace ρ_v in (8) and (9) by the thermodynamical equilibrium distribution :

$$\rho_v^0 = \frac{e^{-\beta E_v}}{\sum_v e^{-\beta E_v}} \tag{12}$$

the right-hand side of eq. (8) and (9) vanishes and we have :

$$\frac{\partial \rho_v}{\partial t} = j_{\|} = 0 .$$

Recalling the expression for E_v we see that ρ_v^0 depends on X_v the mean abscissa of the electron trajectory. That is to say ρ_v^0 describes an inhomogeneous distribution of electrons inside a box of length L where there is an uniform electric field ; electrons accumulate in high electrostatic potential regions exactly as particles of an isothermal neutral gas in a gravity field. In these conditions we understand the reason why $j_{\|}$ is zero : we are in a situation where electrons are not injected at one side of the box and collected at the opposite side. Such an equilibrium configuration is of course not what we want (anyhow, coulomb interaction between electrons makes it unphysical). What we want is an *homogeneous* current carrying state describing electrons in an open system with external contacts that bring charges in and out.

3. **Steady state solution for the master equation and electron temperature.** — We seek a steady state distribution function, that is a solution of the master equation :

$$o = \sum_{v'} (\rho_{v'} W_{v'v} - \rho_v W_{vv'}) . \tag{13}$$

In appendix 2 we present the solutions of the master equation in a special case rather far from realistic conditions. This example is very convenient for finding two solutions of the master equation and for interpreting the result.

The solution of eq. (13) must describe a situation where the electron density is homogeneous and consequently we must look for a k_y-independent solution ρ_v since k_y is the quantum number related to the mean abscissa of the electron in v state. Furthermore ρ must be an even function of k_z. Indeed the electric current density along **B** must be zero ; as the matrix elements of the z component of the electron velocity are :

$$\langle v \mid v_z \mid v' \rangle = \frac{\hbar k_z}{m} \delta_{vv'}$$

the condition $j_z = 0$ implies that ρ_v is a function of k_z^2. We may thus label the occupation ρ_v as $\rho_n(\varepsilon)$, where n is the Landau quantum number, and ε the total *kinetic* energy (excluding, the potential energy $e\mathcal{E}X$). In (13), we may perform the k_y integration. The master equation takes the form :

$$\sum_{n'} \int d\varepsilon' \{ P_{n'n}(\varepsilon', \varepsilon) \rho_{n'}(\varepsilon') - P_{nn'}(\varepsilon, \varepsilon') \rho_n(\varepsilon) \} = 0 . \tag{14}$$

The new transition probabilities, $P_{nn'}(\varepsilon, \varepsilon')$, are k_y integrated. Moreover, they incorporate the density of states $g_n(\varepsilon)$:

$$P_{nn'}(\varepsilon, \varepsilon') = \int g_n(\varepsilon)\, g_{n'}(\varepsilon')\, W_{vv'}\, dk'_y .$$

The detailed form of $P_{nn'}(\varepsilon, \varepsilon')$ depends on the type of scattering. However, we know the order of magnitude of the *energy transfer* $(\varepsilon' - \varepsilon)$ which is the bigger of :

— the phonon energy
— the change in potential energy.

In practice, ΔX is $\sim R$, the cyclotron radius ; for the lowest Landau state,

$$\Delta X \sim \delta = \left(\frac{\hbar}{m\omega_c}\right)^{1/2} .$$

In general a direct solution of (14) is impossible. If it happens that $\rho_n(\varepsilon)$ varies slowly on the scale $(\varepsilon' - \varepsilon)$ of energy transfers, one can transform (14) into a differential equation of the Fokker-Planck type, thereby gaining more insight ; we shall first explore this limiting case without specifying the nature

of $P_{nn'}(\varepsilon, \varepsilon')$. The application to physical situations will be considered later.

We assume that the energy transfer $(\varepsilon' - \varepsilon)$ in a collision is small as compared to the scale of variation of $\rho_n(\varepsilon)$. This may occur either at high temperature ($kT \gg \hbar\omega_q$), or if the electric field is so strong that the average electron energy is $\gg \omega_q$. We may then expand $\rho_{n'}(\varepsilon')$ in a power series in ε', thereby transforming (14) into a set of differential equations.

