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Tome 33 N° 2-3 FÉVRIER-MARS 1972

LE JOURNAL DE PHYSIQUE

Classification
Physics Abstracts :
02.00

THE EVALUATION OF BRANCHING RULES FOR LINEAR GROUPS USING MAPPINGS BETWEEN WEIGHT SPACES

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(*Reçu le 28 août 1971*)

Résumé. — L'étude des règles de branchement associées au plongement du groupe linéaire L(h) dans L(g) conduit à une méthode pour la détermination des pléthysmes $\{ \lambda \}_h \otimes \{ \mu \}_g$ correspondants. Le plongement de L(h) dans L(g) est défini par une correspondance entre vecteurs de base des représentations $\{ 1 \}$ et $\{ \lambda \}$ de L(g) et L(h) respectivement. Ces vecteurs de base sont spécifiés par leur poids, et les règles de branchement associées sont obtenus en établissant une application entre les espaces des poids du groupe. On utilise au maximum la symétrie de Weyl pour établir un algorithme dont on s'est servi pour calculer un grand nombre de pléthysmes.

Abstract. — Consideration of the branching rules associated with the embedding of L(h) in L(g) leads to a method for the determination of the related plethysms $\{ \lambda \} \otimes \{ \mu \}$. The embedding is defined by the correspondence between the basis states of the representations $\{ 1 \}$ and $\{ \lambda \}$ of L(g) and L(h). These basis states are specified by weights, and the branching rules are determined by carrying out mappings between the weight spaces of the groups. Maximum use is made of the Weyl symmetry to establish an algorithm which has been used to calculate a large number of plethysms.

Introduction. — An important problem in the study of representations of groups is the determination of the branching rule associated with the decomposition into irreducible parts of the representation subduced in a subgroup by an irreducible representation of a containing group. The determination of these branching rules is equivalent to the evaluation of plethysms, as defined by Littlewood [1], and this equivalence has already been exploited to some extent [2]-[5].

Of the classical continuous groups of transformation in an n-dimensional space : L(n), O(n) and Sp(n), the key role is played by L(n) since all the finite dimensional, irreducible, tensor representations of O(n) and Sp(n) may be related to those of L(n) by means of fairly simple branching rules [6].

A finite dimensional, irreducible representation of L(n) may be specified by $\{ \lambda \}_n = \{ \lambda_1, \lambda_2, ..., \lambda_p \}$, where $(\lambda) = (\lambda_1, \lambda_2, ..., \lambda_p)$ is a partition of l into p parts, so that $l = \lambda_1 + \lambda_2 + \cdots + \lambda_p$, and

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p > 0 .$$

Corresponding to this representation there exists a Young tableau consisting of l boxes distributed into p rows of lengths $\lambda_1, \lambda_2, ..., \lambda_p$. Conversely every partition (λ) of l into p parts defines an irreducible representation $\{ \lambda \}_n$ of L(n) for all $n \geqslant p$. If $n < p$ then

$$\{ \lambda \}_n = 0 . \tag{1.1}$$

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The defining representation of L(n) is the n-dimensional representation $\{ 1 \}_n$. In general the dimension of the representation $\{ \lambda \}_n$ is denoted $D_n^{\{ \lambda \}}$.

Given a representation $\{ \lambda \}_h$ of L(h) with

$$D_h^{\{ \lambda \}} = g , \tag{1.2}$$

then the corresponding representation matrices form a subset of those of the representation $\{ 1 \}_g$ of L(g). Thus L(g) $\supset$ L(h) and the embedding is defined [7] by the mapping

$$\{ 1 \}_g \to \{ \lambda \}_h . \tag{1.3}$$

With this embedding an arbitrary representation $\{ \mu \}_g$ of L(g) subduces a representation of L(h) which, in the notation of Littlewood [6], [8] is given by the plethysm $\{ \lambda \}_h \otimes \{ \mu \}_g$. Thus under the mapping (1.3) it follows that

$$\{ \mu \}_g \to \{ \lambda \}_h \otimes \{ \mu \}_g \tag{1.4}$$

with

$$D_g^{\{ \mu \}} = D_h^{\{ \lambda \} \otimes \{ \mu \}} . \tag{1.5}$$

The decompostion of the plethysm $\{ \lambda \}_h \otimes \{ \mu \}_g$ into irreducible representations of L(h) corresponds to the evaluation of that plethysm, and furnishes the branching rule associated with L(g) $\supset$ L(h).

It is well known that

$$\{ 1 \}_g . \{ 1 \}_g \ldots \{ 1 \}_g = \sum_\mu f^\mu \{ \mu \}_g , \tag{1.6}$$

where the product on the left contains m factors, and f^{μ} is the degree of the irreducible representation of the symmetric group on m symbols which is specified by the partition (μ) of m. It then follows from (1.3) and (1.4) that [9, p. 66]

$$\{\lambda\}_h.\{\lambda\}_h \dots \{\lambda\}_h = \sum_{\mu} f^{\mu}\{\lambda\}_h \otimes \{\mu\}_g . \quad (1.7)$$

The Littlewood-Richardson rule [6, p. 94], [10] for evaluating Kronecker products of L(h) is such that the left hand side of (1.7) is a linear combination of irreducible representations $\{\nu\}_h$ where (ν) is a partition of ml into not more than mp parts. It is clear from this, and the fact that the product is independent of h for $h \geqslant mp$, that the plethysm $\{\lambda\}_h \otimes \{\mu\}_g$ with g and h satisfying (1.2), is also independent of h provided that $h \geqslant mp$. If use is made of (1.1) and (1.2) in interpreting the terms in a plethysm, then the plethysm is independent of h and g and the subscripts may be dropped. Then

$$\{\lambda\} \otimes \{\mu\} = \sum_{\rho} G^{\{\lambda\}\{\mu\}}_{\{\rho\}}\{\rho\}, \quad (1.8)$$

where (ρ) is a partition of lm and the plethysm coefficients G are independent of the dimensions g and h of the associated groups. The above argument indicates that to evaluate these coefficients it is sufficient to carry out the calculation for $h = mp$. Even this value of h is not necessary if full use is made of conjugacy theorems.

