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ON THE (NON-LINEAR) FOUNDATIONS OF BOUSSINESQ APPROXIMATION APPLICABLE TO A THIN LAYER OF FLUID. (II). VISCOUS DISSIPATION AND LARGE CELL GAP EFFECTS

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Résumé. — Nous présentons ici une généralisation de l'approximation de Boussinesq d'intérêt dans l'étude thermohydrodynamique d'une couche fluide horizontale chauffée par le bas (problème de Rayleigh-Bénard) quand la profondeur de la couche est importante. L'analyse dimensionnelle du problème montre bien sous quelles conditions on ne peut pas négliger, malgré sa petitesse, la compressibilité isotherme. Nous discutons aussi l'éventuelle importance de la dissipation visqueuse ou du gradient thermique adiabatique.

Abstract. — We present here a generalization of the Boussinesq approximation of interest in the thermohydrodynamic study of a horizontal fluid layer heated from below (Rayleigh-Bénard problem) when the depth of the layer is important. A straightforward dimensional analysis of the problem shows under which conditions one cannot neglect, however small, the isothermal compressibility. We discuss also the role played by viscous dissipation and by an adiabatic temperature gradient.

1. Introduction. — In natural convection viscous heating may be important if the body force is large or if the length scale of the problem is large. Such might be the case for convection in the earth's mantle. On the other hand if compressibility effects are of importance they are comparable in magnitude to viscous dissipation effects when Gruneisen's constant is of order unity. Since it is only known empirically that Gruneisen's constant is of order one for fluids, the effects of viscous dissipation and compressibility should be considered together for real substances.

Recently Turcotte *et al.* [1] have discussed the role of viscous dissipation in the convective stability of a horizontal fluid layer heated from below (Rayleigh-Bénard problem). They have shown that both the influence of an adiabatic temperature gradient and of viscous dissipation are governed by the same dimensionless parameter Di (to be defined below). They have considered the case of a quasi-Boussinesq fluid with vanishing isothermal compressibility, χ (for details concerning the Boussinesq approximation see [2], [3], [4] or [5]).

In the present note a dimensional analysis of the kind developed in [3] (hereafter called I) is given. We show that the heuristic quasi-Boussinesq approximation used in [1] is not complete. We critically discuss the relevance of the contribution of an adiabatic temperature gradient, the viscous heating effects as well as the importance of a hydrostatic pressure contribution. This latter term has been arbitrarily disregarded in reference [1]. A most general quasi-Boussinesq description of large cell gap fluid layers is presented following a scheme developed in a previous publication of the present authors [3].

2. Adiabatic temperature gradient and viscous dissipation. — Let us consider a horizontal, single component isotropic Newtonian fluid layer of depth L and infinite horizontal extent. The thermohydrodynamic evolution equations are

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \frac{\partial}{\partial x_i} v_i \qquad (2.1)$$

$$\rho \frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{\partial}{\partial x_i} P + \frac{\partial}{\partial x_j} \tau_{ji} - \rho g \delta_{i3} \qquad (2.2)$$

$$\rho c_{\mathbf{p}} \frac{\mathrm{d}T}{\mathrm{d}t} - \alpha T \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\partial}{\partial x_i} \left(K \frac{\partial}{\partial x_i} T \right) + \tau_{ij} \frac{\partial}{\partial x_j} v_i \,.$$
(2.3)

Here $(^{1})$ a subscript denotes a cartesian component; subscript 3 represents the vertical direction, on occasions also called Z. δ_{ij} is Kronecker's delta and summation convention on repeated indices is used. τ_{ij} is the viscous stress tensor

$$\tau_{ij} = \frac{\mu_2}{3} \frac{\partial v_\lambda}{\partial x_\lambda} \delta_{ij} + \mu_1 \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \frac{\partial v_\lambda}{\partial x_\lambda} \delta_{ij} \right).$$
(2.4)

The remaining symbols have their standard meaning (see however for details [3]). Viscous dissipation is accounted by the last term in the r.h.s. of eq. (2.3).

