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COMPRESSION OF A PLASMA COLUMN OF INFINITE ELECTROCONDUCTIVITY SITUATED IN AN EXTERNAL AXIAL MAGNETIC FIELD (*)

D. ZOLER, I. GROSU, F. LAUR, R. GAZI and D. RUSCANU

Physical and Technical Research Centre, Iassy, Roumania

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Résumé. — On étudie l'effet d'un champ magnétique extérieur, à symétrie axiale sur un plasma autocomprimé. À l'aide d'un modèle qui généralise celui indiqué en [8], nous avons obtenu un système d'équations qui décrit la situation physique considérée.

Les calculs numériques ont été réalisés à l'aide d'un schéma Lax-Wendorff, à double pas spatial et à pas temporel variable. Nous avons également analysé l'effet du champ magnétique extérieur sur différents paramètres : la vitesse du plasma, la température ionique, la température électronique et la densité du plasma.

Les résultats théoriques concordent de manière satisfaisante avec les résultats expérimentaux [1], surtout pour les phases finales de la compression.

À un certain stade, le rayon de la couche de plasma est plus grand en présence du champ magnétique extérieur qu'en son absence. Afin d'obtenir nos résultats, nous avons utilisé un plasma deutérium.

Abstract. — The effect of an external axial magnetic field on a selfcompressed plasma has been studied. Firstly, a set of equations describing the actual physical situation is derived by means of the generalized model described in [8]. Secondly, the results of a numerical calculation using a double spacestep and varying timestep in the Lax-Wendroff scheme are analysed. The effect of the external magnetic field on plasma parameters such as: plasma velocity, ion temperature, electron temperature and plasma density is analysed. The experimental results [1] are compared with theory and are found to be in good agreement especially for the later times of the compression stage.

At one stage, the plasma column radius is greater than the corresponding one for the case with no magnetic field.

All the calculations are performed for a deuterium gas compression.

1. Introduction. — The purpose of this work is to emphasize the selfcompression of a cylindrical plasma column, situated in an axially symmetric and initially homogeneous external magnetic field.

The selection of such a symmetry for our problem was motivated by the need to realize a suitable agreement with the experimental data obtained in some previous papers [1, 2, 3]. According to these results, during the first compression and expansion stages, the most interesting parameters of the plasma depend on only a single coordinate, the column radius.

We shall be working with the M.H.D. plasma model.

It is worth noting that such a model is valid only in the case of a sufficiently dense plasma; this simply means that the mean free path of the particles must be much less than the column radius and the Debye radius much less than the mean free path.

Amongst the dissipative processes we are primarily concerned here with the electron heat conductivity, the ion heat conductivity, the energy exchange between ions and electron and the ion viscosity.

The dissipative coefficients were calculated for a fully ionized plasma.

If the M.H.D. applicability conditions are fulfilled as a result of the column compression, there appears a shock wave, converging towards the column axis. Our object in this paper is to find the shock wave structure or the change of the most important plasma parameters by passing through the shock wave.

The problem of plasma compression for the steady case, where some mean free path phenomena, such as viscosity and heat conduction are neglected was considered in previous papers [4, 7].

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The same problem, in the absence of an external magnetic field, and taking into account the mean free path phenomena was studied in [8].

The consideration of these parameters enables us to study the structure of the shock wave, which is now to be considered not as a gap in the physical quantities, but rather a continuous change in the last ones during the shock.

It is of interest to describe the phenomena in the domain around the column axis for the cumulation moment when all plasma parameters reach very high values.

In order to lay a groundwork for our discussion, it is worth noting that the manner in which the boundary conditions are formulated plays a very important role.

The following remark should be made: the particle losses through the column surface will be neglected.

It is necessary to specify the magnetic field value on the boundary.

The problem treated here is highly simplified for the case of a current carrying region with an infinite electrical conductivity.

This is in good agreement with those experiments for which there is a small penetration of the magnetic field into the plasma.

On the boundary, the magnetic pressure on the plasma field must equal the sum of the plasma pressure, the radial component of the viscous term and the magnetic pressure due to the longitudinal magnetic field in the plasma.

The heat flow through the external surface and the effect of the magnetic field on the transport coefficients will be neglected, except for the electron heat conductivity.

In agreement with the experimental data, the majority of the plasma compression phenomenon occurs during the stage when the electrical current through the plasma changes very little.

In other words, to a good approximation, it lies in the maximal value domain.

Thus, in all the problems, we consider the electrical current to be a constant quantity and, the electro-technical equations may therefore be omitted.

This model is useful in that it provides a detailed description of the hydrodynamical quantities, characterizing plasma compression phenomenon.

2. Physical assumptions and basic equations. — In order to be able to deduce the system of equations for the compression of a plasma column in the presence of an external magnetic field, the following remarks are made [8]:

a) The electron and ion densities are equal

\[ n_e = n_i = Zn_0. \]

b) Initially the electron and ion temperatures are equal \( T_e = T_i = 2 \text{ eV} \).

c) The external magnetic field has only a longitudinal component \( H_z \); the total magnetic field will therefore be \( N(O, H_{\omega}, H_z) \).

d) In all the problems we consider, we shall have a fully ionized plasma, and both energies and pressures are therefore additives. As a state equation, we shall assume and adiabatic law \( (\gamma = \frac{5}{3}) \).

e) The dissipative coefficients are: the ion heat conduction, the electron heat conduction, the ion viscosity and the electroconductivity. They will be deduced for the fully ionized plasma case. None of the dissipative coefficients depend on the magnetic field, except for the ion heat conduction.

