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There are many practical representations of probability families that make them easier to handle in applications. Among them are random sets, possibility distributions, Ferson's p-boxes [4] and Neumaier's clouds [6]. Both for theoretical and practical considerations, it is very useful to know whether one representation can be translated into or approximated by other ones. We first briefly recall formalisms and existing results, before exhibiting relationships between all these representations. In this note, which is a summary of an extended forthcoming paper, we restrict ourselves to representations on a finite set $X = \{x_1, x_2, \ldots, x_n\}$ of n elements.

1 Formalisms

Possibility distribution A possibility distribution is a mapping $\pi : X \to [0,1]$ representing incomplete information about an ill-known parameter v. two dual measures (respectively the possibility and necessity measures) can be defined : $\Pi(A) = \sup_{x \in A} \pi(x)$ and $N(A) = 1 - \Pi(A^c)$. To any normal possibility distribution (π such that $\pi(x) = 1$ for some $x \in X$) can be associated a probability family $\mathcal{P}_\pi$ s.t. $\mathcal{P}_\pi = \{P, \forall A \subseteq X$ measurable, $N(A) \leq P(A) \leq \Pi(A)\}$.

Random Set A random set is defined here as a probability distribution on the power set of X, namely $m : 2^X \to [0,1]$. $m(A)$ is the probability that all is known about v is that $v \in A$. Two dual measures (respectively the plausibility and belief measures) can be defined : $Pl(A) = \sum_{E, E \cap A \neq \emptyset} m(E)$ and $Bel(A) = 1 - Pl(A^c) = \sum_{E, E \subseteq A} m(E)$. To any random set m can be associated a probability family $\mathcal{P}_m$ s.t. $\mathcal{P}_m = \{P | \forall A \subseteq X$ measurable, $Bel(A) \leq P(A) \leq Pl(A)\}$.

Generalized p-box A p-box is usually defined on the real line by a pair of cumulative distributions $[\underline{F}, \overline{F}]$, defining the probability family $\mathcal{P}_{[\underline{F},\overline{F}]} = \{P | \underline{F}(x) \leq F(x) \leq \overline{F}(x) \quad \forall x \in \Re\}$. The notion of cumulative distribution on the real line is based on a natural ordering of numbers. In order to generalize this notion to arbitrary finite sets, we need to define a weak order relation $\leq_R$ on this space. Given $\leq_R$, an R-downset is of the form $\{x_i : x_i \leq_R x\}$, and denoted $(x]_R$. A generalized R-cumulative distribution is defined as the function $F_R : X \to [0,1]$ s.t. $F_R(x) = \Pr((x]_R)$, where Pr is a probability measure on X. We can now define a generalized p-box as a pair $[\underline{F}_R(x), \overline{F}_R(x)]$ of generalized

cumulative distributions defining a probability family $\mathcal{P}_{[\underline{F}_R(x),\overline{F}_R(x)]} = \{P|\forall x, \underline{F}_R(x) \leq F_R(x) \leq \overline{F}_R(x)\}$. Generalized P-boxes can also be represented by a set of constraints

$$\alpha_i \leq P(A_i) \leq \beta_i \qquad i = 1,\ldots,n \tag{1}$$

where $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq 1$, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \leq 1$ and $A_i = (x_i]_R, \forall x_i \in X$ with $x_i \leq_R x_j$ iff $i < j$ (sets A_i form a sequence of nested confidence sets $\emptyset \subset A_1 \subset A_2 \subset \ldots \subset A_n \subset X$).

Cloud Formally, a cloud is described by an Interval-Valued Fuzzy Set (IVF) s.t. $(0,1) \subseteq \cup_{x\in X} F(x) \subseteq [0,1]$, where $F(x)$ is an interval $[\delta(x),\pi(x)]$. A cloud is called *thin* when the two membership functions coincide ($\delta = \pi$). It is called *fuzzy* when the lower membership function δ is 0 everywhere. Let α_i be a sequence of α-cuts s.t. $1 = \alpha_0 > \alpha_1 > \alpha_2 > \ldots > \alpha_n > \alpha_{n+1} = 0$ with A_i, B_i the corresponding α-cut of fuzzy sets π and δ ($A_i = \{x_i, \pi(x_i) > \alpha_{i+1}\}$ and $B_i = \{x_i, \delta(x_i) \geq \alpha_{i+1}\}$). Then, a random variable x is in a cloud if it satisfies the constraints

$$P(B_i) \leq 1 - \alpha_i \leq P(A_i) \text{ and } B_i \subseteq A_i \quad i = 1,\ldots,n. \tag{2}$$

