Skyrme’s interaction in the asymptotic basis

P. Quentin

To cite this version:

P. Quentin. Skyrme’s interaction in the asymptotic basis. Journal de Physique, 1972, 33 (5-6), pp.457-463. <10.1051/jphys:01972003305-6045700>. <jpa-00207271>

HAL Id: jpa-00207271
https://hal.archives-ouvertes.fr/jpa-00207271
Submitted on 1 Jan 1972

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
SKYRME’S INTERACTION IN THE ASYMPTOTIC BASIS

P. QUENTIN (*)
Service de la Métrologie et de la Physique Neutroniques Fondamentales
BP n° 2, 91-Gif-sur-Yvette, France

(Réçu le 12 octobre 1971, révisé le 23 novembre 1971)

Résumé. — Les éléments de matrice des parties à deux et trois corps, et de la partie spin-orbite de l’interaction de Skyrme sont calculés analytiquement dans la base des nombres quantiques asymptotiques. Ces éléments de matrice ne font intervenir que des combinaisons de fonctions simples, ce qui facilite grandement leur évaluation numérique. Nous donnons également, dans cette même base, les éléments de matrice d’une interaction gaussienne. Et enfin nous proposons une méthode de calcul des éléments de matrice d’une interaction de Yukawa (en particulier de Coulomb) à partir de ceux d’une interaction gaussienne.

Abstract. — Matrix elements of the two-body, three-body and spin-orbit parts of Skyrme’s interaction are calculated analytically in the asymptotic basis. These matrix elements only involve combinations of some simple functions which makes their numerical evaluation rather easy. In the asymptotic basis, we have also calculated matrix elements of a gaussian interaction. Finally a method is proposed for the derivation of matrix elements of the Yukawa (and in particular Coulomb) interaction type from gaussian matrix elements.

1 Introduction. — Skyrme’s interaction [1] has been recently used in spherical Hartree Fock calculations [2]. Promising results have been obtained in the study of some super heavy nuclei [3]. It is probably more efficient, when breaking spherical symmetry (but conserving axial symmetry), to perform such calculations in a deformed basis [4]. The purpose of this article is to calculate analytically the matrix elements of Skyrme’s interaction in the basis of the so-called asymptotic quantum numbers [5] (in order to simplify we shall refer to it as the asymptotic basis). With a somewhat analogous method we derive also the matrix elements of a gaussian interaction. This type of interaction has been taken often as an effective interaction in Hartree Fock calculations [6], [7]. But one of its interests is, in particular, that one can derive from the knowledge of its matrix elements, those of a Yukawa or Coulomb interaction.

I.1 THE ASYMPTOTIC BASIS. — The kets of this basis are formed by the eigenvectors of the axially symmetrical harmonic oscillator hamiltonian and the third components of the orbital and spin angular momenta. A ket of this basis will be labelled with obvious notations \( | n_z A > > | \Sigma > > \). Defining \( I = n_z / 2 \) and \( M = A / 2 \), one knows [8] that the ket \( | IM > > \) is a ket of the standard basis (with just a phase factor) of the \( (2I + 1) \) dimensional representation of the rotation group. Besides [4], [8], there exist boson operators \( b^+_z \), \( b^-_z \) in such a way that:

\[
| n_z A > = (-)^I \langle b^+_z \rangle^I \langle b^-_z \rangle^I \frac{1}{\sqrt{\alpha \beta}} \langle 0 >
\]

(1)

where

\[
\alpha = \frac{(n_z + A)}{2}, \quad \beta = \frac{(n_z - A)}{2}
\]

Let us recall also the existence of boson operators \( a_z^+ \) with

\[
| n_z > = \frac{(a_z^+)^n}{\sqrt{n_z !}} | 0 >
\]

(2)

In coordinate representation the corresponding (1) wave functions are:

\[
< z | n_z > = \sqrt{c_z \ u_{n_z}(c_z z)}
\]

\[
< \rho \phi | a^n > = \text{sgn} (\alpha - \beta) \frac{c_z}{\sqrt{\pi}} e^{i(\alpha - \beta) \rho} \times
\]

\[
\times \frac{1}{\sqrt{2}} \sqrt{\frac{c_z^2 \ \rho^2}}
\]