Let us first assume that only the lowest Landau state, $n = 0$, is occupied. (This will be true for very high magnetic fields, such that $\hbar\omega_c$ is much bigger than the effective electron temperature.) In (14) we can drop the n index :

$$\int d\varepsilon' [P(\varepsilon, \varepsilon')\, \rho(\varepsilon') - P(\varepsilon', \varepsilon)\, \rho(\varepsilon)] = 0 . \quad (15)$$

We note that (15) is equivalent to setting $\frac{\partial J(\varepsilon)}{\partial \varepsilon} = 0$, where

$$J(\varepsilon) = \int_{-\infty}^{+\infty} d\varepsilon' \int_{-\infty}^{+\infty} d\varepsilon''\, P(\varepsilon'', \varepsilon')\, \rho(\varepsilon') \,[\theta(\varepsilon - \varepsilon') \times \times \theta(\varepsilon'' - \varepsilon) - \theta(\varepsilon' - \varepsilon)\, \theta(\varepsilon - \varepsilon'')] \quad (16)$$

(θ is the usual step function) is the net *current* of electrons that cross the energy ε per unit time. (flow upward minus flow downward) ([1]). In (16), we expand $\rho(\varepsilon')$ in a Taylor series. We moreover note that ε' and ε'' are both very close to ε. We may assume that P depends only on ε and on the difference $\xi = (\varepsilon' - \varepsilon'')$ in that range (the coefficients being allowed to drift with ε). (16) then takes the simple form :

$$\begin{aligned} J(\varepsilon) &= - a\rho(\varepsilon) - b \frac{\partial \rho}{\partial \varepsilon} \\ a(\varepsilon) &= \int_{-\infty}^{+\infty} P(\xi, \varepsilon)\, \xi\, d\xi \\ b(\varepsilon) &= \int_{-\infty}^{+\infty} P(\xi, \varepsilon) \frac{\xi^2}{2} d\xi . \end{aligned} \quad (17)$$

Let us introduce the net transition probability at energy ε :

$$\Gamma(\varepsilon) = \int d\varepsilon'\, P(\varepsilon, \varepsilon') = \int d\xi\, P(\xi, \varepsilon) .$$

The coefficients a and b are related to the main energy loss per collision

$$a = \Gamma \bar{\xi}$$

and to the corresponding fluctuations

$$b = \Gamma \bar{\xi}^2/2$$

(For our Taylor expansion to be valid, we must have $\bar{\xi}^2 \gg (\bar{\xi})^2$.)

The physical meaning of (17) is then obvious. The particle current along the *energy axis* comprises two parts :

(i) a drift term $- a\rho$, due to the systematic energy loss at each collision,

(ii) a diffusion term $- b \frac{\partial \rho}{\partial \varepsilon}$, due to energy fluctuations, corresponding to a *random walk* in energy space.

A steady solution corresponds to $J(\varepsilon) = 0$ ([2]). The solution of (17) is then trivial :

$$\rho(\varepsilon) \sim \exp\left[- \int_0^{\varepsilon} \frac{a(\varepsilon')}{b(\varepsilon')} d\varepsilon' \right] . \quad (18)$$

The distribution will be maxwellian *if the ratio a/b is energy independent*. If this is not so, the energy profile is model dependent (although one can estimate the average energy of the electrons).

Unfortunately, the simple result (18) breaks down as soon as we include several Landau levels in the picture. Let $J_n(\varepsilon)$ be the current along the ε axis for a given value of n (as provided by n-conserving collisions). We can repeat the preceding argument : for small energy transfers $(\varepsilon - \varepsilon')$,

$$J_n = - a_n \rho_n(\varepsilon) - b_n \frac{\partial \rho_n}{\partial \varepsilon} . \quad (19)$$

where a_n and b_n depend on n as well as on ε. On top of this, we must take account of the cross currents, due to collisions $n \rightleftarrows n'$. For slowly varying ρ_n, we can perform the ε' integration in (14). The condition for a steady solution becomes

$$\frac{\partial J_n}{\partial \varepsilon} + \sum_{n' \neq n} Q_{nn'}[\rho_{n'} - \rho_n] = 0 \quad (20)$$

$$Q_{nn'}(\varepsilon) = \int d\varepsilon'\, P_{nn'}(\varepsilon, \varepsilon')$$

(the energy integrated matrix $Q_{nn'}$ is symmetric if the ε dependence is negligible over a range $\sim (\varepsilon' - \varepsilon)$.