2 Generalized p-boxes

First, let us notice that a generalized upper cumulative distribution $\overline{F}_R$ can be seen as a possibility distribution π_R dominating a probability distribution Pr, since it is a maxitive measure s.t. $\max_{x\in A} \overline{F}_R(x) \geq \Pr(A), \forall A \subseteq X$. In [2], we have shown the following results

Proposition 1. *A family $\mathcal{P}_{[\underline{F}_R(x),\overline{F}_R(x)]}$ described by a generalized P-box can be encoded by a pair of possibility distributions π_1, π_2 s.t. $\mathcal{P}_{[\underline{F}_R(x),\overline{F}_R(x)]} = \mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2}$ with $\pi_1(x) = \overline{F}_R(x)$ and $\pi_2(x) = 1 - \underline{F}_R(x)$*

Proposition 2. *A family $\mathcal{P}_{[\underline{F}_R(x),\overline{F}_R(x)]}$ described by a generalized P-box can be encoded by a random set m s.t. $\mathcal{P}_{[\underline{F}_R(x),\overline{F}_R(x)]} = \mathcal{P}_m$.*

If X is the real line, this last proposition reduces to results already shown in [5].

3 Cloud

In [3], the following relationship linking clouds to possibility distributions is shown

Proposition 3. *A probability family $\mathcal{P}_{\delta,\pi}$ described by the cloud (δ,π) is equivalent to the family $\mathcal{P}_\pi \cap \mathcal{P}_{1-\delta}$ described by the two possibility distributions π and $1-\delta$.*

This result already suggests that clouds and generalized p-boxes are somewhat related. To lay bare this relationship, it is useful to introduce the following special case of clouds:

Definition 1. *A cloud is said to be* comonotonic *if distributions π and δ are comonotonic. If it is not the case, a cloud is called non-comonotonic.*

Fig. 1. Illustration of clouds.

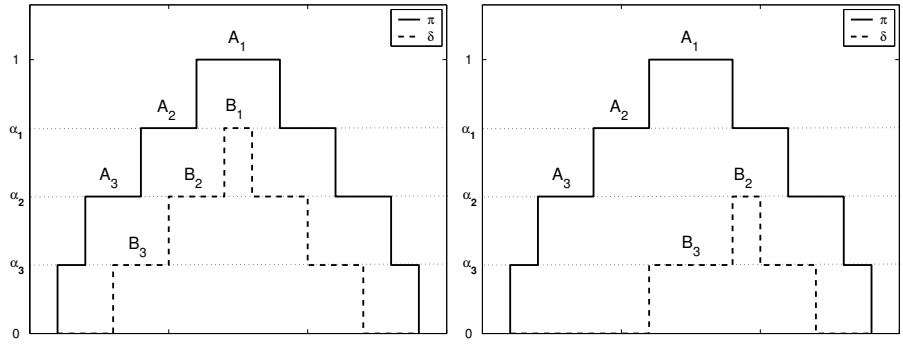

Fig 1.A Comonotonic cloud

Fig 1.B Non-comonotonic cloud

Remark 1. *Thin* and *fuzzy* clouds are special cases of comonotonic clouds.

Figure 1 illustrates comonotonic and non-comonotonic clouds. The two following propositions show why it is useful to make this distinction.

Proposition 4. *The probability family $\mathcal{P}_{\delta,\pi}$ induced by a comonotonic cloud is equivalent to a generalized p-box and can thus be encoded through a random set.*

Proof (sketch). Since comonotonicity imply that sets A_i, B_i $i = 1, \ldots, n$ form a complete sequence of nested sets, one can always retrieve the structure of a generalized p-box from a comonotonic cloud by mapping constraints of the form of equation (2) into constraints of the form of equation (1).