If the plasma electroconductivity is different from zero, we consider only the component of the electroconductivity tensor perpendicular to the azimuthal magnetic field. Such a choice is, initially at least, justified by the smallness of the longitudinal field compared with the azimuthal component.

f) The current density in the plasma has a single component along the column axis.

Thus, Ohm's law is to be written:

\[ J = \sigma_\perp \left( E + \frac{1}{c} [\mathbf{v} \times \mathbf{H}] \right). \] (1)

Because of the fact that \( \sigma_\perp \to \infty \), we obtain:

\[ E + \frac{1}{c} [\mathbf{v} \times \mathbf{H}] = 0. \] (1')

In these conditions, the M.H.D. equation system is of the form [8, 9]:

\[ - \frac{c^2}{4 \pi \sigma_\perp} \Delta \mathbf{H} - \text{rot} [\mathbf{v} \times \mathbf{H}] + \partial \mathbf{H} = \]

\[ = \frac{c^2}{4 \pi \sigma_\perp} \left[ \text{grad} \, \sigma_\perp \cdot \text{rot} \mathbf{H} \right] \]

\[ \rho \frac{dv}{dt} = - \text{grad} \, p + \frac{1}{4 \pi} \left[ \text{rot} \mathbf{H}, \mathbf{H} \right] + \text{Div} \, \sigma' \]

\[ \rho T_i \frac{dS_i}{dt} = \text{div} (\sigma' \mathbf{v}) - \mathbf{v} \cdot \text{Div} \, \sigma' - \text{div} \, \mathbf{F}_i - Q \] (2)

\[ \rho T_e \frac{dS_e}{dt} = - \text{div} \, \mathbf{F}_e + \frac{c^2}{4 \pi \sigma_\perp} (\text{rot} \mathbf{H})^2 + Q \]

\[ \frac{\partial p}{\partial t} + \text{div} \, \rho \mathbf{v} = 0. \]

In the eq. (1), (1') and (2), we denote by: \( \rho \) — the plasma mass density; \( \mathbf{E} \) the electric field in plasma; \( \mathbf{v} \) — the plasma velocity; \( \mathbf{H} \) the magnetic field in plasma; and \( c \) — the velocity of light.

In addition, we consider:

— the ion viscosity tensor,

\[ \sigma'_{jk} = \eta_i \left( \frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3} \delta_{jk} \frac{\partial v_k}{\partial x_k} \right) \] (3)
where $\eta_i$ stands for ion viscosity coefficient;
— the term involving ion heat conduction
\[ F_{ij} = - \kappa_i \frac{\partial T_i}{\partial x_j} \]  
(4)
and electron heat conduction given by
\[ F_{ej} = - \kappa_e \frac{\partial T_e}{\partial x_j} \]  
(5)
— and the term describing the energy exchange between ions and electrons
\[ Q = C \rho \rho_e \frac{k(T_i - T_e)}{(kT_e)^{3/2}} \]  
(6)
where $C$ is a constant whose form will be specified later. $k$ denotes Boltzmann's constant. $\text{Div}$ stands for the tensorial divergence.
We complete the system (2) with a set of thermodynamical equations, and also, according to assumption for a fully ionized plasma with
\[ T_e \cdot \text{d}S_e = \text{d}E_e + p_e \cdot d \left( \frac{1}{\rho} \right) \]
\[ T_i \cdot \text{d}S_i = \text{d}E_i + p_i \cdot d \left( \frac{1}{\rho} \right) \]
$\rho = \rho_i + \rho_e$; $\rho_i = k M \rho T_i$; $\rho_e = Z k M \rho T_e$;  
(7)
$E = E_i + E_e$; $E_i = \frac{3}{2} \rho_i \frac{v_i}{\rho}$; $E_e = \frac{3}{2} \rho_e \frac{v_e}{\rho}$.

The system (2) becomes:
\[ \frac{\partial H_e}{\partial t} = - \frac{\partial}{\partial r} \left( V_r H_e \right) + \frac{\partial}{\partial r} \left[ \frac{\lambda}{r} \frac{\partial}{\partial r} \left( r H_e \right) \right] \]
\[ \frac{\partial H_i}{\partial t} = - \frac{1}{r} \frac{\partial}{\partial r} \left( r V_r H_i \right) + \frac{\lambda}{r} \frac{\partial}{\partial r} \left( r H_i \right) + \frac{\partial}{\partial r} \left( r H_i \right) \]
\[ \frac{\partial V_r}{\partial t} = - \frac{\partial p}{\partial r} - \frac{1}{2 \pi} \frac{\partial}{\partial r} \left( H_i^2 + H_e^2 + H_{0e}^2 + H_{0i}^2 \right) + \frac{4}{3} \frac{\partial}{\partial r} \left( \frac{\partial V_r}{\partial r} \right) + \frac{2}{3 \rho M} \frac{\partial T_i}{\partial t} + \frac{k Z}{M} \frac{\partial T_e}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left[ \rho \left( k T_e \right) \frac{\partial T_e}{\partial r} \right] + \frac{\lambda}{r} \frac{\partial}{\partial r} \left( r H_e \right) \]
\[ \frac{\partial \rho}{\partial t} + \rho \frac{1}{r} \frac{\partial}{\partial r} \left( r V_r \right) = 0 \]  
(8)
In order to perform the numerical calculations, the expressions for the dissipative coefficients in the fully ionized plasma case must be considered.
Their explicit form, showing the dependence on the electron and the ion temperatures and, in case of need also on the magnetic field, can be written as [8],
\[ \lambda = E(Z, L) (kT_e)^{3/2} \]
\[ \eta_i = B(Z, L, M) (kT_i)^{5/2} \]
\[ \kappa_e = D(Z, L, M) (kT_e)^{5/2} \]
\[ \kappa_i = A(Z, L) (kT_i)^{3/2} (H_i^2 + H_{0i}^2) \]
\[ 1 + F(Z, L, M) \frac{(kT_e)^{3/2} (H_i^2 + H_{0i}^2)}{\rho^2} \]  
(9)
In the relations (15), in addition to the known quantities already mentioned, there are also some coefficients $A$, $B$, $D$, $E$, $F$, whose expressions are to be discussed below and $L$ which stands for the coulombian logarithm. In the following, we consider that $L = 20$. Expressions for $A$, $B$, $D$, $E$, $F$ and also for $C$ from (6) can be found in previous publications [10, 11, 12, 13] concerning plasma physics. Thus:

