Some simple flows of a para-magnetic fluid

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Classification
Physics Abstracts :
14.88, 14.19, 14.83

SOME SIMPLE FLOWS OF A PARA-MAGNETIC FLUID

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(*Reçu le 24 mai 1971*)

Résumé. — En utilisant un simple modèle phénoménologique pour le fluide paramagnétique, nous analysons un écoulement parallèle dans un tuyau. Nous examinons aussi la possibilité de maintenir un écoulement circulaire dans un cylindre par rotation d'un champ magnétique homogène.

Abstract. — Using a simple continuum model for a paramagnetic fluid we analyze a simple shearing flow and parallel flow through a pipe. We also examine the possibility of maintaining a steady circular flow in a circular cylinder by rotating a homogeneous magnetic field.

1. **Introduction.** — Recently, fluids have been prepared by suspending sub-micron sized ferro-magnetic grains in a non-magnetic, non-conducting liquid [1]. The grains are treated with a dispersing agent which operates in conjunction with the thermal agitation of the particles to counteract the tendency of the particles to agglomerate and eventually precipitate from solution as a result of the mutual magnetic forces. Consequently, colloidal suspensions of this composition are ultra-stable and behave, for all practical purposes, as non-conducting, magnetizable fluids. Most fluids prepared in this fashion are reported to exhibit no magnetic hysteresis [2] so, although ferro-magnetic fluids have been synthesized [3], we will limit our discussion to para-magnetic fluids.

Because of the marked magnetic properties of these fluids it is possible to influence their behaviour by applying a magnetic field. The application of the magnetic field induces magnetization which interacts with an inhomogeneous magnetic field to produce a body force and, in the event the magnetization and the magnetic field are not co-linear, a body couple. Additionally, static experiments [4], [5] indicate that the pressure distribution in the resting fluid is modified due to the interaction of the magnetization with the magnetic field ; while dynamic experiments [6], [7] show that the general state of stress in a paramagnetic fluid depends upon measures of the magnetization as well as upon measures of the motion and temperature. Conversely, we anticipate that in these materials the local magnetization is influenced by the thermal and kinematic variables as well as by the magnetic field. The magnetic field itself, of course, depends, in part, upon the distribution of magnetization in the fluid.

A continuum theory for magnetic fluids has recently been proposed [8] in order to provide a formulation, including these unique features of para-magnetic fluids, which will serve as a simple, general framework for the interpretation of experimental results. As yet, this theory has only been applied to static, isothermal states of para-magnetic fluids. Here we use the balance laws of this theory in conjunction with particular constitutive assumptions to examine some simple viscometric flows of a model para-magnetic fluid. We also consider the possibility of maintaining a steady rotational flow of this fluid in a cylindrical vessel by rotating an external field in the plane of the circular cross section [9].

2. **Governing equations.** — We use cartesian tensor notation except where otherwise indicated. We consider a non-conducting magnetizable fluid confined in some region V with surface Σ. In the absence of electric fields and for field frequencies which are much less than the ratio of the speed of light c to a typical dimension of V, the integral forms governing the magnetic field $\mathbf{H}$ are, in Gaussian units,

$$\int_B H_k \, dl_k = \frac{4\pi}{c} \int_{\bar{S}} j_k \, dS_k \,, \tag{2.1}$$

and

$$\int_{\hat{S}} H_k \, dS_k = -4\pi \int_{\hat{S}} M_k \, dS_k \,. \tag{2.2}$$

Here, $d\mathbf{l}$ is a differential element of length tangent to the arbitrary closed circuit B, $d\mathbf{S}$ is the differential area element, $\bar{S}$ is any surface capping B, $\mathbf{j}$ is the electric current density, $\hat{S}$ is an arbitrary closed surface and $\mathbf{M}$ is the magnetization per unit volume.

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For sufficiently smooth fields, the local forms of (2.1) and (2.2) are

$$\varepsilon_{ipk} H_{k,p} = \frac{4\pi}{c} j_i \tag{2.3}$$

and

$$H_{k,k} = -4\pi M_{k,k} \tag{2.4}$$

At the surface Σ (2.1) and (2.2) yield the jump conditions

$$[|\varepsilon_{iqp} H_p|] \nu_q = 0 \tag{2.5}$$

and

$$[| H_i + 4\pi M_i |] \nu_i = 0 \tag{2.6}$$

Here $[| A \ldots |] \equiv A^+ \ldots - A^- \ldots$, where $A^+ \ldots (A^- \ldots)$ is the limiting value of the tensor A ... as Σ is approached from the outside (inside). The unit vector $\mathbf{v}$ is the outward normal to Σ.

