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THE CONFIGURATIONS \((d + s)^N\) AND THE GROUP \(R_6\)

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Résumé. — On présente une méthode simple pour calculer les règles de branchement intervenant dans la suite des groupes :
\[ R_{24} \supset SU_3 \times Sp_{12} \supset SU_3 \times SU_3 \times (R_6 \supset R_5 \supset R_3) \]
utilisés dans la classification des états de configurations \((d + s)^N\), ainsi qu’une méthode d’analyse non ambiguë des produits de Kronecker dans \(R_6\) utilisant l’isomorphisme déjà connu entre \(R_6\) et \(U_4\). Plusieurs corrections sont apportées à des travaux antérieurs. On donne enfin une formule explicite des produits de Kronecker dans \(R_4\).

Abstract. — A simple method of computing the branching rules arising in the chain of groups :
\[ R_{24} \supset SU_3 \times Sp_{12} \supset SU_3 \times SU_3 \times (R_6 \supset R_5 \supset R_3) \]
used in classifying the states of the \((d + s)^N\) configurations is presented in addition to an unambiguous method for analyzing Kronecker products for \(R_6\) using the known isomorphism between \(R_6\) and \(U_4\). Several corrections to earlier works are noted. Finally, an explicit formula for the Kronecker products in \(R_4\) is given.

1. Introduction. — The problem of the group-theoretical description of the states and properties of the \((d + s)^N\) configurations was first considered by Elliot [1] who proposed to treat the set of configurations, for a given \(N\), as a single entity and to study their properties with respect to the chain of groups \(U_6 \supset R_6 \supset R_5 \supset R_3\). Recently, Feneuille [2, 3, 4] has made an exhaustive study of their properties using the chain of groups :
\[ R_{24} \supset SU_3 \times Sp_{12} \supset SU_3 \times SU_3 \times (R_6 \supset R_5 \supset R_3) \] (1)
thereby fully exploiting the concepts of quasi-spin and symplectic symmetry [5, 6].

Feneuille used the classical theory of groups [7, 8] to determine the relevant branching rules and to evaluate the Kronecker products of the various irreducible representations. These methods tend to be particularly cumbersome in all but the simplest cases and it is desirable to look for alternative methods. It is the purpose of this paper to show that the branching rules may be simply determined by a theorem that is a direct consequence of Littlewood’s algebra of plethysm [9, 10] and to discuss the computation of the Kronecker products of the irreducible representations of \(R_6\), noting several corrections to Feneuille’s computations. Throughout this article, we shall follow the well-defined notation of Littlewood [11].

2. Branching rules and plethysm. — The basic properties of the algebra of plethysm, as it bears upon problems in atomic spectroscopy, have recently been outlined by Smith and Wybourne [12, 13].

Consider a group \(G\) which corresponds to the full linear group, or to a proper subgroup of the full linear group, and let us suppose \(G\) contains a subgroup \(H\). The characters of the irreducible representations of \(G\) and \(H\) will be labelled by various partitions of integers. The character of the irreducible representation of \(G\) corresponding to the unary partition (1) may always
be trivially decomposed into the characters of its subgroup $H$, the entries in table I being specific to Feneuille's problem.

Let $[\lambda]$ represent the character of $G$ labelled by the partition $(\lambda)$ and $[\rho]$ the character of $H$ labelled by the partition $(\rho)$. Then, the following theorem follows directly from Littlewood's algebra of plethysm and the duality that exists between the symmetric and full linear groups \[11, 14\]:

**Theorem:** If under the restriction $G \to H$ the unary character $[1]$ decomposes as:

$$[1] \to [1/2]' \times 1$$

then the character $[\lambda]$ of $G$ decomposes into the characters of $H$ to yield the characters contained in the plethysm:

$$[[\lambda] + [\rho] + \ldots + [\omega]] \circ [\lambda]. \quad (2)$$

The plethysm of eq. (2) may be evaluated in the usual manner \[9, 13\] by first expressing the characters of $G$ and $H$ in terms of $S$-functions \[11\], evaluating the plethysm to give $S$-functions pertaining to the characters of $H$ and then expressing them in terms of the characters of $H$ to give the final result. The following two examples should suffice to demonstrate the application of the theorem.

Consider the decomposition of the $[11]$ representation of $R_{24}$ into those of $SU_2 \times Sp_{12}$. Inspection of table I shows that the unary character $[1]$ of $R_{24}$ decomposes upon restriction to $SU_2 \times Sp_{12}$ as:

$$[1] \to [1/2]' \times 1$$

Expressing the characters in terms of $S$-functions, we have:

$$\{1\} \to \{1\}' \{1\}.$$

Since the character $[11]$ of $R_{24}$ is the $S$-function $[11]$, it follows from eq. (2) that the relevant plethysm is:

$$\{1\}' \{1\} \circ [11] = \{11\}' \{2\} + \{2\}' \{11\}.$$

Converting the $S$-functions into characters of $SU_2$ and $Sp_{12}$ yields the result:

$$[11] \to 1^2 + 3^2$$

where the superscript attached to the symplectic characters are the multiplicities of the representations of $SU_2$.