Let us restrict now our consideration to the case studied by Turcotte et al. [1]. We make the following assumptions : (i) the volumetric expansion coefficient α , the thermal conductivity K, the specific heat at constant pressure $c_{\rm p}$, the isothermal compressibility χ , and the two viscosity coefficients μ_1 (shear viscosity) and μ_2 (bulk viscosity) are kept constant; (ii) the equation of state is

$$\rho = \rho_0 [1 - \alpha (T - T_0) + \chi (P - P_0)] \quad (2.5)$$

where (ρ_0, T_0, P_0) defines a thermodynamic state. For later convenience we define

$$\pi \equiv P - P_0 + \rho_0 \, gz \,. \tag{2.6}$$

With the above imposed restrictions and definitions the thermohydrodynamic equations reduce to

$$-\alpha\rho_0 \frac{\mathrm{d}T}{\mathrm{d}t} + \chi\rho_0 \frac{\mathrm{d}\pi}{\mathrm{d}t} - \chi g\rho_0^2 v_i \delta_{i3} =$$

=
$$-\rho_0 [1 - \alpha (T - T_0) + \chi (\pi - \rho_0 gz)] \frac{\partial}{\partial x_i} v_i$$

(2.7)

 $(^{1})$ Notice that in eq. (2.3) we have used the specific heat at constant pressure. A better suited description for the dimensional analysis to be developed below would come with the use of the specific heat at constant volume. We shall not dwell on this point here however.

$$\rho_0 [1 - \alpha (T - T_0) + \chi (\pi - \rho_0 gz)] \frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{\partial}{\partial x_i} \pi + \frac{\partial}{\partial x_j} \tau_{ji} + \rho_0 g[\alpha (T - T_0) - \chi (\pi - \rho_0 gz)] \delta_{i3}$$
(2.8)

$$\rho_{0} [1 - \alpha (T - T_{0}) + \chi (\pi - \rho_{0} gz)] c_{p} \frac{dT}{dt} - \alpha T \left[\frac{d\pi}{dt} - \rho_{0} gv_{i} \delta_{i3} \right] = \frac{\partial}{\partial x_{i}} \left(K \frac{\partial}{\partial x_{i}} T \right) + \tau_{ij} \frac{\partial}{\partial x_{j}} v_{i} . \qquad (2.9)$$

We next proceed to a standard transcription to nondimensional fields, along the lines developed in part I [3]. In fact we shall scale here with the same units used by Turcotte et al. [1]. Heights are measured with unit L; velocity with κ/L ; times with L^2/κ ; temperatures with the vertical temperature difference; viscosities with μ_1 and pressure with $\rho_0 v \kappa / L^2$. We have introduced the kinematic viscosity $v \equiv \mu_1/\rho_0$ and the thermal diffusivity coefficient $\kappa \equiv K/\rho_0 c_p$.

These scalings involve the parameters

$$\{ L, K, \rho_0 c_p, \Delta T, \mu_1 \}$$

that together with α , χ , g and T_0 form ten independent quantities. With four basic quantities, application of Buckingham's pi-theorem leaves six independent monomials. We choose the following six pi-nominals $(^2)$:

$$\varepsilon_1 = \alpha \, \Delta T \tag{2.10}$$

$$\varepsilon_2 = \chi \rho_0 \, gL \tag{2.11}$$

$$R = \frac{\alpha g L^3 \Delta T}{v \kappa} \qquad \text{(Rayleigh number)} \qquad (2.12)$$

$$\sigma = v/\kappa \qquad (Prandtl number) \qquad (2.13)$$

$$\rho = \Delta T/T_0 \qquad (2.14)$$

$$Di = \frac{\alpha g L}{c_p}$$
 (Dissipation number). (2.15)

The latter parameter Di is the one used by Turcotte et al. [1] to account for viscous heating effects.

Incorporating the above defined six monomials and using now dimensionless quantities the differential system (2.7) (2.8) and (2.9) reduces to $(^3)$

$$- \varepsilon_1 \frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\varepsilon_1 \varepsilon_2}{R} \frac{\mathrm{d}\pi}{\mathrm{d}t} - \varepsilon_2 V_3 =$$
$$= -\left(1 - \varepsilon_1 \theta + \frac{\varepsilon_1 \varepsilon_2}{R} \pi - \varepsilon_2 z\right) \frac{\partial}{\partial x_i} V_i \quad (2.16)$$

(²) The four basic quantities are mass, length, time and temperature. Di measures the extent to which compression work and frictional heating influence the energy balance in the flow. For large cell gaps ε_2 and *Di* are apparently the most important parameters. A different view is taken here however as we want to consider large values of ΔT .

