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PERCOLATION IN A CONTINUOUS TWO-DIMENSIONAL MEDIUM

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Résumé. — Dans une note précédente, nous avons présenté une méthode de Monte Carlo et des modèles de processus de ramification appliqués à l'étude des amas formés par des atomes répartis aléatoirement et liés par une interaction de portée donnée r_0. Les mêmes techniques sont utilisées ici pour le cas de l'espace à deux dimensions. Un résultat essentiel est que l'amas infini apparaît lorsque le nombre moyen d'atomes contenus dans un cercle de rayon r_0 est 4,1 environ.

Abstract. — In a preceding paper, we presented a Monte Carlo method and two branching process models, used for the study of clusters of atoms randomly distributed and bounded by an interaction of given range r_0. Similar methods are used here for the two-dimensional case. We find that the infinite cluster appears when the mean number of atoms in a circle of radius r_0 is about 4.1.

Introduction. — In a preceding paper [1], we present a study on clusters of atoms randomly distributed in a continuous medium and bounded by long range interaction. The results of a similar study in a two-dimensional medium are given here.

An important parameter for this process is the mean number W of atoms bounded with a given atom :

$$W = \Pi r_0^2 \, c$$

where c is the superficial density of atoms, r_0 the range of interaction.

(Some authors use a parameter proportional to W, $t = \frac{c}{4} r_0^2 = \frac{W}{4\Pi}$).

We denote W_c the critical value of W giving rise to the infinite cluster. Up to now the published values of W_c don't agree very well :

3.2	[2]
3.82	[3]
4.5	[4]

These values are obtained by a Monte Carlo method for [2] and [3], and by an analytical method for [4].

In a critical discussion of these results, Domb notes that the Monte Carlo methods lead to underestimated values of W_c [5]. This may be due to the inadequacy of the linear extrapolation associated with these methods. The method we use is also of Monte-Carlo type. Nevertheless it leads to a fairly good evaluation of W_c for a three-dimensional medium. Compared to the afore-mentioned methods, its two main qualities are the followings :

— first, more important clusters are obtained (size : 256) ;

— second, it is associated with a sharper extrapolation method.

Let us remind of its principle. We choose a starting atom and examine the size of the cluster containing it. For that, we explore the circle of radius r_0 centered on it. The random number n of neighbours in the circle is determined according to a Poisson law of parameter W, and their positions according to a statistical law with uniform density.

If $n = 0$, the starting atom belongs to a cluster of size 1.

If $n \neq 0$, we say we have found n neighbours in the first generation.

Then, we look to see if the atoms of the 1st generation have neighbours themselves, by exploring the circle of each one. However we note that the atoms found in a region previously explored must now be eliminated. The neighbours so obtained belong to the 2nd generation, and the construction of the cluster is pursued until we find an empty generation. The starting atom belongs then to a cluster of size N, where N is the total number of atoms which have been found, plus the origin. In practice, an upper limit for N is imposed by the computer. In our calculations, this limit is 256.

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Results. — 1) We denote P_N^+ the probability that an atom belongs to a cluster of size N of more. For each value of W, 500 to 1 000 clusters have been constructed. The so obtained *experimental* values of P_N^+ are shown in figure 1 in logarithmic scales.

The critical value W_c may be put in evidence from this representation. Indeed when $W < W_c$, all the atoms are in clusters of finite size and P_N^+ must tend toward zero for infinite values of N, on the other hand, when $W > W_c$ a non null proportion of atoms is in the infinite cluster and P_N^+ has a horizontal asymptote, with ordinate $P_\infty(W)$. Each value obtained for P_N^+ is represented with its error interval, the half length of which is twice the standard deviation.

In figure 1, $W = 3.8$ clearly appears as a lower bound of W_c and so the values 3.2 of [2], and even 3.82 of [3], are too weak. On the other hand, $W = 5$ appears as an upper bound.