We shall find it convenient to separate the magnetic field $\mathbf{H}$ into an external field $\mathbf{H}^0$ which would be present even if the magnetic fluid were not and a self-field $\mathbf{H}'$ due to the presence of the magnetic fluid ; so

$$H_i = H_i^0 + H_i' . \tag{2.7}$$

We regard the external magnetic field as determined by known distributions of electric current in the region exterior to V. Then that part of the system (2.3), (2.6) which remains to be satisfied by the self-field is

$$\varepsilon_{ijk} H'_{k,j} = 0 \tag{2.8}$$

everywhere, with

$$H'_{k,k} = -4\pi M_{k,k} \tag{2.9}$$

in V,

$$H'_{k,k} = 0 \tag{2.10}$$

exterior to V and

$$[| \varepsilon_{ijk} H'_{k,j} |] \nu_i = 0 , \quad [| H'_k 4\pi M_k |] \nu_k = 0 \tag{2.11}$$

at Σ.

The integral forms for the conservation laws for mass, linear momentum, bi-vector momentum of momentum and energy are assumed to be slightly modified forms of these obtained by Brown [10] in a direct calculation for magnetic materials.

$$\frac{d}{dt} \int_{\bar{V}} \rho \, dV = 0 \tag{2.12}$$

$$\frac{d}{dt} \int_{\bar{V}} \rho \dot{x}_i \, dV = \int_S t_{ik} \, dS_k - 2\pi \int_{S \cap \Sigma} (M_k \nu_k)^2 \, dS_i + \int_{\bar{V}} \rho (m_k H'_{i,k} + f_i) \, dV \tag{2.13}$$

$$\frac{d}{dt} \int_{\bar{V}} \rho (x_{[i} \dot{x}_{j]} + \beta m_{[i} \dot{m}_{j]}) \, dV = \int_S (x_{[i} t_{j]k} + m_{[i} l_{j]k}) \, dS_k - $$
$$- 2\pi \int_{S \cap \Sigma} (M_p \nu_p)^2 x_{[i} dS_{j]} + \int_V \rho (m_{[i} H_{j]} + m_{[i} H^0_{j]} + x_{[i} m_k H'_{j],k} + x_{[i} f_{j]}) \, dV \tag{2.14}$$

$$\frac{d}{dt} \int_{\bar{V}} \rho \left(\varepsilon + \frac{1}{2} \dot{x}_i \dot{x}_i + \frac{1}{2} \beta \dot{m}_i \dot{m}_i \right) dV = \int_S (\dot{x}_i t_{ik} + \dot{m}_i l_{ik} - q_k) \, dS_k - $$
$$- 2\pi \int_{S \cap \Sigma} (M_p \nu_p)^2 \dot{x}_i \, dS_i + \int_{\bar{V}} \rho (\dot{x}_i m_k H'_{i,k} + \dot{x}_i f_i + \dot{m}_k H'_k + \dot{m}_k H^0_k + r) \, dV . \tag{2.15}$$

Here $\bar{V}$ is an arbitrary part of V, ρ the mass per unit volume, $\rho \mathbf{m} = \mathbf{M}$, S the surface bounding $\bar{V}$, t the stress tensor, $S \cap \Sigma$ the portion of S common to Σ, $\mathbf{f}$ the external body force per unit mass, β a dimensional constant, l the magnetic stress tensor, ε the internal energy per unit mass, $\mathbf{q}$ the heat flux, and r the heat supply per unit mass. Superposed dots indicate the material time derivative and bracketed indices are to be antisymmeterized.

The law of conservation of mass (2.12) assumes its usual form.

In the conservation law for linear momentum (2.13) the additional flux of linear momentum at the material boundary results from the discontinuity of the self-field due to the discontinuity of the magnetization there. The volume force density due to the self-field has been distinguished from the external body force which includes, for example, the volume force due to an external magnetic field.

In the conservation law for bi-vector moment of momentum (2.14) the moment of momentum includes a contribution arising from the spin of the magnetization vector. We ignore the component of the spin which is parallel to the magnetization — a simplification which appears plausible when considering suspensions of either rod-like or spherical grains. In a similar fashion, the flux and external volume sources of moment of momentum includes surface

and volume couples in addition to moments of tractions and body forces which appear in the conservation law for linear momentum.

The conservation law for energy (2.15) includes the kinetic energy associated with the moment of momentum of the magnetization and the rates of working of the generalized surface and volume actions.