(3) We use $\theta = T - T_0$.

$$\left(1 - \varepsilon_1 \theta + \frac{\varepsilon_1 \varepsilon_2}{R} \pi - \varepsilon_2 z\right) \frac{\mathrm{d}V_i}{\mathrm{d}t} = -\sigma \frac{\partial}{\partial x_i} \pi + \sigma \frac{\partial}{\partial x_j} \tau_{ji} + \sigma R \theta \delta_{i3} - \varepsilon_2 \sigma \pi \delta_{i3} + \frac{\varepsilon_2 \sigma R}{\varepsilon_1} z \delta_{i3}\right)$$

$$(2.17)$$

$$\left(1 - \varepsilon_1 \theta + \frac{\varepsilon_1 \varepsilon_2}{R} \pi - \varepsilon_2 z\right) \frac{\mathrm{d}\theta}{\mathrm{d}t} - \varepsilon_1 \frac{Di}{\varphi R} \times \\ \times (1 + \varphi \theta) \left(\frac{\mathrm{d}\pi}{\mathrm{d}t} - \frac{R}{\varepsilon_1} V_3\right) = \left(\frac{\partial}{\partial x_i}\right)^2 \theta \\ + \frac{Di}{R} \tau_{ij} \frac{\partial}{\partial x_j} V_i \,.$$
(2.18)

We next expand in terms of the two *small* parameters ε_1 and ε_2 . Up to the first non trivial order we get (⁴) :

$$\frac{\partial}{\partial x_i} V_i = 0 \tag{2.19}$$

$$\frac{\mathrm{d}V_{i}}{\mathrm{d}t} = -\sigma \frac{\partial}{\partial x_{i}}\pi + \sigma \frac{\partial}{\partial x_{j}}\tau_{ji} + \sigma R\theta \delta_{i3} + \frac{\varepsilon_{2}}{\varepsilon_{1}}\sigma Rz \delta_{i3} \quad (2.20)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{Di}{\varphi}(1 + \varphi\theta)V_3 = \left(\frac{\partial}{\partial x_i}\right)^2\theta + \frac{Di}{R}\tau_{ij}\frac{\partial}{\partial x_i}V_i \quad (2.21)$$

of reference [1]. A minor difference is that we have kept here the time-dependent terms. There is however an important difference between our eq. (2.20) and Turcotte's eq. (11) that we next discuss. At the same time we shall delineate domains of validity of various quasi-Boussinesq approximations when large cell gaps are of importance. This is not a simple task and we propose an approach based on a numerical example.

To fix ideas let us consider the following numerical values, in C.G.S. units { $\rho_0 \sim 1, \alpha \sim 10^{-4}, \chi \sim 10^{-12}, T_0 \sim 10^3, g \sim 10^3, K \sim 10^5, c_p \sim 10^7$ }. We shall let ΔT range between one and 10^3 degrees, and L between 1 and 10^8 cm (in this latter case we have in mind the mantle convection). The Rayleigh number will be restricted however to the case of a *slightly* convecting layer, i.e. $R \sim 10^3$.

Table I gives the values of ε_1 and φ for various values of ΔT . Table II gives ε_2 and *Di* corresponding to values given to the cell gap

TABLE I

Table III provides for each pair { $\Delta T, L$ } (or else { $\varepsilon_1, \varepsilon_2$ }) the corresponding values of v, for a layer assumed to be slightly convecting only i.e. $R \sim 10^3$. Table IV gives the ratio (Di/φ) in terms of values given to { $\Delta T, L$ } and { $\varepsilon_1, \varepsilon_2$ }.