(3)

where

\[
\text{sgn}(A) = (-)^{(A + 1)/2}
\]

\[
c_i = \sqrt{\frac{ma_i}{\hbar}} \quad (i = z, \perp).
\]

(*) Present address : Institut de Physique Nucléaire, Division de physique théorique, 91-Orsay.
The functions \( u_n(x) \) and \( L_m^\pi(x) \) are defined by their generating functions:

\[
\pi^{-1/4} e^{-x^2/2} e^{\sqrt{2} x \lambda} = \sum_{n=0}^{\infty} \lambda^n \quad u_n(x)
\]

\[
\exp \left( - \frac{1}{2} \frac{1 + \lambda}{1 - \lambda} \frac{x^2}{(1 - \lambda)^{1/2}} \right) = \sum_{m=0}^{\infty} \lambda^m \left( \frac{\Gamma(y + m + 1)}{m!} \right)^{1/2} L_m^\pi(x). (4)
\]

I.2 SKYRME'S INTERACTION.

This interaction can be written as

\[
\psi = \psi^{(2)} + \psi^{(3)} + \psi^{s-o}. (5)
\]

i) The interaction \( \psi^{(2)} = \sum_{i<j} \psi_{ij}^{(2)} \) is a two-body interaction, composed of the first terms of the short range expansion of a central force \([9]\). Its expression in coordinate representation will be:

\[
\psi_{12}^{(2)} = a_0 \delta(r) + a_1 [V^+ \delta(r) V] + a_2 [V^2 \delta(r) + \delta(r) V^2] (6)
\]

where \( r = (r_1 - r_2)/\sqrt{2} \); \( V \) is the gradient operator associated with \( r \).

This transformation bracket is just a rotation matrix element \( d_{m',m} \), as it can be shown without any calculations \([13]\) owing to transformation properties of the kets \( |IM> \) (defined previously) under rotations. Using Messiah's phase conventions \([14]\) one finds

\[
< n_1 n_2 || nN > = \delta_{n_1+n_2,n+N} \sqrt{n_1! n_2! n! N!} \frac{\pi^2}{2^{n_1+n_2}} (7)
\]

Transposition of \( n_1 \) and \( n_2 \) involves no change of the transformation bracket \( < n_1 n_2 || nN > \) but a phase factor \((-)^{n} \). This property will be used in the antisymmetrisation of the two-body matrix elements.

An expression of three dimensional transformation brackets is given in reference \([4]\) and reference \([11]\). Nonetheless one can apply the previous result to the case of three independent oscillators corresponding to boson operators \( a_x^+, b_x^+, b_y^+ \) and then one simply writes:

\[
< n_1 \alpha_1 \beta_1, n_2 \alpha_2 \beta_2 || nab, NAB > = \delta_{\alpha_1 \beta_1, \alpha_2 \beta_2} < n_1 n_2 || nN > (8)
\]

\[
= < n_1 n_2 || nN > < \alpha_1 \alpha_2 || AA > < \beta_1 \beta_2 || BB > . (9)
\]

The coefficients \( a_0, a_1, a_2 \), can be spin operators of the following type:

\[
a_i = t_i (1 + x_i P_e) \]

where \( P_e \) is the spin exchange operator.

ii) The interaction \( \psi^{(3)} = \sum_{i<j<k} \psi_{ijk}^{(3)} \) is a three body short range interaction, the role of which is to simulate many body effects:

\[
\psi_{123}^{(3)} = t_3 \delta(r_1 - r_2) \delta(r_1 - r_3). (10)
\]

III. Two-body part of Skyrme's interaction.

III.1 \( a_0 \) TERM.