We introduce, in the form of a balance law, a generalization of the common constitutive assumption relating the magnetization and the magnetic field :

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\bar{V}} \rho\beta\dot{m}_i \,\mathrm{d}V = \int_S l_{ik}\,\mathrm{d}S_k + \int_{\bar{V}} [\rho(H_i^0 + H_i') + \psi_i]\,\mathrm{d}V \quad (2.16)$$

where $\boldsymbol{\psi}$ is called the intrinsic magnetic field. Equation (2.16) expresses that the time rate of change of the magnetization inertia in an arbitrary material volume is balanced by local fields acting over the surface and throughout the volume in addition to the macroscopic magnetic field.

For sufficiently smooth fields local forms of (2.13), (2.16) may be written in the form

$$\rho + \rho\dot{x}_{k,k} = 0\,, \quad (2.17)$$

$$\rho\ddot{x}_i = t_{ik,k} + \rho f_i + \rho m_k H'_{i,k}\,, \quad (2.18)$$

$$-m_{[j}\psi_{k]} + t_{[kj]} + m_{[j,p}l_{k]p} = 0\,, \quad (2.19)$$

$$\rho\dot{\varepsilon} = -\dot{m}_k\psi_k + \dot{x}_{i,k}t_{ik} + \dot{m}_{i,k}l_{ik} - q_{k,k} + \rho r\,, \quad (2.20)$$

$$\rho\beta\ddot{m}_i = l_{ik,k} + \rho H'_i + \rho H_i^0 + \psi_i\,, \quad (2.21)$$

in V, and

$$t_{ik}\nu_k - 2\pi(M_p\nu_p)^2\nu_i = t_i \quad (2.22)$$

$$\varepsilon_{pji}m_j l_{ik}\nu_k = L_p \quad (2.23)$$

on Σ.

Here $\mathbf{t}$ is the appied surface force and $\mathbf{L}$ is the applied surface couple.

Notice that the reduced local form (2.19) of the balance of moment of momentum requires that, in general, the stress tensor be asymmetric.

An entropy production inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\bar{V}} \rho\eta\,\mathrm{d}V + \int_S p_k\,\mathrm{d}S_k - \int_{\bar{V}} \rho\frac{r}{T}\,\mathrm{d}V \geqslant 0 \quad (2.24)$$

applieds to each material volume $\bar{V}$. Here η is the entropy per unit mass, $\mathbf{p}$ the entropy flux vector and T the temperature. With sufficient smoothness the local form of (2.24) is

$$\rho\dot{\eta} + p_{k,k} - \rho\frac{r}{T} \geqslant 0\,. \quad (2.25)$$

The field equations (2.17)-(2.21), the boundary conditions (2.22) and (2.23) and the entropy production inequality (2.25) apply to an arbitrary magnetizable material. We must provide next constitutive assumptions which specify the magneto-mechanical nature of the material. That is, we must express ε, η, $\mathbf{q}$, $\boldsymbol{\psi}$, $\mathbf{p}$, t, and l in terms of measures of the motion, magnetization and the temperature. The particular measures we choose will determine the type of behaviour the material will exhibit. We define a para-magnetic fluid as a material in which ε, η, $\mathbf{q}$, $\boldsymbol{\psi}$, $\mathbf{p}$, t and l are single valued functions of $\mathbf{m}$, $\dot{\mathbf{m}}$, $\nabla\dot{\mathbf{x}}$, T, and ∇T. We restrict the way in which the constitutive functions may depend upon these variables by requiring that these functions have the same form in any two motions of the fluid which differ by a time dependent rigid motion — that is, the material response is unaltered by rigid motions [11]. If we assume that under these transformations the magnetization behaves as an axial vector, this requires that the independent variables $\dot{\mathbf{m}}$ and $\nabla\dot{\mathbf{x}}$ appear only in the combinations

$$\overset{*}{m}_i = \dot{m}_i - W_{ik}m_k \quad (2.26)$$

and

$$D_{ik} = \tfrac{1}{2}(\dot{x}_{i,k} + \dot{x}_{i,k}) \quad (2.27)$$

where

$$W_{ik} = \tfrac{1}{2}(\dot{x}_{i,k} - \dot{x}_{k,i})\,. \quad (2.28)$$

In addition, the constitutive functions are required to satisfy both the form (2.19) of the balance law for moment of momentum and the entropy production inequality (2.25) identically in the independent variables. The first requirement brings the number of field equations into agreement with the number of unknowns ; the second requirement excludes from consideration materials in which a negative local entropy production is possible [12].