$$A = 1.9 \frac{k}{m^{1/2} e^2 L} \frac{1 + 0.59 Z^{-1}}{1 + 0.25 Z + 0.34 Z^{-1}}$$

$$B = 0.81 \frac{M^{1/2}}{e^2 Z^4 L}$$

$$C = 5.0 \frac{m^{1/2}}{e^2} L \left( \frac{Z}{M} \right)^3$$

$$D = A \left( \frac{m}{M} \right)^{1/2} Z^{-1} \frac{13.56 + 7.92 Z^{-1} + 0.43 Z^{-2} + 1.75 Z^{-3} + 0.59 Z^{-4}}{1 + 0.59 Z^{-1}} - \frac{3.34 + 13.56 Z^{-1} + 4.55 Z^{-2}}{0.25 + Z^{-1} + 0.34 Z^{-2}}$$

$$E = 0.08 \frac{m^{1/2} e^2 c^2 Z h}{\gamma(Z)}; \quad F = 0.33 \frac{M^3}{m c^2 e^6 Z^4 L^2}$$

In the relations (16) the new quantities are: $m$ the electron mass, $e$ the electron charge, $\gamma(Z)$ a function whose values are tabulated in reference as [10]. All relations are expressed using the C.G.S. system of units. The form of the equation system [9-14] is complicated, involving a great number of parameters, and for the sake of simplicity, we seek adimensional expressions for the parameters. To solve this problem, it will be necessary to define the following parameters:

$$h_2 = \frac{H_e}{H_o}; \quad h_1 = \frac{H}{H_o}; \quad \sigma = \frac{\rho}{\rho_0}; \quad x = \frac{r}{R_o}; \quad \tilde{p} = \frac{p}{p_0}, \quad \left( p_0 = \frac{H_o}{8 \pi} \right); \quad U = \frac{V}{V_o}, \quad \left( V_o = \sqrt{\rho_0} \right);$$

$$T = \frac{k}{M} \frac{T_1}{p_0}; \quad \theta = \frac{k}{M} \frac{T}{p_0}; \quad \tau = \frac{t}{t_0}, \quad \left( t_0 = \frac{R_o}{\sqrt{\rho_0} \rho_0} \right); \quad H_o = \frac{2 J_o}{c R_o}.$$ (17)

$\rho_0$ is the initial density of the plasma column, $R_o$ is the initial radius of the plasma column, and $J_o$ is the current carried by the plasma. According to the notations (17) we find:

$$\frac{\partial h_1}{\partial t} = - \frac{\partial}{\partial x} (U h_1) + \frac{E}{a \beta} \frac{1}{M^{3/2}} \frac{1}{x} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} (x h_1) \right)$$

$$\frac{\partial h_2}{\partial t} = - \frac{1}{x} \frac{\partial}{\partial x} (x U h_2) + \frac{E}{a \beta} \frac{1}{M^{3/2}} \frac{1}{x} \frac{\partial}{\partial x} (h_2) + \frac{\partial}{\partial x} \left( \frac{1}{a \beta} \frac{\partial}{\partial x} (h_2) \right)$$

$$\sigma \frac{dU}{dt} + \frac{\partial}{\partial x} \left[ \sigma (T + Z \theta) \right] = - \frac{2}{3} \frac{1}{x} \frac{\partial}{\partial x} \left( h_1^2 + h_2^2 \right) + \frac{h_1^2}{x} + \frac{4}{3} B M^{5/2} \frac{\partial}{\partial x} \left( T^{5/2} \frac{\partial U}{\partial x} \right) + \frac{2}{3} \frac{T^{5/2}}{x} \frac{\partial}{\partial x} \left( U \right) - \frac{U}{x} \frac{\partial}{\partial x} \left( T^{5/2} \right)$$

$$\frac{2}{3} \frac{\partial T}{\partial t} + \frac{\sigma T}{x} \frac{\partial (U x)}{\partial x} = \frac{4}{3} B M^{5/2} \frac{\partial (U x)}{\partial x} + \frac{2}{x} \frac{T^{5/2}}{x} \frac{\partial}{\partial x} \left( \frac{U}{x} \right) - \frac{C}{\alpha M^{1/2}} \frac{\sigma^2 (T - \theta)}{\theta^{3/2}}$$

$$\frac{3}{2} \frac{Z \sigma}{x} \frac{\partial \theta}{\partial t} + \frac{Z \sigma}{x} \frac{\partial (U x)}{\partial x} = A M^{5/2} \frac{1}{k} \frac{1}{x} \frac{\partial}{\partial x} \left[ \frac{x \theta^{5/2} \frac{\partial \theta}{\partial x} + 1 + 8 \pi F M^3 x^2 \beta (h_1^2 + h_2^2) \theta}{\sigma^2} \right] +$$

$$\frac{4 E}{M^{3/2}} \frac{1}{a \beta} \left[ \left( \frac{\partial h_2}{\partial x} \right)^2 + \frac{1}{x^2} \left( \frac{\partial}{\partial x} (x h_1) \right)^2 \right] + \frac{C}{M^{1/2}} \frac{\sigma^2 (T - \theta)}{\theta^{3/2}}$$

$$\frac{\partial}{\partial t} + \frac{\partial}{\partial x} (U x) = 0.$$
It may be easily seen that the system (18)-(23) incorporates two parameters \( r_1 \) and \( f_3 \) whose form is:

\[
\alpha = \frac{p_0}{\rho_0 R_0} ; \quad \beta = \rho_0 R_0^2 .
\]  

(24)

It should be noted that, for different initial boundary conditions, it is possible to find more than two parameters.