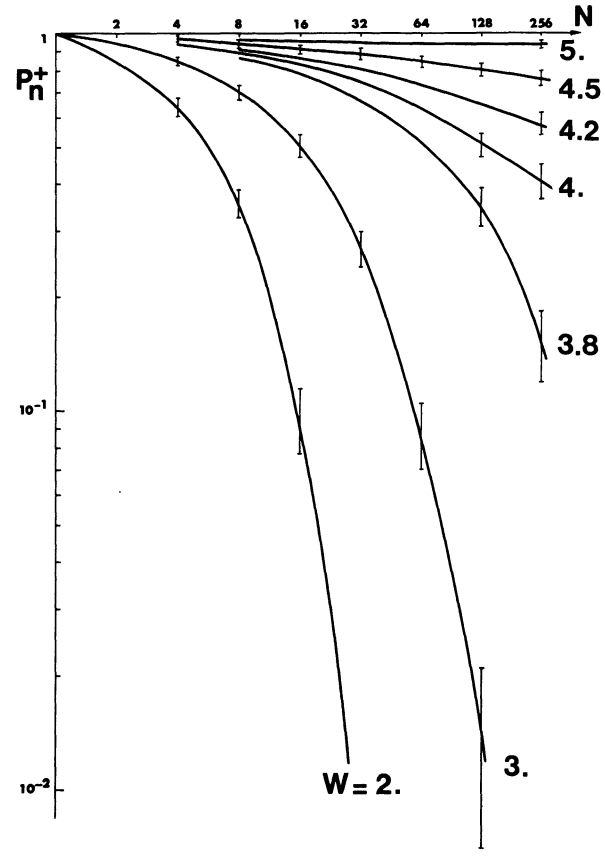

FIG. 1. — Values of P_N^+ — the probability that an atom belongs to a cluster of size N or more — given by the Monte Carlo method.

2) As already noted in the preceding paper the construction of a cluster by successive generations suggests a branching process and hence the following representation of the results : for each value of W, the mean number G_k of atoms belonging to the kth generation is represented versus k (Fig. 2). We have shown [1] that if the quantity

$$W_k = \frac{G_{k+1}}{G_k}$$

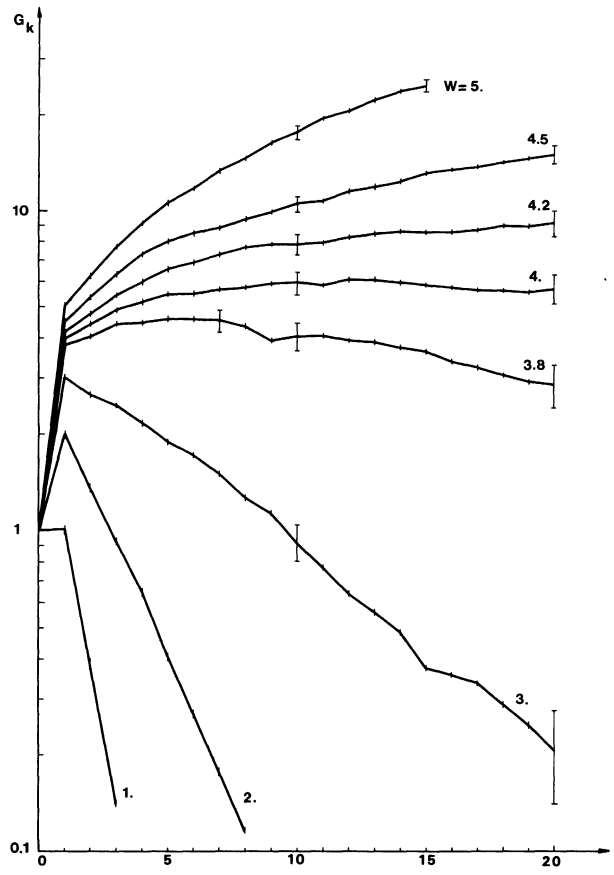

FIG. 2. — G_k : mean population of the kth generation, according to Monte Carlo results.

(mean fertility for the kth generation) has an asymptotic limit W_a, then :

a) $W_a \leqslant 1$ for any value of W;

b) If $W_a < 1$ the infinite cluster does not exist.

This suggests that the critical value W_c is the smallest value of W for which $W_a = 1$. From the curves given in figure 2, one may propose the value :

$$W_c \sim 4.1$$

with an error of about 0.1 or 0.2.

This value is in good agreement with the evaluation proposed by Domb ($W_c \sim 4.0$) [5], as a result of the discussion of the values given in [2], [3] and [4].

It also agrees with the approximate value

$$W_0 = 4.00$$

issued from our theory of the curvature of the cluster frontier [6].

Analytical models. — The construction of a cluster suggests that branching process models may be used to fit the results. We examine the two models already used in the three-dimensional case [1].