These equations correspond to eq. (10), (11) and (12)

TABLE II										
L	1	10	10 ²	10 ³	104	10 ⁵	10 ⁶	107	10 ⁸	
8 ₂	10 ⁻⁹	10^{-8}	10^{-7}	10^{-6}	10^{-5}	10-4	10^{-3}	10^{-2}	10^{-1}	
Di	10^{-8}	10^{-7}	10^{-6}	10^{-5}	10-4	10^{-3}	10^{-2}	10^{-1}	1	

				Тав	le III					
v	L	1	10	10 ²	10 ³	104	10 ⁵	10 ⁶	107	10 ⁸
ΔT	$\varepsilon_1, \varepsilon_2$	10^{-9}	10^{-8}	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
1	10^{-4}	10^{-2}	10	104	107	1010	10 ¹³	10 ¹⁶	10 ¹⁹	10 ²²
10	10^{-3}	10^{-1}	10 ²	10 ⁵	10 ⁸	1011	1014	10 ¹⁷	10 ²⁰	10 ²³
10 ²	10^{-2}	1	10 ³	10 ⁶	10 ⁹	1012	1015	1018	10 ²¹	10 ²⁴
10 ³	10^{-1}	10	104	107	10 ¹⁰	10 ¹³	1016	10 ¹⁹	10 ²²	10 ²⁵
				Тав	le IV					
Di/arphi	L	1	10	10 ²	10 ³	104	10 ⁵	10 ⁶	10 ⁷	10 ⁸
ΔT	$\varepsilon_1, \varepsilon_2$	10^{-9}	10^{-8}	10 ⁻⁷	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
1	10^{-4}	10^{-5}	10-4	10^{-3}	10^{-2}	10^{-1}	1	10	10 ²	10 ³
10	10^{-3}	10^{-6}	10^{-5}	10^{-4}	10-3	10^{-2}	10^{-1}	1	10	10 ²
10 ²	10^{-2}	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10-3	10 ⁻²	10 ⁻¹	1	10
10 ³	10^{-1}	10^{-8}	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10 ⁻¹	1

(4) Notice that now contrary to Turcotte *et al.* [1] we retain here $\chi \neq 0$.

TABLE V

Di/R	L	1	10	10 ²	10 ³	104	10 ⁵	10 ⁶	107	10 ⁸
ΔT	$\varepsilon_1, \varepsilon_2$	10^{-9}	10^{-8}	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10 ⁻²	10 ⁻¹
1	10^{-4}	10^{-11}	10^{-10}	10^{-9}	10^{-8}	10^{-7}	10^{-6}	10^{-5}	10-4	10^-3
10	10^{-3}	10^{-11}	10^{-10}	10^{-9}	10^{-8}	10^{-7}	10^{-6}	10^{-5}	10-4	10^{-3}
10 ²	10^{-2}	10^{-11}	10^{-10}	10^{-9}	10^{-8}	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10-3
10 ³	10^{-1}	10^{-11}	10^{-10}	10^{-9}	10^{-8}	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}

According to tables IV and V we see that for $L \leq 10$ and $\Delta T \gtrsim 1$ (and $v \leq 10^4$) the terms

$$\frac{Di}{\varphi}(1 + \varphi\theta) V_3, \quad \frac{Di}{R} \tau_{ij} \frac{\partial V_i}{\partial x_i} \text{ and } \frac{\varepsilon_2}{\varepsilon_1} \sigma Rz \delta_{i3}$$

are negligibly small. Also $c_p \approx c_v$ and thus eq. (2.19), (2.20) and (2.21) are useless. The standard Boussinesq model is a sufficiently good approximation. This is the case for almost all laboratory-controlled experiments in thin Bénard layers.

For values of $L \gtrsim 10^2$ table IV shows a zone (to the right of the dotted lines) where $Di/\phi > \varepsilon_1$, ε_2 . Here the adiabatic temperature gradient term becomes important, but the viscous heating effects however can be disregarded. Thus between the dotted lines and the heavy lines eq. (2.21) is simply

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{Di}{\varphi} \left(1 + \varphi\theta\right) V_3 = \left(\frac{\partial}{\partial x_i}\right)^2 \theta \qquad (2.22)$$

whereas eq. (2.19) and (2.20) should be taken at the Boussinesq approximation only.

