Quadratic discrete Fourier transform and mutually unbiased bases

Maurice Robert Kibler

0

Quadratic discrete Fourier transform and mutually unbiased bases

Maurice R. Kibler
Université de Lyon, Université Claude Bernard et CNRS (IPNL et IN2P3)
France

1. Introduction, motivations and notations

The use of the discrete Fourier transform (DFT) is quite spread in many fields of physical sciences and engineering as for instance in signal theory. This chapter deals with a quadratic extension of the DFT and its application to quantum information.

From a very general point of view, the DFT can be defined as follows. Let us denote (x_0, x_1, ..., x_{d-1}) a collection of d complex numbers. The transformation

$$x \equiv (x_0, x_1, \ldots, x_{d-1}) \mapsto \tilde{x} \equiv (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{d-1}) \tag{1}$$

defined by

$$\tilde{x}_\alpha = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{i2\pi\alpha n/d} x_n, \quad \alpha = 0, 1, \ldots, d-1 \tag{2}$$

will be referred to as the DFT of x.

Equation (2) can be transcribed in finite quantum mechanics. In that case, x is often replaced by an orthonormal basis $\{|n\rangle : n = 0, 1, \ldots, d-1\}$ of the Hilbert space $\mathbb{C}^d$ (with an inner product noted $\langle\,|\,\rangle$ in Dirac notations). The analog of (2) reads

$$|\tilde{\alpha}\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{i2\pi\alpha n/d} |n\rangle, \quad \alpha = 0, 1, \ldots, d-1 \tag{3}$$

Equation (3) makes it possible to pass from the orthonormal basis $\{|n\rangle : n = 0, 1, \ldots, d-1\}$ to another orthonormal basis $\{|\tilde{\alpha}\rangle : \alpha = 0, 1, \ldots, d-1\}$ and vice versa since

$$\langle n|n'\rangle = \delta(n, n') \Leftrightarrow \langle\tilde{\alpha}|\tilde{\alpha'}\rangle = \delta(\alpha, \alpha') \tag{4}$$

The transformation (3) defines a quantum DFT. In the last twenty years, the notion of quantum DFT has received a considerable attention in connection with finite quantum mechanics and quantum information (Vourdas, 2004).

As an interesting property, the two bases $\{|n\rangle : n = 0, 1, \ldots, d-1\}$ and $\{|\tilde{\alpha}\rangle : \alpha = 0, 1, \ldots, d-1\}$, connected via a quantum DFT, constitute a couple of unbiased bases. Let us recall that two distinct orthonormal bases

$$B_a = \{|a\alpha\rangle : \alpha = 0, 1, \ldots, d-1\} \tag{5}$$

and

$$B_b = \{|b\beta\rangle : \beta = 0, 1, \ldots, d-1\} \tag{6}$$

of the space $\mathbb{C}^d$ are said to be unbiased if and only if

$$\forall \alpha = 0, 1, \ldots, d-1, \ \ \forall \beta = 0, 1, \ldots, d-1 \ : \ |\langle a\alpha | b\beta \rangle| = \frac{1}{\sqrt{d}} \tag{7}$$

The unbiasedness character of the bases $\{|n\rangle : n = 0, 1, \ldots, d-1\}$ and $\{|\tilde{\alpha}\rangle : \alpha = 0, 1, \ldots, d-1\}$ then follows from

$$\langle n|\tilde{\alpha}\rangle = \frac{1}{\sqrt{d}} e^{i2\pi\alpha n/d} \ \ \Rightarrow \ \ |\langle n|\tilde{\alpha}\rangle| = \frac{1}{\sqrt{d}} \tag{8}$$

which is evident from (3).
The determination of sets of mutually unbiased bases (MUBs) in $\mathbb{C}^d$ is of paramount importance in the theory of information and in quantum mechanics. Such bases are useful in classical information (Calderbank et al., 1997), quantum information (Cerf et al., 2002) as well as for the construction of discrete Wigner functions (Gibbons et al., 2004), the solution of the mean King problem (Englert & Aharonov, 2001) and the understanding of the Feynman path integral formalism (Tolar & Chadzitaskos, 2009). It is well-known that the number N_{MUB} of MUBs in $\mathbb{C}^d$ is such that $3 \leq N_{MUB} \leq d+1$ (Durt et al., 2010). Furthermore, the maximum number $N_{MUB} = d+1$ is reached when d is a prime number or a power of a prime number (Calderbank et al., 1997; Ivanović, 1981; Wootters & Fields, 1989). However, when d is not a prime number or more generally a power of a prime number, it is not known if the limiting value $N_{MUB} = d+1$ is attained. In this respect, in the case $d = 6$, in spite of an enormous number of works it was not possible to find more than three MUBs (see for example (Bengtsson et al., 2007; Brierley & Weigert, 2009; Grassl, 2005)).
The main aim of this chapter is to introduce and discuss a generalization of the DFTs defined by (2) and (3) in order to produce other couples of MUBs. The generalization will be achieved by introducing quadratic terms in the exponentials in (2) and (3) through the replacement of the linear term αn by a quadratic term $\xi n^2 + \eta n + \zeta$ with ξ, η and ζ in $\mathbb{R}$. The resulting generalized DFT will be referred to as a quadratic DFT.
The material presented in this chapter is organized in the following way. Section 2 is devoted to the study of those aspects of the representation theory of the group $SU(2)$ in a nonstandard basis which are of relevance for the introduction of the quadratic DFT. The quadratic DFT is studied in section 3. Some applications of the quadratic DFT to quantum information are given in section 4.
Most of the notations in this chapter are standard. Some specific notations shall be introduced when necessary. As usual, $\delta_{a,b}$ stands for the Kronecker delta symbol of a and b, i for the pure imaginary, $\overline{z}$ for the complex conjugate of the number z, $A^\dagger$ for the adjoint of the operator A, and I for the identity operator. We use $[A, B]_q$ to denote the q-commutator $AB - qAB$ of the operators A and B; the commutator $[A, B]_{+1}$ and anticommutator $[A, B]_{-1}$ are noted simply $[A, B]$ and $\{A, B\}$, respectively, as is usual in quantum mechanics. Boldface letters are reserved for squared matrices ($\mathbf{I_d}$ is the d-dimensional identity matrix). We employ a notation of type $|\psi\rangle$, or sometimes $|\psi)$, for a vector in an Hilbert space and we denote $\langle\phi|\psi\rangle$ and $|\phi\rangle\langle\psi|$ respectively the inner and outer products of the vectors $|\psi\rangle$ and $|\phi\rangle$. The symbols $\oplus$ and $\ominus$ refer to the addition and subtraction modulo d or $2j+1$ (with $d = 2j+1 = 2, 3, 4, \ldots$

depending on the context) while the symbol $\otimes$ serves to denote the tensor product of two vectors or of two spaces. Finally $\mathbb{N}$, $\mathbb{N}^*$ and $\mathbb{Z}$ are the sets of integers, strictly positive integers and relative integers; $\mathbb{R}$ and $\mathbb{C}$ the real and complex fields; and $\mathbb{Z}/d\mathbb{Z}$ the ring of integers $0, 1, \ldots, d-1$ modulo d.

2. A nonstandard approach to $SU(2)$

2.1 Quon algebra

The idea of a quon takes its origin in the replacement of the commutation (sign $-$) and anti-commutation (sign $+$) relations

$$a_- a_+ \pm a_+ a_- = 1 \tag{9}$$

of quantum mechanics by the relation

$$a_- a_+ - q a_+ a_- = f(N) \tag{10}$$

where q is a constant and $f(N)$ an arbitrary fonction of a number operator N. The introduction of q and $f(N)$ yields the possibility to replace the harmonic oscillator algebra by a deformed oscillator algebra. For $f(N) = 1$, the case $q = -1$ corresponds to fermion operators (describing a fermionic oscillator) and the case $q = +1$ to boson operators (describing a bosonic oscillator). The other possibilities for q and $f(N) = 1$ correspond to quon operators. We shall be concerned here with a quon algebra or q-deformed oscillator algebra for q a root of unity.

Definition 1. *The three linear operators a_-, a_+ and N_a such that*

$$[a_-, a_+]_q = I, \quad [N_a, a_+] = a_+, \quad [N_a, a_-] = -a_-, \quad (a_+)^k = (a_-)^k = 0, \quad (N_a)^\dagger = N_a \tag{11}$$

where

$$q = \exp\left(\frac{2\pi i}{k}\right), \quad k \in \mathbb{N} \setminus \{0, 1\} \tag{12}$$

define a quon algebra or q-deformed oscillator algebra denoted $A_q(a_-, a_+, N_a)$ or simply $A_q(a)$. The operators a_- and a_+ are referred to as quon operators. The operators a_-, a_+ and N_a are called annihilation, creation and number operators, respectively.

Definition 1 differs from the one by Arik and Coon (Arik & Coon, 1976) in the sense that we take q as a primitive kth root of unity instead of $0 < q < 1$. In Eq. (12), the value $k = 0$ is excluded since it would lead to a non-defined value of q. The case $k = 1$ must be excluded too since it would yield trivial algebras with $a_- = a_+ = 0$. We observe that for $k = 2$ (i.e., for $q = -1$), the algebra $A_{-1}(a)$ corresponds to the ordinary fermionic algebra and the quon operators coincide with the fermion operators. On the other hand, we note that in the limiting situation where $k \to \infty$ (i.e., for $q = 1$), the algebra $A_1(a)$ is nothing but the ordinary bosonic algebra and the quon operators are boson operators. For k arbitrary, N_a is generally different from $a_+ a_-$; it is only for $k = 2$ and $k \to \infty$ that $N_a = a_+ a_-$. Note that the nilpotency relations $(a_+)^k = (a_-)^k = 0$, with k finite, are at the origin of k-dimensional representations of $A_q(a)$ (see section 2.2).

