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LOCAL CHIRAL AXIS IN A DIRECTIONAL MEDIUM
AND WEDGE COMPONENT OF A DISCLINATION

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Résumé. — On démontre l'existence d'un ensemble de surfaces caractéristiques d'une singularité donnée (disinclinaison, point singulier) et ayant pour propriété de contenir le directeur. Il y a autant de tels ensembles qu'il y a de singularités. La normale en chaque point à ces surfaces définit un axe d'hélicité local. On définit en outre un processus de mesure de la composante dièdre de la disclinaison à l'aide d'un circuit qui l'entoure.

Abstract. — We demonstrate the existence of a set of surfaces containing the director and related to each singularity (disclination, point disclination) of the director field. There can be as many sets as there are singularities. The normal to these surfaces defines a chiral axis at each point. A Burgers' circuit is introduced to measure the wedge component of the disclination.

1. Introduction. — The geometry of directional media, well-known for its complexity, presents two possibilities of simplification, either when the director $\mathbf{n}$ is everywhere perpendicular to a set of surfaces $\Sigma$, or when there is a set of surfaces $\Sigma$ to which it is everywhere tangent. The first case is of practical importance for smectics A which grow in layers of constant thickness: this last condition makes the surfaces $\Sigma$ parallel surfaces, from which it follows that a family of straight normals belong to these surfaces. The second case refers to nematics and, more specifically, cholesterics; in the ground state of cholesterics the surfaces $\Sigma$ are the so-called cholesteric planes, and it is a question we may ask whether these planes transform continuously into well-defined surfaces in a distorted cholesteric, whether they lose their individuality, or whether, more simply, we can still define $\Sigma$ surfaces in the distorted state, not necessarily correlated to the cholesteric planes. If we can define such transformed surfaces, and indeed we shall prove we can, the question of the definition of a local chiral axis is therefore solved: the local chiral axis is along the normal to $\Sigma$ and the local pitch is defined unambiguously. Of course, since cholesterics are nothing else but twisted nematics, all the results apply equally to both cases.

In a former article [1] we have proposed a general description of directional media in which we attach to each point of the medium a triple $\mathbf{N}$, $\mathbf{n}$, $\mathbf{m}$ of three orthonormal vectors. This defines an infinity of fields of directors, of constant components along the axes of the moving frame defined by the triple. If $\mathbf{n}$ is the physical director, $\mathbf{N}$, for instance, is a geometrical director whose study can be of interest for the distribution of $\mathbf{n}$. Hence, if we get a family of surfaces to which $\mathbf{n}$ is tangent, this is also a family of surfaces to which $\mathbf{N}$ is perpendicular. In this point of view, the two problems cited above are therefore correlated.

In this article, we shall show that there are always particular sets of surfaces $\Sigma$ to which $\mathbf{n}$ is tangent, and that their consideration leads to a generalization of the notion of Burgers' circuit to the case of disclinations (rotation dislocations), which are the typical linear defects of directional media. Most of the problems raised by this approach are just formulated here, but we have the feeling that their solution should give a strong foundation to the geometrical concepts involved in the physics of directional media, for which liquid crystals seem to be the simplest physical image today.

2. Existence of a congruence of normal lines in a directional medium. — Take an unrestricted continuous distribution $\mathbf{n}(r)$: the reader will easily admit that the directors $\mathbf{n}$ envelop a set of lines (L) depending on two parameters: they form a congruence of lines. We shall here summarize some well-known results of differential geometry (Julia, 1954) whose

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evidence will anyway be clear if the reader refers to his intuitive knowledge of the distribution of the directors in a medium containing disclination lines.

In a set of lines (L) depending on two parameters \( \alpha \) and \( \beta \), it is always possible to find a division into subsets depending on one parameter such that each subset has an envelope; to this requirement corresponds a particular relation \( \alpha = \alpha(\beta) \), and each subset is scribed on a surface \( \Sigma \). The outline of the proof is as follows.

Define a line (L) by the intersection of two surfaces
\[
\phi_1(r, \alpha, \beta) = 0, \quad \phi_2(r, \alpha, \beta) = 0.
\]
The points of contact with the envelope obey the relations
\[
\frac{\partial \phi_1}{\partial \alpha} + \frac{\partial \alpha}{\partial \alpha} \frac{\partial \phi_1}{\partial \beta} = 0, \quad \frac{\partial \phi_2}{\partial \alpha} + \frac{\partial \alpha}{\partial \alpha} \frac{\partial \phi_2}{\partial \beta} = 0.
\]
Elimination of \( r = (x, y, z) \) between there four equations leads to a differential equation of the first order \( \Phi(\alpha, \beta, \beta/\alpha) = 0 \). The solutions of this equation correspond to the subset.