To each partition (λ) there corresponds a conjugate partition $(\tilde{\lambda})$ which is such that the number of boxes in the ith row of the Young tableau specified by $(\tilde{\lambda})$ is just the number of boxes in the ith column of the Young tableau specified by (λ). Correspondingly there exist mutually conjugate representations $\{\lambda\}$ and $\{\tilde{\lambda}\}$. Plethysms satisfy the theorem of conjugates [8] which implies that :

$$\begin{aligned} G^{\{\lambda\}\{\mu\}}_{\{\rho\}} &= G^{\{\tilde{\lambda}\}\{\mu\}}_{\{\tilde{\rho}\}} \quad \text{if} \quad l \text{ is even} \\ &= G^{\{\tilde{\lambda}\}\{\tilde{\mu}\}}_{\{\tilde{\rho}\}} \quad \text{if} \quad l \text{ is odd} \end{aligned} \quad (1.9)$$

where (λ) is a partition of l.

These relations reduce the number of independent plethysms by a factor of about two and they may be used to reduce the value of h needed in the evaluation of a plethysm below the limit mp.

Many methods have been developed for evaluating plethysms of which the most efficient appears to be Littlewood's third method [6, p. 291], [8] taken in combination with some principal part theorems developed by Ibrahim [11]. Unfortunately the calculations necessarily become very cumbersome and this particular technique is not well suited to machine calculation. Two other distinct methods [12], [13] have however been programmed for the computer. These give, very rapidly, those plethysms $\{l\} \otimes \{m\}$ in which $\{l\}$ and $\{m\}$ correspond to symmetric tensor representations. From these all other plethysms $\{l\} \otimes \{\mu\}$ may be obtained using Littlewoods algebra of plethysms [6, p. 290], [8]. This stage of the calculation has also been programmed successfully.

Despite this advance it seems worthwhile calculating plethysms from first principles by evaluating branching rules and simply interpreting the results in terms of plethysms.

In section 2 the irreducible representations of L(h) are discussed and the basis states of these representations are defined by means of weights. A method is given for the calculation of a multiplicity matrix each of whose elements is the multiplicity, in a particular irreducible representation, of a set of weights all related by the Weyl symmetry of weight space. In addition a method is given of calculating the inverse of this multiplicity matrix which enables the representation corresponding to any given set of weights to be determined.

The method of calculating plethysms is then explained in some detail in section 3. It consists essentially of three steps : Firstly the mapping (1.3) is defined more precisely using the theory of weights developed in section 2. Secondly this mapping is applied to the decomposition (1.4) to give a set of weights. Thirdly the inverse multiplicity matrix is used to generate from this set of weights the corresponding set of irreducible representations.

This technique of calculating branching rules has been used for particular group-sub-group decompositions in both nuclear [14] and atomic [15] physics. More generally a formula involving weights, their multiplicities and the Weyl symmetry operations of the groups has been derived [16] which determines branching rules. This formula is rather unwieldy but has been used to determine some specific branching rules [17], [18]. The role played by the mapping between weight spaces in specifying the embedding of one Lie group in another has been emphasized by many authors [7], [17], [19] and the importance of the inverse multiplicity matrix has also been stressed [20]. Despite these developments the use of mappings between weight spaces has not been systematically applied to the embedding of L(h) in L(g) to give branching rules which are essentially independent of h and g, and which therefore furnish the evaluation of plethysms.

Care has been taken in section 3 to develop a method suited to such a systematic application using a computer. Using this method all the plethysms $\{\lambda\} \otimes \{\mu\}$ have been calculated for which $lm \leqslant 18$ and for $l = 10$, $m = 2$ and $l = 2$, $m = 10$.

In section 4 plethysms on a restricted number of variables are discussed. These furnish the branching rules appropriate to L(g) $\supset$ L(h) is cases for which h is too small for the result to give the complete plethysm even using the conjugacy relations (1.9). It is shown that the method of section 3 is well suited to calculations of such restricted plethysms. In particular a generating function is found for the coefficients $G^{\{l\}\{m\}}_{\{\rho\}}$ in the case when $h = 2$, i. e. (ρ) is restricted to be a partition into not more than two parts. More generally the plethysms $\{\lambda\} \otimes \{\mu\}$ on 2 variables have been calculated with $l \leqslant 12$, $m \leqslant 12$, $m \leqslant 100$. For $h = 3$ and $h = 4$ other tables have been compiled for values of l and m beyond the range for the complete plethysms.

2. **Weights and their multiplicities.** — The basis states of an irreducible representation $\{\lambda\}_h$ of L(h) may be enumerated by inserting in the boxes of the Young tableau corresponding to this representation the numbers 1, 2, ..., h in any combination such that reading across any row the numbers are non-decreasing whilst reading down any column they are increasing [21, p. 385]. Distinct arrays of numbers then define distinct basis states.

The weight associated with such a basis state is then defined to be $[\alpha] = [1^{\alpha_1} 2^{\alpha_2} \dots h^{\alpha_h}]$ where α_i is the number of times the number i appears in that array. Several distinct arrays may give rise to the same weight and it is necessary to specify $M_{[\alpha]}^{\{\lambda\}_h}$, the multiplicity of the weight $[\alpha]$ in the representation $\{\lambda\}_h$. The rules associated with the insertion of numbers in the boxes of the Young tableau are just those rules which imply that $M_{[\alpha]}^{\{\lambda\}_h}$ is the number of times $\{\lambda\}_h$ is contained in the product of symmetric representations

$$\{\alpha_1\}.\{\alpha_2\}\dots\{\alpha_h\}, \quad \text{i. e.}$$

$$\{\alpha_1\}.\{\alpha_2\}\dots\{\alpha_h\} = \sum_{\lambda} M_{[\alpha]}^{\{\lambda\}_h}\{\lambda\}_h . \tag{2.1}$$

Since the Kronecker product of representations of L(h) is both a commutative and an associative operation it is clear that the weights $[\alpha]$ and $[S\alpha]$ defined by

$$[1^{\alpha_1} 2^{\alpha_2} \dots 2^{\alpha_h}] \quad \text{and} \quad [1^{\alpha_{s_1}} 2^{\alpha_{s_2}} \dots h^{\alpha_{s_h}}],$$

where $s_1 s_2 \dots s_h$ is any permutation S of 1, 2, ..., h, have the same multiplicity in every representation $\{\lambda\}_h$. This symmetry of the weights is known as the Weyl symmetry. For every weight $[\alpha]$ there exists one weight in the set $[S\alpha]$, namely $[T\alpha]$, such that

$$[1^{\alpha_{t_1}} 2^{\alpha_{t_2}} \dots h^{\alpha_{t_h}}] = [1^{\mu_1} 2^{\mu_2} \dots q^{\mu_q}(q+1)^0 \dots h^0]$$

where $(\mu) = (\mu_1, \mu_2, \dots, \mu_q)$ is a partition of l. It is convenient to denote this particular weight not by $[T\alpha]$ but by the partition symbol (μ).