Jenkins [6] shows that constitutive functions consistent with these requirements have the form

$$F = \varepsilon - T\eta = F(T, m) \quad (2.29)$$

$$\eta = -\frac{\partial F}{\partial T} \quad (2.30)$$

$$\psi_k = -\rho\frac{\partial F}{\partial m_k} + \hat{\psi}_k \quad (2.31)$$

$$Tp_k = q_k = \hat{q}_k \quad (2.32)$$

$$t_{ik} = -p\delta_{ik} + \hat{t}_{ik} \quad (2.33)$$

$$l_{ik} = 0 \quad (2.34)$$

where F is the Helmholtz free energy per unit mass. Quantities bearing hats are the non-equilibrium parts which are functions of $\mathbf{m}$, $\overset{*}{\mathbf{m}}$, D, T and ∇T vanishing when $\overset{*}{\mathbf{m}} = \nabla T = 0$, $D = 0$. These quantities must satisfy, identically, the relations

$$-m_{[j}\hat{\psi}_{k]} + \hat{t}_{[kj]} = 0 \quad (2.35)$$

and

$$- \hat{q}_k T_k + \hat{t}_{ik} D_{ik} - \hat{\psi}_i \overset{*}{m}_i \geqslant 0 . \qquad (2.36)$$

3. **A model paramagnetic fluid.** — We here restrict attention to isothermal phenomena and ignore the energy equation (2.20) which is, in general, incompatible with this assumption.

We shall consider a simple continuum model of an incompressible paramagnetic fluid. The constitutive relations are designed to represent a dilute homogeneous suspension of spherical ferromagnetic grains in which rotational motion of the grains due to thermal agitation is significant.

The magneto-static behavious of the fluid is assumed to be governed by the free energy density

$$F = - 4 \pi \rho \frac{m_s}{\chi_0} \left[m + m_s \ln \left(\frac{m_s - m}{m_s} \right) \right] . \qquad (3.1)$$

Here the constants χ_0 and m_s are called the initial magnetic susceptibility and the saturation magnetization per unit mass respectively. For $\dot{\mathbf{m}} = 0$, $\nabla \dot{\mathbf{x}} = 0$ we combine (2.21) and (2.31) to obtain

$$H_i = 4 \pi \rho \frac{m_s}{\chi_0} \frac{m_i}{m_s - m} , \qquad (3.2)$$

or

$$M = M(H) = \frac{M_s}{4 \pi \dfrac{M_s}{\chi_0 H} + 1} . \qquad (3.3)$$

Consequently, the magnetization in the resting fluid increases, monotonically with H, so that

$$M = 0, \quad 4 \pi M' = \chi_0 ,$$

when $H = 0$, and $M \to M_s$ as $H \to \infty$.

For the non-equilibrium parts of the stress and the intrinsic magnetic field we take

$$\hat{t}_{ik} = 2 \mu D_{ik} + 2 \gamma m_{[k} \overset{*}{m}_{i]} \qquad (3.4)$$

and

$$\hat{\psi}_i = 2 \gamma \overset{*}{m}_i \qquad (3.5)$$

respectively. In general, the coefficients γ and μ are functions of m. The forms (3.4) and (3.5) satisfy the identity (2.35) while the entropy production inequality (2.36) requires that

$$\mu > 0 \quad \text{and} \quad \gamma < 0 . \qquad (3.6)$$

In the model paramagnetic fluid which we consider we specify that

$$\mu = \text{Cte} \quad \text{and} \quad \gamma = - \rho \frac{\alpha^2}{m} \qquad (3.7)$$

where α is a constant. A consequence of the assumption (3.7) is that in a steady simple shearing flow the viscous torque at a point in the material (the antisymmetric part of the stress tensor) is proportional to the number of orientated grains at that point.

4. **Two parallel flows.** — Simple shear. The parallel flow easest to treat is that of the homogeneous shear of an unbounded material. That is

$$\dot{x}_1 = k x_2 , \quad \dot{x}_2 = \dot{x}_3 = 0 , \quad k = \text{Cte} . \qquad (4.1)$$

In this flow, the stretching tensor D and the spin tensor W are given by

$$D_{ik} = \frac{1}{2} \begin{vmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

and

$$W_{ik} = \frac{1}{2} \begin{vmatrix} 0 & k & 0 \\ -k & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} .$$

We consider an external magnetic field which is homogeneous and acting in the plane of shear

$$H_i^0 = H^0 (\cos \psi, \sin \psi, 0) . \qquad (4.2)$$

Then the external body force per unit mass $\mathbf{f}$, given by

$$f_i = m_k H^0_{i,k} , \qquad (4.3)$$

vanishes.