The difficulty is now apparent : even if a_n/b_n is ε-independent, it will usually depend on n. The different Landau levels would thus be *heated* at different temperatures if $Q_{nn'}$ were 0. Cross-transitions tend to equilibrate the energy distributions of the various Landau states — but doing so they spoil the max-

([1]) More generally, conservation of particles implies

$$\frac{\partial \rho(\varepsilon)}{\partial t} + \frac{\partial J(\varepsilon)}{\partial \varepsilon} = 0 .$$

([2]) The continuity equation implies only $\frac{\partial J}{\partial \varepsilon} = 0$. But the flow is bounded at the lower ε end, and thus J is necessarily zero.

wellian nature of the distribution. The steady solution is now model dependent. It will be obtained by solving (19) and (20), subject to the boundary conditions that $J_n = 0$ at the bottom of each n-band. Since the densities of states (hidden in $P_{nn'}(\varepsilon, \varepsilon')$) depend on n, the structure of $Q_{nn'}$ is complicated, and one cannot draw any simple conclusion (although a numerical solution appears possible).

In conclusion, we see that we may expect a maxwellian distribution with an effective electron temperature only in the extreme quantum limit, when only one Landau state is occupied (even in this simple case, the maxwellian distribution will not hold at the very bottom of the band ($\varepsilon \lesssim \hbar\omega_q$, $e\mathcal{E}\delta$), where our assumption of slow variation of the density of states is not valid).

4. **The extreme quantum limit.** — We now discuss the case where $n = 0$ is the only level which might be occupied by electrons; then the expression of $\gamma_{\nu\nu'}(q)$ is highly simplified. Let us call ν_0 the set of the three indices $(0, k_y, k_z)$; we get

$$|\gamma_{\nu_0\nu_0'}(\mathbf{q})|^2 = C^2(q) \exp\left[-(q_x^2 + q_y^2)\frac{\delta^2}{2}\right] \times \times \delta(k_y' - k_y + q_y)\,\delta(k_z' - k_z + q_z)\,. \quad (21)$$

Thus $|\gamma_{\nu_0\nu_0'}(\mathbf{q})|^2$ is appreciable only if q_y (and also q_x, but we are not interested in this point) is varied in an interval of order $[-\delta^{-1}, \delta^{-1}]$. $W_{\nu_0\nu_0'}$ will be important only for transitions in which the hopping length due to the collision

$$|X_{\nu_0} - X_{\nu_0'}| = \frac{\hbar(k_y' - k_y)}{m\omega_c}$$

is less than or of the order of the radius δ.

In practice, the dispersion of the optical phonons is small on the scale of q_x, $q_y \sim \delta^{-1}$ and we neglect it. Assuming that there is only one phonon mode, we are left with an Einstein model, in which ω_q = constant = ω_0. Along the same lines, we neglect the variation of $|C(q)|^2$ with q. These simplifications are admittedly very crude : we make them in order to have a simpler algebra. The basic transition probability takes the form :

$$W_{\nu_0\nu_0'} \simeq A \exp -\frac{(X_{\nu_0} - X_{\nu_0'})^2}{2\delta^2} \times \times [N(\omega_0)\,\delta(\varepsilon_{\nu_0} - \varepsilon_{\nu_0'} - e\mathcal{E}(X_{\nu_0} - X_{\nu_0'}) + \hbar\omega_0) + (1 + N(\omega_0))\,\delta(\varepsilon_{\nu_0} - \varepsilon_{\nu_0'} - e\mathcal{E}(X_{\nu_0} - X_{\nu_0'} - \hbar\omega_0))] \quad (22)$$

where A is a constant of proportionality.

If we neglect the variation of density of states over the range $(\varepsilon - \varepsilon')$, we see that the transition probability appears as a function $W(\varepsilon' - \varepsilon + e\mathcal{E}x, x)$ where $x = X_{\nu_0'} - X_{\nu_0}$ is the electron hopping length at the collision. Note that for a given x, W obeys the detailed balance condition :

$$W(\varepsilon - \varepsilon' - e\mathcal{E}x, x) = W(\varepsilon' - \varepsilon + e\mathcal{E}x, -x)\, e^{\beta(\varepsilon - \varepsilon' - e\mathcal{E}x)}\,.$$