1) In the first model, the fertility of an atom depends only on its generation number : the number of descendants of an atom belonging to the kth generation follows a Poisson law of parameter W_k. The values W_k

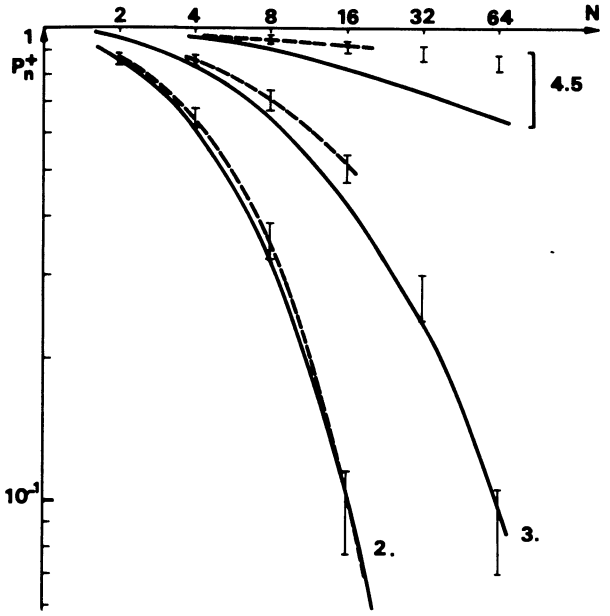

FIG. 3. — $P_N^+ = f(N)$. Vertical segments : Monte Carlo results. ——— Process with fertilities depending on the generation number. --------- Vital space process.

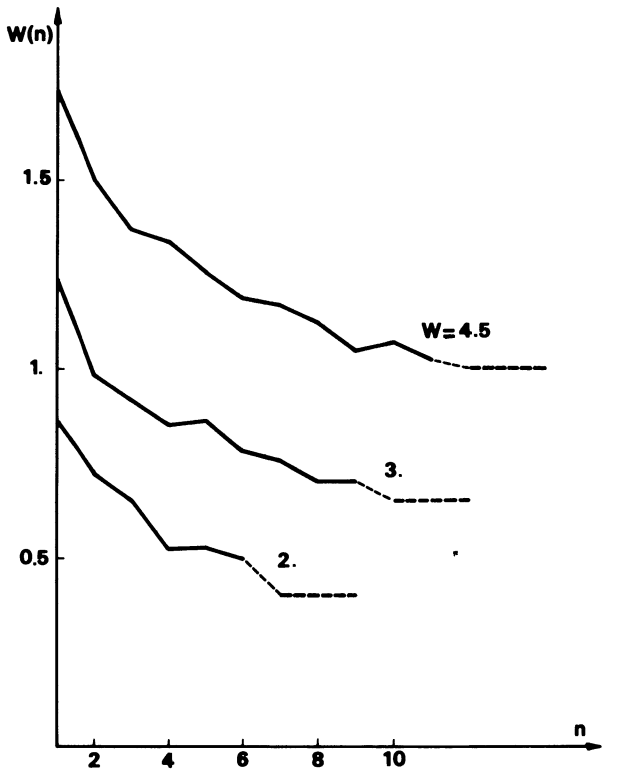

FIG. 4. — $W(n)$: fertility of an atom versus the number n of atoms belonging to its own generation, computed from the Monte Carlo results, and used in the calculation of P_N^+ in the vital space process. Dashed lines represent estimated asymptotic values.

used for the computation are deduced from the experimental G_k. The theoretical P_N^+ are shown in figure 3 (solid line). For the low values of W, the agreement with the experimental P_N^+ is fairly good ; for W greater than 2, the theoretical P_N^+ are quite weaker. The same phenomena has been observed in the three-dimensional case. It may be attributed to the fact that, when W is not small, a mean fertility W_k does not correctly take into account the *congestion effect* due to the finite dimensionality of the space.

We try to describe this effect in a more correct manner by the following model called *vital space process* in our preceding paper.

2) In this second model, the fertility $W(n)$ of an atom depends on the number n of atoms belonging to its own generation. The values of $W(n)$ are computed from the results of the Monte Carlo method, and given in figure 4. In fact the precision of these results is not sufficient to put on an effect of the generation number k on $W(n)$, and we have taken a mean value on the ten first generations.

Finally, the theoretical P_N^+, following this model, are shown in figure 3 (dashed line). The agreement with the experimental results in fairly good.

Conclusion. — 1) The results of our Monte Carlo method suggest for the critical value of $W (= \Pi r_0^2 c)$:

$$W_c \sim 4.1 .$$

2) The two models of branching process — already used in the three-dimensional case — are tested and show the same essential properties :

— for low values of W, the model with fertilities depending only on the generation number is quite good ;

— for higher values, the congestion effect is described in a more correct manner by a process where the fertility of an atom depends on the number of atoms in its generation.

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