Furthermore the Kronecker product in (2.1) is independent of h so that without loss of generality

$$M_{[\alpha]}^{\{\lambda\}_h} = M_{(\mu)}^{\{\lambda\}}, \tag{2.2}$$

where $M_{(\mu)}^{\{\lambda\}}$ is an element of a square matrix M since both (μ) and (λ) are partitions of l. Using

$$\{\mu_1\}.\{\mu_2\}\dots\{\mu_q\} = \sum_{\lambda} M_{(\mu)}^{\{\lambda\}}\{\lambda\} \tag{2.3}$$

it is a straight forward task to enumerate those representations $\{\lambda\}$ containing the weight (μ) and the corresponding multiplicities. All other weights and their multiplicities may then be generated by the use of the Weyl symmetry operators S.

It is worth pointing out that (2.3) implies that $M_{(\mu)}^{\{\lambda\}}$ may also be obtained by a process involving division. In fact this multiplicity is given by

$$\frac{\{\lambda\}}{\{\mu_1\}.\{\mu_2\}\dots\{\mu_q\}} = M_{(\mu)}^{\{\lambda\}}\{0\}. \tag{2.4}$$

The evaluation of $M_{(\mu)}^{\{\lambda\}}$ by means of this procedure corresponds exactly to the Method B given by Delaney and Gruber [17]. The advantage of using (2.3) rather than (2.4) is that (2.3) furnishes a complete row of the matrix M whilst (2.4) furnishes only a single element. For example (2.4) gives

$$\frac{\{831\}}{\{72\,^21\}} = 5\{0\} \tag{2.5}$$

whilst (2.3) gives

$$\begin{aligned}&\{7\}.\{2\}.\{2\}.\{1\} = \\ &= \{12\} + 3\{11.1\} + 5\{10.2\} + 5\{93\} + \\ &\quad + 3\{84\} + \{75\} + 3\{10.1^2\} + 6\{921\} \\ &\quad + 5\{831\} + 2\{741\} + 3\{82^2\} + 2\{732\} \\ &\quad + \{91^3\} + 2\{821^2\} + \{731^2\} + \{72^2\,1\}.\end{aligned} \tag{2.6}$$

A weight $[\alpha] = [1^{\alpha_1} 2^{\alpha_2} \dots h^{\alpha_h}]$ is conventionally said to be higher than a weight $[\beta] = [1^{\beta_1} 2^{\beta_2} \dots h^{\beta_h}]$ if the *first* non-vanishing term $\alpha_i - \beta_i$ is *positive*. With this definition the highest weight of the representation $\{\lambda\}$ is just $[\lambda]$, which may be denoted by the partition (λ). This weight clearly has multiplicity 1 and may be used to specify the representation.

It is convenient to introduce a different ordering scheme for weights and representations defined by partitions. A partition $(\mu) = (\mu_1, \mu_2, \dots, \mu_q)$, involving q parts, is said to precede a partition $(\nu) = (\nu_1, \nu_2, \dots, \nu_r)$, involving r parts if the *last* non-vanishing term $\mu_i - \nu_i$ is *negative*. With this definition all partitions into q parts precede those corresponding to partitions into r parts if $q < r$. If the partition (μ) precedes the partition (ν) then correspondingly the weight (μ) is said to precede the weight (ν), and similarly the representation $\{\mu\}$ is said to precede the representation $\{\nu\}$.

Clearly the highest weight of the representation $\{\lambda\}$, given by (λ), precedes all others. Furthermore if both the representations $\{\lambda\}$ and the weights (μ) are listed in order of precedence it follows that the matrix $M_{(\mu)}^{\{\lambda\}}$ will be a lower triangular square matrix with each diagonal element 1. The inverse matrix B is also lower triangular and may easily be evaluated. The existence of the matrix B implies that if a set of weights of a representation is known then it is possible to determine the irreducible constituents of that representation.

From a computational point of view the task of evaluating the product (2.3) where q is greater than about 12 is very lengthy and it is simpler to calculate the inverse multiplicity matrix B by a different method.

The matrix element $B_{\{\lambda\}}^{(\mu)}$ is just the coefficient of

$\{ \mu_1 \}.\{ \mu_2 \} ... \{ \mu_q \}$ in the formula expressing $\{ \lambda \}$ as a sum of products of symmetric representations. The relevant formula is the identity due to Littlewood [6, p. 98]

$$\{ \lambda \} = \{ \lambda_1, \lambda_2, ..., \lambda_p \} = \left| \{ \lambda_i - i + j \} \right| \quad (2.7)$$

which expresses $\{ \lambda \}$ as a p^{th} order determinant whose elements are symmetric representations $\{ k \}$. Expanding the determinant in (2.7) gives

$$\{ \lambda \} = \sum_{\mu} B^{(\mu)}_{\{\lambda\}} \{ \mu_1 \}.\{ \mu_2 \} ... \{ \mu_q \} . \quad (2.8)$$

In the determinant the elements along the diagonal are $\{ \lambda_i \}$ and decrease to the left and increase to the right in steps of one across each row. The convention to be adopted is that $\{ 0 \} = 1$ and $\{ k \} = 0$ for all $k < 0$. If the determinant is expanded with respect to the elements in the extreme right-hand column which are

$$\{ \lambda_1 - 1 + p \}, \{ \lambda_2 - 2 + p \}, ..., \{ \lambda_{p-1} + 1 \}, \{ \lambda_p \} \quad (2.9)$$

then $\{ \lambda \}$ is obtained by adding together all those products of non-negative terms obtained by subtracting from the numbers in (2.9) all possible permutations of the numbers

$$p - 1, p - 2, ..., 1, 0$$

and assigning a factor $+ 1$ or $- 1$ to each term according as the permutations is even or odd.