For arbitrary k, the quon operators a_- and a_+ are not connected via Hermitian conjugation. It is only for $k = 2$ or $k \to \infty$ that we may take $a_+ = (a_-)^\dagger$. In general (i.e., for $k \neq 2$ or $k \not\to \infty$), we have $(a_\pm)^\dagger \neq a_\mp$. Therefore, it is natural to consider the so-called k-fermionic algebra Σ_q with the generators a_-, a_+, $a_+^+ = (a_+)^\dagger$, $a_-^+ = (a_-)^\dagger$ and N_a (Daoud et al., 1998). The defining relations for Σ_q correspond to the ones of $A_q(a_-, a_+, N_a)$ and $A_{\bar{q}}(a_+^+, a_-^+, N_a)$ complemented by the relations

$$a_- a_+^+ - q^{-\frac{1}{2}} a_+^+ a_- = 0, \quad a_+ a_-^+ - q^{\frac{1}{2}} a_-^+ a_+ = 0 \tag{13}$$

Observe that for $k = 2$ or $k \to \infty$, the latter relation corresponds to an identity. The operators a_-, a_+, a_+^+ and a_-^+ are called k-fermion operators and we also use the terminology k-fermions in analogy with fermions and bosons. They clearly interpolate between fermions and bosons. In passing, let us mention that the k-fermions introduced in (Daoud et al., 1998) share some common properties with the parafermions of order $k-1$ discussed in (Beckers & Debergh, 1990; Durand, 1993; Khare, 1993; Klishevich & Plyushchay, 1999; Rubakov & Spiridonov, 1988). The k-fermions can be used for constructing a fractional supersymmetric algebra of order k (or parafermionic algebra of order $k-1$). The reader may consult (Daoud et al., 1998) for a study of the k-fermionic algebra Σ_q and its application to supersymmetry.

2.2 Quon realization of $su(2)$

Going back to quons, let us show how the Lie algebra $su(2)$ of the group $SU(2)$ can be generated from two quon algebras. We start with two commuting quon algebras $A_q(a)$ with $a = x, y$ corresponding to the same value of the deformation parameter q. Their generators satisfy Eqs. (11) and (12) with $a = x, y$ and $[X, Y] = 0$ for any X in $A_q(x)$ and any Y in $A_q(y)$. Then, let us look for Hilbertian representations of $A_q(x)$ and $A_q(y)$ on k-dimensional Hilbert spaces $\mathcal{F}_x$ and $\mathcal{F}_y$ spanned by the bases $\{|n_1) : n_1 = 0, 1, \ldots, k-1\}$ and $\{|n_2) : n_2 = 0, 1, \ldots, k-1\}$, respectively. These two bases are supposed to be orthonormal, i.e.,

$$(n_1|n_1') = \delta(n_1, n_1'), \quad (n_2|n_2') = \delta(n_2, n_2') \tag{14}$$

We easily verify the following result.
Proposition 1. *The relations*

$$\begin{aligned} x_+|n_1) &= |n_1 + 1), \quad x_+|k-1) = 0 \\ x_-|n_1) &= [n_1]_q \, |n_1 - 1), \quad x_-|0) = 0 \\ N_x|n_1) &= n_1|n_1) \end{aligned} \tag{15}$$

and

$$\begin{aligned} y_+|n_2) &= [n_2 + 1]_q \, |n_2 + 1), \quad y_+|k-1) = 0 \\ y_-|n_2) &= |n_2 - 1), \quad y_-|0) = 0 \\ N_y|n_2) &= n_2|n_2) \end{aligned} \tag{16}$$

define k-dimensional representations of $A_q(x)$ and $A_q(y)$, respectively. In (15) and (16), we use the notation

$$\forall n \in \mathbb{N}^* \; : \; [n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \ldots + q^{n-1}, \quad [0]_q = 1 \tag{17}$$

which is familiar in q-deformations of algebraic structures.
Definition 2. *The cornerstone of the quonic approach to $su(2)$ is to define the two linear operators*

$$h = \sqrt{N_x \left(N_y + 1\right)}, \quad v_{ra} = s_x s_y \tag{18}$$

with

$$s_x = q^{a(N_x+N_y)/2}x_+ + e^{i\phi_r/2}\frac{1}{[k-1]_q!}(x_-)^{k-1} \tag{19}$$

$$s_y = y_-q^{-a(N_x-N_y)/2} + e^{i\phi_r/2}\frac{1}{[k-1]_q!}(y_+)^{k-1} \tag{20}$$

In (19) and (20), we take

$$a \in \mathbb{Z}/d\mathbb{Z}, \quad \phi_r = \pi(k-1)r, \quad r \in \mathbb{R} \tag{21}$$

and the q-deformed factorials are defined by

$$\forall n \in \mathbb{N}^* \; : \; [n]_q! = [1]_q[2]_q \dots [n]_q, \quad [0]_q! = 1 \tag{22}$$

Note that the parameter a might be taken as real. We limit ourselves to a in $\mathbb{Z}/d\mathbb{Z}$ in view of the applications to MUBs.
The operators h and v_{ra} act on the states

$$|n_1, n_2) = |n_1) \otimes |n_2) \tag{23}$$

of the k^2-dimensional Fock space $\mathcal{F}_x \otimes \mathcal{F}_y$. It is straightforward to verify that the action of v_{ra} on $\mathcal{F}_x \otimes \mathcal{F}_y$ is governed by

$$\begin{aligned} v_{ra}|k-1, n_2) &= e^{i\phi_r/2}|0, n_2-1), \quad n_2 \neq 0 \\ v_{ra}|n_1, n_2) &= q^{n_2 a}|n_1+1, n_2-1), \quad n_1 \neq k-1, \quad n_2 \neq 0 \\ v_{ra}|n_1, 0) &= e^{i\phi_r/2}|n_1+1, k-1), \quad n_1 \neq k-1 \end{aligned} \tag{24}$$

and

$$v_{ra}|k-1, 0) = e^{i\phi_r}|0, k-1) \tag{25}$$

As a consequence, we can prove the identity

$$(v_{ra})^k = e^{i\phi_r} I \tag{26}$$

The action of h on $\mathcal{F}_x \otimes \mathcal{F}_y$ is much more simple. It is described by

$$h|n_1, n_2) = \sqrt{n_1(n_2+1)}|n_1, n_2) \tag{27}$$

which holds for $n_1, n_2 = 0, 1, \dots, k-1$. Finally, the operator v_{ra} is unitary and the operator h Hermitian on the space $\mathcal{F}_x \otimes \mathcal{F}_y$.
We are now in a position to introduce a realization of the generators of the non-deformed Lie algebra $su(2)$ in terms of the operators v_{ra} and h. As a preliminary step, let us adapt the trick used by Schwinger in his approach to angular momentum via a coupled pair of harmonic oscillators (Schwinger, 1965). This can be done by introducing two new quantum numbers J and M

$$J = \frac{1}{2}(n_1 + n_2), \quad M = \frac{1}{2}(n_1 - n_2) \tag{28}$$

and the state vectors

$$|J,M\rangle = |n_1,n_2) = |J+M,J-M) \;\; \Rightarrow \;\; \langle J,M|J',M'\rangle = \delta_{J,J'}\delta_{M,M'} \tag{29}$$

Note that

$$j = \frac{1}{2}(k-1) \tag{30}$$

is an admissible value for J. We may thus have $j = \frac{1}{2}$, 1, $\frac{3}{2}$, ... (since $k = 2, 3, 4, \ldots$). For the value j of J, the quantum number M can take the values $m = j, j-1, \ldots, -j$. Then, let us consider the $(2j+1)$-dimensional subspace $\epsilon(j)$ of the k^2-dimensional space $\mathcal{F}_x \otimes \mathcal{F}_y$ spanned by the basis

$$B_{2j+1} = \{|j,m\rangle : m = j, j-1, \ldots, -j\} \tag{31}$$

with the orthonormality property

$$\langle j,m|j,m'\rangle = \delta_{m,m'} \tag{32}$$

We guess that $\epsilon(j)$ is a space of constant angular momentum j. As a matter of fact, we can check that $\epsilon(j)$ is stable under h and v_{ra}.
Proposition 2. *The action of the operators h and v_{ra} on $\epsilon(j)$ is given by*

$$h|j,m\rangle = \sqrt{(j+m)(j-m+1)}|j,m\rangle \tag{33}$$

$$v_{ra}|j,m\rangle = \delta_{m,j}e^{i2\pi jr}|j,-j\rangle + (1-\delta_{m,j})q^{(j-m)a}|j,m+1\rangle \tag{34}$$

where q is given by (12) with $k = 2j+1$, $r \in \mathbb{R}$ and $a \in \mathbb{Z}/(2j+1)\mathbb{Z}$.
It is sometimes useful to use the Dirac notation by writing

$$h = \sum_{m=-j}^{j} \sqrt{(j+m)(j-m+1)}|j,m\rangle\langle j,m| \tag{35}$$

$$v_{ra} = e^{i2\pi jr}|j,-j\rangle\langle j,j| + \sum_{m=-j}^{j-1} q^{(j-m)a}|j,m+1\rangle\langle j,m| \tag{36}$$

$$(v_{ra})^\dagger = e^{-i2\pi jr}|j,j\rangle\langle j,-j| + \sum_{m=-j+1}^{j} q^{-(j-m+1)a}|j,m-1\rangle\langle j,m| \tag{37}$$

It is understood that the three preceding relations are valid as far as the operators h, v_{ra} and $(v_{ra})^\dagger$ act on the space $\epsilon(j)$. It is evident that h is an Hermitian operator and v_{ra} a unitary operator on $\epsilon(j)$.
Definition 3. *The link with $su(2)$ can be established by introducing the three linear operators j_+, j_- and j_z through*

$$j_+ = hv_{ra}, \quad j_- = (v_{ra})^\dagger h, \quad j_z = \frac{1}{2}\left[h^2 - (v_{ra})^\dagger h^2 v_{ra}\right] \tag{38}$$

For each couple (r,a) we have a triplet (j_+, j_-, j_z). It is clear that j_+ and j_- are connected via Hermitian conjugation and j_z is Hermitian.

Proposition 3. *The action of j_+, j_- and j_z on $\epsilon(j)$ is given by the eigenvalue equation*

$$j_z|j,m\rangle = m|j,m\rangle \tag{39}$$

and the ladder equations

$$\begin{aligned} j_+|j,m\rangle &= q^{(j-m+s-1/2)a}\sqrt{(j-m)(j+m+1)}|j,m+1\rangle && (40) \\ j_-|j,m\rangle &= q^{-(j-m+s+1/2)a}\sqrt{(j+m)(j-m+1)}|j,m-1\rangle && (41) \end{aligned}$$

where $s = 1/2$.