With (4.1) and (4.2) it is natural to consider magnetization fields $\mathbf{m}$ which are homogeneous and in the plane of shear

$$m_i = m(\cos \lambda, \sin \lambda, 0) \qquad (4.4)$$

where $m = m(t)$ and $\lambda = \lambda(t)$. In this event, the self-field $\mathbf{H}'$ is zero and the momentum equation (2.18) is satisfied if the pressure p is constant.

Equation (2.21) governing the magnetization is, with (3.2) and (3.1),

$$\beta \ddot{m}_i = - \frac{4 \pi M_S}{\chi_0} \frac{m_i}{m_S - m} - \frac{2 \alpha^2}{m} \overset{*}{m}_i + H_i^0 \qquad (4.5)$$

in what follows we neglect the inertia associated with the magnetization. Setting $\beta = 0$ and using (4.2) and (4.4), we may rewrite (4.5) as

$$\frac{\overline{m}'}{\overline{m}} = \frac{- \overline{M}_S \overline{m}}{1 - \overline{m}} + \overline{H}^0 \cos \varphi \qquad (4.6)$$

and

$$\overline{m} \varphi' = \overline{m} (1 + \overline{H}^0 \sin \varphi) \qquad (4.7)$$

where

$$\varphi \equiv \psi - \lambda , \qquad \overline{m} \equiv \frac{m}{m_S} ,$$

$$\overline{M}_S \equiv \frac{4 \pi M_S}{\chi_0 a^2 k} , \qquad \overline{H}^0 \equiv \frac{H^0}{k \alpha^2} \qquad (4.8)$$

and a dash indicates derivatives taken with respect to $\tau = 2\,t/k$.

Steady solutions to the system (4.6), (4.7) are, for $|\,\overline{H}^0\,| \geqslant 1$,

$$\varphi = \sin^{-1}\left(\frac{1}{\overline{H}^0}\right), \qquad \overline{m} = \frac{1}{1 + \overline{M}_S \tan\varphi}, \tag{4.9}$$

and

$$\varphi = \sin^{-1}\left(\frac{1}{\overline{H}^0}\right) + \pi\,, \quad \overline{m} = \frac{1}{1 + \overline{M}_S \tan\varphi}\,. \tag{4.10}$$

Solution (4.10) can be shown to be an unstable singular point of (4.6), (4.7), while (4.9) is the stable steady solution. Note that for this constitutive theory the angle between the magnetization and the external magnetic field is, in this simple flow, independent of the magnitude of the magnetization.

The non-vanishing steady components of the stress associated with the solution (4.9) are

$$t_{12} = \left(\mu + \rho\frac{\alpha^2}{2}\,m\right)k\,, \tag{4.11}$$

$$t_{21} = \left(\mu - \rho\frac{\alpha^2}{2}\,m\right)k\,, \tag{4.12}$$

and

$$t_{11} = t_{22} = t_{33} = -\,p\,. \tag{4.13}$$

The apparent viscosity ν, defined by

$$\nu \equiv \frac{t_{12}}{k} = \nu(H^0, k)\,, \tag{4.14}$$

is, for a fixed external field, a decreasing function of the rate of shear. For a fixed rate of shear, ν increases with the magnitude of the external field.

Equation (4.13) indicates that in para magnetic fluids described by this model the differences $t_{11} - t_{33}$ and $t_{22} - t_{33}$ are zero. Thus, experiments to determine the existence of normal stress effects [13] will test the validity of the constitutive theory (3.1), (3.4), (3.5) and (3.7).

For $\overline{H}^0 = 0$ equation (4.6) shows that $\overline{m}' \leqslant 0$ and an initial magnetization field will always vanish. In this case, the off-diagonal components of the steady stress are given by

$$t_{12} = t_{21} = \mu k\,. \tag{4.15}$$

When $0 < |\,\overline{H}^0\,| < 1$ the magnetization and the stress are time dependent.

Poiseuille flow.

McTague [6] has experimentally investigated the flow of a paramagnetic fluid through a thin tube for orientations of the external magnetic field parallel to and perpendicular to the axis of the tube. Here we analyse his experiment, in terms of our model, for the longitudinal orientation of the field.

In this case, we assume that the flow through a cylindrical tube of radius R is steady and parallel. In a cylindrical coordinate system the physical components of the velocity field are

$$u_{<r>} = u_{<\theta>} = 0\,, \quad u_{<z>} = u(r)\,, \tag{4.16}$$

with the conditions

$$u(R) = 0 \quad \text{and} \quad u'(0) = 0\,. \tag{4.17}$$

When the external magnetic field is perpendicular to the axis of the tube we do not anticipate that the velocity field has the simple form (4.16).