A trivial illustration is provided by the case of unidimensional single glide in a solid crystal [2]. Consider the unit vectors perpendicular to the bent surfaces as the directors \( n \): in single glide the surfaces are transformed to parallel surfaces and the normals, which constitute the lines (L), envelop the common evolute (E) to these surfaces (Fig. 1); the planes containing the glide direction and the lines (L) are the surfaces \( \Sigma \). The singularity defined by this process is a surface.

Fig. 1. — Unidimensional single glide in a solid crystal.

A first example relating to liquid crystals is the case of a Frank wedge disclination [3] (Fig. 2A); the planes perpendicular to the disclination line are clearly the subsets \( \Sigma \): they contain the lines (L), which envelop the intersection of the disclination line (F) with the plane (L). The subsets (E) are made of cones with vertices located on (FE), and basis (FH): on each cone the lines (L) envelop one point of (FH). The subsets (H) are made of cones, with vertices situated on (FH), and basis (FE): on each cone the lines (L) envelop one point of (FH).

Another example is provided by confocal domains (Fig. 2C): here we have \( T W O \) lines, the ellipse (FE) and the hyperbola (FH), and \( T W O \) divisions of the set (L) in subsets (E) and (H). The subsets (E) are made of cones with vertices located on (FE), and basis (FH) on each cone, the lines (L) envelop one point of (FH). The subsets (H) are made of cones, with vertices situated on (FH), and basis (FE): on each cone the lines (L) envelop one point of (FH).

More generally, there are as many divisions in subsets \( i \) as there are sheets on the locus of the here above mentioned envelopes, the so-called focal surface \( F \). In liquid crystals, this surface is generally degenerated and the different sheets \( F_i \) reduce to one or several lines (disclinations). The degeneracy can go further, and lines become singular points.
In the general case, where the focal surface is not degenerate, the lines \((L)\) are tangent to all the sheets of \((F)\). Hence we have to consider that the lines of force \((L)\) of the directors encounter all the disclination lines and points. Since this condition is obviously generally not fulfilled in a practical case (liquid crystals), this means that we have to divide the medium in domains; each of which contains a whole set of lines \((L)\) and focal surface \((F)\) attached to them: the problem is therefore how to cover space with such different domains; this difficulty arises for example in the case of the confocal domains and has not been solved yet.

Consider a domain and assume that the \(F_{r,s}\) are lines. Since all the \(F_{r,s}\) are encountered by the lines \((L)\), they must intersect any surface \(\Sigma_j\) of a partitioning \((j)\), or belong to these surfaces. If \((F_j)\) belongs to a given surface \(\Sigma_j\) of the subset \((j)\), it must for the same reason belong to all the \(\Sigma_j\) of the subset, and appear therefore as a common intersection to all the \(\Sigma_j\). Consider the surfaces \(\Sigma_j\) as transformed cholesteric planes: the \(F_{r,s}\) which intersect the \(\Sigma_j\), have to be considered as \textit{wedge disclinations} for this set, and the \(F_{r,s}\) which belong to the \(\Sigma_j\) as \textit{twist} disclinations. (However, wedge and twist quantities can be defined independently of the chosen set.)

The confocal domains quoted above give us an example: \((F_{10})\) intersects the subset \((H)\) and lies on each surface of the subset \((E)\).

All these properties provide us with local frames of reference: we are led, in the study of a given disclination (line or point) \(F_{r,s}\), to consider the corresponding surfaces \(\Sigma_i\). \(N\) will be the normal to \(\Sigma_i\) in each point \(M\); \(n\) the director is in the plane tangent to \(\Sigma_i\) in \(M\); \(m = N \wedge n\).

The variation of this frame of reference is described through the medium with the help of a curvature tensor \(K_{ij}(r)\) (cf. [1]), such that the infinitesimal rotation \(d\omega\) which relates the orientation of the frame in \(r\) to the orientation in \(r + dr\) is given by the Pfaffian form, generally non-integrable,

\[
d\omega_{ij} = K_{ij} \, dx_i. \tag{1}
\]

An important relation, consequence of eq. (1), is the following

\[
p_{i,j} = e_{ipq} \, K_{jp} \, p_q, \tag{2}
\]

where \(p\) is any unit vector of constant components in the moving frame of reference. \(p\) can be therefore \(N\), \(n\) or \(m\).