On the other hand, the same quantity integrated over x,

$$P(\varepsilon - \varepsilon') \approx \int dx\, W(\varepsilon - \varepsilon' - e\mathcal{E}x, x)$$

does *not* obey a global detailed balance relation when $\mathcal{E} \neq 0$:

$$P(\varepsilon - \varepsilon') \neq P(\varepsilon' - \varepsilon)\, e^{\beta(\varepsilon - \varepsilon')}\,. \quad (23)$$

Let $P_0(\varepsilon - \varepsilon')$ be the transition probability in the absence of an electric field (which does obey the detailed balance (23)). According to (22), the corresponding quantity $P(\varepsilon - \varepsilon')$ for finite $\mathcal{E}$ is obtained by *convoluting* P_0 with a gaussian function

$$\varphi_{\mathcal{E}}(\varepsilon - \varepsilon') \sim \exp\left[-\frac{(\varepsilon - \varepsilon')^2}{2\,e^2\,\mathcal{E}^2\,\delta^2}\right]. \quad (24)$$

In our simplified model, the procedure for taking account of the electric field is thus very simple : $\mathcal{E}$ simply acts to *blur* the phonon energies, in a gaussian *symmetrical* way. So doing, it breaks detailed balance (23) : hence the heating.

In our present case, $P_0(\xi)$ and $P(\xi)$ are quite simple. P_0 involves only two discrete peaks, an *emission* peak at $\xi = -\hbar\omega_0$, with weight $(1 + N(\omega_0))$ and an *absorption* peak at $\xi = +\hbar\omega_0$ with weight $N(\omega_0)$. The symmetrical gaussian broadening does not affect the center of these peaks, and thus the average energy loss is field independent :

$$\bar{\xi} = \hbar\omega_0 \,\mathrm{th}\,\beta\frac{\hbar\omega_0}{2}\,. \quad (25)$$

The amplitude of fluctuations, $\overline{\xi^2}$, is however modified by the field

$$\overline{\xi^2} = \frac{\int P(\xi)\,\xi^2\,d\xi}{\int P(\xi)\,d\xi} = \hbar^2\,\omega_0^2 + e^2\,\mathcal{E}^2\,\delta^2\,. \quad (26)$$

We note that both (25) and (26) are ε-independent. On making use of (17) and (18) we conclude that the distribution of electrons is indeed *maxwellian*, with a temperature

$$kT' = \frac{\overline{\xi^2}}{2\,\bar{\xi}} = \frac{\hbar^2\,\omega_0^2 + e^2\,\mathcal{E}^2\,\delta^2}{2\,\hbar\omega_0\,\mathrm{th}\,\dfrac{\beta\hbar\omega_0}{2}}\,. \quad (27)$$

(27) is the central result of our paper. It is valid whenever the Fokker-Planck type analysis is acceptable,

i.e. if $kT' \gg \hbar\omega_0, e\mathcal{E}\delta$. Such a condition is always met in the high temperature limit $\hbar\omega_0 \ll kT$. The electron temperature is then :

$$T' = T\left[1 + \left(\frac{e\mathcal{E}\delta}{\hbar\omega_0}\right)^2\right] \tag{28}$$

for arbitrary values of the field $\mathcal{E}$. In the low temperature limit ($\hbar\omega_0 \gg kT$), (27) is only correct for high electric fields

$$T' \sim \frac{e^2\,\mathcal{E}^2\,\delta^2}{2\,k\hbar\omega_0} \quad (e\mathcal{E}\delta \gg \hbar\omega_0)\,. \tag{29}$$

At low or intermediate fields, the Fokker-Planck differential equation would not apply (and anyhow, departures from the maxwellian are expected for $\varepsilon \sim \hbar\omega_0, e\mathcal{E}\delta$, due to density of state effects).

Finally we can say that the E.D.F. has a maxwellian expression characterized by an electron temperature T' in the two following limits : 1) at low temperature and high electric field. Then T' does not depend on the phonon temperature T and varies as $1/B$; 2) at high temperatures ; then T' depends on T and $1/B$.