In this paper, an attempt is made to arrive at a theoretical understanding of the effect of a longitudinal external magnetic field on the compression of a fully plasma column of infinite electrical conductivity.

In these conditions, the system of eq. (18)-(23) can be essentially simplified.

Firstly, the terms involving the factor \( E \) vanish identically. Let us fix the initial physical situation in order to deduce all other simplifications.

Initially, the plasma column is maintained in a homogeneous longitudinal magnetic field \( h_{10} \) and also an azimuthal field only inside a layer infinitely small thickness, near the plasma boundary.

According to a well known theorem the magnetic field freezes into the plasma layer, as a result of the infinite conductivity of the plasma and the azimuthal field cannot penetrate into the plasma. In other words, the azimuthal field component is vanishingly small throughout the plasma and differs from zero only the boundary. In the same way, from the eq. (23) and (19) we infer

\[
\frac{h_2}{\sigma} = \frac{h_{20}}{\sigma_0} \quad (\sigma_0 = 1)
\]  

(24')

and therefore

\[
h_2 = h_{20} \sigma .
\]  

(25)

Thus, system (18)-(23) reduces to a set of four equations with three parameters \( \alpha, \beta, h_{20} \), as follows:

\[
d\frac{U}{d\tau} + \frac{\partial}{\partial x}[\sigma(T + \theta)] =
\]

\[
= - 2 h_{20} \sigma \frac{d\sigma}{dx} + 3.76 \times 10^{-35} \frac{x}{\sigma} \left[ \frac{\partial}{\partial x} \left( T^{5/2} \frac{U}{x} \right) + \frac{T^{5/2}}{x} \frac{U}{x} \right]
\]

(26)

\[
\frac{3}{2} \frac{d^2 T}{d\tau^2} + \frac{T}{x} \frac{\partial}{\partial x} (Ux) = 3.76 \times 10^{-35} \frac{x}{\sigma} \left[ \frac{\partial U}{\partial x} - \frac{U}{x} \left( \frac{\partial U}{\partial x} - \frac{U}{x} \right) \right] +
\]

\[
+ 1.12 \times 10^{-34} \frac{x}{\sigma} \frac{\partial}{\partial x} \left( T^{5/2} \frac{\partial T}{\partial x} \right) - 2.35 \times 10^{33} \frac{1}{\sigma} \frac{T - \theta}{\theta^{3/2}}
\]

(27)

\[
\frac{3}{2} \frac{d^2 \theta}{d\tau^2} + \frac{\theta a Z}{x} \frac{\partial}{\partial x} (Ux) = \frac{4.02 \times 10^{-33} \frac{x}{\sigma}}{1 + 8.68 \times 10^{58} a^2 \beta h_{20} \theta^3} \frac{1}{x} \frac{\partial}{\partial x} \left( T^{5/2} \frac{\theta}{x} \right) + 2.35 \times 10^{33} \frac{x}{\sigma} \frac{T - \theta}{\theta^{3/2}}
\]

(28)

\[
\frac{d\sigma}{d\tau} + \frac{\sigma}{x} \frac{\partial}{\partial x} (Ux) = 0 .
\]

Here, we have used:

\[
\frac{AM^{7/2}}{k} = 4.02 \times 10^{-33} ; \quad \frac{BM^{5/2}}{k} = 3.76 \times 10^{-35} ; \quad \frac{C}{M^{1/2}} = 2.35 \times 10^{33} ;
\]

\[
\frac{DM^{7/2}}{k} = 1.12 \times 10^{-34} ; \quad \frac{E}{M^{3/2}} = 2.77 \times 10^{24} ; \quad 8 \pi FM^3 = 8.68 \times 10^{-58} .
\]

(29)

These quantities have been calculated for \( D_2 \) gas \( (Z = 1 ; \quad m = 3.34 \times 10^{-24} \text{ g/cm}^3) \).

Obviously, there is an important difference between our problem, and the treatments in which the external magnetic field is absent.

System (26)-(28) incorporates three parameters : \( \alpha, \beta, h_{20} \).
3. Initial conditions and boundary conditions. — In order to find some correct and suitable results, from the physical point of view, using a numerical approach for models of the compression of a plasma column the problem of initial and boundary conditions is important.

In formulating our assumptions, we consider for the moment, the dimensional form of the initial and boundary conditions in order to emphasize the physical features.

The initial conditions are most conveniently expressed as follows:

a) at the initial moment \( t = 0 \), the plasma velocity towards the column axis is identically zero;
b) we assume that we are dealing with a cold plasma and that therefore \( T_e = T_i = 0 \);
c) we assume a homogeneous plasma \( \rho_0 = Cte \);
d) the azimuthal field \( H_\phi \) = 0;
e) \( H_\zeta = H_{\zeta 0} = Cte \); and
f) the column radius is \( R = R_0 \).

