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Classification
Physics Abstracts
1.680 — 7.480

$d \to d'$ CROSSOVER IN THE ANISOTROPIC FREE BOSE GAS

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(*Reçu le 23 septembre, révisé le 17 décembre 1974, accepté le 19 décembre 1974*)

Résumé. — On étudie le passage de d à d' dimensions pour un gaz libre anisotrope de Bose. Pour $d' \geqslant 2$, les résultats obtenus sont en accord avec la solution que donne le groupe de renormalisation. Ils sont étendus à la région $d' < 2$. Ceci permet de penser que l'exposant critique de crossover et égal à deux pour $d' = 1$, $n = 2$.

Abstract. — The $d \to d'$ crossover in the anisotropic free Bose gas is considered. The results agree with the renormalization group solution in the range of $d' \geqslant 2$, where this solution exists and extend its validity to values of $d' < 2$. It is conjectured that for $d' = 1$, $n = 2$ the crossover critical index equals two.

1. **General.** — This paper treats the simple and exactly soluble model of a phase transition in the d-dimensional anisotropic Bose gas at a constant volume. The spectrum of the gas is assumed to be

$$E_\alpha(\mathbf{p}) = \sum_{i=1}^{d'} \frac{p_i^2}{2\,m} + \alpha \sum_{i=d'+1}^{d} \frac{p_i^2}{2\,m} = \frac{p_1^2}{2\,m} + \alpha \frac{p_2^2}{2\,m} . \qquad 0 < \alpha < 1 \tag{1}$$

It was recognized some time ago [1, 2] that the Bose-Einstein condensation of the isotropic $\alpha = 0$ gas corresponds, by virtue of universality, to the $n = \infty$ vector order parameter phase transition in a d'-dimensional system with short range forces. E.g. in both cases the critical index of the susceptibility is

$$\gamma = \frac{2}{d' - 2} \qquad 2 < d' \leqslant 4 . \tag{2}$$

It is then expected that the crossover $\alpha \to 0$ in the system (1) corresponds to a $d \to d'$ crossover of a $n = \infty$ vector model.

The crossover occurs in strongly anisotropic systems. Above a certain *crossover* temperature, T^* such a system behaves according to the laws appropriate to the lower d' dimensionality, below the crossover temperature, it becomes *d-dimensional* and eventually undergoes a phase transition at T_c. Both T^* and T_c depend on the anisotropy parameter α in a way specified by the exponents ψ and φ respectively. It is generally believed that these two exponents are equal. However, it should be mentioned that in this paper we are concerned only with the exponent φ, related to T_c.

The renormalization group approach [3, 4] furnishes the solution of the crossover problem for an arbitrary n, provided that the renormalization group converges in d' dimensions. It converges above the logarithmic line in the n, d' diagram. This line is at $d' = 2$ for $n = \infty$. For $n < \infty$ it bends to a lower d'. The renormalization group result is

$$T_c(\alpha) - T_c(0) \sim \alpha^{1/\varphi} , \tag{3}$$

where

$$\varphi = \gamma(n, d') , \tag{4}$$

and

$$\psi = \varphi .$$

Our first objective is to show that the universality holds in the case of the anisotropic Bose gas (1). Thus, we first rederive eq. (3) and (4) for $d' > 2$. Furthermore, we extend our crossover calculation into the region $d' < 2$, where the renormalization group does not converge, i. e. where eq. (3) and (4) are not obeyed.

The region below the logarithmic line (i. e. $d' < 2$ for $n = \infty$) is the one in which some exact results were recently derived [4] for n arbitrary and $d' = 0,1$. Two conjectures were formulated [5] on the basis of these results. First, it was proposed that some of the quantities which are singular above the logarithmic line at finite T_c, remain singular below this line, but at $T_c(d') = 0$. In particular, the critical index γ exists. Second, it was proposed that eq. (4) remains valid in the region below the logarithmic line, with $\varphi = \gamma = 1$ for $d' = 1$.

Our ($n = \infty$) results for $d' < 2$ support both conjectures : $T_c(\alpha)$ is given by eq. (3) with $T_c(d') = 0$, provided that $d > 2$. The critical index γ exists and

$\varphi = \gamma$. However, some care is required in the definition of γ for $d' < 2$: since $T_c(\alpha = 0) = 0$, the usual T factor in the nominator of the correlation function becomes critical below $d' = 2$. This factor cancels out of the susceptibility. Therefore, γ deduced from the correlation function is smaller by one than that deduced from the susceptibility. It is this latter γ which has to be set equal to φ. Thus in the case $d' = 1$, $\varphi = \gamma = 2$.