For $a = 0$, Eqs. (39), (40) and (41) give relations that are well-known in angular momentum theory. Indeed, the case $a = 0$ corresponds to the usual Condon and Shortley phase convention used in atomic and nuclear spectroscopy. As a corollary of Proposition 3, we have the following result.

Corollary 1. *The operators j_+, j_- and j_z satisfy the commutation relations*

$$[j_z, j_+] = j_+, \quad [j_z, j_-] = -j_-, \quad [j_+, j_-] = 2j_z \tag{42}$$

and thus span the Lie algebra of $SU(2)$.

The latter result does not depend on the parameters r and a. The writing of the ladder operators j_+ and j_- in terms of h and v_{ra} constitutes a two-parameter polar decomposition of the Lie algebra $su(2)$. Thus, from two q-deformed oscillator algebras we obtained a polar decomposition of the non-deformed Lie algebra of $SU(2)$. This decomposition is an alternative to the polar decompositions obtained independently in (Chaichian & Ellinas, 1990; Lévy-Leblond, 1973; Vourdas, 1990).

2.3 The $\{j^2, v_{ra}\}$ scheme

Each vector $|j,m\rangle$ is a common eigenvector of the two commuting operators j_z and

$$j^2 = \frac{1}{2}(j_+j_- + j_-j_+) + j_3^2 = j_+j_- + j_3(j_3-1) = j_-j_+ + j_3(j_3+1) \tag{43}$$

which is known as the Casimir operator of $su(2)$ in group theory or as the square of a generalized angular momentum in angular momentum theory. More precisely, we have the eigenvalue equations

$$j^2|j,m\rangle = j(j+1)|j,m\rangle, \quad j_z|j,m\rangle = m|j,m\rangle, \quad m = j, j-1, \ldots, -j \tag{44}$$

which show that j and m can be interpreted as angular momentum quantum numbers (in units such that the rationalized Planck constant $\hbar$ is equal to 1). Of course, the set $\{j^2, j_z\}$ is a complete set of commuting operators. It is clear that the two operators j^2 and v_{ra} commute. As a matter of fact, the set $\{j^2, v_{ra}\}$ provides an alternative to the set $\{j^2, j_z\}$ as indicated by the next result.

Theorem 1. *For fixed j (with $2j \in \mathbb{N}^*$), r (with $r \in \mathbb{R}$) and a (with $a \in \mathbb{Z}/(2j+1)\mathbb{Z}$), the $2j+1$ common eigenvectors of the operators j^2 and v_{ra} can be taken in the form*

$$|j\alpha; ra\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} q^{(j+m)(j-m+1)a/2 - jmr + (j+m)\alpha}|j,m\rangle, \quad \alpha = 0, 1, \ldots, 2j \tag{45}$$

where

$$q = \exp\left(\frac{2\pi i}{2j+1}\right) \tag{46}$$

The corresponding eigenvalues are given by

$$j^2|j\alpha; ra\rangle = j(j+1)|j\alpha; ra\rangle, \quad v_{ra}|j\alpha; ra\rangle = q^{j(r+a)-\alpha}|j\alpha; ra\rangle, \quad \alpha = 0, 1, \ldots, 2j \tag{47}$$

so that the spectrum of v_{ra} is nondegenerate and $\{j^2, v_{ra}\}$ does form a complete set of commuting operators. The inner product

$$\langle j\alpha; ra|j\beta; ra\rangle = \delta_{\alpha,\beta} \tag{48}$$

shows that

$$B_{ra} = \{|j\alpha; ra\rangle : \alpha = 0, 1, \ldots, 2j\} \tag{49}$$

is a nonstandard orthonormal basis for the irreducible matrix representation of $SU(2)$ associated with j. For fixed j, there exists a priori a $(2j+1)$-multiple infinity of orthonormal bases B_{ra} since r can have any real value and a, which belongs to the ring $\mathbb{Z}/(2j+1)\mathbb{Z}$, can take $2j+1$ values ($a = 0, 1, \ldots, 2j$).
Equation (45) defines a unitary transformation that allows to pass from the standard orthonormal basis B_{2j+1}, quite well-known in angular momentum theory and group theory, to the nonstandard orthonormal basis B_{ra}. For fixed j, r and a, the inverse transformation of (45) is

$$|j, m\rangle = q^{-(j+m)(j-m+1)a/2+jmr}\frac{1}{\sqrt{2j+1}}\sum_{\alpha=0}^{2j} q^{-(j+m)\alpha}|j\alpha; ra\rangle, \quad m = j, j-1, \ldots, -j \tag{50}$$

which looks like an inverse DFT up to phase factors. For $r = a = 0$, Eqs. (45) and (50) lead to

$$|j\alpha; 00\rangle = \frac{1}{\sqrt{2j+1}}\sum_{m=-j}^{j} q^{(j+m)\alpha}|j, m\rangle, \quad \alpha = 0, 1, \ldots, 2j \tag{51}$$

$$\Leftrightarrow \quad |j, m\rangle = \frac{1}{\sqrt{2j+1}}\sum_{\alpha=0}^{2j} q^{-(j+m)\alpha}|j\alpha; 00\rangle, \quad m = j, j-1, \ldots, -j \tag{52}$$

Equations (51) and (52) correspond (up to phase factors) to the DFT of the basis B_{2j+1} and its inverse DFT, respectively.
Note that the calculation of $\langle j\alpha; ra|j\beta; sb\rangle$ is much more involved for $(r \neq s,\ a = b)$, $(r = s,\ a \neq b)$ and $(r \neq s,\ a \neq b)$ than the one of $\langle j\alpha; ra|j\beta; ra\rangle$ (the value of which is given by (48)). For example, the overlap between the bases B_{ra} and B_{sa}, of relevance for the case $(r \neq s,\ a = b)$, is given by

$$\langle j\alpha; ra|j\beta; sa\rangle = \frac{1}{2j+1}\,\frac{\sin[j(s-r)+\alpha-\beta]\pi}{\sin[j(s-r)+\alpha-\beta]\frac{\pi}{2j+1}} \tag{53}$$

The cases $(r = s,\ a \neq b)$ and $(r \neq s,\ a \neq b)$ need the use of Gauss sums as we shall see below.
The representation theory and the Wigner-Racah algebra of the group $SU(2)$ can be developed in the $\{j^2, v_{ra}\}$ quantization scheme. This leads to Clebsch-Gordan coefficients and $(3-j\alpha)_{ra}$

symbols with properties very different from the ones of the usual $SU(2) \supset U(1)$ Clebsch-Gordan coefficients and $3 - jm$ symbols corresponding to the $\{j^2, j_z\}$ quantization scheme. For more details, see Appendix which deals with the case $r = a = 0$.

The nonstandard approach to the Wigner-Racah algebra of $SU(2)$ and angular momentum theory in the $\{j^2, v_{ra}\}$ scheme is especially useful in quantum chemistry for problems involving cyclic symmetry. This is the case for a ring-shape molecule with $2j+1$ atoms at the vertices of a regular polygon with $2j+1$ sides or for a one-dimensional chain of $2j+1$ spins of $\frac{1}{2}$-value each (Albouy & Kibler, 2007). In this connection, we observe that the vectors of type $|j\alpha; ra\rangle$ are specific symmetry-adapted vectors. Symmetry-adapted vectors are widely used in quantum chemistry, molecular physics and condensed matter physics as for instance in ro-vibrational spectroscopy of molecules (Champion et al., 1977) and ligand-field theory (Kibler, 1968). However, the vectors $|j\alpha; ra\rangle$ differ from the symmetry-adapted vectors considered in (Champion et al., 1977; Kibler, 1968; Patera & Winternitz, 1976) in the sense that v_{ra} is not an invariant under some finite subgroup (of crystallographic interest) of the orthogonal group $O(3)$. This can be clarified as follows.

Proposition 4. *From (36), it follows that the operator v_{ra} is a pseudo-invariant under the cyclic group C_{2j+1}, a subgroup of $SO(3)$, whose elements are the Wigner operators $P_{R(\varphi)}$ associated with the rotations $R(\varphi)$, around the quantization axis Oz, with the angles*

$$\varphi = p\frac{2\pi}{2j+1}, \quad p = 0, 1, \ldots, 2j \tag{54}$$

More precisely, v_{ra} transforms as

$$P_{R(\varphi)} v_{ra} \left(P_{R(\varphi)}\right)^\dagger = e^{-i\varphi} v_{ra} \tag{55}$$

Thus, v_{ra} belongs to the irreducible representation class of C_{2j+1} of character vector

$$\chi^{(2j)} = (1, q^{-1}, \ldots, q^{-2j}) \tag{56}$$

In terms of vectors of $\epsilon(j)$, we have

$$P_{R(\varphi)}|j\alpha; ra\rangle = q^{jp}|j\beta; ra\rangle, \quad \beta = \alpha \ominus p \tag{57}$$

so that the set $\{|j\alpha; ra\rangle : \alpha = 0, 1, \ldots, 2j\}$ is stable under $P_{R(\varphi)}$. The latter set spans the regular representation of C_{2j+1}.