From the system (26)-(29) it is evident, that the initial condition for the axial magnetic field is not a necessary one because of the fact that \( H_\zeta \) is not explicitly expressed in the equations, but only by means of the parameter \( h_{\zeta 0} \).

In addition, we comment on the role of the electron temperature in the denominator of the last term of the equations (27) and (28).

If we start with \( T_{\zeta 0} = 0 \). To to avoid confusion, a divergence of the numerical computational scheme would occur. The condition of cold plasma is to be replaced here with the relation \( T_{\zeta 0} = T_{\zeta 0} = 2 eV \).

It is worth noting that such a choice [4, 5] is well justified from the physical point of view.

From the above considerations, it may be concluded that we are dealing with initial conditions of the form:

\[
\begin{align*}
t &= 0; & \rho &= \rho_0; & T_i &= T_e = 2 eV; \\
V_r &= 0; & H_\phi &= 0; & R &= R_0; & H_\zeta &= H_{\zeta 0}. 
\end{align*}
\]

Before formulating the boundary conditions, we must specify what we mean by boundary conditions in this case.

In our case, the boundary notion involves both the external surface of the plasma column and the column axis and the limiting conditions for both ranges of plasma column must be considered. It has already been mentioned that the plasma column is thermally isolated, and heat loss across the external surface is therefore absent.

As a result, both the electron and the ion components of the heat flow cancel out on the boundary \( R_0 \), in other words:

\[
\left. \kappa_e \frac{\partial T_i}{\partial r} \right|_{r=R} = 0; \quad \left. \kappa_e \frac{\partial T_\zeta}{\partial r} \right|_{r=R} = 0. \quad (31)
\]

Furthermore, through the use of the simplifying assumption that there is no particle flow across the external boundary (i.e. the plasma mass is to be conserved during the compression stage) it can be concluded that the internal pressure equals the pressure on the boundary.

The internal pressure consists of the total gas pressure (i.e. the pressure together with the viscous heat terms) plus the magnetic pressure of the axial field.

The total pressure equals the pressure of the azimuthal magnetic field on the boundary. Therefore, we can write:

\[
\rho = \frac{4}{3} \eta_1 \frac{\partial V_r}{\partial r} + \frac{2}{3} \eta_1 \frac{\partial V_r}{\partial r} + \frac{H_{\zeta 0} \rho^2}{\rho_0} \frac{\partial^2}{\partial r^2} = \frac{H_{\zeta 0}^2}{8 \pi}. \quad (32)
\]

It can readily be checked that, according to conditions on the internal boundary (and therefore on the plasma column axis), the plasma velocity on the axis, the azimuthal field and the heat flow must be zero.

In order to characterize these statements analytically, we write:

\[
r = 0; \quad V_r = 0; \quad H_\phi = 0; \quad \frac{\partial T_i}{\partial r} = \frac{\partial T_\zeta}{\partial r} = 0. \quad (33)
\]

It would be quite practical to express the system of initial and boundary conditions (30)-(33) adimensionally, as follows:

\[
\begin{align*}
t &= 0; & \sigma &= 1; & U &= 0; & T &= \theta = 0; & x &= 1; & h_3 &= h_{\zeta 0} \\
x &= 0; & U &= 0; & \frac{\partial T}{\partial x} &= \frac{\partial \theta}{\partial x} = 0 \\
x &= X; & \left( \frac{X}{R_0} \right) &= \frac{\partial T}{\partial x} = \frac{\partial \theta}{\partial x} = 0 \\
(T + Z \theta) &= 3.76 \times 10^{-35} \cdot a. T^{5/2} \left( \frac{\partial U}{\partial x} - \frac{1}{2} \frac{U}{x} \right) + \\
&+ h_{\zeta 0} \sigma^2 = \frac{1}{x^2}. \quad (34)
\end{align*}
\]

The eq. (26)-(29) which are consistent with the initial and the boundary conditions (34)-(37) form a closed mathematical system which may be conveniently solved.

It is appropriate to choose typical values for several initial quantities, depending on the real typical situation.

4. Results of the numerical calculation. — 4.1 Analysis of the results in the first stage compression.

It has already been noted in section (2) that, on grounds of simplicity, the finite difference system is to be written in an adimensional form. This enables us to reduce the number of the parameters to three \((\alpha, \beta, h_{\zeta 0})\). In order to solve several cases consistent
In order to describe the compression stage of a D₂ gas, the initial conditions, in dimensional variables, are the following:

- the initial radius of the plasma column 18.47 cm,
- the axial magnetic field \( H_z = 0 \); 540 uCGS; 810 uCGS,
- \( T_e = T_i = 2 \) eV,
- \( \rho_0 = 1.25 \times 10^{-7} \) g/cm³ which corresponds approximately to a particle density \( n = 3.75 \times 10^{16} \) cm⁻³.

- \( J_0 = 1.25 \times 10^6 \) A.

Under these conditions, the parameters \( \alpha, \beta \) and \( h_{20} \) will be:

\[
\alpha = 1.47 \times 10^{33}; \quad \beta = 4.27 \times 10^{-5};
\]

\[
h_{20} = 0; \quad 4.02 \times 10^{-2}; \quad 6.03 \times 10^{-2} \quad (38)
\]

and the initial values are:

\[
\tau = 0 \quad \begin{cases} T = \theta' = 0.018 \\ \sigma = 1; \quad U = 0 \end{cases} \quad (39)
\]

From the relations (24) we note that \( \alpha \) is inversely proportional with the density cube.

Therefore, for a constant current, as our case, \( \alpha \) decreases with the density increase.