Thus for example

$$\{ 5\, 2^2\, 1 \} = \begin{vmatrix} \{5\} & \{6\} & \{7\} & \{8\} \\ \{1\} & \{2\} & \{3\} & \{4\} \\ \{0\} & \{1\} & \{2\} & \{3\} \\ 0 & 0 & \{0\} & \{1\} \end{vmatrix}, \quad (2.10)$$

and the corresponding terms obtained by the subtraction procedure are :

8431		8431	
+ 3210	5221	− 3210	5230
+ 2130	6301	− 2031	6400
+ 1320	7111	− 0321	8110
− 2310	6121	+ 2301	6130
− 1230	7201	+ 0231	8200
− 3120	5311	+ 3021	5410

where terms involving negative numbers have been omitted. Thus

$$\begin{aligned} \{ 5\, 2^2\, 1 \} = &+\{ 8 \}.\{ 2 \}-\{ 6 \}.\{ 4 \}-\{ 8 \}.\{ 1 \}.\{ 1 \} \\ &-\{ 7 \}.\{ 2 \}.\{ 1 \}+2\{ 6 \}.\{ 3 \}.\{ 1 \} \\ &+\{ 5 \}.\{ 4 \}.\{ 1 \}-\{ 5 \}.\{ 3 \}.\{ 2 \} \\ &+\{ 7 \}.\{ 1 \}.\{ 1 \}.\{ 1 \} \\ &-\{ 6 \}.\{ 2 \}.\{ 1 \}.\{ 1 \} \\ &-\{ 5 \}.\{ 3 \}.\{ 1 \}.\{ 1 \} \\ &+\{ 5 \}.\{ 2 \}.\{ 2 \}.\{ 1 \} \end{aligned} \quad (2.11)$$

and the corresponding row of the matrix B is given by

$$0, 0, 1, 0 - 1, 0, - 1, - 1, 2, 1, 0, - 1, 0, 0, 1, - 1, - 1, 0, - 1, 0, 0, ...$$

where the zeros correspond to partitions (μ) of 10 not contained in (2.11), and terms omitted on the right are all zero. The partitions are enumerated in order of precedence i. e. (10), (91), (82), (73), (64), (55), (811), (721)...

This method of evaluating the expansion of $\{ \lambda \}$ is due to Murnaghan [22] and is well-suited to machine calculation. Blaha [20] has independently derived the same procedure for the determination of the matrices B and has tabulated these matrices for partitions of l with $l \leqslant 6$. These results are also contained in the work of Murnaghan. For use in connection with the calculation of plethysms, the matrices M and B have been computed in their entirety up to $l = 16$ and in part up to $l = 32$.

3. **The evaluation of plethysms.** — It is convenient to consider in the first instance that embedding of the linear group L(h) into another L(g) which is defined by the symmetric representation $\{ l \}_h$ of L(h) subduced by the defining representation $\{ 1 \}_g$ of L(g), i. e. under the reduction L(g) $\supset$ L(h)

$$\{ 1 \}_g \rightarrow \{ l \}_h \quad (3.1)$$

with

$$g = D_g^{\{1\}} = D_h^{\{l\}} . \quad (3.2)$$

The mapping (3.1) can be made explicit by enumerating the weights of $\{ 1 \}_g$ and $\{ l \}_h$ and placing them in one to one correspondence. For example the reduction L(4) $\supset$ L(2) is defined by the mapping

$$\{ 1 \}_4 \rightarrow \{ 3 \}_2 \quad (3.3)$$

with the weights of these representations in the one-to-one correspondence

$$\begin{aligned} (1) = 1 &= [1] \leftrightarrow [1^3] &= 111 = (3) \\ 2 &= [2] \leftrightarrow [1^2\, 2] &= 112 = (21) \\ 3 &= [3] \leftrightarrow [12^2] &= 122 \\ 4 &= [4] \leftrightarrow [2^3] &= 222 \end{aligned} \quad (3.4)$$

The ordering of these weights is arbitrary, as can be shown by applying the Weyl symmetry to the weights of $\{ 1 \}_4$, although here the lexigraphical rule is followed in drawing up both lists of weights of $\{ 1 \}_4$ and $\{ 3 \}_2$.

The weights of a symmetric representation $\{ m \}_g$ of L(g) can be obtained in a straightforward manner from the weights of the defining representation $\{ 1 \}_g$, and their images can than be found under the mapping (3.1). These will be the weights of the representation $\{ l \}_h \otimes \{ m \}_g$ which is in general reducible, For example under the mapping (3.3) defined more precisely by (3.4) the image of $\{2\}_4$ is given by

$$\{2\}_4 \to \{3\}_2 \otimes \{2\}_4 \tag{3.5}$$

$$\begin{array}{rcl}
(2) = 11 = [1^2] \leftrightarrow \frac{1\,1\,1}{1\,1\,1} = [1^6] & = (6) \\
(1^2) = 12 = [12] \leftrightarrow \frac{1\,1\,2}{1\,1\,1} = [1^5\,2] & = (51) \\
13 = [13] \leftrightarrow \frac{1\,2\,2}{1\,1\,1} = [1^4\,2^2] & = (42) \\
14 = [14] \leftrightarrow \frac{2\,2\,2}{1\,1\,2} = [1^3\,2^3] & = (3^2) \\
22 = [2^2] \leftrightarrow \frac{1\,1\,2}{1\,1\,2} = [1^4\,2^2] & = (42) \\
23 = [23] \leftrightarrow \frac{1\,2\,2}{1\,1\,2} = [1^3\,2^3] & = (3^2) \\
24 = [24] \leftrightarrow \frac{2\,2\,2}{1\,2\,2} = [1^2\,2^4] & \\
33 = [3^2] \leftrightarrow \frac{1\,2\,2}{1\,2\,2} = [1^2\,2^4] & \\
34 = [34] \leftrightarrow \frac{2\,2\,2}{2\,2\,2} = [12^5] & \\
44 = [4^2] \leftrightarrow \frac{2\,2\,2}{2\,2\,2} = [2^6]\,. &
\end{array} \tag{3.6}$$

The weights of $\{3\}_2 \otimes \{2\}_4$ have been written in the first instance as two weights from $\{3\}_2$ placed one above the other to form a 2×3 rectangular array.