Proposition 5. *The lower probability of the probability family $\mathcal{P}_{\delta,\pi}$ induced by a non-comonotonic cloud is not even a 2-monotone capacity (i.e. $\exists A, B \subset X$ s.t. $\underline{P}(A \cap B) + \underline{P}(A \cup B) \leq \underline{P}(A) + \underline{P}(B)$)*

Proof (sketch). For each non-comonotonic cloud, there exist two sets B_i, A_j with $i > j$ and s.t. $B_i \cap A_j \neq \emptyset$, $B_i \not\subseteq A_j$ and $A_j \not\subseteq B_i$. Using a result from Chateauneuf [1] and the fact that $\mathcal{P}_{\delta,\pi}$ is the intersection of two families corresponding to belief functions, we can show that the following inequality holds

$$\underline{P}(A_j \cap B_i) + \underline{P}(A_j \cup B_i) < \underline{P}(A_j) + \underline{P}(B_i)$$

and this concludes the proof.

Remark 2. δ must not be trivially reduced to a single set B_n s.t. $B_n \cap A_{n-1} = \emptyset$, otherwise the cloud can still be encoded by a random set (and is thus a capacity of order ∞), even if it is no longer equivalent to a generalized p-box.

To our knowledge, non-comonotonic clouds are the only simple models (in the finite case, we need at most $2|X|$ values to fully specify a cloud) of imprecise probabilities that induce capacities that are not 2-monotone.

Let us also notice that if proposition 4 holds in the continuous case, we have a nice way to characterize probability families induced by comonotonic clouds. Namely, a continuous belief function [7] with uniform mass density, whose focal elements would be disjoint sets of the form $[x(\alpha),u(\alpha)]\cup[v(\alpha),y(\alpha)]$ where $\{x:\pi(x)\geq\alpha\}=[x(\alpha),y(\alpha)]$ and $\{x:\delta(x)\geq\alpha\}=[u(\alpha),v(\alpha)]$. In particular, for *thin* clouds, focal sets would be doubletons of the form $\{x(\alpha),y(\alpha)\}$.

Computing upper and lower probability bounds $\underline{P}(A),\overline{P}(A)$ of non-comonotonic clouds appear not to be so easy a task. Thus, one may wish to work with inner or outer approximations of the family $\mathcal{P}_{\delta,\pi}$. The two following propositions provide such bounds, which are easy to compute.

Proposition 6. *If $\mathcal{P}_{\delta,\pi}$ is the probability family described by the cloud (δ,π) on a referential X, then, the following bounds provide an outer approximation :*

$$\max(N_\pi(A),N_\delta(A))\leq P(A)\leq\min(\Pi_\pi(A),\Pi_\delta(A))\ \forall A\subset X \tag{3}$$

Remark 3. These bounds are the ones considered by Neumaier in [6], and the fact that they are outer approximations explain why they are poorly related to random sets or to Walley's natural extensions.

But clouds can be approximated by random sets:

Proposition 7. *Given sets $\{B_i,A_i,i=1,\ldots,n\}$ and the corresponding confidence values α_i, associated to the distributions (δ,π) of a cloud, the belief and plausibility measures of the random set s.t. $m(A_i\setminus B_{i-1})=\alpha_{i-1}-\alpha_i$ are inner approximations of $\mathcal{P}_{\delta,\pi}$.*

Remark 4. If the cloud is comonotonic, this random set is the one corresponding to the family $\mathcal{P}_{\delta,\pi}$

Our results show that clouds generalize p-boxes and possibility distributions as representations of imprecise probabilities, but are generally not a special case of random set. Even if they look more complex to deal with than p-boxes and possibility distributions, clouds are more expressive and remain relatively simple representations. Moreover, results presented here may allow for easier computations in various cases. We thus think that using clouds can be potentially interesting in various applications, but that more work is needed to fully assess this potential.

4 Open questions and problems

There remain many open questions and problems related to clouds, some of them being already emphasized by Neumaier. Among them are :

- Testing the mathematical and the computational tractability of clouds
- Testing clouds as descriptive models of uncertainty
- Extending existing results to more general frameworks (unbounded variables, lower/upper previsions)
- Studying under which operations the cloud representation is preserved (joint distributions, fusion, extension, ...)

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