According to some workers [8], the large values of \( \alpha \) correspond to large ion free mean paths and the whole compression process is carried out continuously, without any shock wave.

There is a temperature equalisation throughout the entire plasma column.

For the limiting case \( \alpha \to \infty \) we obtain the steady solution of the problem where the temperatures are supposed to be equal throughout radial dimension of the plasma column.

The value of \( \alpha \) chosen by us lies in the range where the ion free mean path is sufficiently small that the compression phenomenon will be associated with the appearance of a convergent shock wave.

In the front of this shock wave the characteristic parameters are undergoing important increases.

It is worth noting that for smaller values than the characteristic one, we are also looking for solutions of the typical form of a convergent shock wave.

Before considering the analysis of the results for the first compression stage, concerning the numerical calculations, some remarks are to be made.

The code to be used in this paper is a two step, Lax-Wendroff method [15], for the M.H.D. equation system. The time-steps \( \Delta t \) that can be used in an explicit calculation are variable [16]. The explicit code is indicated in [17]. One of the striking observations is that the initial width of the shock wave front has a value of the order of a few microns. It can be readily checked that this result is in perfect agreement with the experimental values reported for the ion mean free path in the deuterium plasma case.

In order to extend magnetohydrodynamic calculation to the description of real experimental devices to find the structure of the collisional shock, the wave shock must lengthen out in at least three spacetimesteps. Initially, according to the value of the boundary radius \( R_0 = 18.47 \) cm and to the spacetimesteps number of 200, a spacetimestep will be 0.92 mm long which greatly exceeds the width of the front shock wave.

One can compute right through the shock, using an artificially enhanced viscosity to broaden the numerical solution.

Such an artificial viscosity is frequently used in the hydrodynamic and magnetohydrodynamic models for the plasma simulation. It should be stressed that the artificial viscosity does not affect the calculated values of the parameters as regards their maximum value. Its effect is of lesser interest for the later stage of the compression.

To obtain a valid model for the structure of a collisional shock one must choose an appropriate set of enhanced transport coefficients. Thus for the ion viscosity we consider an additional term in the temperature of 14 eV (adimensionally 0.126) and for the electron heat conduction, a term of 2 eV (adimensionally 0.036). These terms occur in the eq. (27), (28) and in the boundary conditions (37).

The results concerning the first compression stage are given by the parameters:

\[
a) \quad \alpha = 1.47 \times 10^{33}; \quad h_{20} = 0; \quad \beta = 4.27 \times 10^{-5}
\]

\[
b) \quad \alpha = 1.47 \times 10^{33}; \quad h_{20} = 4.02 \times 10^{-2}; \quad 6.03 \times 10^{-2} \quad (40)
\]

\[\beta = 4.27 \times 10^{-5}\]

for the cases when the magnetic field is present, or absent, respectively.

The plasma column radius is divided into 200 spacetimesteps. The first ten timesteps are chosen to be \( \Delta t = 5 \times 10^{-5} \).

Such a choice is motivated by the convergence requirement. The eighteenth timestep is \( 3.5 \times 10^{-4} \), and then, for any step, the timestep diminishes by \( 10^{-7} \). The dimensional value of the eighteen timesteps is:

\[
3.5 \times 10^{-4} \cdot 0.977 \times 10^{-6} =
\]

\[
= 3.419 \times 10^{-10} \text{s} = 0.3419 \text{ ns}.
\]

For the sake of simplicity, we examine the adimensional form of the variables.

In order to obtain the dimensionalisation, it is necessary to seek the quantities for which this process is carried out.

In figures 1-4, the variations of the quantities: plasma velocity \(- U\), density \(- \sigma\), ion temperature \( T\), and electron temperature \( \theta \), are plotted for the situation (40b) with axial magnetic field.

We look for the variation of these quantities at several instants of time \(0.00275; 0.0527; 0.1050;\)
From figure 1 we remark on the shock wave advance towards the plasma column centre that results in the increase of the plasma quantities with velocity different from zero.

For the first moments of the compression (and in our case, for the moment 0.00275) the adimensional velocity reaches the value — 11, corresponding to a very strong shock of narrow front.

In that front, for the quantities $\sigma$, $T$, $\theta$ a very strong increase would appear only in some space points of the mesh.

For the subsequent time moments, the shock wave front broadens, as expected, because of the ion viscosity and also because of the electron and ion heat conductivities.

If we exclude the peak values for several time moments we observe a similar shape for all the $\sigma$, $T$, $\theta$ curves.

This is due to the fact that the shock wave has a nearly similar structure to that of a steady state one.

Figure 2 shows an increase of $\sigma$ from the centre of the plasma column towards the edge and also a large peak of the density is found near the boundary.

The curves in the figure 3 show the ion temperature behaviour for the later period of the compression stage and include two maxima, the second being less distinctly marked.

We also remark that two peaks occur in the electron temperature curves of figure 4. For the electron temperature, the second peak is more pronounced than that corresponding to the ion temperature.

There is a plausible explanation for this behaviour in terms of the presence of the term describing the exchange energy between ions and electrons. This term is proportional to $(T - \theta)$ and depending on the temperature difference in several points of the plasma column, there is a certain amount which is to be added or subtracted from the temperature values leading thus to the second maximum occurrence.

In the same way, the appearance of the second maximum, more pronounced for the electron temperature, is to be explained in terms of the greater effect of the exchange term, in the eq. (28).

Apart from this term, the ion temperature estimation also contains the viscosity.

0.1302; 0.2327) depending on the radius of the plasma column.

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In the same way, the appearance of the second maximum, more pronounced for the electron temperature, is to be explained in terms of the greater effect of the exchange term, in the eq. (28).