Beyond the heavy line in table IV the ratio $\varepsilon_2/\varepsilon_1 \gtrsim 1$. Thus here the term $\frac{\varepsilon_2}{\varepsilon_1} \sigma Rz$ should be relevant in eq. (2.20). Table IV also shows that $(Di/\varphi) > \varepsilon_1, \varepsilon_2$, in that same range. Table V shows however that $(Di/R) \geq \varepsilon_1 \varepsilon_2$. In this range of values the correct quasi-Boussinesq model may be given by eq. (2.19), (2.20) and (2.21) [or alternatively (2.22)].

In the right upper corner of table V we see that $\varepsilon_2 > \frac{Di}{R} \gtrsim \varepsilon_1$. Here the correct quasi-Boussinesq approximation is given by eq. (2.19), (2.20) and (2.21).

Tables I and II provide estimates of the *smallness* of parameters ε_1 and ε_2 . We have now introduced these two parameters as we wanted to discuss in the most straightforward way the effects of both temperature and pressure variations. They have a direct influence upon the buoyancy force. Two other parameters like ε_1 and *Di* may give on occasion a more suitable choice.

For values $L \gtrsim 10^8$ and $\Delta T \gtrsim 10^3$ the parameters ε_1 and ε_2 reach order unity and no longer are useful for our two-parameter perturbative scheme. As we now want to discuss a quasi-Boussinesq model valid for large cell gaps ($L \gtrsim 10^8$) we turn to a slightly

different choice of parameters. We now consider the following six monomials : { ε_1 , ε_2 , $\eta \equiv \varepsilon_2/R\varepsilon_1^2$, σ , Di and φ }. We insert then in eq. (2.16), (2.17) and (2.18) and expand the system of equations in powers of ε_1 and η . We get

$$\varepsilon_2 V_3 = (1 - \varepsilon_2 z) \frac{\partial V_i}{\partial x_i}$$
 (2.23)

• • • •

$$(1 - \varepsilon_2 z) \frac{dV_i}{dt} = -\sigma \frac{\partial \pi}{\partial x_i} + \sigma \frac{\partial}{\partial x_j} \tau_{ji} + \sigma R \theta \delta_{i3} - \varepsilon_2 \sigma \pi \delta_{i3} + \frac{\varepsilon_2}{\varepsilon_1} \sigma R z \delta_{i3} \quad (2.24)$$
$$(1 - \varepsilon_1 z) \frac{d\theta}{dt} + \frac{Di}{\varphi} (1 + \varphi \theta) V_3 = \Delta^2 \theta . \quad (2.25)$$

Comparison of eq. (2.23) to (2.25) with the system (2.19), (2.20) and (2.21) shows : (i) hydrostatic pressure terms ($\varepsilon_2 z$) are present in both systems; (ii) a new term ($-\varepsilon_2 \sigma \pi \delta_{i3}$) appears in eq. (2.24); (iii) viscous dissipation is a second order effect here.

The case $L \sim 10^8$, $\Delta T \sim 10^3$, $T_0 \sim 10^3$ should be of relevance to mantle convection [6]. With data provided by McKenzie *et al.* [6] in C.G.S. units { $\alpha \sim 3 \times 10^{-5}$, $\nu \sim 2 \times 10^{21}$, $\kappa \sim 10^{-2}$, $\Delta T \sim 2 \times 10^3$, $L \sim 7 \times 10^7$ } we arrive at the following estimates

{
$$Di \sim 2 \times 10^{-1}$$
, $R \sim 7 \times 10^{5}$, $\varepsilon_{2} \sim 5$,
 $\varphi \sim 1$, $\sigma \sim 10^{23}$, $\varepsilon_{1} \sim 6 \times 10^{-2}$,
 $\eta \sim 2 \times 10^{-3}$ }.

We have $Di/\phi \gg Di/R$ and $\varepsilon_1 \gtrsim \eta$. Thus in our opinion eq. (2.23), (2.24) and (2.25) contain all relevant quasi-Boussinesq contributions to the thermohydrodynamic problem. The adiabatic gradient term shows its importance whereas viscous heating appears negligible. On the other hand eq. (2.24) shows the relevance of an hydrostatic pressure contribution for large L. It also clearly appears that the velocity field cannot be considered solenoidal as already remarked in reference [6]. Notice that in deriving the system (2.23) to (2.25) we have considered perturbations upon a constant density reference hydrostatic field. A more transparent deduction and improved on quantitative grounds arises with the use of a reference adiabatic hydrostatic field. To this task we devote the next section following ideas already developed in 1 (see also ref. [7]).