Starting with the generating function of \( u_n(x) \) defined in the Relation (4), one can show

\[
< n \delta(x) | n' > = \frac{e^n}{\sqrt{\pi}} A(n) A(n') \]

where:

\[
A(n) = \delta_{n, \text{even}} \left( \frac{-n^{1/2} \sqrt{n}}{2^{n/2}(n/2)!} \right)
\]

\[
\delta_{n, \text{even}} = O(n \text{ odd}) ; = 1 (n \text{ even}) .
\]

In the same way, one obtains for the remainder of the wave-function (\( x \) being the projection of \( r \) on \( xOy \) plane):

\[
< ab | \delta(x) | a' b' > = \frac{e^n}{\pi} \delta_{a,b} \delta_{a',b'} . (14)
\]

(2) Of course in this case, \( R \) is only proportional to the real center of mass coordinate. This coordinate is just taken for analytical convenience.
One then deduces \( i \) standing for \( \{ n, \alpha, \beta, \Sigma \} \) \(^{(1)}\)

\[
< 12 \mid a_0 \delta(r) \mid 34 > = \frac{a_0^3}{\pi^{3/2}} \left( \sum_n f^n A(n) A(n') \right) \sum_{a'b} g^{n,b} \delta_{a,b} \delta_{a',b'}
\]

(15)

with

\[
f^n = f^n(n_1, n_2, n_3, n_4) = \langle n_1 n_2 \| nN > \langle n_3 n_4 \| n'N > \]

\[
g^{n,b} = g^{n,b}(\alpha_1, \beta_1, x_2, \alpha_2, \beta_2, x_3, \alpha_3, \beta_3, x_4, \beta_4) = \langle \alpha_1 x_2 \| aA > \langle \beta_1 \beta_2 \| bB > \langle \alpha_3 x_4 \| a' A > \langle \beta_3 \beta_4 \| b' B >
\]

where \( \omega_0^0 = \sqrt{m_0 \omega_0 / \hbar} \) is the spherical harmonic oscillator constant ; and where \( \omega_0^0 \) is defined by \( \omega_0^0 = \omega_{1/2} \omega_e \).

From a given \( n \) (resp. \( a \) and \( b \)) one deduces \( n' \), \( N \) (resp. \( a' \), \( A \) and \( b' \), \( B \)) by means of selection rules (defined previously in Formula (11)) for the Moshinsky coefficients.

It is worthwhile to notice that this matrix element is independent of the deformation of the basis. Besides, because of selection rules valid for the Moshinsky coefficients, one can easily verify the conservation of angular momentum projection.

II.2 \( a_1 \) TERM. — To evaluate this matrix element one can use the following relations between \( \delta_{a,n} \) operators and bosons operators.

\[
\delta_x = \frac{c_2}{\sqrt{2}} (- a_1^+ + a_1),
\]

\[
\delta_x = \frac{c_1}{\sqrt{2}} (- b_1^+ - b_1' + a_1 + b_1),
\]

(16)

\[
\delta_y = \frac{ic_1}{\sqrt{2}} (b_1^+ - b_1' + a_1 - b_1).
\]

First of all, we apply the gradient operator on bras and kets and then we use the quadratures (13) and (14) :

\[
< 12 \mid a_1 \mathbf{V}^+ \delta(r) \mathbf{V} \mid 34 > = \frac{2 c_0^3 a_1}{\pi^{3/2}} \times
\]

\[
\left[ c_2^2 \left( \sum_n f^n A(n - 1) A(n' - 1) \right) \sqrt{n} \sqrt{n'} \left( \sum_{a,b} g^{n,b} \delta_{a,b} \delta_{a',b'} \right) + c_2^2 \left( \sum_n f^n A(n) A(n') \right) \times \right.
\]

\[
\times \sum_{a,b} g^{n,b} \left( \sqrt{a + 1} \sqrt{a'} + 1 \delta_{a + 1,b} \delta_{a' + 1,b'} + \sqrt{a} \sqrt{a'} \delta_{a - 1,b} \delta_{a' - 1,b'} \right) \right].
\]

III.3 \( a_2 \) TERM. — From relations (16) one easily demonstrates

\[
\mathbf{V}^2 = - c_2^2 (1 + N_+ - N_- + N_-) + \frac{c_2^2}{2} (a_2^+ + a_2^2 - 1 - 2 N_+)
\]

where

\[
N_+ = b_1^+ b_1', \quad N_- = b_1 b_1'.
\]