Apart from this term, the ion temperature estimation also contains the viscosity.

0.1302; 0.2327) depending on the radius of the plasma column.

From figure 1 we remark on the shock wave advance towards the plasma column centre that results in the increase of the plasma quantities with velocity different from zero.
In other words, we seek to compare the variation of the above mentioned quantities in the presence of a magnetic field with the variation in the absence of a field. It is easily observed that the dotted curves correspond to the case with no axial magnetic field. There is one very important conclusion that can be already drawn. Provided that the shape of the pairs of curves in figures 5-8 is essentially the same, it may be concluded that, in the first compression stage, the magnetic field presence does not qualitatively affect the variation of the parameters.

Also in this stage, the parameter variation is little affected by the axial magnetic field from a quantitative point of view.

As a general statement, one must emphasize that, in the presence of a magnetic field for the same moments and the same spatial position, the absolute values of all magnitudes are greater than the corresponding ones in the case with no magnetic field.

In figures 5-8 the dotted curves illustrate the case for the absence of an axial magnetic field. These results would be expected to hold according to the physical situation considered by us, namely a deuterium plasma of infinite conductivity.

In such a plasma, the axial magnetic field affects the compression phenomenon by means of a magnetic pressure which, together with plasma kinetic pressure, opposes the compression due to the magnetic piston created by the azimuthal field.

In this respect, as expected, the influence of the additional magnetic pressure of the axial magnetic field is not an appreciable one, as long as the compression does not reach great enough values. In the next figure 9 we have plotted the values of \( n, T, \theta, \sigma \) for the same moment 0.2327. For the sake of simplicity, several scales for the four quantities have been used.

The values refer to the case with an axial magnetic field. As it has already been mentioned, in the case with no magnetic field, the variation is qualitatively the same.

As a first statement, we remark that both electron and ion temperatures reach maximum values for points much closer to the axis as compared to the corresponding ones for the density. It is also worth noting that for a structure such as that of a plane stationary shock wave that results as a characteristic of the first compression stages, the same remark has been made in [8] and [16].

The differences from the plane, stationary wave are caused by the cylindrical structure of our problem [8, 9]. It is noteworthy that the density peak coincides to a velocity level that is in good agreement with the general theory of the shock waves. In other words, behind the shock, where the velocity has a nearly constant value, the density takes a maximum value.
In addition, we remark that the peak of $\theta$ is greater than the peak of $T$, in this compression stage. In view of this possible interpretation we mention that up until the moment 0.2327 and also in this moment, the term corresponding to the electron heat conduction exceeds the contributions of the dissipative term in the ion temperature.

Obviously, this fact can easily be deduced from the first term of the second member of eq. (28) and the second term of eq. (27), corresponding to both heat conductivities.

4.2 Parameter maximum values on the plasma column axis. — In the previous section we were mainly concerned with the cases where the maximum values reached by the ion and electron temperatures and also by the plasma density in the shock wave lie in zones situated far enough from the plasma column axis. The calculations were stopped at the moment $\tau = 0.32$, because of the fact that the numerical scheme is not accurate enough for greater times. For that reason, some nonphysical values for the ion temperature were obtained. To overcome this the analysis of the finite difference scheme suggest that the timestep should be diminished. This condition leads to the improvement of the convergence requirement for the calculation scheme.

With that point in view, we decided to choose a varying timestep $\Delta \tau$, more precisely a $\Delta \tau$ ever smaller. We started with a $\Delta \tau = 3.2 \times 10^{-4}$ and after any 200 steps this timestep decreased with $10^{-5}$.

This program modification enables us to reach the time when the maximum values for several parameters lie on the plasma column axis or very near the axis.

In addition this provides a valuable estimation of the axial magnetic field effect.

Let us consider a physical situation, characterized by the following choice of the parameters :

\begin{enumerate}
  \item $\alpha = 1.47 \times 10^{33}$; $h_{20} = 0$; $\beta = 4.27 \times 10^{-2}$
  \item $\alpha = 1.47 \times 10^{33}$; $h_{20} = 6.03 \times 10^{-2}$; $\beta = 4.27 \times 10^{-2}$.
\end{enumerate}

Provided that for the first compression stages the results are essentially the same as in the section 4.1, we restrict ourselves to the analysis at times close to those for which the maximum values are reached on the axis.

Figures 10-12 show the behaviour of the parameters $\sigma$, $T$, $\theta$, at the times : 0.3003; 0.3543; 0.4063. The characteristic feature of these graphs is the much greater value of the parameters as compared to the corresponding ones mentioned in the previous section. Also it is to be noted that the peak is found near the axis or even on the axis.

Figures 10-12 refer to the case when the axial magnetic field is different from zero.

On the curves plotted in these figures one observes a pronounced increase of the maximum value of
several parameters, for the transition from the time 0.3543 to the time 0.4063 as compared to the transition from the time 0.3003 to the time 0.3543.

For instance, for the ion temperature during the transition to the time 0.3003, the maximum value is $T_{max} = 0.440$; while at the time 0.3543, $T_{max} = 0.495$. Their difference is about 0.045. In exchange, the difference between the ion temperatures for the moments, 0.3453 and 0.4063 is about 0.550. In other words, in the period separating these two moments, the ion temperature increases more than twice. Such important increases may also be seen in the curves corresponding to the density and the electron temperature.