2.4 Examples

Example 1: The $j = \frac{1}{2}$ case. The eigenvectors of v_{ra} are

$$|\frac{1}{2}\alpha; ra\rangle = \frac{1}{\sqrt{2}} e^{i\pi(a/2 - r/4 + \alpha)}|\frac{1}{2}, \frac{1}{2}\rangle + \frac{1}{\sqrt{2}} e^{i\pi r/4}|\frac{1}{2}, -\frac{1}{2}\rangle, \quad \alpha = 0, 1 \tag{58}$$

where $r \in \mathbb{R}$ and a can take the values $a = 0, 1$. In the case $r = 0$, Eq. (58) gives the two bases

$$B_{00} \ : \ |\frac{1}{2}0; 00\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}, \frac{1}{2}\rangle + |\frac{1}{2}, -\frac{1}{2}\rangle\right), \quad |\frac{1}{2}1; 00\rangle = -\frac{1}{\sqrt{2}}\left(|\frac{1}{2}, \frac{1}{2}\rangle - |\frac{1}{2}, -\frac{1}{2}\rangle\right) \tag{59}$$

and

$$B_{01} \ : \ |\frac{1}{2}0; 01\rangle = \frac{i}{\sqrt{2}}\left(|\frac{1}{2}, \frac{1}{2}\rangle - i|\frac{1}{2}, -\frac{1}{2}\rangle\right), \quad |\frac{1}{2}1; 01\rangle = -\frac{i}{\sqrt{2}}\left(|\frac{1}{2}, \frac{1}{2}\rangle + i|\frac{1}{2}, -\frac{1}{2}\rangle\right) \tag{60}$$

The bases (59) and (60) are, up to phase factors, familiar bases in quantum mechanics for $\frac{1}{2}$-spin systems.
Example 2: The $j = 1$ case. The eigenvectors of v_{ra} are

$$|1\alpha; ra\rangle = \frac{1}{\sqrt{3}} q^r \left(q^{a+2\alpha-2r}|1,1\rangle + q^{a+\alpha-r}|1,0\rangle + |1,-1\rangle \right), \quad \alpha = 0,1,2 \tag{61}$$

where $r \in \mathbb{R}$ and a can take the values $a = 0,1,2$. In the case $r = 0$, Eq. (61) gives the three bases

$$\begin{aligned} B_{00} \ : \ |10;00\rangle &= \frac{1}{\sqrt{3}} (|1,-1\rangle + |1,0\rangle + |1,1\rangle) \\ |11;00\rangle &= \frac{1}{\sqrt{3}} \left(|1,-1\rangle + q|1,0\rangle + q^2|1,1\rangle \right) \\ |12;00\rangle &= \frac{1}{\sqrt{3}} \left(|1,-1\rangle + q^2|1,0\rangle + q|1,1\rangle \right) \end{aligned} \tag{62}$$

$$\begin{aligned} B_{01} \ : \ |10;01\rangle &= \frac{1}{\sqrt{3}} (|1,-1\rangle + q|1,0\rangle + q|1,1\rangle) \\ |11;01\rangle &= \frac{1}{\sqrt{3}} \left(|1,-1\rangle + q^2|1,0\rangle + |1,1\rangle \right) \\ |12;01\rangle &= \frac{1}{\sqrt{3}} \left(|1,-1\rangle + |1,0\rangle + q^2|1,1\rangle \right) \end{aligned} \tag{63}$$

$$\begin{aligned} B_{02} \ : \ |10;02\rangle &= \frac{1}{\sqrt{3}} \left(|1,-1\rangle + q^2|1,0\rangle + q^2|1,1\rangle \right) \\ |11;02\rangle &= \frac{1}{\sqrt{3}} (|1,-1\rangle + |1,0\rangle + q|1,1\rangle) \\ |12;02\rangle &= \frac{1}{\sqrt{3}} (|1,-1\rangle + q|1,0\rangle + |1,1\rangle) \end{aligned} \tag{64}$$

It is worth noting that the vectors of the basis B_{00} exhibit all characters

$$\chi^{(\alpha)} = \left(1, q^{\alpha}, q^{2\alpha} \right), \quad \alpha = 0,1,2 \tag{65}$$

of the three vector representations of C_3. On another hand, the bases B_{01} and B_{02} are connected to projective representations of C_3 because they are described by the pseudo-characters

$$\chi_1^{(\alpha)} = \left(1, q^{1+\alpha}, q^{1-\alpha} \right), \quad \alpha = 0,1,2 \tag{66}$$

and

$$\chi_2^{(\alpha)} = \left(1, q^{2+\alpha}, q^{2-\alpha} \right), \quad \alpha = 0,1,2 \tag{67}$$

respectively.

3. Quadratic discrete Fourier transforms

We discuss in this section two quadratic extensions of the DFT, namely, a quantum quadratic DFT that connects state vectors in a finite-dimensional Hilbert space, of relevance in quantum information, and a quadratic DFT that might be of interest in signal analysis.

3.1 Quantum quadratic discrete Fourier transform

Relations of section 2 concerning $SU(2)$ can be transcribed in a form more adapted to the Fourier transformation formalism and to quantum information. In this respect, let us introduce the change of notations

$$d = 2j + 1, \quad n = j + m, \quad |n\rangle = |j, -m\rangle \tag{68}$$

and

$$|a\alpha; r\rangle = |j\alpha; ra\rangle \tag{69}$$

so that (49) becomes

$$B_{ra} = \{|a\alpha; r\rangle : \alpha = 0, 1, \ldots, d-1\} \tag{70}$$

(Note that d coincides with the dimension k of the spaces $\mathcal{F}_x$ and $\mathcal{F}_y$ of section 1.) Then from Eq. (45), we have

$$|a\alpha; r\rangle = q^{(d-1)^2 r/4} \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} q^{n(d-n)a/2 + n[\alpha - (d-1)r/2]} |d-1-n\rangle, \quad \alpha = 0, 1, \ldots, d-1 \tag{71}$$

or equivalently

$$|a\alpha; r\rangle = q^{(d-1)^2 r/4} \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} q^{(d-1-n)(n+1)a/2 + (d-1-n)[\alpha - (d-1)r/2]} |n\rangle, \quad \alpha = 0, 1, \ldots, d-1 \tag{72}$$

where

$$q = \exp\left(\frac{2\pi i}{d}\right) \tag{73}$$

The inversion of (71) gives

$$|d-1-n\rangle = q^{-n(d-n)a/2 - (d-1)^2 r/4 + n(d-1)r/2} \frac{1}{\sqrt{d}} \sum_{\alpha=0}^{d-1} q^{-n\alpha} |a\alpha; r\rangle, \quad n = 0, 1, \ldots, d-1 \tag{74}$$

By introducing

$$(\mathbf{F_{ra}})_{n\alpha} = \frac{1}{\sqrt{d}} q^{n(d-n)a/2 + (d-1)^2 r/4 + n[\alpha - (d-1)r/2]}, \quad n, \alpha = 0, 1, \ldots, d-1 \tag{75}$$

equations (71) and (74) can be rewritten as

$$|a\alpha; r\rangle = \sum_{n=0}^{d-1} (\mathbf{F_{ra}})_{n\alpha} \, |d-1-n\rangle, \quad \alpha = 0, 1, \ldots, d-1 \tag{76}$$

and

$$|d-1-n\rangle = \sum_{\alpha=0}^{d-1} \overline{(\mathbf{F_{ra}})_{n\alpha}} \, |a\alpha;r\rangle, \quad n = 0,1,\ldots,d-1 \tag{77}$$

respectively. For $r = a = 0$, Eqs. (76) and (77) yield

$$|0\alpha;0\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{i2\pi\alpha n/d} |d-1-n\rangle, \quad \alpha = 0,1,\ldots,d-1$$

$$\Leftrightarrow \quad |d-1-n\rangle = \frac{1}{\sqrt{d}} \sum_{\alpha=0}^{d-1} e^{-i2\pi n\alpha/d} |0\alpha;0\rangle, \quad n = 0,1,\ldots,d-1 \tag{78}$$

which corresponds (up to a change of notations) to the DFT described by (3). For $a \neq 0$, Eq. (76) can be considered as a quadratic extension (quadratic in n) of the DFT of the basis $\{|n\rangle : n = 0,1,\ldots,d-1\}$ and Eq. (77) thus appears as the corresponding inverse DFT. This can be summed up by the following definition.

Definition 4. *Let* $\mathbf{H_{ra}}$ *be the* $d \times d$ *matrix defined by the matrix elements*

$$(\mathbf{H_{ra}})_{n\alpha} = \frac{1}{\sqrt{d}} q^{(d-1-n)(n+1)a/2+(d-1)^2 r/4+(d-1-n)[\alpha-(d-1)r/2]}, \quad n,\alpha = 0,1,\ldots,d-1 \tag{79}$$

where, for a fixed value of d *(with* $d \in \mathbb{N} \setminus \{0,1\}$*),* r *and* a *may have values in* $\mathbb{R}$ *and* $\mathbb{Z}/d\mathbb{Z}$*, respectively. In compact form*

$$(\mathbf{H_{ra}})_{n\alpha} = \frac{1}{\sqrt{d}} e^{2\pi i\nu/d} \tag{80}$$

with

$$\nu = -\frac{1}{4}(d-1)^2 r + \frac{1}{2}(d-1)a + (d-1)\alpha - \frac{1}{2}[2\alpha + 2a - da - (d-1)r]n - \frac{1}{2}an^2 \tag{81}$$

The expansion

$$|a\alpha;r\rangle = \sum_{n=0}^{d-1} (\mathbf{H_{ra}})_{n\alpha} \, |n\rangle, \quad \alpha = 0,1,\ldots,d-1 \tag{82}$$

defines a quadratic quantum DFT of the orthonormal basis

$$B_d = \{|n\rangle : n = 0,1,\ldots,d-1\} \tag{83}$$

This transformation produces another orthonormal basis, namely, the basis B_{ra} *(see Eq. (70)). The inverse transformation*

$$|n\rangle = \sum_{\alpha=0}^{d-1} \overline{(\mathbf{H_{ra}})_{n\alpha}} \, |a\alpha;r\rangle, \quad n = 0,1,\ldots,d-1 \tag{84}$$

gives back the basis B_d.

For fixed d, r and a, each of the d vectors $|a\alpha; r\rangle$, with $\alpha = 0, 1, \ldots, d-1$, is a linear combination of the vectors $|0\rangle, |1\rangle, \ldots, |d-1\rangle$. The vector $|a\alpha; r\rangle$ is an eigenvector of the operator

$$v_{ra} = e^{i\pi(d-1)r}|d-1\rangle\langle 0| + \sum_{n=0}^{d-2} q^{(d-1-n)a}|d-2-n\rangle\langle d-1-n| \tag{85}$$

or

$$v_{ra} = e^{i\pi(d-1)r}|d-1\rangle\langle 0| + \sum_{n=1}^{d-1} q^{na}|n-1\rangle\langle n| \tag{86}$$

(cf. Eq. (36)). The operator v_{ra} can be developed as

$$v_{ra} = e^{i\pi(d-1)r}|d-1\rangle\langle 0| + q^{a}|0\rangle\langle 1| + q^{2a}|1\rangle\langle 2| + \ldots + q^{(d-1)a}|d-2\rangle\langle d-1| \tag{87}$$

Then, the action of v_{ra} on the state $|n\rangle$ is described by

$$v_{ra}|n\rangle = \delta_{n,0}e^{i\pi(d-1)r}|d-1\rangle + (1-\delta_{n,0})q^{na}|n-1\rangle \tag{88}$$

(cf. Eq. (34)). Its eigenvalues are given by

$$v_{ra}|a\alpha; r\rangle = q^{(d-1)(r+a)/2-\alpha}|a\alpha; r\rangle, \quad \alpha = 0, 1, \ldots, d-1 \tag{89}$$

(cf. Eq. (47)).