3. Quasi-Boussinesq approximation for very large cell gaps. — Peltier [8] has recently given a similar analysis to the one described below. He uses ideas earlier advanced by Ostrach [9]. Peltier's analysis lacks however the rigor of a two-parameter perturbative scheme as set forth by the present authors ([3, 7], see also [2] and [10]). The present section is aimed at assessing in general terms the role of large cell gap effects in the thermohydrodynamic description of Newtonian fluid layers.

A critical discussion of Peltier's approach [8] is now given. We start writing eq. (2.1) to (2.3) incorporating the equation of state (2.5). We get $(^5)$:

$$-\alpha \frac{\mathrm{d}T}{\mathrm{d}t} + \chi \frac{\mathrm{d}P}{\mathrm{d}t} = -(1 - \alpha \Delta T + \chi \Delta P) \frac{\partial V_i}{\partial x_i}$$
(3.1)

$$\rho_0(1 - \alpha \,\Delta T + \chi \,\Delta P) \frac{\mathrm{d}V_i}{\mathrm{d}t} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \tau_{ji} - \rho_0 \,g(1 - \alpha \,\Delta T + \gamma \,\Delta P) \,\delta_{i3} \quad (3.2)$$

$$\rho_{0}(1 - \alpha \Delta T + \chi \Delta P) c_{p} \frac{dT}{dt} - T\alpha \frac{dP}{dt} =$$
$$= \frac{\partial}{\partial x_{i}} K \frac{\partial}{\partial x_{i}} T + \tau_{ij} \frac{\partial}{\partial x_{j}} V_{i} . \quad (3.3)$$

According to prescriptions given already in 1 (see also [7] for related details) we consider a reference a.h.f. (denoted with subscript a)

$$T_{a} = T_{0} - \beta z \tag{3.4}$$

$$P_{a} = P_{0} + \frac{1}{\chi} \left[\left(1 - \frac{\alpha \beta}{\chi g \rho_{0}} \right) (e^{-g \rho_{0} \chi z} - 1) - \alpha \beta z \right]$$
(3.5)

$$\rho_{a} = \rho_{0} \left[\left(1 - \frac{\alpha \beta}{\chi g \rho_{0}} e^{-g \rho_{0} \chi z} + \frac{\alpha \beta}{\chi g \rho_{0}} \right) \dots \right]. \quad (3.6)$$

We now denote with *tilde* perturbations upon this a.h.f.

$$\tilde{\rho} = \rho - \rho_a \tag{3.7}$$

$$\tilde{T} = T - T_{a} \tag{3.8}$$

$$\tilde{P} = P - P_{a} \,. \tag{3.9}$$

We now introduce the tilded quantities in eq. (3.1) to (3.3) and assume, as done by Peltier, that $c_{\rm p}$, α , χ , K and μ are held constants. We get

$$(-g\rho_0^2\chi + \rho_0\alpha\beta) e^{-g\rho_0\chi z} V_3 + \frac{d\tilde{\rho}}{dt} = -\tilde{\rho}\frac{\partial}{\partial x_i}V_i - \rho_0 \left[\left(1 - \frac{\alpha\beta}{\rho_0\chi g}\right)e^{-g\rho_0\chi z} + \frac{\alpha\beta}{\chi g\rho_0}\right]\frac{\partial}{\partial x_i}V_i \quad (3.10)$$

$$\left\{\rho_0\left[\left(1-\frac{\alpha\beta}{\chi g\rho_0}\right)e^{-g\rho_0\chi z}+\frac{\alpha\beta}{\chi g\rho_0}\right]+\tilde{\rho}\right\}\frac{\mathrm{d}V_i}{\mathrm{d}t}=-\frac{\partial}{\partial x_i}\tilde{P}+\frac{\partial}{\partial x_j}\tau_{ji}-g\tilde{\rho}\delta_{i3}$$
(3.11)