4.1 INTERPRETATION IN A RANDOM WALK MODEL. — The previous results may be understood qualitatively by considering the motion of the electron as a random walk in energy space. Let us first investigate the case where the electric field is zero. Then eq. (15) gives a two-term product :

$$W_{\mathcal{E}=0}(x, \varepsilon - \varepsilon') = A\,\mathrm{e}^{-x^2/\delta^2}[N(\omega_0)\,\delta(\varepsilon - \varepsilon' + \hbar\omega_0) + (1 + N(\omega_0))\,\delta(\varepsilon - \varepsilon' - \hbar\omega_0)]\,.$$

If we imagine that transitions occur at equal intervals of time τ, we can say that the first term $\exp[-x^2/\delta^2]$ is proportional to the probability for an electron to jump over a distance x during a transition and that the second is the probability for the electron to vary its kinetic energy from ε to ε' during the same transition. The length a of the most likely jumps is of the order of δ and the mean kinetic energy variation during a transition is

$$\varepsilon' - \varepsilon \simeq -\hbar\omega_0 \,\mathrm{th}\,\frac{\beta\hbar\omega_0}{2} = -\hbar\Omega_0\,.$$

When $\mathcal{E} \neq 0$, the change of the kinetic energy and the length of the jump during a transition are no longer independent. In a model where at each collision an electron jumps over a distance $\pm a$ in the $\mathcal{E}$ direction with the same probability, the kinetic energy variation will be equal to : $\varepsilon' - \varepsilon = -\hbar\Omega_0 \pm e\mathcal{E}a$. Equivalently, we can say that in the kinetic energy space, reduced to a half positive axis, the representative point of the electron's kinetic energy makes a succession of equiprobable jumps of length $\pm e\mathcal{E}a - \hbar\Omega_0$. The determination of the electron's kinetic energy distribution after m such jumps, in the limit $m \to \infty$, is reduced to an unsymmetric random walk problem on a half axis, i.e., with a reflection at the zero of the kinetic energy. This reflection combined with the asymmetry introduced by the systematic loss of the energy $\hbar\Omega_0$ at each jump prevents us from solving this random walk problem exactly. However, in the limit where the kinetic energy reached after m jumps is much larger than the maximum length of a jump : $e\mathcal{E}a + \hbar\Omega_0$, we can neglect the reflection and use the very well known results on random walks [6]. The probability that after m transitions, i.e. after a time $t = m\tau$, an electron starting with a kinetic energy ε_0 acquires the kinetic energy ε is given by :

$$\mathcal{P}^m(\varepsilon) = C_m \exp\left[-\frac{(\varepsilon - \varepsilon_0 + m\hbar\Omega_0)^2}{2\,me^2\,\mathcal{E}^2\,a^2}\right]. \tag{30}$$

As we have already said this eq. (30) is valid only for sufficiently large values of ε. In the limit $m \to \infty$, the E.D.F. is independent of both ε_0 and m and becomes :

$$\rho(\varepsilon) = \lim_{m\to\infty} \mathcal{P}^m(\varepsilon) = C \exp\left[-\frac{\hbar\Omega_0\,\varepsilon}{e^2\,\mathcal{E}^2\,a^2}\right].$$

This result is only valid when the range of $\rho(\varepsilon)$, i.e. $\frac{e^2\,\mathcal{E}^2\,a^2}{\hbar\Omega_0}$ is much larger than $e\mathcal{E}a - \hbar\Omega_0$. In short, this random walk model can be applied in the low temperature limit : $\beta\hbar\omega_0 \gg 1$. For high electric fields such that

$$e\mathcal{E}a \gg \hbar\Omega_0 \sim \hbar\omega_0$$

the E.D.F. is a maxwellian with the temperature (29).

4.2 GENERALIZATION TO OTHER QUANTUM NUMBERS. — Changing the quantum number n does not affect the preceeding analysis. The *intraband* transition probability, $P_n(\varepsilon - \varepsilon')$, is obtained in the same way : at zero field, it is still given by two discrete peaks at $\pm\hbar\omega_0$, with weight $N(\omega_0)$ and $(1 + N(\omega_0))$. The probability at finite field is still found by convoluting P_0 with a gaussian. The only difference is in the *width* of the gaussian, which departs from (24). It is easy to see that the width in question varies as $(n + \frac{1}{2})^{1/2}$. Physically, such an n-dependence is obvious : the bigger n, the bigger the orbit radius (which varies as $(n + \frac{1}{2})^{1/2}$), and consequently the bigger the change in potential energy upon scattering. Without any detailed calculation, we see at once that the mean energy loss, which is not affected by the gaussian, is n-independent

$$\overline{\xi} = \hbar\Omega_0\,.$$

On the other hand, the fluctuation term *is* modified

$$\overline{\xi}^2 \sim \hbar^2\,\omega_0^2 + e^2\,\mathcal{E}^2\,\delta^2(2n + 1)\,.$$

As a result, the ratio a_n/b_n depends on n : the *temperatures* of different n-bands are different, and we

encounter all the difficulties described in section 3 : cross transitions will spoil the maxwellian distribution and no simple conclusion emerges. We note only that the dependence of T'_n upon n is again physically obvious : collisions transfer into the z motion the potential energy gained by the electron when it scatters further along in the x-direction : the bigger n, the bigger the jump, and thus T_n increases.