3.1.1 Diagonalization of v_{ra}

Let $\mathbf{V_{ra}}$ be the $d \times d$ unitary matrix that represents the linear operator v_{ra} (given by (87)) on the basis B_d. Explicitly, we have

$$\mathbf{V_{ra}} = \begin{pmatrix} 0 & q^{a} & 0 & \ldots & 0 \\ 0 & 0 & q^{2a} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & q^{(d-1)a} \\ e^{i\pi(d-1)r} & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{90}$$

where the lines and columns are arranged in the order $0, 1, \ldots, d-1$. Note that the nonzero matrix elements of V_{0a} are given by the irreducible character vector

$$\chi^{(a)} = (1, q^{a}, \ldots, q^{(d-1)a}) \tag{91}$$

of the cyclic group C_d.

Proposition 5. *The matrix $\mathbf{H_{ra}}$ reduces the endomorphism associated with the operator v_{ra}. In other words*

$$(\mathbf{H_{ra}})^{\dagger}\, \mathbf{V_{ra}}\mathbf{H_{ra}} = q^{(d-1)(r+a)/2} \begin{pmatrix} q^{0} & 0 & \ldots & 0 \\ 0 & q^{-1} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & q^{-(d-1)} \end{pmatrix} \tag{92}$$

in agreement with Eq. (47).

Concerning the matrices in (90) and (92), it is important to note the following convention. According to the tradition in quantum mechanics and quantum information, all the matrices in this chapter are set up with their lines and columns ordered from left to right and from top to bottom in the range $0, 1, \ldots, d-1$. Different conventions were used in some previous works by the author. However, the results previously obtained are equivalent to those of this chapter.
The eigenvectors of the matrix $\mathbf{V_{ra}}$ are

$$\phi(a\alpha; r) = \sum_{n=0}^{d-1} (\mathbf{H_{ra}})_{n\alpha} \, \phi_n, \quad \alpha = 0, 1, \ldots, d-1 \tag{93}$$

where the ϕ_n with $n = 0, 1, \ldots, d-1$ are the column vectors

$$\phi_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \phi_{d-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{94}$$

representing the state vectors $|0\rangle, |1\rangle, \ldots, |d-1\rangle$, respectively. These eigenvectors are the column vectors of the matrix $\mathbf{H_{ra}}$. They satisfy the eigenvalue equation (cf. 89)

$$\mathbf{V_{ra}} \phi(a\alpha; r) = q^{(d-1)(r+a)/2-\alpha} \phi(a\alpha; r) \tag{95}$$

with $\alpha = 0, 1, \ldots, d-1$.

3.1.2 Examples

Example 3: The $d = 2$ case. For $d = 2$, there are two families of bases B_{ra}: the B_{r0} family and the B_{r1} family (a can take the values $a = 0$ and $a = 1$). In terms of matrices, we have

$$\mathbf{H_{ra}} = \frac{1}{\sqrt{2}} \begin{pmatrix} q^{a/2-r/4} & -q^{a/2-r/4} \\ q^{r/4} & q^{r/4} \end{pmatrix}, \quad \mathbf{V_{ra}} = \begin{pmatrix} 0 & q^a \\ q^r & 0 \end{pmatrix}, \quad q = e^{i\pi} \tag{96}$$

The matrix $\mathbf{V_{ra}}$ has the eigenvectors (corresponding to the basis B_{ra})

$$\phi(a\alpha; r) = \frac{1}{\sqrt{2}} (q^{a/2-r/4+\alpha} \phi_0 + q^{r/4} \phi_1), \quad \alpha = 0, 1 \tag{97}$$

where

$$\phi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{98}$$

For $r = 0$, we have

$$V_{00} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V_{01} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{99}$$

the eigenvectors of which are (cf. (97))

$$\phi(00; 0) = \frac{1}{\sqrt{2}} (\phi_1 + \phi_0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \phi(01; 0) = \frac{1}{\sqrt{2}} (\phi_1 - \phi_0) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{100}$$

and

$$\phi(10;0) = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_0) = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \phi(11;0) = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_0) = -\frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{101}$$

which correspond to the bases B_{00} and B_{01}, respectively. Note that (100) and (101) are, up to unimportant multiplicative phase factors, qudits used in quantum information.
Example 4: The $d = 3$ case. For $d = 3$, we have three families of bases, that is to say B_{r0}, B_{r1} and B_{r2}, since a can be 0, 1 and 2. In this case

$$\mathbf{H_{ra}} = \frac{1}{\sqrt{3}}\begin{pmatrix} q^{a-r} & q^{a+2-r} & q^{a+1-r} \\ q^a & q^{a+1} & q^{a+2} \\ q^r & q^r & q^r \end{pmatrix}, \quad \mathbf{V_{ra}} = \begin{pmatrix} 0 & q^a & 0 \\ 0 & 0 & q^{2a} \\ q^{3r} & 0 & 0 \end{pmatrix}, \quad q = e^{i2\pi/3} \tag{102}$$

and $\mathbf{V_{ra}}$ admits the eigenvectors (corresponding to the basis B_{ra})

$$\phi(a\alpha;r) = \frac{1}{\sqrt{3}}q^r\left(q^{a+2\alpha-2r}\phi_0 + q^{a+\alpha-r}\phi_1 + \phi_2\right), \quad \alpha = 0,1,2 \tag{103}$$

where

$$\phi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{104}$$

In the case $r = 0$, we get

$$V_{00} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad V_{01} = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & q^2 \\ 1 & 0 & 0 \end{pmatrix}, \quad V_{02} = \begin{pmatrix} 0 & q^2 & 0 \\ 0 & 0 & q \\ 1 & 0 & 0 \end{pmatrix} \tag{105}$$

The eigenvectors of V_{00}, V_{01} and V_{02} follow from Eq. (103). This yields

$$\begin{aligned} \phi(00;0) &= \frac{1}{\sqrt{3}}(\phi_2 + \phi_1 + \phi_0) \\ \phi(01;0) &= \frac{1}{\sqrt{3}}\left(\phi_2 + q\phi_1 + q^2\phi_0\right) \\ \phi(02;0) &= \frac{1}{\sqrt{3}}\left(\phi_2 + q^2\phi_1 + q\phi_0\right) \end{aligned} \tag{106}$$

or

$$\phi(00;0) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \phi(01;0) = \frac{1}{\sqrt{3}}\begin{pmatrix} q^2 \\ q \\ 1 \end{pmatrix}, \quad \phi(02;0) = \frac{1}{\sqrt{3}}\begin{pmatrix} q \\ q^2 \\ 1 \end{pmatrix} \tag{107}$$

corresponding to B_{00},

$$\begin{aligned} \phi(10;0) &= \frac{1}{\sqrt{3}}(\phi_2 + q\phi_1 + q\phi_0) \\ \phi(11;0) &= \frac{1}{\sqrt{3}}\left(\phi_2 + q^2\phi_1 + \phi_0\right) \\ \phi(12;0) &= \frac{1}{\sqrt{3}}\left(\phi_2 + \phi_1 + q^2\phi_0\right) \end{aligned} \tag{108}$$

or

$$\phi(10;0) = \frac{1}{\sqrt{3}} \begin{pmatrix} q \\ q \\ 1 \end{pmatrix}, \quad \phi(11;0) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ q^2 \\ 1 \end{pmatrix}, \quad \phi(12;0) = \frac{1}{\sqrt{3}} \begin{pmatrix} q^2 \\ 1 \\ 1 \end{pmatrix} \tag{109}$$

corresponding to B_{01}, and

$$\begin{aligned} \phi(20;0) &= \frac{1}{\sqrt{3}} \left(\phi_2 + q^2\phi_1 + q^2\phi_0 \right) \\ \phi(21;0) &= \frac{1}{\sqrt{3}} \left(\phi_2 + \phi_1 + q\phi_0 \right) \\ \phi(22;0) &= \frac{1}{\sqrt{3}} \left(\phi_2 + q\phi_1 + \phi_0 \right) \end{aligned} \tag{110}$$

or

$$\phi(20;0) = \frac{1}{\sqrt{3}} \begin{pmatrix} q^2 \\ q^2 \\ 1 \end{pmatrix}, \quad \phi(21;0) = \frac{1}{\sqrt{3}} \begin{pmatrix} q \\ 1 \\ 1 \end{pmatrix}, \quad \phi(22;0) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ q \\ 1 \end{pmatrix} \tag{111}$$

corresponding to B_{02}. Note that (107), (109) and (111) are, up to unimportant multiplicative phase factors, qutrits used in quantum information.

3.1.3 Decomposition of V_{ra}

The matrix $\mathbf{V_{ra}}$ can be decomposed as

$$\mathbf{V_{ra}} = \mathbf{P_r X Z}^a \tag{112}$$

where

$$\mathbf{P_r} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & e^{i\pi(d-1)r} \end{pmatrix} \tag{113}$$

and

$$\mathbf{X} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & q & 0 & \dots & 0 \\ 0 & 0 & q^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & q^{d-1} \end{pmatrix} \tag{114}$$

The matrices $\mathbf{X}$ and $\mathbf{Z}$ can be derived from particular $\mathbf{V_{ra}}$ matrices since

$$\mathbf{X} = \mathbf{V_{00}}, \quad \mathbf{Z} = \left(\mathbf{V_{00}}\right)^{\dagger} \mathbf{V_{01}} \tag{115}$$

which emphasize the important role played by the matrix $\mathbf{V_{ra}}$.