As the second example, consider the decomposition of the character $[22]$ of $R_6$ upon the restriction $R_6 \to R_5$. From table I we conclude that the relevant plethysm, in terms of $S$-functions, is:

$$\{(1) + \{0\}\} \circ \{(22) - \{0\}\}$$

$$= \{22\} + \{21\} - \{1\} - \{0\}.$$ 

Expressing the $S$-functions in terms of the characters of $R_5$ yields immediately the result:

$$[22] \to [22] + [21] + [2].$$

3. **Kronecker products for $R_6$.** — Feneuille \[3, 4\] has discussed the symmetry properties of the Coulomb interaction and scalar three-particle effective operators under the chain of groups of eq. (1). In particular, he has shown that operators may be constructed that have well-defined transformation properties under the group of rotations in six dimensions, $R_6$. If the symbol $W$ is used to designate the irreducible representations of $R_6$, then the determination of the number of times $C(WW'W'')$ the identity representation occurs in the triple Kronecker product $W \times W' \times W''$, or equivalently the number of times $W''$ occurs in $W \times W'$, plays a central role in the detailed numerical analysis of the matrix elements of operators acting in the $(d + s)^N$ configurations. It is to the analysis of these triple Kronecker products we shall be turning our attention after first making a few remarks on the properties of the irreducible representations of the group $R_6$.

The decomposition of the Kronecker products of the irreducible representations of the orthogonal or symplectic groups presents no problems and Littlewood \[15\] has presented an elegant, yet very simple, solution. Special difficulties arise in the case of the rotation group $R_n$ in an even number of dimensions. If $n = 2u$ and in the partition $(1..1, 1..2, \ldots, 1..u)$ labeling the character $[\lambda_1, \lambda_2, \ldots, \lambda_u]$ of $O_n$ we have $\lambda_u = 0$ then the simple characters of the orthogonal group $O_n$ are also simple characters of $R_n$, but if $\lambda_u \neq 0$ the representation of $O_n$ is self-associated and separates into two irreducible representations of $R_n$.

Thus, the corresponding character $[\lambda]$ of $O_n$ may be expressed as the sum of two simple conjugate characters $[\lambda_1] + [\lambda_2]$ of $R_n$. The two representations are interchanged by a transformation of negative determinant. The distinction between these two representations is indicated by making the separation of the character $[\lambda_1, \lambda_2, \ldots, \lambda_u]$ of $O_n$ under restriction to $R_n$ according to the equation \[16\]:

$$[\lambda_1, \lambda_2, \ldots, \lambda_u] \to [\lambda_1, \lambda_2, \ldots, \lambda_u] + [\lambda_1, \lambda_2, \ldots, -\lambda_u]. \quad (3)$$

Separations of this type are given by Feneuille \[2\].

Clearly in forming Kronecker products of irreducible representations, a careful distinction must be made between those of the group $O_{2u}$ and those...
Thus table X of Feneuille’s paper [4] is really appropriate to the group $O_6$ and not $R_6$.

The analysis of the Kronecker products of the irreducible representations of $R_6$ may be most readily made in terms of the basic spin representation $[1/2 1/2 \pm 1/2]$ of $R_6$ and making use of the isomorphism that exists between the characters of $R_6$ and those of the four-dimensional unitary group [16, 17].

It is not difficult to establish the correspondences [16]:

$$[1/2 1/2 1/2] \otimes \{abc\} = [1/2(a + b - c), 1/2(a - b + c), 1/2(a - b - c)].$$

and:

$$[1/2 1/2 - 1/2] \otimes \{abc\} = [1/2(a + b - c), 1/2(a - b + c), 1/2(b + c - a)].$$

where $\{abc\}$ is an $S$-function for the unitary group in four dimensions. Equivalently:

$$[1/2 1/2 1/2] \otimes \{a + b, a - c, b - c\} = [abc]$$

$$[1/2 1/2 - 1/2] \otimes \{a + b, a + c, b + c\} = [abc].$$

The basic multiplication law for the operation of plethysm:

$$(A \otimes B) (A \otimes C) = A \otimes (BC)$$

leads directly to the following expressions for the Kronecker product of the characters of $R_6$:

$$[abc] \times [def] = [1/2 1/2 1/2] \otimes \{(a + b, a - c, b - c) \{d + e, d - f, e - f\}\}$$

and:

$$[abc] \times [def] = [1/2 1/2 - 1/2] \otimes \{(a + b, a + c, b + c) \{d + e, d + f, e + f\}\}.$$

Thus, the Kronecker products of the characters of both the true and the spin representations of $R_6$ may be unambiguously analyzed by the use of either eq. (9) or (10) followed by multiplication of the $S$-functions appearing in the right hand side products. The resulting $S$-functions having more than four parts are discarded and those with four parts reduced to three or less parts by use of the equivalence:

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \equiv \{\lambda_1 - \lambda_4, \lambda_2 - \lambda_4, \lambda_3 - \lambda_4, 0\}.$$  

These remaining $S$-functions are then converted back into characters of $R_6$ by use of eq. (4) or (5) as appropriate.