5. **Conclusion.** — In this paper we have determined the electron distribution function in crossed electric and magnetic field, lying along the x and z axis respectively, in the limit of high magnetic fields and weak electron-phonon coupling. The E.D.F. is then a solution of the Pauli master equation and depends only on the electron kinetic energy ε. The transition probability between electronic states was assumed to depend only on the change of energy and of abscissa of the electron during a collision. This assumption which is essential for the resolution of our problem, has been shown to be entirely correct for Einstein phonons having an electron coupling independent of their wave vector. But it may be extended at least approximately to less restrictive cases, for example to acoustical phonons. In both cases the transition probability will have finite abscissa and energy ranges and therefore the total probability $P(\varepsilon - \varepsilon')$, for changing ε to ε' whatever the abscissa change is, will have also a finite range. It is then possible to transform the master equation into a Fokker-Planck equation under some restrictive conditions. We thus obtain for the Einstein phonons a maxwellian E.D.F. involving an electron temperature T' proportional to the ratio of the second to the first moment of $P(\varepsilon - \varepsilon')$: $T' = \bar{\xi}^2/2\,\bar{\xi}$; this result may also be extended to the acoustical phonons. It is valid only if $\bar{\xi}^2 \gg (\bar{\xi})^2$ a condition which is satisfied for the optical phonons, considered here, in the following regions ; low temperature and high electric field, compared to the phonon energy, or high temperature whatever the electric field.

The low temperature and low field region cannot be investigated by the same technique. In contrast to the preceding cases all the attempts we made to solve directly the master equation gave results strongly dependent on the model used to describe phonons and their interactions with electrons.

Appendix 1. — Eq. (9) may be cast in the equivalent form

$$j_{\|} = \sum_{\nu\nu'} (X_{\nu'} - X_{\nu})\, W_{\nu\nu'}\, \rho_{\nu} .$$

In this form, it hardly needs a demonstration, expressing as it does an obvious physical fact : conduction along E is due to *hopping* of the orbit at each phonon collision. The center of the orbit (or rather of the cycloid) goes from X_{ν} to $X_{\nu'}$. The net current is the hopping probability $\rho_{\nu}\, W_{\nu\nu'}$, times the displacement at each hop : (9) follows at once. This simple argument was given by Titeïca [5] forty years ago.

A microscopic proof of (9) is given by Budd in ref. [2] and we might simply refer the reader to it. However, an objection has been repeatedly raised to our paper-namely that (9) involves only the diagonal part of the density matrix, while the velocity operator v_x is known to have only off diagonal components. This apparent paradox is fictitious. In order to show it explicitly, we briefly sketch a demonstration of (9), analogous to that of Budd [2] (see also the analysis of Kahn and Frederickse [7]).

We start from the Liouville equation obeyed by the density matrix

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho] \tag{31}$$

where $H = H_e + H_p + H_{ep}$ is the total hamiltonian. ρ is an operator in electron space (basis : the eigenstates ν of a H_e including the electric field), and in phonon space (basis : the eigenstates λ of H_p). Following Kohn and Luttinger, we assume that the (rapidly fluctuating) off diagonal part of ρ is forced by the slow drift of the diagonal element $\rho^{\lambda\lambda}_{\nu\nu} = \rho^{\lambda}_{\nu}$. The solution of (31) can then be expanded in powers of H_{ep}, yielding, for a *static process*