The problem of the crossover into the region below the logarithmic line is interesting in many respects. The above mentioned conjectures [5] were formulated in connection with the Peierls transition in a 3-d system of parallel linear $d' = 1$ chains. It was argued [5] that according to whether the deformation wavelength is commensurate or incommensurate with the interatomic distance, the $d = 3 \to d = 1$ crossover occurs respectively at $n = 1$ or $n = 2$. The first $n = 1$, $d' = 1$ crossover [6] corresponds to going to the logarithmic line, i. e.

$$T_c(\alpha) \sim (\ln \alpha)^{-1} . \tag{5}$$

The same type of result holds here ($n = \infty$) for $d' = 2$. Concerning the $n = 2$, $d' = 1$ crossover, it is very likely that below the logarithmic line the relation $\varphi = \gamma$, with the properly defined γ, is valid, not only for $n = \infty$, as shown here, but also for general n. Benefiting then from the fact that γ is independent [4] of n for $d' = 1$ and $n > 1$, our results indicate that in the case $n = 2$, $d' = 1$, $\varphi = \gamma = 2$. The power law is again conjectured, only φ is larger by one than that proposed previously [5]. T_c starts from zero with an infinite slope, when the interchain coupling increases progressively from zero.

Let us finally mention that the crossover in the anisotropic Bose gas was already considered [7] in connection with the superconducting ($n = 2$) fluctuations in the A-15 systems. The authors of this work considered only the $d = 3 \to d = 1$ crossover, but with a more complicated spectrum than eq. (1). This spectrum was believed to be more appropriate to the A-15 systems. In carrying out such analogies, one must be aware of the fact that besides the difference in the value of n between the Bose ($n = \infty$) and the superconductivity ($n = 2$) problem, a more complicated Bose spectrum may correspond to a change in the range of effective forces in the superconductivity problem.

2. **Calculations.** — The critical temperature $T_c(\alpha)$ of the d-dimensional anisotropic Bose gas is the highest temperature for which the chemical potential in the expression for the number of particles is equal to zero, i. e.

$$N = \frac{2s+1}{h^d} V_d \int d^d p \frac{1}{\exp\left[\dfrac{E_\alpha}{k_b T_c(\alpha)}\right] - 1} . \tag{2.1}$$

Here, N is the total number of bosons in the volume V_d. The energy E_α is given by eq. (1.1).

One has to be careful with the limits of integration in eq. (2.1). For the isotropic Bose gas one can integrate over the momentum up to infinity, the smooth cut-off being assured by the exponential decrease of the Bose function. However, when α tends to zero in eq. (1.1) the smooth cut-off of the corresponding momentum components tend to be infinite. Therefore, it has to be replaced by the α-independent *Brillouin-zone* cut-off. Obviously, only such a Bose gas is analogous to the Wilson $n = \infty$ anisotropic vector model [8]. In view of what is said here, we have to solve the equation

$$N = \frac{2s+1}{h^d} V_d \int_{-\infty}^{\infty} d^{d'} p_1 \int_{-P}^{P} \frac{d^\Delta p_2}{\exp\left[\dfrac{p_1^2 + \alpha p_2^2}{2 m k_b T_c(\alpha)}\right] - 1} \tag{2.2}$$

where $\Delta = d - d'$.

The cut-off is conveniently determined by the requirement $T_c(\alpha = 0) = T_c(d')$ as

$$P = \left[\frac{h^\Delta}{V_\Delta S_\Delta} \Delta^{1-(\Delta/2)}\right]^{1/\Delta} , \tag{2.3}$$

where $S_\Delta = 2\pi^{\Delta/2}/\Gamma(\Delta/2)$, while V_d is kept constant in the process of changing α.

Performing the integrations in eq. (2.2) we get

$$C = \frac{1}{2} \tau_\alpha^{d/2} \sum_{n=1}^{\infty} \frac{1}{n^{d'/2}} (\alpha n)^{-\Delta/2} \gamma\left(\frac{\Delta}{2}, \frac{n\alpha}{\tau_\alpha}\right) , \tag{2.4}$$

where we have introduced the abbreviations

$$\tau_\alpha = \frac{2 m k_b T_c(\alpha)}{P^2 \Delta} , \tag{2.5}$$

$$C = N \Big/ \frac{2s+1}{h^d} V_d S_{d'} S_\Delta (P^2 \Delta)^{d/2} , \tag{2.6}$$

and $\gamma\left(\dfrac{\Delta}{2}, \dfrac{n\alpha}{\tau_\alpha}\right)$ is the usual incomplete gamma function [9].