As explained in section 2, given a set of weights corresponding to a representation such as $\{l\}_h \otimes \{m\}_g$ the irreducible constituents of this representation may be established by a consideration of only those weights defined by a partition of $n = lm$. All other weights may be obtained from these by the use of the Weyl symmetry. Thus in the example given the relevant weights of L(2) are such that their frequencies of occurrence in the list (3.6) are specified by the column matrix W:

$$\begin{array}{cc}
 & W \\
\begin{array}{c}(6)\\(51)\\(42)\\(3^2)\end{array} & \begin{pmatrix}1\\1\\2\\2\end{pmatrix}
\end{array} \tag{3.7}$$

The corresponding inverse multiplicity matrix, B, is

$$\begin{array}{cc}
 & \begin{array}{cccc}(6) & (51) & (42) & (3^2)\end{array} \\
\begin{array}{c}\{6\}\\\{51\}\\\{42\}\\\{3^2\}\end{array} & \begin{pmatrix}1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1\end{pmatrix}
\end{array}. \tag{3.8}$$

The product BW gives the column matrix, G:

$$\begin{array}{cc}
 & G \\
\begin{array}{c}\{6\}\\\{51\}\\\{42\}\\\{3^2\}\end{array} & \begin{pmatrix}1\\0\\1\\0\end{pmatrix}
\end{array}. \tag{3.9}$$

The elements of G specify the number of times a particular irreducible representation of L(2) is contained in $\{3\}_2 \otimes \{2\}_4$. Thus

$$\{3\}_2 \otimes \{2\}_4 = \{6\}_2 + \{42\}_2\,. \tag{3.10}$$

It should be stressed that the mapping

$$\{1\}_g \to \{3\}_h \tag{3.11}$$

may be associated with the reduction $L(g) \supset L(h)$ where

$$D_g^{\{1\}} = D_h^{\{3\}}\,,$$

i. e.

$$g = \tfrac{1}{6}\,h(h+1)(h+2)\,. \tag{3.12}$$

In this general case it is necessary to extend the list (3.4) to include g terms and the list (3.6) to include $D_g^{\{2\}}$ terms. However this extension is such that all the additional weights of $\{3\}_h \otimes \{2\}_g$ defined by partitions necessarily involve partitions of 6 into 3 or more parts. These partitions are all preceded by those of (3.7), so that $\{3\}_h \otimes \{2\}_g$ must contain, as before, the terms $\{6\}_h$ and $\{42\}_h$. It is then a straight forward matter to verify that

$$D_g^{\{2\}} = D_h^{\{6\}} + D_h^{\{42\}}$$

if g and h satisfy (3.12). This implies that

$$\{3\} \otimes \{2\} = \{6\} + \{42\}\,, \tag{3.13}$$

where the redundant subscripts g and h have been dropped.

Since the enumeration problem increases rapidly with increasing values of g and h it is essential to choose the lowest value of h consistent with the resulting plethysm containing all possible terms. The discussion of section 1 indicates that h may be chosen in the evaluation of $\{l\} \otimes \{m\}$ as low as m. This implies that in the example given here of the evaluation of $\{3\} \otimes \{2\}$, the complete answer (3.13) follows immediately from (3.10) without the necessity of applying any dimensionality check. Clearly in this example it was indeed necessary to choose $h = 2$. Choosing $h = 1$ does not give enough information. In fact if $h = 1$ then $g = 1$ and

$$\{3\}_1 \otimes \{2\}_1 = \{6\}_1\,. \tag{3.14}$$

It is clearly inefficient to carry out the enumeration procedure without making full use of the Weyl symmetry, which, even in the trivial example of (3.5), implies that the last four lines of (3.6) are redundant. It is best to construct the mapping from right to left i. e. to list all possible weights of $\{l\} \otimes \{m\}$ corresponding to partitions (ν) of ml, and to find the combinations of weights of $\{l\}$ from which these can be derived.

From all possible rectangular $m \times l$ arrays corresponding to a partition (ν) containing ν_1 1's, ν_2 2's, etc. it is necessary to select only those arrays ν_A satisfying the following conditions : A(i) Each row of the array must be a possible weight $[\alpha]$ of $\{ l \}$, so that the numbers in it must be non-decreasing from left to right. A(ii) The set of rows of the array must correspond to a weight $[\beta]$ of $\{ m \}$, so that they too must be non-regressive relative to the lexigraphical ordering.

If the total number of ways of forming such an array corresponding to a partition (ν) is $W_{(\nu)}^{\{l\}\{m\}}$, then the plethysm coefficients of (1.8) are given by :

$$G_{\{\rho\}}^{\{l\}\{m\}} = \sum_{(\nu)} B_{\{\rho\}}^{(\nu)} W_{(\nu)}^{\{l\}\{m\}} . \qquad (3.15)$$

The problem of enumerating the arrays is simply combinatorial and can be regarded as an application of Pólya's Theorem. Permutations within the rows, and of the rows, are regarded as inessential so the symmetry group relating equivalent arrays is the wreath product $\Sigma_m[\Sigma_l]$. The cycle index of this group determines the number of arrays ν_A for each partition (ν). So, as Read [23] shows, the cycle index $Z(\Sigma_m[\Sigma_l])$ is equivalent to the plethysm $\{ l \} \otimes \{ m \}$. Unfortunately the recognition of this equivalence does not help to evaluate plethysms as the number of cycle indices of wreath products which have been evaluated is very limited.

If the enumeration of arrays only yielded plethysms of the type $\{ l \} \otimes \{ m \}$, this technique would be of no great interest since other methods of evaluating these particular plethysms are available. The approach is much more powerful than this however, for it enables all plethysms $\{ \lambda \} \otimes \{ \mu \}$ to be evaluated, where (λ) and (μ) are any partitions of l and m.