Using the techniques defined in the previous subsection one finds

\[
a_2 < 12 \mid \mathbf{V}^2 \delta(r) + \delta(r) \mathbf{V}^2 \mid 34 > = \frac{2 c_0^3 a_1}{\pi^{3/2}} \times
\]

\[
\times \sum_n f^n A(n) A(n') \sum_{a,b} g^{n,b} \delta_{a,b} \delta_{a',b'} c_2^2 (n + n' + 1) + 2 c_2^2 (a + a' + 1).\]

IV. Three-body part of Skyrme's interaction. — IV.1 ONE DIMENSIONAL PROBLEM. — We have to calculate

\[
< n_1 n_2 n_3 | \delta(z_1 - z_2) \delta(z_1 - z_3) | n_4 n_5 n_6 > = c_2^2 \int u_{n_1}(z) u_{n_2}(z) u_{n_3}(z) u_{n_4}(z) u_{n_5}(z) u_{n_6}(z) dz.
\]

(20)

By means of the generating functions of \( u_n(x) \) defined in relation (4), the following expansion can be derived

\[
u_{n_1}(x) u_{n_2}(x) = \pi^{-1/4} \sum_n C(n_1, n_2, n) e^{-x^2/2} u_n(x)
\]

(21)

\( (1) \) In Sections III, IV, VI and VII we will ignore trivial spin dependence, of the form : \( \delta_{z_1 z_2} \delta_{z_3 z_4} \).
Moreover the $C$ coefficients are equal to zero except for three numbers $n, n_1, n_2$, which verify the triangle conditions.

We can now rewrite eq. (20) as

$$< n_1 n_2 n_3 | \delta(z_1 - z_2) \delta(z_1 - z_3) | n_4 n_5 n_6 > = \frac{c^2}{\pi} \sum_{n a p l} C(n_1, n_4, m) C(n_2, n_5, n) C(n_3, n_6, p) \times C(m, n, l) C(i, p, j) \delta_{j, p, j} \left( \frac{-j/2}{3(j+1/2)(j/2)!} \right).$$  \hspace{1cm} (22)

Let us notice that parity conditions on $C$ coefficients involve in particular that the previous matrix element will be equal to zero for an odd $\sum_{i=1}^{6} n_i$ quantity.

IV.2 TWO DIMENSIONAL PROBLEM. — We shall use a method, already proposed in an other context [13], to reduce the two dimensional problem to the single dimensional one.

One can rewrite the defining Relation (1) as:

$$| IM > = (\cdots)^{I-M} \frac{(b_{+})^{I-M}(b_{-})^{I-M}}{\sqrt{(I + M)! \sqrt{(I - M)!}}} | 00 > .$$

The operators $b_{+}^*, b_{-}^*$ are defined as linear combinations of the operators $a_{+}^*, a_{-}^*$ [8]:

$$b_{+}^* = \frac{1}{\sqrt{2}}(a_{+}^* + ia_{+}^*); \qquad b_{-}^* = \frac{1}{\sqrt{2}}(a_{+}^* - ia_{+}^*).$$ \hspace{1cm} (24)

Replacing the $b_{+}^*$ by the $a_{+}^*$ operators, we transform kets $| IM >$ into a sum of kets $| n_x = I + m, n_y = I - m >$.

The coefficients of such a transformation are, by definition [15], the Wigner rotation matrix elements $D_{m_{n_{x}, n_{y}}}$.

Using Messiah's phase conditions [14], one gets:

$$| IM > = \sum_{m} (i)^{I-m}(-1)^{I+m} \times d^{m}_{m_{n_{x}, n_{y}}} \left( \frac{\pi}{2} \right) | I + m, I - m > .$$ \hspace{1cm} (25)