It is worth noting that figures 10-12 correspond to the presence of an axial magnetic field. The phenomenon becomes even more visible in the case with no magnetic field.
All the remarks are the expected ones for moments close to the cumulation and focalisation time. This moment is characterized by large increases of the parameters on the plasma column axis. As in the first two sections of this chapter, we have plotted $a$, $T$, $\theta$ values for the time $0.4063$, both in $h_{20} = 0$ and $h_{20} = 6.03 \times 10^{-2}$ cases, on figures 13-15.

A first striking observation valid for all the three quantities is that an axial magnetic field retards the time when the maximum value reaches the column axis.

From table I, it can easily be seen that a noticeable difference is found between the maximum values in the two different physical situations.

While in the earlier compression stages we find small differences between the maximum temperature values both in the presence and absence of the axial magnetic field (nearly 1 eV) and also for the density; for the times when the maximum values approach the axis, the differences reach 24.6 eV for $T$, 14.8 eV for $\theta$ and $3.36 \times 10^{-7}$ g/cm$^3$ for the density. It is noteworthy that while in the earlier stages an increasing effect of the axial magnetic field was found as we advance towards the plasma external region (without approaching too close to the boundary), in this stage for radii exceeding 0.2 for $T$ and $\theta$ and 0.1 for $\sigma$, the differences are to be left unchanged.

This rule holds for all cases except for the region lying in the immediate neighbourhood of the boundary. Here, as in the previous section, a decrease of plasma parameters and also, apart from the ion temperature, one finds that for $h_{20} = 0$ and

$$h_{20} = 6.03 \times 10^{-2}$$

their values approach one another.

Figure 16 suggests a further difference between the curves reported for the earlier compression stages and those corresponding to the near compression stage, namely the large ion temperature values compared with the electron ones.

Undoubtedly, the great difference is due to the ion viscosity whose contribution to the temperature increase in the axis region is a noticeable one. Our statement can be well justified from the mathematical point of view, by comparing the positive terms with the negatives ones in the quantities including the viscosity in eq. (2) and also by taking into account the velocity curve in figure 16.

Figure 16 shows a sudden variation of $U$ in the column axis zone; therefore the inequality

$$\left( \frac{\partial U}{\partial x} \right)^2 \gg \frac{\partial U}{\partial x}.$$

$U$ will hold.

Provided that the term $\left( \frac{\partial U}{\partial x} \right)^2$ contributes to the ion temperature increase, the greater value of $T$ in this region is well justified. The greater values of the electron temperatures compared with those of the ion temperature for regions situated far off the axis, account for the validity of such a statement if we consider the velocity variation in these regions.

It can be readily checked that $U$ is nearly constant and we may therefore neglect the contributions of the terms $U \cdot \frac{\partial U}{\partial x}$ and $\left( \frac{\partial U}{\partial x} \right)^2$.

<table>
<thead>
<tr>
<th>$h_{20}$</th>
<th>$T_{\text{max}}$</th>
<th>$T_{\text{bound.}}$</th>
<th>$\theta_{\text{max}}$</th>
<th>$\theta_{\text{bound.}}$</th>
<th>$\rho_{\text{max}}$</th>
<th>$\rho_{\text{bound.}}$</th>
<th>$X_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.255 eV</td>
<td>0.428</td>
<td>1.013</td>
<td>0.4279</td>
<td>10.8</td>
<td>5.26</td>
<td>0.472</td>
</tr>
<tr>
<td>6.03 x 10^{-2}</td>
<td>114.8 eV</td>
<td>46.13 eV</td>
<td>97.7 eV</td>
<td>46.1 eV</td>
<td>10.14 x 10^{-7} g/cm^3</td>
<td>6.39 x 10^{-7} g/cm^3</td>
<td>8.90 cm</td>
</tr>
</tbody>
</table>

**TABLE I**
This accounts for the effect of the electron conductivity which is greater than the ion one. The second peak of the electron temperature becomes smaller than the main one as compared with the earlier stages.

The calculations reported here enable us to draw some conclusions concerning the plasma boundary behaviour. From table I and figure 17 we remark that the plasma column radius is always greater in presence of an axial magnetic field than in its absence.

The effect of an external magnetic field is therefore to increase the plasma compression time, i.e. the time during which the plasma column reaches a minimum radius.

5. Conclusions. — The calculation and data analysis is useful in that it provides a valuable model for the detailed study of the effect of an external axial magnetic field on the compression of a cylindrical plasma column. The theoretical results refer to the case when the azimuthal field is taken into account as a factor responsible for the compression. Our calculations consider an external axial magnetic field; we suppose, as in the previous papers, that the plasma field has only an azimuthal component.

Let us now summarise several results concerning the effect of the external magnetic field on the plasma compression:

- The axial magnetic field does not qualitatively change the variation of the main plasma parameters such as the ion temperature, the electron temperature and the density.
- From a quantitative point of view, the axial magnetic field leads to a decrease of the parameters for several times, and especially with respect to the maximum value.
- The effect of the magnetic field increases with time, after the beginning of the compression.
- For the same times, the calculations show that the value of the plasma boundary radius is greater in the presence of the magnetic field than in its absence.
- The plasma velocity, and especially the velocity of the boundary elements is smaller in the presence of the magnetic field.
- Provided that we are dealing with a plasma column of infinite conductivity, the variation of the axial magnetic field will approach the density variation. This enables us to conclude that its maximum value reached for the time \( \tau = 0.4063 \) is 8.109 times greater than its initial value.

The theoretical results performed with this model are in a good agreement with the experimental data.

Of course, by adding to our model some terms describing other processes of interest at one time, it is possible to get a more suitable description of the plasma column compression.

But these are not the only conditions that are required and the literature and also our own experience indicate the great dependence of the data accuracy on the calculation scheme and also on its manner of use.

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