Eq. (2.4) is a closed equation for τ_α as a function of α for arbitrary $\alpha < 1$. Here, we shall investigate this equation only for small α. In doing that, we shall distinguish several regimes, according to the values of d and d'.

2.1 $d' > 2$. — For $\alpha = 0$

$$C = \tau_0^{d'/2} \frac{1}{\Delta} \sum_{n=1}^{\infty} \frac{1}{n^{d'/2}} . \tag{2.7}$$

In order to obtain the form (1.3) we examine the relation

$$C\left(\frac{1}{\tau_\alpha^{d'/2}} - \frac{1}{\tau_0^{d'/2}}\right) = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n^{d'/2}} \frac{(-1)^k (nx)^k}{k! \left(\dfrac{\Delta}{2} + k\right)} . \tag{2.8}$$

In eq. (2.8) we put $\alpha/\tau_\alpha = x$ and use the series expansion

$$\gamma\left(\frac{\Delta}{2}, nx\right) = \sum_{k=0}^{\infty} \frac{(-1)^k (nx)^k}{k!\left(\frac{\Delta}{2}+k\right)}. \quad (2.9)$$

Limiting our consideration to the case of $\alpha \to 0$, we solve the double infinite sum in eq. (2.9) replacing the sum over n by an integral over $nx = t$. This procedure is justified since $x \to 0$ when $\alpha \to 0$ for dimensions $d' > 2$, where $\tau_0 \neq 0$. Thus we have for small x

$$C\left(\frac{1}{\tau_\alpha^{d'/2}} - \frac{1}{\tau_0^{d'/2}}\right) \approx \frac{1}{2} x^{(d'/2)-1} \int_x^\infty t^{-d'/2} \sum_{k=1}^{\infty} \times \frac{(-1)^k t^k}{k!\left(\frac{\Delta}{2}+k\right)} \mathrm{d}t. \quad (2.10)$$

The sum over k is uniformely convergent and integrating term by term we get

$$C\left(\frac{1}{\tau_\alpha^{d'/2}} - \frac{1}{\tau_0^{d'/2}}\right) = \frac{1}{2} x^{(d'/2)-1} . S - \frac{1}{2} x^{(d'/2)-1} \sum_{k+1}^{\infty}{}' \times \frac{(-1)^k}{k!\left(\frac{\Delta}{2}+k\right)} \frac{x^{k+1-(d'/2)}}{k+1-\frac{d'}{2}}\,[1], \quad (2.11)$$

where

$$S = \lim_{r\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\left(\frac{\Delta}{2}+k\right)} \frac{r^{k+1-(d'/2)}}{k+1-\frac{d'}{2}} = \begin{cases} \dfrac{(-1)^{(d'/2)-1}}{\left(\frac{d'}{2}-1\right)!} \dfrac{2}{d-2} & \text{for } d' \text{ even} \\[2ex] \Gamma\left(1-\dfrac{d'}{2}\right) \dfrac{2}{d-2} & \text{for } d' \text{ odd}. \end{cases} \quad (2.12)$$

Within our limitation of ($\alpha \to 0$) we take only the term of the lowest order in the second sum of eq. (2.11). Then we obtain :

- $d' > 4$

$$C\left(\frac{1}{\tau_\alpha^{d'/2}} - \frac{1}{\tau_0^{d'/2}}\right) = \frac{\alpha/\tau_\alpha}{2\left(\frac{\Delta}{2}+1\right)\left(2-\frac{d'}{2}\right)}. \quad (2.13)$$

Taking again only the lowest order terms in α we get

$$T_c(\alpha) - T_c(0) = \frac{P^2}{2\,mk_b\,C}\frac{\Delta}{d'} \frac{\alpha \tau_0^{d'/2}}{\left(\frac{\Delta}{2}+1\right)\left(\frac{d'}{2}-2\right)} \quad (2.14)$$

and from definition (1.3)

$$\varphi = 1 \quad (2.15)$$

- $2 < d' \leqslant 4$.