We can then deduce the two dimensional matrix element. Making use of selection rules for the one dimensional matrix elements and of the angular momentum conservation condition, one obtains these matrix elements as functions of the $\alpha$ and $\beta$ numbers:

$$< \alpha_1 \beta_1 \alpha_2 \beta_2 \beta_3 \beta_4 | \delta(x_1 - x_2) \delta(x_1 - x_3) | \alpha_4 \beta_4 \alpha_5 \beta_5 \alpha_6 \beta_6 > = $$

$$\sum_{\alpha_1 ... \alpha_6} (-1)^{\frac{\pi}{2} a_{1}^{*} b_{1}^{*}/2} \cdots d_{(\alpha_{1} - \alpha_{2})/2, \beta_{1} - \alpha_{2}}(\alpha_{2} - \alpha_{3})/2, \beta_{2} - \alpha_{3})/2 \frac{\pi}{2} \times< b_1 b_2 b_3 | \delta(x_1 - x_2) \delta(x_1 - x_3) | b_4 b_5 b_6 > \times$$

$$\times< a_1 a_2 a_3 | \delta(x_1 - x_2) \delta(x_1 - x_3) | a_4 a_5 a_6 > \times$$

where

$$\lambda = \frac{\sum j^2 b_j}{2} + \sum_{j=1}^{3} b_j \text{ and } \forall j, a_j + b_j = \alpha_j + \beta_j .$$

V. Spin orbit part of Skyrme's interaction. — Let us define $S = S_{z} + S_{x} ; \: L = iV^{+} \times \delta(r) V$.

Thus the considered interaction can be written as:

$$V^{S.O.} = WS.L = \frac{W}{2}(L_{+} S_{-} + L_{-} S_{+}) + WL_{z}S_{z}$$ \hspace{1cm} (27)

where

$$S_{z} = S_{z} + iS_{y} ; \quad S_{-} = S_{x} - iS_{y}$$

$$L_{+} = \delta_{x}^{+} \delta(r) (i \delta_{y}) + (i \delta_{y})^{+} \delta(r) \delta_{x} \times$$

$$L_{-} = L_{+}^{*} .$$

By means of Relation (16) one can deduce

$$L_{+} = c_{s}^{-} c_{t}^{+} \left[ - (a_{z}^{+} + a_{y}^{+}) \delta(r) (b_{x}^{+} + b_{y}) +$$

$$+ (b_{x}^{+} + b_{y})^{+} \delta(r) (a_{z}^{+} + a_{y}) \right].$$ \hspace{1cm} (28)
Thus one gets:
\[ < n a b | L_+ | n' a' b' > = \frac{2 \sqrt{2} c_2 c_1 c_3^2}{\pi^{3/2}} \]
\[ \left[ - \delta_{a,b} \delta_{a'+1,b'} \sqrt{a'} + 1 \sqrt{n A(n-1) A(n')} + \delta_{a-1,b} \delta_{a'-1,b'} \sqrt{a'} \sqrt{n A(n'-1) A(n)} \right]. \quad (29) \]
(It can be noticed that the minus sign in front of the first term in the sum, comes from the phase factor sgn introduced in eq. 3).

From this expression the corresponding matrix elements of \( L^- \) are just obtained by interchanging \((n, a, b)\) with \((n', a', b')\).

When coupling bras and kets with the Moshinsky coefficients
\[ < \alpha_1 \alpha_2 || a A > < \beta_1 \beta_2 || b B > < \alpha_3 \alpha_4 || a' A > < \beta_3 \beta_4 || b' B > \]
we obtain the following selection rule (with \( A_i = \alpha_i - \beta_i \))
\[ (A_1 + A_2) - (A_1 + A_2) = (a - b) - (a' - b') \quad (30) \]

In a Hartree Fock calculation, one has only to calculate matrix elements with \( Q_1 = Q_3 \) and \( Q_2 = Q_4 \) (\( \Omega \) being the third projection of the total angular momentum).

The relation (30) can be written in that particular case
\[ (2S + S_4) - (2S + S_2) = (a - b) - (a' - b') . \quad (30 \text{bis}) \]

VI. Matrix elements of a gaussian interaction. —

We shall give now, the analytical expression for the matrix elements of a gaussian interaction. This interaction, as previously noticed, is interesting for itself and also for the calculation of matrix elements of the Yukawa (and in particular Coulomb) interaction type. We shall treat this question in the next section.