The matrices $\mathbf{P_r}$, $\mathbf{X}$ and $\mathbf{Z}$ (and thus $\mathbf{V_{ra}}$) are unitary. They satisfy

$$\mathbf{V_{ra}Z} = q\mathbf{Z V_{ra}} \tag{116}$$

$$\mathbf{V_{ra}X} = q^{-a}\mathbf{X V_{ra}} \tag{117}$$

Equation (116) can be iterated to give the useful relation

$$(\mathbf{V_{ra}})^m \mathbf{Z}^n = q^{mn}\mathbf{Z}^n (\mathbf{V_{ra}})^m \tag{118}$$

where $m, n \in \mathbb{Z}/d\mathbb{Z}$. Furthermore, we have the trivial relations

$$e^{-i\pi(d-1)r}(\mathbf{V_{r0}})^d = \mathbf{Z}^d = \mathbf{I_d} \tag{119}$$

More generally, we can show that

$$\forall n \in \mathbb{Z}/d\mathbb{Z} \; : \; (\mathbf{V_{ra}})^n = q^{-n(n-1)a/2}(\mathbf{V_{r0}})^n \mathbf{Z}^{an} \tag{120}$$

Consequently

$$(\mathbf{V_{ra}})^d = e^{i\pi(d-1)(r+a)}\mathbf{I_d} \tag{121}$$

in agreement with the obtained eigenvalues for $\mathbf{V_{ra}}$ (see Eq. (95)).

3.1.4 Weyl pairs

The relations in sections 3.1.1 and 3.1.3 can be particularized in the case $r = a = 0$. For example, Eq. (118) gives the useful relation

$$\mathbf{X}^m \mathbf{Z}^n = q^{mn}\mathbf{Z}^n \mathbf{X}^m, \quad (m, n) \in \mathbb{N}^2 \tag{122}$$

The fundamental relationship between the matrices $\mathbf{X}$ and $\mathbf{Z}$ is emphasized by the following proposition.
Proposition 6. *The unitary matrices* $\mathbf{X}$ *and* $\mathbf{Z}$ *satisfy the q-commutation relation*

$$[\mathbf{X}, \mathbf{Z}]_q = \mathbf{XZ} - q\mathbf{ZX} = 0 \tag{123}$$

and the cyclicity relations

$$\mathbf{X}^d = \mathbf{Z}^d = \mathbf{I_d} \tag{124}$$

In addition, they are connected through

$$(\mathbf{F_{00}})^\dagger \mathbf{X F_{00}} = \mathbf{Z} \tag{125}$$

that indicates that $\mathbf{X}$ *and* $\mathbf{Z}$ *are related by an ordinary DFT transform.*
According to Proposition 6, the matrices $\mathbf{X}$ and $\mathbf{Z}$ constitute a Weyl pair $(\mathbf{X}, \mathbf{Z})$. Weyl pairs were introduced at the beginning of quantum mechanics (Weyl, 1931) and used for building operator unitary bases (Schwinger, 1960). We shall emphasis their interest for quantum information and quantum computing in section 4.
Let x and z be the linear operators associated with $\mathbf{X}$ and $\mathbf{Z}$, respectively. They are given by

$$x = v_{00}, \quad z = (v_{00})^\dagger v_{01} \;\Rightarrow\; xz = v_{01} \tag{126}$$

as functions of the operator v_{ra}. Each of the relations involving **X** and **Z** can be transcribed in terms of x and z.
The properties of x follow from those of v_{ra} with $r = a = 0$. The unitary operator x is a shift operator when acting on $|j,m\rangle$ or $|n\rangle$ (see (34) and (88)) and a phase operator when acting on $|j\alpha;00\rangle = |0\alpha;0\rangle$ (see (47) and (89)). More precisely, we have

$$x|j,m\rangle = |j,m \oplus 1\rangle \;\Leftrightarrow\; x|n\rangle = |n \ominus 1\rangle \tag{127}$$

and

$$x|0\alpha;0\rangle = q^{-\alpha}|0\alpha;0\rangle \tag{128}$$

The unitary operator z satisfies

$$z|j,m\rangle = q^{j-m}|j,m\rangle \;\Leftrightarrow\; z|n\rangle = q^{n}|n\rangle \tag{129}$$

and

$$z|a\alpha;0\rangle = q^{-1}|a\alpha_1;0\rangle, \quad \alpha_1 = \alpha \ominus 1 \tag{130}$$

It thus behaves as a phase operator when acting on $|j,m\rangle$ or $|n\rangle$ and a shift operator when acting on $|a\alpha;0\rangle$.
In view of (128) and (129), the two cyclic operators x and z (cf. $x^d = z^d = I$) are isospectral operators. They are connected via a discrete Fourier transform operator (see Eq. (125)).
Let us now define the operators

$$u_{ab} = x^a z^b, \quad a,b = 0,1,\ldots,d-1 \tag{131}$$

The d^2 operators u_{ab} are unitary and satisfy the following trace relation

$$\mathrm{tr}\left((u_{ab})^{\dagger} u_{a'b'}\right) = d\,\delta_{a,a'}\,\delta_{b,b'} \tag{132}$$

where the trace is taken on the d-dimensional space $\epsilon(d) = \epsilon(2j+1)$. This trace relation shows that the d^2 operators u_{ab} are pairwise orthogonal operators so that they can serve as a basis for developing any operator acting on the Hilbert space $\epsilon(d)$. Furthermore, the commutator and the anticommutator of u_{ab} and $u_{a'b'}$ are given by

$$[u_{ab}, u_{a'b'}] = \left(q^{-ba'} - q^{-ab'}\right) u_{a''b''}, \quad a'' = a \oplus a', \quad b'' = b \oplus b' \tag{133}$$

and

$$\{u_{ab}, u_{a'b'}\} = \left(q^{-ba'} + q^{-ab'}\right) u_{a''b''}, \quad a'' = a \oplus a', \quad b'' = b \oplus b' \tag{134}$$

Consequently, $[u_{ab}, u_{a'b'}] = 0$ if and only if $ab' \ominus ba' = 0$ and $\{u_{ab}, u_{a'b'}\} = 0$ if and only if $ab' \ominus ba' = (1/2)d$. Therefore, all anticommutators $\{u_{ab}, u_{a'b'}\}$ are different from 0 if d is an odd integer. From a group-theoretical point of view, we have the following result.
Proposition 7. *The set* $\{u_{ab} = x^a z^b : a,b = 0,1,\ldots,d-1\}$ *generates a* d^2*-dimensional Lie algebra. This algebra can be seen to be the Lie algebra of the general linear group* $GL(d,\mathbb{C})$*. The subset* $\{u_{ab} : a,b = 0,1,\ldots,d-1\} \setminus \{u_{00}\}$ *thus spans the Lie algebra of the special linear group* $SL(d,\mathbb{C})$*.*

A second group-theoretical aspect connected with the operators u_{ab} concerns a finite group, the so-called finite Heisenberg-Weyl group $WH(\mathbb{Z}/d\mathbb{Z})$, known as the Pauli group P_d in quantum information (Kibler, 2008). The set $\{u_{ab} : a, b = 0, 1, \ldots, d-1\}$ is not closed under multiplication. However, it is possible to extend the latter set in order to have a group as follows.
Proposition 8. *Let us define the operators* w_{abc} *via*

$$w_{abc} = q^a u_{bc}, \quad a, b, c = 0, 1, \ldots, d-1 \tag{135}$$

Then, the set $\{w_{abc} = q^a x^b z^c : a, b, c = 0, 1, \ldots, d-1\}$*, endowed with the multiplication of operators, is a group of order* d^3 *isomorphic with the Heisenberg-Weyl group* $WH(\mathbb{Z}/d\mathbb{Z})$*. This group, also referred to as the Pauli group* P_d*, is a nonabelian (for* $d \geq 2$*) nilpotent group with nilpotency class equal to 3. It is isomorphic with a finite subgroup of the group* $U(d)$ *for* d *even or* $SU(d)$ *for* d *odd.*
Proposition 8 easily follows from the composition law

$$w_{abc} w_{a'b'c'} = w_{a''b''c''}, \quad a'' = a \oplus a' \ominus cb', \quad b'' = b \oplus b', \quad c'' = c \oplus c' \tag{136}$$

Note that the group commutator of the two elements w_{abc} and $w_{a'b'c'}$ of the group $WH(\mathbb{Z}/d\mathbb{Z})$ is

$$w_{abc} w_{a'b'c'} (w_{abc})^{-1} (w_{a'b'c'})^{-1} = w_{a''00}, \quad a'' = bc' \ominus cb' \tag{137}$$

which can be particularized as

$$u_{ab} u_{a'b'} (u_{ab})^{-1} (u_{a'b'})^{-1} = q^{ab' \ominus ba'} I \tag{138}$$

in terms of the operators u_{ab}.
All this is reminiscent of the group $SU(2)$, the generators of which are the well-known Pauli matrices. Therefore, the operators u_{ab} shall be referred as generalized Pauli operators and their matrices as generalized Pauli matrices. This will be considered further in section 4.

3.1.5 Link with the cyclic group C_d

There exists an interesting connection between the operator v_{ra} and the cyclic group C_d (see section 2). The following proposition presents another aspect of this connection.
Proposition 9. *Let* R *be a generator of* C_d *(e.g., a rotation of* $2\pi/d$ *around an arbitrary axis). The application*

$$R^n \mapsto \mathbf{X}^n \ : \ n = 0, 1, \ldots, d-1 \tag{139}$$

defines a d*-dimensional matrix representation of* C_d*. This representation is the regular representation of* C_d*.*
Thus, the reduction of the representation $\{\mathbf{X}^n : n = 0, 1, \ldots, d-1\}$ contains once and only once each (one-dimensional) irreducible representation

$$\chi^{(a)} = (1, q^a, \ldots, q^{(d-1)a}), \quad a = 0, 1, \ldots, d-1 \tag{140}$$

of C_d.