For the flow (4.16) the non-vanishing physical components of the stretching tensor and the spin tensor are

$$D_{<rz>} = D_{<zr>} = \tfrac{1}{2}\,u'\,, \tag{4.18}$$

and

$$W_{<rz>} = -\,W_{<zr>} = -\,\tfrac{1}{2}\,u' \tag{4.19}$$

respectively.

The external field $\mathbf{H}^0$ is

$$H^0_{<r>} = H^0_{<\theta>} = 0\,, \quad H^0_{<z>} = H^0 = \text{Cte}\,. \tag{4.20}$$

The magnetization field is taken to be

$$m_{<r>} = m\sin\varphi\,, \quad m_{<\theta>} = 0\,,$$
$$m_{<z>} = m\cos\varphi\,; \tag{4.21}$$

where $\varphi = \varphi(r, t)$ and $m = m(r, t)$.

The magnetization field (4.20) has a non-vanishing divergence and a component normal to the boundary. In this case, equations (2.8)-(2.11) indicate that a self-field will be present. For an cylinder of great length there will be only a radial component of the self-field $\mathbf{H}'$ given as a solution to (2.8) and (2.9)-(2.10) by

$$H'_{<r>} = -\,4\,\pi\rho m_{<r>} = -\,4\,\pi\rho m\sin\varphi \tag{4.22}$$

within the cylinder, and

$$H'_{<r>} = 0 \tag{4.23}$$

outside of the cylinder.

With the assumptions (4.17) and (4.20) and the self-field (4.22) the linear momentum equation (2.18) becomes

$$\frac{\partial t_{<rr>}}{\partial r} + \frac{1}{r}\frac{\partial t_{<r\theta>}}{\partial\theta} + \frac{\partial t_{<rz>}}{\partial z} + \frac{t_{<rr>} - t_{<\theta\theta>}}{r} = -\,\rho m_{<r>}\frac{\partial H'_{<r>}}{\partial r} \tag{4.24}$$

$$\frac{\partial t_{<\theta r>}}{\partial r} + \frac{1}{r}\frac{\partial t_{<\theta\theta>}}{\partial\theta} + \frac{\partial t_{<\theta z>}}{\partial z} + \frac{t_{<r\theta>} + t_{<\theta r>}}{r} = 0 \tag{4.25}$$

$$\frac{\partial t_{<zr>}}{\partial r} + \frac{1}{r}\frac{\partial t_{<z\theta>}}{\partial\theta} + \frac{\partial t_{<zz>}}{\partial z} + \frac{1}{r}\,t_{<zr>} = 0\,. \tag{4.26}$$

If we ignore, for the moment, the self-field (4.22) and use (4.19), (4.20) and (4.21) in equation (4.5) with $\beta = 0$, we obtain, when k is replaced by $-u'$ in the definitions $(4.8)_3$ and $(4.8)_4$, equations governing the magnetization of the same form as (4.6) and (4.7). Thus, in this approximation, a stable steady magnetization is possible if

$$|\overline{H}^0| = \left| \frac{H^0}{u' \alpha^2} \right| \geqslant 1$$

and is given by (4.9). Note that this condition governing the possibility of obtaining steady solutions depends upon $u' = u'(r)$ and may be satisfied at some values of r but not at others.

If we include the influence of the self-field the character of the steady solution is not essentially altered. Physically, the self-field directly surpresses the magnitude of the radial component of the magnetization and induces a couple tending to align the magnetization vector in the direction of flow. Thus the self-field effectively augments the external field and acts contrary to the orienting influence of the flow. In what follows, we shall assume that the steady field of magnetization is adaquately described by (4.9).

The non-vanishing physical components of the stress are

$$t_{<rz>} = \mu u' - \rho \frac{\alpha^2 m u'}{2}, \tag{4.27}$$

$$t_{<zr>} = \mu u' + \rho \frac{\alpha^2 m u'}{2}, \tag{4.28}$$

and

$$t_{<rr>} = t_{<\theta\theta>} = t_{<zz>} = -p. \tag{4.29}$$

With (4.26)-(4.28) equations (4.23)-(4.25) give

$$\frac{\partial p}{\partial r} = -2\pi\rho^2 \frac{\partial m^2_{<r>}}{\partial r}, \quad \frac{\partial p}{\partial \theta} = 0, \tag{4.30}$$

and

$$\frac{1}{r}\left(r\left(\mu + \rho\frac{\alpha^2 m}{2}\right)u'\right)' - \frac{\partial p}{\partial z} = 0; \tag{4.31}$$

or,

$$p = -2\pi\rho^2 m^2 \sin^2\varphi - az + \text{Cte}, \qquad a = \text{Cte} > 0, \tag{4.32}$$

and

$$\left(\mu + \rho\frac{\alpha^2 m}{2}\right)u' = -\frac{ar}{2}. \tag{4.33}$$

The integration constant in (4.33) is zero by $(4.17)_2$.