Let us define an operator \( \vartheta \) by \( L_\vartheta = \vartheta + \vartheta^+ \); thus \( \vartheta = \delta_\varphi^+ \delta_\varphi (i \varphi) \). By means of relation (16), \( \vartheta \) can be written as:
\[ \vartheta = \frac{c_4^2}{4} (-b_a^+ - b_b^+ + b_a + b_b)^+ \times \delta(x) \times (-b_a^+ + b_b^+ - b_a + b_b). \quad (31) \]

Making use of the selection rule (30), we get for the matrix element of \( \vartheta \) in the plane \( xOy \):
\[ < ab | \vartheta | a' b' > = \frac{c_4^2}{4} \left[ \sqrt{a + 1} \sqrt{a' + 1} \delta_{a+1,b} \delta_{a'+1,b'} - \delta_{a,b} \right]. \quad (32) \]

Owing to the reality and symmetry of these matrix elements, one can deduce easily that the matrix elements of \( L_\vartheta \) will be given by:
\[ < n a b | L_\vartheta | n' a' b' > = \frac{2 c_3^2}{\pi^{3/2}} A(n) A(n') \]
\[ \left( \sqrt{a + 1} \sqrt{a' + 1} \delta_{a+1,b} \delta_{a'+1,b'} - \delta_{a,b} \right). \quad (32\text{bis}) \]

When including the spin dependence, the matrix elements of \( \vartheta^{+A} \) are found equal to:

\[ < 12 | \vartheta^{+A} | 34 > = W_{31, 32} \delta_{22, 4}(\Sigma_3 + \Sigma_4) \sum_n f_n \sum_{a,b} g_{n a b} < n a b | L_\vartheta | n' a' b' > + \]
\[ + \frac{W}{2} (\delta_{31, 32} \delta_{42, 41} - \delta_{32, 31} \delta_{41, 42}) \sum_n f_n \sum_{a,b} g_{n a b} < n a b | L_\vartheta | n' a' b' > + \]
\[ + \frac{W}{2} (\delta_{31, 32} \delta_{42, 41} + \delta_{32, 31} \delta_{41, 42}) \sum_n f_n \sum_{a,b} g_{n a b} < n a b | L_\vartheta | n' a' b' >. \quad (33) \]

In the asymptotic basis the matrix elements of \( v_{12} \) are factorized into elements on \( Oz \) axis and elements on \( xOy \) plane. In the one dimensional case from relations (21) and (4) one can easily show that
\[ v_{12} = \frac{1}{\sigma} e^{-r/\sigma^2}. \quad (34) \]

In the asymptotic basis the matrix elements of \( v_{12} \) are factorized into elements on \( Oz \) axis and elements on \( xOy \) plane. In the one dimensional case from relations (21) and (4) one can easily show that
\[ \frac{1}{\sigma} e^{-r/\sigma^2} \]

In the same way, one finds for the two-dimensional matrix elements:
\[ \frac{e^{-\frac{1}{2} r^2 \sigma^2}}{\sigma^2} - b_a^+ - b_b^+ + b_a + b_b)^+ \times \delta(x) \times (-b_a^+ + b_b^+ - b_a + b_b). \quad (31) \]

Making use of the selection rule (30), we get for the matrix element of \( \vartheta \) in the plane \( xOy \):
\[ < ab | \vartheta | a' b' > = \frac{c_4^2}{4} \left[ \sqrt{a + 1} \sqrt{a' + 1} \delta_{a+1,b} \delta_{a'+1,b'} - \delta_{a,b} \right]. \quad (32) \]

Owing to the reality and symmetry of these matrix elements, one can deduce easily that the matrix elements of \( L_\vartheta \) will be given by:
\[ < n a b | L_\vartheta | n' a' b' > = \frac{2 c_3^2}{\pi^{3/2}} A(n) A(n') \]
\[ \left( \sqrt{a + 1} \sqrt{a' + 1} \delta_{a+1,b} \delta_{a'+1,b'} - \delta_{a,b} \right). \quad (32\text{bis}) \]

When including the spin dependence, the matrix elements of \( \vartheta^{+A} \) are found equal to:

\[ < 12 | \vartheta^{+A} | 34 > = W_{31, 32} \delta_{22, 4}(\Sigma_3 + \Sigma_4) \sum_n f_n \sum_{a,b} g_{n a b} < n a b | L_\vartheta | n' a' b' > + \]
\[ + \frac{W}{2} (\delta_{31, 32} \delta_{42, 41} - \delta_{32, 31} \delta_{41, 42}) \sum_n f_n \sum_{a,b} g_{n a b} < n a b | L_\vartheta | n' a' b' > + \]
\[ + \frac{W}{2} (\delta_{31, 32} \delta_{42, 41} + \delta_{32, 31} \delta_{41, 42}) \sum_n f_n \sum_{a,b} g_{n a b} < n a b | L_\vartheta | n' a' b' >. \quad (33) \]