The lines of force of \(N\) form a congruence of lines normal to the surfaces \(\Sigma\) (we drop hereafter the index \(i\) referring to the chosen subset). This property reads (cf. for ex. Bilby et al. [4])

\[
N \cdot \text{curl} \, N = 0 \tag{3}
\]
or, using the \(K_{ij}\) symbols

\[
K_{ij} \, N_i \, N_j = K = K. \tag{4}
\]

We give to eq. (3) the name of condition of absence of \textit{generalized twist}. It does not mean that the medium is not twisted in the sense of the physics of liquid crystals, i. e. does not exclude \(n, \text{curl} \, n \neq 0\).

This paragraph has finally established the existence of local frames of reference in each point of the medium, and pertaining to each disclination. The way we have established this property shows that there is only one solution to the problem. We have at the same time solved the problem of a local helical axis; \(N\) plays this role, and the local pitch is clearly given by \(q = 2 \pi / P = K_{ij} \, N_i \, N_j = K\). Note however that the local rotation \(\Omega_j = K_{ij} \, N_j\) is generally not parallel to \(N_j\).

Experimentally (4), this method of uniquely identifying a transformed cholesteric plane and local chiral axis succeeds in any complete \textit{domain} if that domain is completely observed. In the more usual practical case of a limited observation region only some of the cholesteric planes present can thus be identified in general: those whose connection to two disclination lines is observed; one of the line being of twist character for the chosen set of transformed cholesteric planes, the other of wedge character. Presumably we must postulate external disclinations, chosen by somewhat arbitrary conventions, to make this specification possible for all the cholesteric planes in the region of observation.

### 3. Wedge component of a disclination line.

We have proved in reference [1] that the condition of uniqueness of the moving frame reads:

\[
\theta_{ij} = e_{ikj} \, K_{ij,k} - \frac{1}{2} \, e_{ipq} \, e_{jim} \, K_{pl} \, K_{qm} = 0. \tag{5}
\]

Let us write:

\[
M_{ij} = \frac{1}{2} \, e_{ipq} \, e_{jim} \, K_{pl} \, K_{qm} \tag{6}
\]

\(M_{ij}\) is the minor \((ij)\) of the \(K_{ij}\) matrix, and is related, as shown in (1), to the non-commutativity of the small rotations between neighboring frames.

Condition (5) is satisfied everywhere, except on disclinations. We shall consider two cases:

a) The point \(P\) where the line intersects \(\Sigma\) is regular on \(\Sigma\). The director is therefore necessarily singular at \(P\), and there is a \textit{core}.

b) The point \(P\) is singular on \(\Sigma\). The director may therefore be non-singular (coreless disclination), or singular (conical point on a focal line of a confocal domain).

#### 3.1 Disclination with a core, regular point on \(\Sigma\).

The geometrical properties of \(\Sigma\) can be expressed in function of the \(K_{ij}\). The Gaussian curvature \(\sigma = 1/R_1 \, R_2\) is given by the expression (cf. ref. [5]):

\[
\sigma = \frac{1}{2} \, \text{div} \{ N \, \text{div} \, N + N \wedge \text{curl} \, N \} \tag{7}
\]

(7) We are indebted to Pr. Frank for the following remarks which conclude this paragraph.
or, using the $K_{ij}$s

$$\sigma = M_{ij} N_i N_j . \quad (8)$$

Expression (8) can be derived straightforwardly from the considerations exposed in [1]: the gaussian curvature of a surface $\Sigma$ is defined usually as the ratio of an element of area on the surface to the corresponding element of area on Dupin's indicatrix of $\Sigma$ [6]. Dupin's indicatrix appears in (1) as the particular case of the indicatrix of Feldtkeller (IF) applied to the geometrical director $N$.

Consider a closed circuit $(y)$ on $\Sigma$. At each point of $(y)$ we define a local frame of reference $(R)$ consisting of the tangent $t$ to $(y)$, the normal $N$, and the geodesic normal $G = N \wedge t$ (Fig. 3). This is the so-called Darboux-Ribaucour triple [6]. Let us write $\theta$ the oriented angle between $t$ and $n$ at point $M$ on $(y)$, $s$ the arc length on $(y)$. The component on $N$ of the instantaneous rotation of $(R)$ is, using eq. (1)

$$\rho_G = K_{ij} N_i t_i + \frac{d\theta}{ds} \quad (9)$$

$\rho_G$ is called the geodesic curvature of $(y)$ at $M$.