3.1.6 Link with the W_∞ algebra

Let us define the matrix

$$\mathbf{T}_{(\mathbf{n_1},\mathbf{n_2})} = q^{\frac{1}{2}n_1 n_2}\mathbf{Z}^{n_1}\mathbf{X}^{n_2}, \quad (n_1, n_2) \in \mathbb{N}^2 \tag{141}$$

It is convenient to use the abbreviation

$$(n_1, n_2) \equiv n \quad \Rightarrow \quad \mathbf{T}_{(\mathbf{n_1},\mathbf{n_2})} \equiv \mathbf{T_n} \tag{142}$$

The matrices $\mathbf{T_n}$ span an infinite-dimensional Lie algebra. This may be precised as follows.
Proposition 10. *The commutator* $[\mathbf{T_m}, \mathbf{T_n}]$ *is given by*

$$[\mathbf{T_m}, \mathbf{T_n}] = -2i \sin\left(\frac{\pi}{d} m \times n\right) \mathbf{T_{m+n}} \tag{143}$$

where

$$m \times n = m_1 n_2 - m_2 n_1, \quad m + n = (m_1 + n_1, m_2 + n_2) \tag{144}$$

The matrices $\mathbf{T_m}$ *can be thus formally viewed as the generators of the infinite-dimensional Lie algebra* W_∞.
The proof of (143) is easily obtained by using (122). This leads to

$$\mathbf{T_m}\mathbf{T_n} = q^{-\frac{1}{2}m\times n}\mathbf{T_{m+n}} \tag{145}$$

which implies (143). Thus, we get of Lie algebra W_∞ (or sine algebra) investigated in (Fairlie et al., 1990).

3.2 Quadratic discrete Fourier transform

3.2.1 Generalities

We are now prepared for discussing analogs of the transformations (82) and (84) in the language of classical signal theory.
Definition 5. *Let us consider the transformation*

$$x = \{x_m \in \mathbb{C} : m = 0, 1, \ldots, d-1\} \;\leftrightarrow\; y = \{y_n \in \mathbb{C} : n = 0, 1, \ldots, d-1\} \tag{146}$$

defined by

$$y_n = \sum_{m=0}^{d-1} (\mathbf{F_{ra}})_{mn}\, x_m \;\Leftrightarrow\; x_m = \sum_{n=0}^{d-1} \overline{(\mathbf{F_{ra}})_{mn}}\, y_n \tag{147}$$

where

$$(\mathbf{F_{ra}})_{nm} = \frac{1}{\sqrt{d}} q^{n(d-n)a/2+(d-1)^2 r/4+n[m-(d-1)r/2]}, \quad n, m = 0, 1, \ldots, d-1 \tag{148}$$

For $a \neq 0$*, the bijective transformation* $x \leftrightarrow y$ *can be thought of as a quadratic DFT.*
In Eq. (147), we choose the matrix $\mathbf{F_{ra}}$ as the quadratic Fourier matrix instead of the matrix $\mathbf{H_{ra}}$ because the particular case $r = a = 0$ corresponds to the ordinary DFT (see also (Atakishiyev et al., 2010)). Note that the matrices $\mathbf{F_{ra}}$ and $\mathbf{H_{ra}}$ are interrelated via

$$(\mathbf{F_{ra}})_{nm} = (\mathbf{H_{ra}})_{n'm}, \quad n' = d - 1 - n \tag{149}$$

Therefore, the lines of $\mathbf{F_{ra}}$ in the order $0, 1, \ldots, d-1$ coincide with those of $\mathbf{H_{ra}}$ in the reverse order $d-1, d-2, \ldots, 0$.
The analog of the Parseval-Plancherel theorem for the ordinary DFT can be expressed in the following way.
Theorem 2. *The quadratic transformations $x \leftrightarrow y$ and $x' \leftrightarrow y'$ associated with the same matrix $\mathbf{F_{ra}}$, with $r \in \mathbb{R}$ and $a \in \mathbb{Z}/d\mathbb{Z}$, satisfy the conservation rule*

$$\sum_{n=0}^{d-1} \overline{y_n}\, y'_n = \sum_{m=0}^{d-1} \overline{x_m}\, x'_m \tag{150}$$

where both sums do not depend on r and a.

3.2.2 Properties of the quadratic DFT matrix

In order to get familiar with the quadratic DFT defined by (147), we now examine some of the properties of the quadratic DFT matrix $\mathbf{F_{ra}}$.
Proposition 11. *For d arbitrary, the matrix elements of $\mathbf{F_{ra}}$ satisfies the useful symmetry properties*

$$(\mathbf{F_{ra}})_{d-1\alpha} = q^{(d-1)(r+a)/2-\alpha} e^{-i\pi(d-1)r}\, (\mathbf{F_{ra}})_{0\alpha}, \quad \alpha = 0, 1, \ldots, d-1 \tag{151}$$

$$(\mathbf{F_{ra}})_{n-1\alpha} = q^{(d-1)(r+a)/2-\alpha+na}\, (\mathbf{F_{ra}})_{n\alpha}, \; n = 1, 2, \ldots, d-1, \; \alpha = 0, 1, \ldots, d-1 \tag{152}$$

which can be reduced to the sole symmetry relation

$$(\mathbf{F_{0a}})_{n\ominus 1\alpha} = q^{(d-1)a/2-\alpha+na}\, (\mathbf{F_{0a}})_{n\alpha}, \quad n, \alpha = 0, 1, \ldots, d-1 \tag{153}$$

when $r = 0$.
Proposition 12. *For d arbitrary, the matrix $\mathbf{F_{ra}}$ is unitary.*
The latter result can be checked from a straightforward calculation. It also follows in a simple way from

$$\langle j\alpha; ra | j\beta; sb \rangle = \langle a\alpha; r | b\beta; s \rangle = ((\mathbf{F_{ra}})^{\dagger} \mathbf{F_{sb}})_{\alpha\beta} \tag{154}$$

It is sufficient to put $s = r$ and $b = a$ in (154) and to use (48).
For d arbitrary, in addition to be unitary the matrix $\mathbf{F_{ra}}$ is such that the modulus of each of its matrix elements is equal to $1/\sqrt{d}$. Thus, $\mathbf{F_{ra}}$ can be considered as a generalized Hadamard matrix (we adopt here the normalization of Hadamard matrices generally used in quantum information and quantum computing (Kibler, 2009)). In the case where d is a prime number, we shall prove in section 4 from (154) that the matrix $(\mathbf{F_{ra}})^{\dagger}\mathbf{F_{rb}}$ is another Hadamard matrix for $b \neq a$. Similar results hold for the matrix $\mathbf{H_{ra}}$.
Proposition 13. *For d arbitrary, the matrix $\mathbf{F_{ra}}$ can be factorized as*

$$\mathbf{F_{ra}} = \mathbf{D_{ra}}\mathbf{F}, \quad \mathbf{F} = \mathbf{F_{00}} \tag{155}$$

where $\mathbf{D_{ra}}$ is the $d \times d$ diagonal matrix with the matrix elements

$$(\mathbf{D_{ra}})_{mn} = q^{m(d-m)a/2+(d-1)^2 r/4 - m(d-1)r/2} \delta_{m,n} \tag{156}$$

and $\mathbf{F}$ is the well-known ordinary DFT matrix.
For fixed d, there is one d-multiple infinity of Gaussian matrices $\mathbf{D_{ra}}$ (and thus $\mathbf{F_{ra}}$) distinguished by $a \in \mathbb{Z}/d\mathbb{Z}$ and $r \in \mathbb{R}$. The matrix $\mathbf{F}$ was the object of a great number of studies.

The main properties of the ordinary DFT matrix **F** are summarized in (Atakishiyev et al., 2010). Let us simply recall here the fundamental property

$$\mathbf{F}^4 = \mathbf{I_d} \tag{157}$$

of interest for obtaining the eigenvalues and eigenvectors of **F**.
Proposition 14. *The determinant of* $\mathbf{F_{ra}}$ *reads*

$$\det \mathbf{F_{ra}} = e^{i\pi(d^2-1)a/6} \det \mathbf{F} \tag{158}$$

where the value of $\det \mathbf{F}$ *is well-known (Atakishiyev et al., 2010; Mehta, 1987).*
Proposition 15. *The trace of* $\mathbf{F_{ra}}$ *reads*

$$\mathrm{tr}\, \mathbf{F_{ra}} = e^{i\pi(d-1)^2/(2d)} \frac{1}{\sqrt{d}} S(u,v,w) \tag{159}$$

where $S(u,v,w)$ *is*

$$S(u,v,w) = \sum_{k=0}^{|w|-1} e^{i\pi(uk^2+vk)/w} \tag{160}$$

with

$$u = 2-a, \quad v = d(a-r)+r, \quad w = d \tag{161}$$

(note that v *is not necessarily an integer).*
Let us recall that the sum defined by (160) is a generalized quadratic Gauss sum. It can be calculated easily in the situation where u, v and w are integers such that u and w are mutually prime, uw is not zero, and $uw+v$ is even (Berndt et al., 1998).
Note that the case $a=2$ deserves a special attention. In this case, the quadratic character of $\mathrm{tr}\, \mathbf{F_{ra}}$ disappears. In addition, if $r=0$ we get

$$\mathrm{tr}\, \mathbf{F_{02}} = \sqrt{d} \tag{162}$$

as can be seen from a direct calculation.
Example 5: In order to illustrate the preceding properties, let us consider the matrix

$$\mathbf{F_{02}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ q^5 & 1 & q & q^2 & q^3 & q^4 \\ q^2 & q^4 & 1 & q^2 & q^4 & 1 \\ q^3 & 1 & q^3 & 1 & q^3 & 1 \\ q^2 & 1 & q^4 & q^2 & 1 & q^4 \\ q^5 & q^4 & q^3 & q^2 & q & 1 \end{pmatrix} \tag{163}$$

corresponding to $d=6$ ($\Rightarrow q = e^{i\pi/3}$), $r=0$ and $a=2$. It is a simple matter of trivial calculation to check that the properties given above for $\mathbf{F_{ra}}$ are satisfied by the matrix $\mathbf{F_{02}}$.

