Analytic phase-shift analysis
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Surprising though it may seem, a measurement of the differential cross-sections and polarizations for $\pi^+ p$ and $\pi^- p$ elastic scattering (at a given energy below the inelastic threshold) may be sufficient to determine completely all the phase-shifts, and therefore in particular the charge-exchange cross-section and polarization, as well as the $A$ and $R$ parameters for all charge states. On the other hand, above the inelastic threshold, a complete set of measurements (differential cross-sections, polarizations, $A$ and $R$ parameters, and total cross-sections, for all charge states), leaves a continuum ambiguity in the determination of the phase-shifts.

Since the scattering is described by four complex functions, which are analytic in the complex cos $\theta$-plane, and which are constrained by four real unitarity equations below the inelastic threshold, one might indeed expect that the measurement of only four real quantities should suffice for the complete determination of the system. We shall show presently that this is true, and in particular that the four real quantities may be taken to be the above two cross-sections and two polarizations. An interesting alternative is to take the experimentally accessible cross-sections, $\pi^+ p \rightarrow \pi^+ p$, $\pi^- p \rightarrow \pi^- p$, $\pi^- p \rightarrow \pi^0 n$, and the $\pi^+ p$ polarization [1]. Above the inelastic threshold, the unitarity conditions are only inequality constraints, and we can show that a complete set of measurements leaves undetermined one real function of $x = \cos \theta$. The corresponding phenomenon for scattering without spin or isospin has been discussed recently [2], [3]. We consider first $\pi^+ p$ scattering, since this is not encumbered by isospin complications. (Exactly the same equations, and so the same conclusions, obtain for $K^+ p$ scattering.) We write the amplitude as

$$f(x) = g(x) + i\sigma \cdot n h(x),$$

the differential cross-section (multiplied by $q^2$) as

$$\sigma(x) = |g(x)|^2 + |h(x)|^2,$$

and the polarization as

$$P(x) = \frac{2 \text{Im} g^\dagger(x) h(x)}{\sigma(x)}.$$
We consider the elastic unitarity constraint as a mapping of the imaginary parts, \(a\) and \(b\), of \(g\) and \(h\), into \(a'\) and \(b'\) [4]:

\[
a_i = \frac{1}{2} \int_{-1}^{1} dx P_i(x) a(x)
\]

\[
b_i = \frac{1}{2} \int_{-1}^{1} dx P_i^1(x) b(x)
\]

\[
A_i = \frac{1}{2} \int_{-1}^{1} dx P_i(x) - \frac{1}{2} \sigma(x) P(x) b(x) + E(x) a(x) \frac{D(x)}{D_i}
\]

\[
B_i = \frac{1}{2} \int_{-1}^{1} dx P_i^1(x) \frac{1}{2} \sigma(x) P(x) a(x) \pm E(x) b(x) \frac{D(x)}{D_i}
\]

where

\[
a'(x) = \sum_{i=0}^{\infty} (2l+1) P_i(x) \times
\]

\[
\left[ a_i^2 + \frac{l(l+1)}{(2l+1)^2} b_i^2 + A_i^2 + \frac{B_i^2}{l(l+1)} \right],
\]

\[
b'(x) = \sum_{i=1}^{\infty} P_i^1(x) \left[ \frac{b_i^2}{2l+1} + 2a_i b_i +
\right.
\]

\[
+ \frac{2(l+1)(l+2)}{l(l+1)} A_i B_i + \frac{2(l+1)}{l^2(l+1)^2} B_i^2),
\]

and similarly for eq. (7). We thereby guarantee the exponential decrease of \(A_i\) and \(B_i\), for large \(l\), and consequently analyticity in the \(z\)-plane. For further details, we refer to a forthcoming paper [6].

The introduction of other charge-states is straightforward, and for definiteness we consider as given the \(\pi^- p \rightarrow \pi^- p\) cross-section, \(\sigma_\pi\), and polarization, \(P_\pi\). We define a new mapping for the isospin \(\frac{1}{2}\) amplitudes, which has the same form as eqs. (4)-(11), except that in eq. (4), (5), (8) and (9), \(a\), \(b\), \(A\) and \(B\) refer to the \(I = \frac{1}{2}\) state, whereas in eqs. (6), (7), (10) and (11) they refer to the \(\pi^- p \rightarrow \pi^- p\) isospin combination, namely \([(I = \frac{1}{2}) + 2(I = \frac{1}{2})]/3\). (In the case \(K^0 p \rightarrow K^0 p\), one simply has to take the \(I = 0\) amplitudes in the first, and the combination

\[
\sum_{i=0}^{\infty} \left[ (l+1) \frac{1}{4} - \frac{\eta^2_{l+1}}{4} + l \frac{1}{4} - \frac{\eta^2_l}{4} \right] P_i(x)
\]

in the second set of equations). When the cross-sections \(\sigma_\pi\) and \(\sigma_\pi\) are sufficiently close to the pure \(S\)-wave case, and the polarizations are small enough, there is a locally unique fixed point of the equations, as may be shown by applying the Contraction Mapping Principle in our Banach space (12).

Above the inelastic threshold, the terms

\[
\sum_{i=0}^{\infty} \left[ (l+1) \frac{1}{4} - \frac{\eta^2_{l+1}}{4} + l \frac{1}{4} - \frac{\eta^2_l}{4} \right] P_i(x)
\]

and

\[
\sum_{i=1}^{\infty} \frac{\eta^2_{l+1} - \eta^2_l}{4} P_i^1(x)
\]

must be added respectively to the right-hand sides of eq. (8) and (9) (for each of the two isospin states), where \(\eta_{l+1}\) are the usual inelasticity parameters. It can now be shown that, even when a complete set of independent quantities is measured (which amounts to seven real functions), still one of the four functions (14) and (15) is left undetermined. This indeterminacy corresponds precisely to a global, \(x\)-dependent phase. The constraints of the optical theorem are only finite-dimensional, and can easily be met, and this still leaves an infinite number of the \(\eta_l\) undetermined.

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Note added in proof: The fact that a measurement of four real quantities may suffice for the complete determination of the \(\pi N\) system below the inelastic threshold does not require the use of cos \(\theta\)-analyticity. A treatment employing merely continuity in cos \(\theta\) can be found in a recent preprint of R. F. Alvarez-Estrada and B. Carreras [7].
References and Footnotes

[1] The fact that a complete set of measurements is not necessary for the determination of all the scattering amplitudes below the inelastic threshold will always be encountered wherever a symmetry group like spin or isospin acts.