In principle, we may substitute for $m = m(u')$ from (4.9) using the modified definitions $(4.8)_2$ and $(4.8)_3$, solve (4.33) for u', and integrate the resulting differential form to obtain the steady velocity field.

However, equation (4.33) integrates directly in the limit of large external magnetic fields. The solution (4.9) gives $m \to m_S$ as $H^0 \to \infty$; so upon replacing m by m_S in (4.33) and integrating, we obtain

$$u = \frac{a}{2} \frac{1}{(2\mu + \rho\alpha^2 m_s)} (R^2 - r^2). \tag{4.34}$$

The volume rate of flow Q associated with this velocity field is

$$Q = \frac{\pi a}{(2\mu + \rho\alpha^2 m_s)} \frac{R^4}{4}. \tag{4.35}$$

Thus, for a magnetic fluid described by this model, the coefficient μ may be measured in the absence of an external magnetic field. With this, the coefficient α^2 may be determined by comparing the flow rates (for a fixed capillary and pressure gradient) in the saturated and in the unmagnetized fluid.

5. **The experiment of Moskowitz and Rosensweig [9].** — Moskowitz and Rosensweig subjected a paramagnetic fluid contained in a cylinder to the influence of a homogeneous magnetic field rotating uniformly in the circular cross-section. They observed that an apparent steady rotation of the fluid resulted. They determined how a measure of this induced rotational velocity field was related to the magnitude and angular velocity of the external field. In an attempt to explain this phenomenon Moskowitz and Rosensweig assumed that the paramagnetic material was an incompressible Newtonian fluid characterized by a symmetric stress tensor linear in the stretching tensor. They further assumed that a body couple was exerted on the fluid when the induced magnetization and the external magnetic field were not parallel. However, in this case, if the equation expressing the conservation of linear momentum is satisfied, the assumption of a symmetric stress tensor and the existence of external body couples are not consistent with the conservation of angular momentum.

Conversely, if a solution to the differential equation expressing the conservation of angular momentum is obtained it will be, in general, inconsistent with the conservation of linear momentum Moskowitz and Rosensweig (and earlier Tsvetkov [14] to explain a similar experiment performed with nematic liquid crystals) obtain a steady rotational velocity field as a solution to the balance law for angular momentum which fails to conserve linear momentum.

To repair this defect in approach we utilize the more general constitutive theory (3.1)-(3.7) which satisfies the angular momentum balance (2.19) identically. We attempt to determine whether, using this model, steady rotational flow can be induced in the cylindrical vessel by the homogeneous rotating magnetic field.

The physical components of the velocity field in cylindrical coordinates are assumed to be of the form

$$u_{<r>} = 0\,, \quad u_{<\theta>} = rW(r)\,, \quad u_{<z>} = 0\,. \tag{5.1}$$

Then the non-vanishing physical components of the stretching tensor and the rate of deformation tensor are

$$D_{<r\theta>} = D_{<\theta r>} = \tfrac{1}{2}\, rW' \tag{5.2}$$

and

$$W_{<r\theta>} = -\, W_{<\theta r>} = -\frac{1}{2\,r}(r^2\, W)'\,. \tag{5 3}$$

The homogeneous, rotating external field is

$$H^0_{<r>} = H\cos(\theta - \Omega t)\,,$$

$$H^0_{<\theta>} = -\, H\sin(\theta - \Omega t)\,, \quad H^0_{<z>} = 0\,. \tag{5.4}$$

The physical components of the magnetization field are assumed to have the same time dependence as the external field

$$m_{<r>} = m\cos(\theta - \Omega t - \varphi)\,,$$

$$m_{<\theta>} = -\, m\sin(\theta - \Omega t - \varphi)\,, \quad m_{<z>} = 0 \tag{5.5}$$

where $m = m(r)$ and $\varphi = \varphi(r)$.

Again we ignore the magnetization inertia ; and, in this case, we neglect entirely the effects of the self field introduced by the distribution of magnetization (5.5).