$$\begin{cases} \rho_0 \left[\left(1 - \frac{\alpha \beta}{\chi g \rho_0} \right) e^{-g \rho_0 \chi z} + \frac{\alpha \beta}{\chi g \rho_0} \right] + \tilde{\rho} \\ & \times \begin{cases} \frac{d \tilde{P}}{dt} - g \rho_0 \left[\left(1 - \frac{\alpha \beta}{\chi g \rho_0} \right) e^{-g \rho_0 \chi z} + \frac{\alpha \beta}{\chi g \rho_0} \right] V_3 \\ & \end{cases} = \frac{\partial}{\partial x_i} \left(K \frac{\partial}{\partial x_i} \tilde{T} \right) + \tau_{ij} \frac{\partial V_i}{\partial x_j}. \quad (3.12)$$

Notice that the density perturbation takes a simple form. We have

$$\tilde{\rho} = \rho_0(-\alpha \tilde{T} + \chi \tilde{P}). \qquad (3.13)$$

We now proceed to non-dimensionalize the quantities. The only differences here with section 2 above are the following : (i) $\Delta T_{\rm a}$ and \tilde{T} are scaled with βL ; (ii) \tilde{P} is scaled with $\rho_0 \frac{\nu \kappa}{L^2}$; (iii) we introduce the new monomial

$$\Omega \equiv \frac{\tilde{T}_{\max}}{\beta L} \tag{3.14}$$

where \tilde{T}_{max} denotes an upper bound for \tilde{T} ; $\Omega \leq 1$. Thus using the monomials defined in the latter part of section 2 above the new monomial Ω and the remarks just given above we get up to first order in ε_1 and η ,

$$\varepsilon_2 V_3 = \frac{\partial}{\partial x_i} V_i$$
 (3.15)

$$e^{-\varepsilon_{2}z} \frac{\mathrm{d}V_{i}}{\mathrm{d}t} = -\sigma \frac{\partial}{\partial x_{i}} \tilde{P} + \sigma \frac{\partial}{\partial x_{j}} \tau_{ji} + (\sigma R \Omega \tilde{T} - \sigma \varepsilon_{2} \tilde{P}) \delta_{i3} \quad (3.16)$$

$$\Omega \frac{\mathrm{d}\tilde{T}}{\mathrm{d}t} - V_3 + Di \left(\frac{1}{\varphi} - z + \Omega \tilde{T}\right) V_3 =$$
$$= \mathrm{e}^{\varepsilon_{2Z}} \Omega \left(\frac{\partial}{\partial x_i}\right)^2 \tilde{T} . \quad (3.17)$$

These eq. (3.15), (3.16) and (3.17) constitute in our opinion a most general nonlinear quasi-Boussinesq model for a large cell gap (Newtonian fluid)

⁽⁵⁾ Eq. (3.6) corresponds with eq. (8) [8] correcting for an unfortunate misprint in the sign of the exponent there.

layer. These equations correspond *in principle* to eq. (12), (13) and (14) of reference [8]. This correspondence in practice is not accurate. This is due to the fact that Peltier [8] considers infinitesimal perturbations upon the reference a.h.f. Thus the convective $\begin{pmatrix} \partial \\ \partial \end{pmatrix}$

terms (like $V_j \frac{\partial}{\partial x_j} V_i$ and alike) do not appear in [8].

Comparison now with the system (2.23), (2.24) and (2.25) shows that considering perturbations upon a constant density reference hydrostatic field indeed gives altogether the qualitative contributions. The difference is a matter of quantitative evaluation. For ε_2 now need not be small. When $\varepsilon_2 < 1$ the exponentials in (3.16) and (3.17) when Taylor expanded yield back eq. (2.24) and (2.25).

Lastly we remark that holding constant α , χ , $c_{\rm p}$, K and μ as done by Peltier [8] may on occasion represent too a strong restriction. If one considers all

these quantities as functions of T and P as they should in general be, a first order Taylor approximation may or may not be enough. In mantle convection pressure and/or temperature (viz. height) dependence of some of the parameters may be of importance. In such cases the scheme developped in 1 ought to be used.

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