Eq. (8) and (9) lead to a very simple geometrical interpretation of the incompatibility relation (5). Multiply both sides of eq. (5) by $N_i N_j$ and integrate on the part of $\Sigma$ enclosed by $(y)$. The term $\nabla \wedge \mathbf{K}$ is integrated by parts. We find

$$\int \int \epsilon_{kl} K_{ij,k} N_i N_j \, d\Sigma = 2 \int \int M_{ij} N_i N_j \, d\Sigma + \oint K_{ij} N_j t_i \, ds .$$

Hence:

$$\int \int \theta_{ij} N_i N_j \, dS = \int \int \frac{d\Sigma}{R_1 R_2} + \oint \rho_G \, ds - \oint d\theta . \quad (10)$$

Let us first apply (10) to a region where the lines (L) are regular ($\theta_{ij} = 0$), $\oint d\theta = 2 \pi$. Hence we get the celebrated theorem by Ossian Bonnet [7] (eq. (11)) which generalizes the well-known result of Gauss according to which the sum of the angles of a geodesic triangle is smaller than $2 \pi$ on a surface where $\sigma < 0$, larger than $2 \pi$ on a surface where $\sigma > 0$

$$\int \int \sigma \, d\Sigma + \oint \rho_G \, ds = 2 \pi . \quad (11)$$

Eq. (11) applies to any circuit enclosing a regular region on $\Sigma$, since $\sigma$ and $\rho_G$ are independent of the distribution of $n$ on $\Sigma$.

Assume now that $(y)$ encloses a singular point of the lines (L). We have, by definition,

$$\int d\theta = 2 \pi (1 - S) , \quad (12)$$

where $S$ is the strength of the singularity ($n$ rotates by an angle $2 \pi S$ about $F$); the angle is measured on $\Sigma$, which solves the difficulty raised with the non-commutativity of small rotations. Using eq. (10), (11) and (12), we are now left with the relation

$$S = \frac{1}{2 \pi} \int \int \theta_{ij} N_i N_j \, d\Sigma . \quad (13)$$

Apply (10) to a closed orientable surface, in which we achieve a cut of finite length (C) along a geodesic line (along which $\rho_G = 0$). We integrate over the whole surface, limited by (C). The extremities of the cut contribute to the integral $\oint \rho_G \, ds$ by a quantity $2 \pi$, which is the jump of the instantaneous rotations. The sum of the strengths of all disclinations intersecting $\Sigma$ is therefore given by

$$C = \sum_{i} S_i = \frac{1}{2 \pi} \int \int \frac{d\Sigma}{R_1 R_2} . \quad (14)$$

A torus gives $C = 0$ and a sphere $C = 2$. This result was already obtained with the help of topological arguments [8], [9].

3.2 SINGULAR POINT ON $\Sigma$ (Fig. 4). --- We have illustrated in figure 4 three kinds of singular points on $\Sigma$: in one case the director becomes parallel to the disclination line, and the singular point is a cusp; in the other case, the singular point is conical, as occurs in cofocal domains. We can enter in the same paragraph the case of a singular point (Fig. 4C) where all the surfaces $\Sigma$ are in contact: this is a disclination point, which can be considered as end of a line (non-singular). When the disclination point is rejected to infinity, we are anyway left with a line (coreless), of the type described in [10].

In all these cases it is possible to draw on $\Sigma$ a closed circuit $(y)$ which encloses the disclination. Moreover the curvature $\sigma$ does not have any singularity anywhere ($\sigma = 0$ at the singular point on $\Sigma$), and we can consider the singular points as limits of small spherical portions of small curvature. Hence the above results continue to hold, with the same meaning attached to each quantity. In particular
3.3 Extension of Eq. (13). — Eq. (13) relates the strength of a line to a particular surface $\Sigma$. But $\theta_{ij} N_j$ is divergenceless. This follows directly from eqs. (2), (5) and (6). Hence the integration does not depend on the surface bounded by $(\gamma)$ if we write eq. (13) as follows

$$S = \frac{1}{2\pi} \int_{(\gamma)} \theta_{ij} N_j d\Sigma_i.
$$

(15)

Also, moving $(\gamma)$ along the tube of force of $N$ does not change the value of the right side of eq. (13), at least as long as $(\gamma)$ does not cross $(F)$ during this variation. Hence formula (15) applies to any kind of circuit $(\gamma)$ enclosing the line $(F)$.

In formula (15), the vector

$$\Omega_l = \frac{1}{2\pi} \theta_{ij} N_j
$$

appears as the true quantity to be related to each point of the line $(F)$. Its relation with the densities of dislocations and disclinations proposed by Kléman and Friedel [11] in a different approach of the problem will not be discussed here. It is however important to stress that the quantity $S$ which is conserved all along $(F)$ is a scalar (true scalar) and not a vector, in contradistinction to the case of dislocations. $S$ is related to the wedge component of the line. The twist component will be the subject of a forthcoming paper.

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