When viewed from a frame rotating with the external magnetic field

$$\bar{\theta} = \theta - \Omega t \tag{5.6}$$

the fields (5.1), (5.4) and (5.5) are independent of the time :

$$u_{<\bar{\theta}>} = (W - \Omega)\, r \tag{5.7}$$

$$D_{<r\bar{\theta}>} = D_{<\bar{\theta}r>} = \frac{1}{2}\, rW' \tag{5.8}$$

$$W_{<r\bar{\theta}>} = -\, W_{<\bar{\theta}r>}$$

$$= -\frac{1}{2\,r}\left(r^2(W - \Omega)\right)' \equiv -\frac{F}{2} \tag{5.9}$$

$$H^0_{<r>} = H^0\cos\bar{\theta}\,,$$

$$H^0_{<\bar{\theta}>} = -\, H^0\sin\bar{\theta} \tag{5.10}$$

$$m_{<r>} = m\cos(\bar{\theta} - \varphi)\,,$$

$$m_{<\bar{\theta}>} = -\, m\sin(\bar{\theta} - \varphi)\,. \tag{5.11}$$

Using equation (4.5) a straight-forward calculation shows that in this coordinate system the values of m and φ corresponding to the steady stable magnetization are, when k is replaced by F in the definitions $(4.8)_2$ and $(4.8)_3$, given by (4.9).

In this case, the non vanishing physical components of the stress are

$$t_{<r\theta>} = \mu rW' - \rho\frac{\alpha^2\, mF}{2} \tag{5.12}$$

$$t_{<\theta r>} = \mu rW' + \rho\frac{\alpha^2\, mF}{2} \tag{5.13}$$

$$t_{<rr>} = t_{<\theta\theta>} = t_{<zz>} = -\, p\,. \tag{5.14}$$

The divergence of the stress tensor, given in cylindrical coordinates by the left hand side of (4.24), (4.26), must be balanced by the radial centripital acceleration

$$\rho\frac{u^2_{<\theta>}}{r}\,.$$

Using (5.12), (5.14) we obtain

$$\frac{\partial p}{\partial r} = \rho rW^2\,, \tag{5.15}$$

$$\frac{\partial t_{<\theta r>}}{\partial r} + \frac{t_{<r\theta>} + t_{<\theta r>}}{r} - \frac{1}{r}\frac{\partial p}{\partial\theta} = 0\,, \tag{5.16}$$

and

$$\frac{\partial p}{\partial z} = 0\,. \tag{5.17}$$

Thus the single valued pressure in given by

$$p = \int_0^r \xi W(\xi)\,\mathrm{d}\xi + \mathrm{Cte} \tag{5.18}$$

and

$$\left(\mu rW' + \rho\frac{\alpha^2\, mF}{2}\right)' + 2\,\mu W' = 0 \tag{5.19}$$

or

$$\left(\mu + \rho\frac{\alpha^2\, m}{2}\right)\frac{1}{r}(r^2\, W)' - \rho\alpha^2\, m\Omega = b = \mathrm{Cte}\,. \tag{5.20}$$

Again, upon replacing m by m_S, we may integrate equation (5.20) to determine the flow field in the limiting case of a large external field. It is

$$u = rW = \frac{b + \rho\alpha^2\, m_s\,\Omega}{\left(\mu + \rho\dfrac{\alpha^2\, m_s}{2}\right)}\,\frac{r}{2} + \frac{c}{r} \tag{5.21}$$

where $c = \mathrm{Cte}$. However, requireing the velocity to be zero at the wall and finite at the center yields

$$u \equiv 0\,. \tag{5.22}$$

De Gennes [15] was the first to mention the apparent impossibility of inducing a rotating flow in a saturated paramagnetic fluid by means of an external magnetic field. The calculation here is in agreement with his remarks. Inversion and integration of equation (5.20) will determine whether a steady rotational velocity can be maintained in the general case for the model considered here. Judging from the behaviour in the limiting case, it seems inprobable that in the general

case a steady velocity will result. The experimental observations of Moskowitz and Rosensweig might be explained by examining in detail the effects of the self-field or by introducing a more complicated constitutive theory ; however, before multiplying the mathematical difficulties it may be wise to repeat the experiment under conditions which insure negligible interaction between grains and a homogeneous external magnetic field.

Acknowledgements. — I am grateful to Professor P. G. de Gennes and Dr. A. Martinet for their interest in this work. This research was sponsored by the Direction des Recherches et Moyens d'Essais.

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