Each of the distinct rectangular arrays ν_A corresponding to a partition of (ν) and an arrangement A of the set of ν_1 1's, ν_2 2's, ... etc. within an $m \times l$ rectangle, subject to the conditions A(i) and A(ii), may be thought of as a weight of a representation of L(h) which is the image of a weight of the representation $\{ \mu \}$ of L(g) under the mapping,

$$\{ 1 \} \rightarrow \{ \lambda \} \qquad (3.17)$$

associated with the reduction L(g) $\supset$ L(h). Each row of ν_A represents a weight $[\alpha]$ associated with the representation $\{ \lambda \}$ of L(h) and corresponds to just one of the weights of the representation $\{ 1 \}$ of L(g). The number of distinct basis states of $\{ \lambda \}$ having identical weight $[\alpha]$ is just $M_{(\sigma)}^{\{\lambda\}}$ where (σ) is a partition of l and $[T\alpha] = (\sigma)$. In general ν_A may contain β_α rows corresponding to the weight $[\alpha]$. These rows are the image in the weight space of L(h) of a set of β_α weights of the representation $\{ 1 \}$ of L(g), and they constitute a basis of a representation $\{ \eta \}$ of L(g) where (η) is a partition of β_α. Since each one of the weights $[\alpha]$ may be any one of a set of $M_{[\alpha]}^{\{\lambda\}} = M_{(\sigma)}^{\{\lambda\}}$ weights, the number of ways in which they can form a basis of $\{ \eta \}$ is just $D_{M_{(\sigma)}^{\{\lambda\}}}^{\{\eta\}}$.

The addition of sets of weights corresponding to different weights $[\alpha]$ to produce the array ν_A is exactly equivalent to forming the Kronecker product of the sets of representations of L(g) associated with each $[\alpha]$. This Kronecker product gives

$$\prod_{\sigma} \Big(\sum_{\eta} D_{M_{(\sigma)}^{\{\lambda\}}}^{\{\eta\}} \{ \eta \} \Big) = \sum_{\mu} V_{\nu_A}^{\{\lambda\}\{\mu\}} \{ \mu \} \qquad (3.18)$$

where the coefficients $V_{\nu_A}^{\{\lambda\}\{\mu\}}$ are the number of ways a particular rectangular array ν_A, corresponding to the partition (ν), may be formed from the weights of the representation $\{ \mu \}$ of L(g) by virtue of the mapping (3.17), associated with L(g) $\supset$ L(h).

Summing over all arrays A gives the coefficients

$$W_{(\nu)}^{\{\lambda\}\{\mu\}} = \sum_{A} V_{\nu_A}^{\{\lambda\}\{\mu\}} , \qquad (3.19)$$

which is the number of times the weight (ν) occurs in the plethysm $\{ \lambda \} \otimes \{ \mu \}$. Then the plethysm coefficients of (1.8) are given by

$$G_{\{\rho\}}^{\{\lambda\}\{\mu\}} = \sum_{\nu} B_{\{\rho\}}^{(\nu)} W_{(\nu)}^{\{\lambda\}\{\mu\}} . \qquad (3.20)$$

As an example consider the 6×5 array ν_A corresponding to the partition $(\nu) = (14, 7, 5, 2^2)$ of 30 given by :

ν_A	(σ)	β_α	$\{\lambda\} \rightarrow \{5\}$	$\{41\}$	$\{32\}$	$\{31^2\}$	$\{2^2 1\}$	$\{21^3\}$	$\{1^5\}$
11122	(32)	1	1	1	1	0	0	0	0
11123	(31^2)	3	1	2	1	1	0	0	0
11123	(31^2)	3	1	2	1	1	0	0	0
11123	(31^2)	3	1	2	1	1	0	0	0
12345	(1^5)	2	1	4	5	6	5	4	1
12345	(1^5)	2	1	4	5	6	5	4	1

The rows of this array correspond to the weights $[\alpha]$, and (σ) is defined by the condition that $[T\alpha] = (\sigma)$. The weights $[\alpha]$ occur β_α times in the array and the multiplicities $M_{(\sigma)}^{\{\lambda\}}$ of each of the weights in all possible representations $\{ \lambda \}$ have been indicated. Corresponding with this array is a weight $[\beta_\alpha] = [1^1\, 2^3\, 3^2]$ of $\{ \mu \}$ and this weight is associated through the Weyl symmetry with the partition $(\tau) = (321)$.

Using, for example, the information in the columns labelled by β_α and $\{41\}$, it then follows from (3.18) that

$$\sum_{\mu} V_{\nu_A}^{\{41\},\{\mu\}}\{\mu\} =$$

$$= [D_1^{\{1\}}\{1\}][D_2^{\{3\}}\{3\} + D_2^{\{21\}}\{21\} + D_2^{\{1^3\}}\{1^3\}][D_4^{\{2\}}\{2\} + D_4^{\{1^2\}}\{1^2\}]$$

$$= 40\{6\} + 124\{51\} + 156\{42\} + 72\{3^2\} + 140\{41^2\} + 160\{321\} + 68\{31^3\} + 44\{2^2 1^2\} + 12\{21^4\}.$$

In this example the products which have to be evaluated are all fairly small, but even this much calculation is seldom necessary.

First of all, the terms of $V_{\nu_A}^{\{\lambda\}\{\mu\}}$ with $\{\lambda\} = \{31^2\}$, $\{2^2 1\}$, $\{21^3\}$ and $\{1^5\}$ are, in this example, all zero since for at least one partition (σ), corresponding to a row of ν_A, $M_{(\sigma)}^{\{\lambda\}} = 0$, and $D_0^{\{\eta\}} = 0$ for all $\{\eta\}$.

Secondly, if for any $\{\lambda\}$, ν_A is such that $M_{(\sigma)}^{\{\lambda\}} = 1$ for all the row partitions (σ), as in this example in the case of $\{\lambda\} = \{6\}$, then

$$\sum_{\mu} V_{\nu_A}^{\{\lambda\}\{\mu\}}\{\mu\} = \prod_{\sigma}\Big(\sum_{\eta} D_1^{\{\eta\}}\{\eta\}\Big) = \prod_{\sigma}\{\tau_j\} = M_{\{\tau\}}^{\{\mu\}}\{\mu\}$$

where $(\tau) = (\tau_1, \tau_2, ...)$ is the partition whose elements suitably ordered are the set of β_α associated with ν_A.