The choice of whether to use eq. (9) or (10) is usually made on the basis of the simplicity of the multiplication of the resultant $S$-functions appearing in the product. The following two results may be readily deduced:

If:

$$[abc] \times [def] = \Sigma[xyz]$$

then:

$$[ab - c] \times [de - f] = \Sigma[xyz]$$

and if:

$$[abc] \times [def] = \Sigma[pqr]$$

then:

$$[ab - c] \times [def] = \Sigma[pq - r].$$

The multiplication of the $S$-functions may be checked by calculating the degrees of the corresponding representations for the symmetric groups using the method of hook graphs [14, 18] and noting that if:

$$\{\rho\} \times \{\sigma\} = \Sigma g_{\rho\sigma \tau\nu} \{\upsilon\}$$

then:

$$f^{(\rho)}(\sigma) = \frac{(m+n)!}{m! n!} f^{(\rho)} f^{(\sigma)} = \Sigma g_{\rho\sigma \tau\nu} f^{(\upsilon)}$$

where $f^{(\rho)}$, $f^{(\sigma)}$ and $f^{(\upsilon)}$ are the degrees of the representations $\{\rho\}$, $\{\sigma\}$ and $\{\upsilon\}$ of the symmetric groups on $m$, $n$, and $m + n$ symbols respectively. In practice, the computation is frequently simplified by considering only those parts of the multiplication of $S$-functions that give rise to $S$-functions involving not more than four parts and then checking the final results in terms of the dimensional formula for the irreducible representations of $R_6$.

As an example of the techniques outlined, consider the Kronecker products involving the irreducible representations $[111]$ and $[11-1]$ of $R_6$. It follows from eq. (4) and (9) that:

$$[111] \times [111] = [1/2 1/2 1/2] \otimes \{(2) \{2\}\} = [1/2 1/2 1/2] \otimes \{(4) + \{22\} + \{31\}\}. $$

Using eq. (4) gives the final result:

$$[111] \times [111] = [200] + [211] + [222].$$

Equation (12) then establishes the result:

$$[11-1] \times [11-1] = [200] + [21-1] + [22-2].$$

The Kronecker product $[111] \times [11-1]$ is obtained by use of eq. (9) followed by eq. (11) and (4) to yield:

$$[111] \times [11-1] = [000] + [110] + [220].$$

The multiplication of two $S$-functions of weights $m$ and $n$ can only yield $S$-functions of weight $(m+n)$. Thus, if in the formation of the Kronecker product of two characters of $R_6$ only $S$-functions of weight $m + n$ arise, then the Kronecker product can only be decomposed into representations of $R_6$ whose $S$-function representatives in $U_4$ are of weight $m + n$. For example, the Kronecker product $[210] \times [220]$ involves the multiplication of the $S$-functions $\{321\} \{422\}$ which must yield $S$-functions of weight 14, whereas the character $[310]$ can only be derived from the $S$-functions $\{431\}$, $\{542\}$, $\{653\}$, etc., which are of weights 8, 12, 16, etc., and hence we may conclude that $C([210] \times [220] [310]) = 0$.

The method outlined for calculating the Kronecker products for $R_6$ has the advantage of yielding results
### TABLE II

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### THE CONFIGURATION \((d + s)^N\) AND THE GROUP \( R_4 \)

185
TABLE II (continued)

c(ww' [420])

\[
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\end{array}
\]

The striking simplifications that arise in recognizing the isomorphism that exists between \( R_6 \) and \( U_4 \) evidently have no analogues in the groups \( R_{2v} \), where \( v \geq 4 \), but do point to a need for a careful analysis of the properties of the even-dimensional rotation groups. For example, once the isomorphism between \( R_4 \) and the double binary group is recognized [11] it is a simple matter to derive the following expression for the Kronecker products of the characters of \( R_4 \):

\[
[ab] \times [cd] = \sum_{\alpha=0}^{t} \sum_{\beta=0}^{u} [a + \epsilon + \alpha - \beta, b + d - \alpha + \beta]
\]

(15)

where \( t \) is the lesser of \((a + b)\) and \((c + d)\), and \( u \) is the lesser of \((a - b)\) and \((c - d)\). This result holds for both the true and the spin representations of \( R_4 \).

Acknowledgements. — We are grateful to the University of Canterbury Computing Center for use of their IBM 360-44 computer, to Julian Brown for assistance in developing the programmes and to Serge Feneuille for several profitable discussions.

REFERENCES