In this case, eq. (2.11) becomes

$$C\left(\frac{1}{\tau_\alpha^{d'/2}} - \frac{1}{\tau_0^{d'/2}}\right) = \frac{1}{2} \frac{\alpha^{(d'/2)-1}}{\tau_\alpha^{(d'/2)-1}} \cdot S, \quad (2.16)$$

which leads to

$$T_c(\alpha) - T_c(0) = -\frac{P^2\,S\,\Delta\tau_0^2}{2\,mk_b\,Cd'} \alpha^{(d'/2)-1} \quad (2.17)$$

and gives :

$$\varphi = \frac{2}{d'-2}. \quad (2.18)$$

One can see that, as expected, the critical exponent depends only on d' i. e. on the lower dimension. Eq. (2.18) and (2.15) agree with the result obtained within the framework of renormalization group where one has for $n = \infty$,

$$\gamma = \begin{cases} 1 & \text{for } d' > 4 \\ \dfrac{2}{d'-2} & \text{for } 2 < d' \leqslant 4 \end{cases} \quad (2.19)$$

and relation (1.4) is valid.

2.2 $0 < d' \leqslant 2, (d > 2)$. — As already mentioned, this case is of special interest since the renormalization group solution is not available.

We proceed in a way analogous to the one used above. Since, in the present case $\tau_0 = 0$, we do not subtract $1/\tau_0$ as in eq. (2.8), but consider directly the form (2.4). Replacing the sum by an integral, as we did in eq. (2.10), we have to make an aditional assumption that $x = \alpha/\tau_\alpha \to 0$ when $\alpha \to 0$. This assumption is verified *a posteriori* by the solution. By a completely analogous method to that described above we obtain the following results :

- $d' = 2$

$$T_c(\alpha) = -\frac{P^2\,\Delta^2 C}{2\,mk_b} (\ln \alpha)^{-1} \quad (2.20)$$

$$\varphi = \infty \quad (2.21)$$

- $0 < d' < 2$

$$T_c(\alpha) = \frac{P^2\,\Delta C}{2\,mk_b\,S} \alpha^{(2-d')/2} \quad (2.22)$$

where

$$S = \frac{2}{d-2} \Gamma\left(1-\frac{d'}{2}\right)$$

$$\varphi = \frac{2}{2-d'} \quad (2.23)$$

2.3 $d' = 0, (d > 2)$. — This case is to be treated separately, since according to eq. (2.23) $x = \alpha/\tau_\alpha$

[1] The prime over $\sum_k$ means that the term $k = \frac{d'}{2} - 1$ for even $d' \geqslant 4$ has the form const. $\ln x$. For $d' = 4$ this term gives the leading contribution.

does not tend to zero when $\alpha \to 0$. Thus we have to solve

$$C = \left(\frac{\alpha}{\tau}\right)^{-\Delta/2} \gamma\left(\frac{\Delta}{2}, \frac{\alpha n}{\tau_\alpha}\right). \qquad (2.24)$$

Obviously, the solution of this equation is $\tau \sim \alpha$, i. e. $\varphi = 1$. Therefore, although for $d' = 0$ x is finite for $\alpha \to 0$, the limiting value of eq. (2.23) for $d' \to 0$ agrees with the value which follows from eq. (2.24) for $d' = 0$. Unlike eq. (2.23) in this latter equation the assumption $x \to 0$ is not introduced.

One can easily verify that the coefficient γ for the Bose system takes the same values as φ even for $d' < 2$, so the equality (1.4) remains valid at $d' < 2$.

Finally, let us mention that nowhere in the above derivation was it required that d and d' are integers. Therefore, all the results are valid for the general d and d'.

3. **Conclusion.** — We have considered the crossover $d \to d'$ for the anisotropic Bose gas, with an arbitrary d and d'.

In the limit of $\alpha \to 0$ it was found that the exponent φ is given by :

$$\varphi = \gamma = \begin{cases} \dfrac{2}{\mid d' - 2 \mid} & \text{for} \quad 0 \leqslant d' \leqslant 4 \\ 1 & \text{for} \quad d' > 4 \end{cases}$$

where $\varphi = \infty$ for $d' = 2$ corresponds to the logarithmic dependence $T_c(\alpha) \sim (\ln \alpha)^{-1}$.

This result agrees for $d' \geqslant 2$ whith the renormalization group result $\varphi = \gamma$, and extends its validity into the regime of $d' < 2$.

The results which concern, more specifically, the anisotropic Bose gas are :

(i) The closed expression for T_c as a function of α for arbitrary α.

(ii) The coefficient which premultiplies the $\alpha^{1/\varphi}$ dependence. This coefficient depends on both d and d'.

Acknowledgement. — One of the authors (S. B.) enjoyed several discussions with Dr. G. Toulouse.

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