Let us consider such an interaction:
\[ v_{12} = \frac{1}{\sigma} e^{-r/\sigma^2}. \quad (34) \]

In the asymptotic basis the matrix elements of \( v_{12} \) are factorized into elements on \( Oz \) axis and elements on \( xOy \) plane. In the one dimensional case from relations (21) and (4) one can easily show that
\[ \frac{1}{\sigma} e^{-r/\sigma^2} \]

In the same way, one finds for the two-dimensional matrix elements:
\[ \frac{e^{-\frac{1}{2} r^2 \sigma^2}}{\sigma^2} - b_a^+ - b_b^+ + b_a + b_b)^+ \times \delta(x) \times (-b_a^+ + b_b^+ - b_a + b_b). \quad (31) \]

Making use of the selection rule (30), we get for the matrix element of \( \vartheta \) in the plane \( xOy \):
\[ < ab | \vartheta | a' b' > = \frac{c_4^2}{4} \left[ \sqrt{a + 1} \sqrt{a' + 1} \delta_{a+1,b} \delta_{a'+1,b'} - \delta_{a,b} \right]. \quad (32) \]
By means of the generating function defined in relation (4) the W coefficients can be calculated, as follows (4):

\[
W_{(a,b,\alpha,\beta,\gamma)} = \delta_{a-b,\alpha-\beta} \left[ \frac{(c_\alpha \sigma)^{(a-b)}}{(1 + c_\alpha^2 \sigma^2)^{(a+b+\alpha+\beta+\gamma)/2}} \right] \left[ \frac{(a + \alpha' + b + \beta')/2! \prod_{k=0}^{\min(b,\beta')} (c_\alpha^2 \sigma^2 - 1)^k C_k^b C_{\beta'}^k}{C_{(a+b+\alpha+\beta+\gamma)/2}} \right]
\]

where \( C_n^p \) (\( n \geq p \)) is a coefficient of the binomial expansion. Finally one deduces for the total matrix element

\[
\left< 12 \left| e^{-r_1^2/r_2^3} \right| 34 \right> = e^{-r_1^2/r_2^3} \sum_{n} \sum_{p} C(n, n', p) A(p) \left( \frac{1}{\sqrt{1 + c_\alpha^2 \sigma^2}} \right)^{p+1} \sum_{a,b} g_{a,b} W_{(a,b,\alpha,\beta,\gamma)}
\]

VII. Yukawa interaction and Coulomb interaction. — The Yukawa interaction can be reduced to an integral of gaussian interactions [13] in the following way

\[
C(n, n', p) A(p) \sum_{a,b} g_{a,b} W_{(a,b,\alpha,\beta,\gamma)}
\]

with \( r \) relative coordinate as defined in Relation (6).

Thus the matrix elements of the Yukawa interaction can be written as

\[
e^{-r_1^2/r_2^3} \int_{0}^{\infty} e^{-\mu r_1^2/2} x^{p-1} e^{-x^2/2 \sigma^2} dx
\]

where \( p \) is always even

The integral \( I_{q,l,m,n}^{(\mu,c_\alpha,\beta,\gamma)} \) is defined by:

\[
I_{q,l,m,n}^{(\mu,c_\alpha,\beta,\gamma)} = \int_{0}^{\infty} \sigma^{q-1} e^{-\mu r_1^2/2} x^{p-1} e^{-x^2/2 \sigma^2} dx
\]

Let us put some restrictions on the integer numbers \( q, l, m, n \) when using the integrals \( I_{q,l,m,n}^{(\mu,c_\alpha,\beta,\gamma)} \) found in Formula (40):

- \( q, l, m, n \) are positive numbers
- \( l + m \equiv n - 1 \)

The remaining integrals can be calculated by recurrence relations. Two cases have to be distinguished: the prolate deformation case \( (\alpha < 1) \) and the oblate deformation case \( (\alpha > 1) \). We write \( \alpha - 1 = -\varepsilon \delta^2 \) (\( \varepsilon = 1 \) for \( \alpha < 1 \), \( \varepsilon = -1 \) for \( \alpha > 1 \)).