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \rho^{\lambda\lambda'}_{\nu\nu'} = 0 = {} & [E_{\nu} - E_{\nu'} + \omega_{\lambda} - \omega_{\lambda'}]\, \rho^{\lambda\lambda'}_{\nu\nu'} + \\ & + (H_{ep})^{\lambda\lambda'}_{\nu\nu'} [\rho^{\lambda'}_{\nu'} - \rho^{\lambda}_{\nu}] \\ & + (H_{ep})^{\lambda\lambda''}_{\nu\nu''} \left\{ \frac{\rho^{\lambda''}_{\nu''} - \rho^{\lambda'}_{\nu'}}{E_{\nu''} - E_{\nu'} + \omega_{\lambda''} - \omega_{\lambda'} - i\eta} (H_{ep})^{\lambda''\lambda'}_{\nu''\nu'} \right\} \\ & - \left\{ \frac{\rho^{\lambda}_{\nu} - \rho^{\lambda''}_{\nu''}}{E_{\nu} - E_{\nu''} + \omega_{\lambda} - \omega_{\lambda''} - i\eta} (H_{ep})^{\lambda\lambda''}_{\nu\nu''} \right\} (H_{ep})^{\lambda''\lambda'}_{\nu''\nu'} . \end{aligned} \tag{32}$$

(We did not drop any indices in order to avoid any ambiguity.) We now extract from (32) the *off diagonal* elements $\rho^{\lambda\lambda'}_{\nu\nu'}$, and we take its trace over the phonon coordinates (assumed to be in thermal equilibrium), in order to obtain the reduced electron distribution :

$$\rho_{\nu\nu'} = \sum_{\lambda} \rho^{\lambda\lambda}_{\nu\nu'} .$$

In the energy denominators, we retain only the δ-function part, describing the dissipative processes which control $j_{\|}$. In doing the phonon average, the term linear in H_{ep} disappears ; in the quadratic term, we separate phonon emission and absorption. (32) thus becomes

$$\begin{aligned} \rho_{\nu\nu'} = {} & \frac{i\pi}{E_{\nu} - E_{\nu'}} \sum_{q\nu''} \gamma_{\nu\nu''}(q)\, \gamma_{\nu''\nu'}(-q) \times \\ & \times \{ [\rho_{\nu}(N_q + 1) - \rho_{\nu''} N_q]\, \delta(E_{\nu''} - E_{\nu} + \omega_q) \\ & + [\rho_{\nu} N_q - \rho_{\nu''}(N_q + 1)\, \delta(E_{\nu''} - E_{\nu} - \omega_q) + id\,(\nu \to \nu') \} \end{aligned} \tag{33}$$

(33) is quite similar to eq. (2.32) of ref. [7], except that we deal with phonons instead of static impurities.

We want to calculate the current along the field

$$j_{\|} = \mathrm{Tr}\ \rho v_x \,. \tag{34}$$

It is true that v_x is off diagonal : in the form (34), $j_{\|}$ only involves the off diagonal part of ρ. However, we may replace ρ by its explicit expression (33) : the current will then involve ρ_v, and (9) will follow at once, despite the fact that v_x is non diagonal. In order to see this equivalence, we note that

$$[H_e, v_y] = -\, i\hbar\omega_0\, v_x \,.$$

Thus

$$\frac{(v_x)_{v'v'}}{E_{v'} - E_v} = \frac{i}{\hbar\omega_0}\,(v_y^{nd})_{v'v} \tag{35}$$

where v_y^{nd} is the non diagonal part of v_y :

$$v_y^{nd} = \frac{P_y}{m} + \omega_0\, x \,.$$

We combine (33), (34) and (35). After some straightforward relabeling of indices, we find

$$j_{\|} = \frac{\pi}{\hbar\omega_0}\,\gamma_{vv''}\,\{\,[\rho_v(N_q+1) - \rho_{v''}\,N_q]\,\delta(E_{v''} - E_v + \omega_q) -$$

$$-\,(v \rightleftarrows v'')\,\}\,\times\,\{\,\gamma_{v''v'}(v_y^{nd})_{v'v} - (v_y^{nd})_{v''v'}\,\gamma_{v'v}\,\} \tag{36}$$

(34) involves the commutator $[\gamma, v_y^{nd}]$. Because the phonons couple only to the electron density, only p_y/m contributes to the commutator ; and we have

$$[\gamma, v_y^{nd}]_{v''v'} = \frac{\hbar}{m}\,(k_y - k_y'')\,\gamma_{v''v'} = \omega_0(X_{v''} - X_v)\,\gamma_{v''v'} \,. \tag{37}$$

Taken together, (36) and (37) are equivalent to (9) and (10).