Thirdly if the rows of ν_A are all distinct so that $\beta_\alpha = 1$ for all α, then $\{\eta\} = \{1\}$ and

$$D_{M_{(\sigma)}^{\{\lambda\}}}^{\{1\}} = M_{(\sigma)}^{\{\lambda\}},$$

so that

$$\sum_{\mu} V_{\nu_A}^{\{\lambda\}\{\mu\}}\{\mu\} = \sum_{\mu}\Big(\prod_{\sigma} M_{(\sigma)}^{\{\lambda\}}\Big) M_{\{1^m\}}^{\{\mu\}}\{\mu\}.$$

Thus in these three cases

$$\sum_{\mu} V_{\nu_A}^{\{\lambda\}\{\mu\}}\{\mu\} = \sum_{\mu}\Big(\prod_{\sigma} M_{(\sigma)}^{\{\lambda\}}\Big) M_{(\tau)}^{\{\mu\}}\{\mu\}, \tag{3.21}$$

so that the calculation merely involves manipulation of elements of multiplicity matrices and no Kronecker products need be evaluated.

The conjugacy relations (1.9) between plethysm coefficients are usually used to obtain $\{\tilde{\lambda}\} \otimes \{\mu\}$ (or $\{\tilde{\lambda}\} \otimes \{\tilde{\mu}\}$), from $\{\lambda\} \otimes \{\mu\}$ so that only about half of the plethysms for given l and m need be evaluated. Since in the approach used here the complete set of plethysms is found at once, the conjugacy relations may be used to restrict the representations $\{\rho\}$ that need be considered. It is necessary to evaluate $G_{\{\rho\}}^{\{\lambda\}\{\mu\}}$ for only one of each conjugate pair $\{\rho\}$ and $\{\tilde{\rho}\}$ and it is simplest to take the preceding representation of each pair.

Since $B_{\{\rho\}}^{(\nu)}$ is lower triangular this means it is necessary to evaluate $W_{(\nu)}^{\{\lambda\}\{\mu\}}$ for all (ν) preceding and including the last partition which precedes its conjugate. Some of these partitions (ν) may nevertheless be preceded by their conjugate partition, in which case $W_{(\nu)}^{\{\lambda\}\{\mu\}}$ is more easily found from

$$W_{(\nu)}^{\{\lambda\}\{\mu\}} = \sum_{\rho} M_{(\nu)}^{\{\rho\}} G_{\{\rho\}}^{\{\lambda\}\{\mu\}} \tag{3.22}$$

and the conjugacy relations (1.9), than from the enumeration procedure.

The details of the calculation of the plethysms $\{\lambda\} \otimes \{\mu\}$ for (λ) and (μ) partitions of 3 and 2 respectively are given in Table I. The partitions (ν) which need to be considered run up to (321) so that the calculation is effectively carried out in L(h) with $h = 3$. Without the use of the conjugacy relations it would be necessary to carry out the calculation, in the notation of section I, with $h = lm = 3.2 = 6$. For each row of all the possible arrays ν_A the multiplicity $M_{(\sigma)}^{\{\lambda\}}$ is given, where (σ) is determined by the weight of each row and $\{\lambda\}$ is $\{3\}$, $\{21\}$ or $\{1^3\}$. Similarly the multiplicity $M_{\{\tau\}}^{\{\mu\}}$ is given for each array, where (τ) is determined by the weight $[\beta_\alpha]$ corresponding to the frequencies of repetitions of rows in the array and $\{\mu\}$ is $\{2\}$ or $\{1^2\}$. These multiplicities are such that $V_{\nu_A}^{\{\lambda\}\{\mu\}}$ may be calculated using (3.21). Summing over ν_A for each (ν) then gives $W_\nu^{\{\lambda\}\{\mu\}}$.

TABLE I

(ν)	ν_A	β_α	$M_{(\sigma)}^{\{\lambda\}}$	$M_{(\tau)}^{\{\mu\}}$	$V_{\nu_A}^{\{\lambda\}\{\mu\}}$	$W_{(\nu)}^{\{\lambda\}\{\mu\}}$
(6)	111	2	100			
	111	2	100	10	100000	100000
(51)	111	1	100			
	112	1	110	11	110000	110000
(42)	111	1	100			
	122	1	110	11	110000	
	112	2	110			
	112	2	110	10	101000	211000
(33)	111	1	100			
	222	1	100	11	110000	
	112	1	110			
	122	1	110	11	111100	221100
(411)	111	1	100			
	123	1	121	11	110000	
	112	1	110			
	113	1	110	11	111100	221100
(321)	111	1	100			
	223	1	110	11	110000	
	112	1	110			
	123	1	121	11	112200	
	113	1	110			
	122	1	110	11	111100	333300

The ordering of the terms in V and W corresponds to

$$\{\lambda\}\{\mu\} = \{3\}\{2\}; \{3\}\{1^2\}; \{21\}\{2\}; \{21\}\{1^2\}; \{1^3\}\{2\}; \{1^3\}\{1^2\}.$$

Then (3.15) gives in matrix notation

$$
G_{\{\rho\}}^{\{\lambda\}\{\mu\}} = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ 0 & -1 & 1 & & & \\ 0 & 0 & -1 & 1 & & \\ 1 & -1 & -1 & 0 & 1 & \\ 0 & 1 & 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 1 & 1 & & & \\ 2 & 2 & 1 & 1 & & \\ 2 & 2 & 1 & 1 & & \\ 3 & 3 & 3 & 3 & 0 & 0 \end{pmatrix} =
\begin{array}{c|cccccc|c}
 & \{3\}\otimes\{2\} & \{3\}\otimes\{1^2\} & \{21\}\otimes\{2\} & \{21\}\otimes\{1^2\} & \{1^3\}\otimes\{2\} & \{1^3\}\otimes\{1^2\} & \\
\{6\} & 1 & & & & & & \{1^6\} \\
\{51\} & 0 & 1 & & & & & \{21^4\} \\
\{42\} & 1 & 0 & 1 & & & & \{2^2 1^2\} \\
\{33\} & 0 & 1 & 0 & 1 & & & \{2^3\} \\
\{411\} & 0 & 0 & 0 & 1 & & & \{31^3\} \\
\{321\} & 0 & 0 & 1 & 1 & 0 & 0 & \{321\}\,. \\
 & \{1^3\}\otimes\{1^2\} & \{1^3\}\otimes\{2\} & \{21\}\otimes\{1^2\} & \{21\}\otimes\{2\} & \{3\}\otimes\{1^2\} & \{3\}\otimes\{2\} &
\end{array}
$$