In the particular case of the coulomb force \( (\mu = 0) \), we have to evaluate the following integrals

\[
I_{q,l,m,n}^{(\mu,c_\alpha,\beta,\gamma)} = \left( \frac{\delta^{l-m} (-\delta^{l-m})^{m+1}}{c_\alpha^2} \right)^l C_l^m C_{m-1}^{n+1} (1 + \alpha)^{n-1} \int_{1}^{\infty} u^{2(l+j-q)} (u^2 + \alpha - 1)^{n-1} e^{-\mu u^2/2c_\alpha^2} du
\]

From these conditions, one can deduce that integrals of type I are always convergent (even in the case \( \mu = 0 \)). Introducing the deformation parameter \( \alpha = c_\alpha^2/2 = \omega_{\parallel}/\omega_{\perp} \) one gets for the integrals of type I:

\[
\int_{1}^{\infty} u^{2(l+j-q)} du = \delta^{(l+j-q-n)+1} F_{l+j-q,n}^{e}(\delta)
\]

with

\[
F_{m,n}^{e}(\delta) = \int_{1/\delta}^{\infty} u^{2m} du
\]

The restrictions previously recalled involve that \( m \leq n - 1 \) and that \( n > 0 \).

The following recurrence relations for the \( J \) integrals are easily found:

\[(*) \quad This \ result \ is \ found \ for \ a-b \geq 0. \ If \ a-b < 0, \ one \ can \ use \ the \ invariance \ of \ the \ W \ coefficient \ in \ the \ interchange \ of \ a \ (resp. \ a') \ and \ b \ (resp. \ b') \ (as \ it \ can \ be \ seen \ in \ relation (36)).\]
and in the limiting case $m = 0; n = 1$:

$$\begin{align*}
J_{0,1}^{n+1}(\delta) &= -\frac{1}{2} \log \frac{1 - \delta}{1 + \delta} \\
J_{0,1}^{n-1}(\delta) &= \pi - \arctan \frac{1}{\delta}.
\end{align*}$$

(VIII. conclusion. —) The definition of the asymptotic basis in terms of boson occupation numbers has allowed us to calculate very easily an analytic expression of matrix elements in the case of the Skyrme’s interaction. This analytic expression is rather interesting from a numerical computation point of view. Indeed, such matrix elements only involve simple combinations of functions independent of force range and deformation parameters; (precisely: functions $A(p)$ and $C(n, m, p)$ of the text and matrix elements $d_{n,m}(n/2)$). The main part of such a simplification lies in the breaking up of the Moshinsky coefficients into three similar parts. This simplicity is also due to the force we have considered. For example this interaction gives a rather simple deformation dependence of its matrix elements (5).

Gaussian interaction and, all the more, the Yukawa (or Coulomb) interaction involve in addition to the former functions, functions depending on the range of the force or on the deformation of the basis (functions $W_0^{(a, b', n', v)}$ and $J_{0,n}^{e}(\delta)$). Nevertheless, for a given interaction, and for a given basis, these latter functions can be tabulated rather easily in consideration of the small number of arguments they possess.

The author would very much like to thank Dr. M. Gaudin for his initiation to some mathematical techniques used in this work. He is also greatly indebted to Professor M. Veneroni who has suggested this study. He acknowledges many stimulating discussions with R. Babinet, H. Flocard and D. Gogny. He would like, at last, to express his thanks to C. Titin-Schnaider for a critical reading of the manuscript.

(5) For a given set of states $1, 2, 3, 4$, we have some selection rules. First of all, those derived from relation (30) which is just the conservation of angular momentum projection. For the matrix elements we have considered here, we also get on the $Oz$ axis: $n_1 + n_2 + n_3 + n_4 = \text{even}$ (excepted for $L+$ and $L-$ operators, where this sum $= \text{odd}$).

References