Once again, let us emphasize that we give this lengthy calculation only to convince those who are not satisfied with the obvious physical argument of Titeïca. Note that this may also be written as

$$j_{\|} = \sum_v X_v[\rho_{v'}\,W_{v'v} - \rho_v\,W_{vv'}] = \sum_v X_v\,\frac{\partial\rho_v}{\partial t} \,.$$

In a steady state, we would conclude that $j_{\|} = 0$. Actually this result is valid for a box of finite size L whereas it is incorrect when $L \to \infty$ because in this case X_v may be infinite and $X_v\,\dfrac{\partial\rho_v}{\partial t}$ may reach a finite value even in a steady state. By allowing an infinite value for L we describe electrons moving between the two plane electrodes of a generator, one of them being the electron source, the other the electron sink. In short the calculation of the current is only possible by taking properly into account the boundary conditions.

Appendix 2. — Solutions of the master equation for a system with one degree of freedom. — We assume that the transition probability $W_{vv'}$ has non vanishing values only between states v and v' such that $n = n'$ and $k_z = k_z'$. Thus k_y is the only quantum number which can vary. This implies that the electrons have one degree of freedom and that we have to look for the dependence of $\rho_{nkz}(k_y)$ in k_y when n and k_z have given values.

In addition to the detailed balance relation :

$$W_{vv'} = W_{v'v}\,\exp\beta e\mathcal{E}(X_v - X_{v'}) \tag{38}$$

and as consequence of the homogeneity of the medium we have :

$$W_{n,k_y,k_z;n,k_y+a,k_z} = W_{n,k_y-a,k_z;n,k_y,k_z} \quad \forall a \,. \tag{39}$$

Owing to eq. (38) we already known that ρ_v^0 is a solution of the master equation. Due to eq. (39) $\rho_{nk_z}(k_y) = \text{const.}$ is also a solution of the master equation ; unlike ρ_v^0 it leads to a non vanishing $j_{\|}$ current. These two results may be well understood in the following way. Let us imagine that the electrons are moving along the $\mathcal{E}$ direction by a succession of jumps between points with abscissa X, ordered increasingly as X_0, X_1, ..., X_i, X_{i+1} ... The property (39) shows that the probability for a jump from X_i to X_{i+1} is the same as that from X_{i-1} to X_i. On the other hand property (38) shows that the probability for a jump from X_i to X_{i+1} is greater than that from X_{i+1} to X_i. The homogeneous solution $\rho_{nk_z}(k_y) = \text{const.}$ implies that the electron fluxes forward and backward have different intensities and that they give a net current $j_{\|} \neq 0$. On the contrary the inhomogeneous solution $\rho_{nk_z}(k_y) = \rho_v^0$ satisfies the relation :

$$\rho^0_{nk_z}(X_i)\,W_{nk_z}(X_i, X_{i+1}) = \rho^0_{nk_z}(X_{i+1})\,W_{nk_z}(X_{i+1}, X_i) \tag{40}$$

so that the two preceding electrons fluxes are equal.

In conclusion the master equation associated to this simple one degree of freedom case has two solutions describing respectively the situation with a non vanishing $j_{\|}$ current and the thermal equilibrium state with $j_{\|} = 0$.

In the first situation we can easily be convinced that energy conservation is satisfied. Electrons receive the power $j_{\|}\,\mathcal{E}$ from the electric field and give back the power P to phonons :

$$P = \sum_v \sum_{v'} \sum_{\mathbf{q}} \hbar\omega_q\,\rho_v(W^E_{vv'}(\mathbf{q}) - W^A_{vv'}(\mathbf{q})) \tag{41}$$

where $W^E_{vv'}(\mathbf{q})$ and $W^A_{vv'}(\mathbf{q})$ are respectively the probability per unit time of emitting or absorbing a

phonon of wavevector $\mathbf{q}$ and energy $\hbar\omega_q$. By using energy conservation we replace $\hbar\omega_q$ by $E_v - E_{v'}$ in the first case and by $-(E_v - E_{v'})$ in the second. P is then a function of the total transition probability

$$W_{vv'} = W^E_{vv'} + W^A_{vv'} :$$

$$P = \sum_v \sum_{v'} \rho_v (E_v - E_{v'})\, W_{vv'} . \qquad (42)$$

By noting that in the present one dimensional case we have $E_v - E_{v'} = e\mathcal{E}(X_v - X_{v'})$ and by inverting v and v' in eq. (37), we obtain :

$$P = e\mathcal{E} \sum_v \sum_{v'} \frac{X_v - X_{v'}}{2} (\rho_v\, W_{vv'} - \rho_{v'}\, W_{v'v}) = e\mathcal{E} j_{\|} .$$

Q.E.D.

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