The terms of the matrix G must be read both ways, i. e. first using the labels at the top and on the left and then using the labels at the bottom and on the right, except for self conjugate partitions (ν) such as (321) which must only be counted once. This gives the results

$$
\begin{aligned}
&\{3\} \otimes \{2\} \quad \{6\} + \{42\} \\
&\{3\} \otimes \{1^2\} = \{51\} + \{3^2\} \\
&\{21\} \otimes \{2\} = \{42\} + \{321\} + \{31^3\} + \{2^3\} \\
&\{21\} \otimes \{1^2\} = \{3^2\} + \{41^2\} + \{321\} + \{2^2 1^2\} \\
&\{1^3\} \otimes \{2\} = \{2^3\} + \{21^4\} \\
&\{1^3\} \otimes \{1^2\} = \{2^2 1^2\} + \{1^6\}.
\end{aligned}
$$

Further results will be published elswhere.

4. **Plethysms on a restricted number of variables.** — The method of section 3 is especially valuable as it is readily adapted to calculations with a restricted number of variables. For the evaluation of $\{\lambda\} \otimes \{\mu\}$ in L(h), the weights of both $\{\lambda\}$ and $\{\lambda\} \otimes \{\mu\}$ must correspond to partitions into not more than h parts. So both B and W need be found only for such partitions, and these are always the easier ones.

In L(2), the result is particularly simple. The rectangular arrays contain only 1's and 2's and the conditions A(i) and A(ii) of Section 3 imply that all the 1's must be adjacent whilst the 2's complete the rectangular array, e. g.

1111
1112
1112
1222
1222.

The number of 2's must be less than or equal to the number of 1's. So the pattern of 2's corresponds to a partition of $p \leqslant lm/2$ with no part greater than l and not more than m parts. For each of these arrays the associated multiplicities are easily obtained, so giving W.

For L(2)

$$
\begin{aligned}
M_{(\mu)}^{\{\lambda\}} &= 1 \quad \text{if } (\lambda) \text{ precedes } (\mu) \text{ or if } (\lambda) = (\mu) \\
&= 0 \quad \text{otherwise}.
\end{aligned} \tag{4.1}
$$

Therefore

$$
\begin{aligned}
B_{\{\rho\}}^{(\nu)} &= \quad 1 \quad \text{if} \quad (\rho) = (\nu) \\
&= -1 \quad \text{if} \quad (\rho)' = (\nu) \\
&= 0 \quad \text{otherwise}
\end{aligned} \tag{4.2}
$$

where $(\rho)'$ is the partition immediately preceding (ρ). It then follows from (3.20) that

$$
G_{(\nu)}^{\{\lambda\}\{\mu\}} = W_{(\nu)}^{\{\lambda\}\{\mu\}} - W_{(\nu)'}^{\{\lambda\}\{\mu\}}. \tag{4.3}
$$

Results have been computed in L(2) for $l \leqslant 12$, $m \leqslant 12$ and $lm \leqslant 100$.

For the special case of $\{l\} \otimes \{m\}$ all the multiplicities of the weights of both $\{l\}$ and $\{m\}$ are 1, so that in L(2), $W_{(n-p,p)}^{\{l\}\{m\}}$ is simply the number of partitions of p with no part greater than l and not more than m parts. This is well known [24, p. 5] to be the coefficient of x^p in

$$
\frac{(1 - x^{l+1})(1 - x^{l+2}) \dots (1 - x^{l+m})}{(1 - x)(1 - x^2) \dots (1 - x^m)}.
$$

So, using (4.3), $G_{(n-p,p)}^{\{l\}\{m\}}$ is the coefficient of x^p in

$$
\frac{(1 - x^{l+1})(1 - x^{l+2}) \dots (1 - x^{l+m})}{(1 - x^2)(1 - x^3) \dots (1 - x^m)}. \tag{4.4}
$$

This generating function for the plethysm $\{l\} \otimes \{m\}$ in L(2) differs slightly from the erroneous result of Littlewood [6, p. 208].

Apart from the calculations of plethysms in L(2) which are of immediate relevance [5] to atomic and nuclear spectroscopy, plethysms $\{\lambda\} \otimes \{\mu\}$ in L(3) have been calculated for these pairs of values of (l, m) :

(10,3) ; (9,3) ; (8,4) ; (7,4) ; (6,4) ; (6,5) ; (5,5) ; (4,6) ; (4,7) ; (3,7) ; (3,8) ; (3,9) ; (3,10). The results include plethysms relevant to the $U(3)$ nuclear shell model of Elliott [2]. Plethysms in L(4) have also been calculated for (12,2) ; (11,2) ; (8,3) ; (7,3) ; (5,4) ; (4,5) ; (2,11) ; (2,12).

5. **Conclusion.** — It has been demonstrated that it is feasible to calculate plethysms by making use of their connection with the branching rules associated with $L(g) \supset L(h)$ and a « first-principles » method of evaluating such branching rules. The method has yielded, through the use of a computer, tables of all plethysms $\{ \lambda \} \otimes \{ \mu \}$ for $lm \leqslant 18$, and for $l = 10$, $m = 2$ and $l = 2$, $m = 10$. This exceeds slightly the work of Ibrahim [25], [26] and of Butler and Wybourne [12].

The method has the advantage of yielding for each l and m all plethysms $\{ \lambda \} \otimes \{ \mu \}$ in a single calculation, unlike any other method developed to date. Moreover the method is such that for each l and m the calculation depends on no results obtained for any other values of l and m. Finally in contrast to some methods of calculation [13], [25], [26] this method is very well suited to the calculation of plethysms on a restricted number of variables.

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