4. Application to quantum information

4.1 Computational basis and standard $SU(2)$ basis

In quantum information science, we use qubits which are indeed state vectors in the Hilbert space $\mathbb{C}^2$. The more general qubit

$$|\psi_2\rangle = c_0|0\rangle + c_1|1\rangle, \quad c_0 \in \mathbb{C}, \quad c_1 \in \mathbb{C}, \quad |c_0|^2 + |c_1|^2 = 1 \tag{164}$$

is a linear combination of the vectors $|0\rangle$ and $|1\rangle$ which constitute an orthonormal basis

$$B_2 = \{|0\rangle, |1\rangle\} \tag{165}$$

of $\mathbb{C}^2$. These two vectors can be considered as the basis vectors for the fundamental irreducible representation class of $SU(2)$, in the $SU(2) \supset U(1)$ scheme, corresponding to $j = 1/2$ with

$$|0\rangle \equiv |1/2, 1/2\rangle, \quad |1\rangle \equiv |1/2, -1/2\rangle \tag{166}$$

More generally, in d dimensions we use qudits of the form

$$|\psi_d\rangle = \sum_{n=0}^{d-1} c_n|n\rangle, \quad c_n \in \mathbb{C}, \quad n = 0, 1, \ldots, d-1, \quad \sum_{n=0}^{d-1} |c_n|^2 = 1 \tag{167}$$

where the vectors $|0\rangle, |1\rangle, \ldots, |d-1\rangle$ span an orthonormal basis of $\mathbb{C}^d$ with

$$\langle n|n'\rangle = \delta_{n,n'} \tag{168}$$

By introducing

$$j = \frac{1}{2}(d-1), \quad m = n - \frac{1}{2}(d-1), \quad |j, m\rangle = |d-1-n\rangle \tag{169}$$

(a change of notations equivalent to (68)), the qudits $|n\rangle$ can be viewed as the basis vectors $|j, m\rangle$ for the irreducible representation class associated with j of $SU(2)$ in the $SU(2) \supset U(1)$ scheme. More precisely, the correspondence between angular momentum states and qudits is

$$|0\rangle \equiv |j, j\rangle, \quad |1\rangle \equiv |j, j-1\rangle, \quad \ldots, \quad |d-1\rangle \equiv |j, -j\rangle \tag{170}$$

where $|j, j\rangle, |j, j-1\rangle, \ldots, |j, -j\rangle$ are common eigenvectors of angular momentum operators j^2 and j_z. In other words, the basis B_d (see (83)), known in quantum information as the computational basis, may be identified to the $SU(2) \supset U(1)$ standard basis or angular momentum basis B_{2j+1} (see (31)). We shall see in section 4.2 that such an identification is very useful when d is a prime number and does not seem to be very interesting when d is not a prime integer. Note that the qudits $|0\rangle, |1\rangle, \ldots, |d-1\rangle$ are often represented by the column vectors $\phi_0, \phi_1, \ldots, \phi_{d-1}$ (given by (94)), respectively.

4.2 Mutually unbiased bases

The basis B_{ra} given by (70) can serve as another basis for qudits. For arbitrary d, the couple (B_{ra}, B_d) exhibits an interesting property. For fixed d, r and a, Eq. (71) gives

$$\forall n \in \mathbb{Z}/d\mathbb{Z},\ \forall \alpha \in \mathbb{Z}/d\mathbb{Z}\ :\ |\langle n|a\alpha;r\rangle| = \frac{1}{\sqrt{d}} \tag{171}$$

Equation (171) shows that B_{ra} and B_d are two unbiased bases.
Other examples of unbiased bases can be obtained for $d = 2$ and 3. We easily verify that the bases B_{r0} and B_{r1} for $d = 2$ given by (58) are unbiased. Similarly, the bases B_{r0}, B_{r1} and B_{r2} for $d = 3$ given by (61) are mutually unbiased. Therefore, by combining these particular results with the general result implied by (171) we end up with three MUBs for $d = 2$ and four MUBs for $d = 3$, in agreement with $N_{MUB} = d + 1$ when d is a prime number. The results for $d = 2$ and 3 can be generalized in the case where d is a prime number. This leads to the following theorem (Albouy & Kibler, 2007; Kibler, 2008; Kibler & Planat, 2006).
Theorem 3. *For $d = p$, with p a prime number, the bases B_{r0}, B_{r1}, ..., B_{rp-1}, B_p corresponding to a fixed value of r form a complete set of $p + 1$ MUBs. The p^2 vectors $|a\alpha;r\rangle$ or $\phi(a\alpha;r)$, with $a, \alpha = 0, 1, \ldots, p-1$, of the bases $B_{r0}, B_{r1}, \ldots, B_{rp-1}$ are given by a single formula, namely, Eq. (72) or (93). The index r makes it possible to distinguish different complete sets of $p + 1$ MUBs.*
The proof is as follows. First, according to (171), the computational basis B_p is unbiased with any of the p bases $B_{r0}, B_{r1}, \ldots, B_{rp-1}$. Second, we get

$$\langle a\alpha;r|b\beta;r\rangle = \frac{1}{p}\sum_{k=0}^{p-1} q^{k(p-k)(b-a)/2+k(\beta-\alpha)} \tag{172}$$

or

$$\langle a\alpha;r|b\beta;r\rangle = \frac{1}{p}\sum_{k=0}^{p-1} e^{i\pi\{(a-b)k^2+[p(b-a)+2(\beta-\alpha)]k\}/p} \tag{173}$$

The right-hand side of (173) can be expressed in terms of a generalized quadratic Gauss sum. This leads to

$$\langle a\alpha;r|b\beta;r\rangle = \frac{1}{p}S(u,v,w) \tag{174}$$

where the Gauss sum $S(u, v, w)$ is given by (160) with the parameters

$$u = a - b, \quad v = -(a-b)p - 2(\alpha - \beta), \quad w = p \tag{175}$$

which ensure that $uw + v$ is even. The generalized Gauss sum $S(u, v, w)$ in (174)-(175) can be calculated from the methods described in (Berndt et al., 1998). We thus obtain

$$|\langle a\alpha;r|b\beta;r\rangle| = \frac{1}{\sqrt{p}} \tag{176}$$

for all a, b, α, and β in $\mathbb{Z}/p\mathbb{Z}$ with $b \neq a$. This completes the proof.
Theorem 3 renders feasible to derive in one step the $(p + 1)p$ qupits (i.e., qudits with $d = p$ a prime integer) of a complete set of $p + 1$ MUBs in $\mathbb{C}^p$. The single formula (72) or (93), giving the p^2 vectors $|a\alpha;r\rangle$ or $\phi(a\alpha;r)$, with $a, \alpha = 0, 1, \ldots, p-1$, of the bases $B_{r0}, B_{r1}, \ldots, B_{rp-1}$, is easily codable on a classical computer.

Example 6: The $p = 2$ case. For $r = 0$, the $p + 1 = 3$ MUBs are

$$\begin{aligned}
B_{00} &: \quad \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad -\frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
B_{01} &: \quad i\frac{|0\rangle - i|1\rangle}{\sqrt{2}}, \quad -i\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \\
B_{2} &: \quad |0\rangle, \quad |1\rangle
\end{aligned} \tag{177}$$

cf. (59), (60), (100) and (101). The global factors -1 in B_{00} and $\pm i$ in B_{01} arise from the general formula (72); they are irrelevant for quantum information and can be omitted.

Example 7: The $p = 3$ case. For $r = 0$, the $p + 1 = 4$ MUBs are

$$\begin{aligned}
B_{00} &: \quad \frac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}, \quad \frac{q^2|0\rangle + q|1\rangle + |2\rangle}{\sqrt{3}}, \quad \frac{q|0\rangle + q^2|1\rangle + |2\rangle}{\sqrt{3}} \\
B_{01} &: \quad \frac{q|0\rangle + q|1\rangle + |2\rangle}{\sqrt{3}}, \quad \frac{|0\rangle + q^2|1\rangle + |2\rangle}{\sqrt{3}}, \quad \frac{q^2|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}} \\
B_{02} &: \quad \frac{q^2|0\rangle + q^2|1\rangle + |2\rangle}{\sqrt{3}}, \quad \frac{q|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}, \quad \frac{|0\rangle + q|1\rangle + |2\rangle}{\sqrt{3}} \\
B_{3} &: \quad |0\rangle, \quad |1\rangle, \quad |2\rangle
\end{aligned} \tag{178}$$

with $q = e^{i2\pi/3}$, cf. (62), (63), (64), (106), (108) and (110).

As a simple consequence of Theorem 3, we get the following corollary which can be derived by combining Theorem 3 with Eq. (154).

Corollary 2. *For $d = p$, with p a prime number, the $p \times p$ matrix $(\mathbf{F_{ra}})^{\dagger}\mathbf{F_{rb}}$ with $b \neq a$ ($a, b = 0, 1, \ldots, p-1$) is a generalized Hadamard matrix.*

Going back to arbitrary d, it is to be noted that for a fixed value of r, the $d + 1$ bases B_{r0}, B_{r1}, …, B_{rd-1}, B_d do not provide in general a complete set of $d + 1$ MUBs even in the case where d is a power p^e with $e \geq 2$ of a prime integer p. However, it is possible to show (Kibler, 2009) that the bases B_{0a}, $B_{0a\oplus 1}$ and B_d are three MUBs in $\mathbb{C}^d$, in agreement with $N_{MUB} \geq 3$. Therefore for d arbitrary, given two Hadamard matrices $\mathbf{F_{ra}}$ and $\mathbf{F_{sb}}$, the product $\mathbf{F_{ra}}^{\dagger}\mathbf{F_{sb}}$ is not in general a Hadamard matrix.

In the case where d is a power p^e with $e \geq 2$ of a prime integer p, tensor products of the unbiased bases B_{r0}, B_{r1}, …, B_{rp-1} can be used for generating $p^e + 1$ MUBs in dimension $d = p^e$. This can be illustrated with the following example.

Example 8: The $d = 2^2$ case. This case corresponds to a spin $j = 3/2$. The application of (45) or (72) yields four bases B_{0a} ($a = 0, 1, 2, 3$). As a point of fact, the five bases B_{00}, B_{01}, B_{02}, B_{03} and B_4 do not form a complete set of $d + 1 = 5$ MUBs ($d = 4$ is not a prime number). Nevertheless, it is possible to find five MUBs because $d = 2^2$ is the power of a prime number. This can be achieved by replacing the space $\epsilon(4)$ spanned by

$$B_4 = \{|3/2, m\rangle : m = 3/2, 1/2, -1/2, -3/2\} \quad \text{or} \quad \{|n\rangle : n = 0, 1, 2, 3\} \tag{179}$$

by the tensor product space $\epsilon(2) \otimes \epsilon(2)$ spanned by the canonical or computational basis

$$B_2 \otimes B_2 = \{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\} \tag{180}$$

The space $\epsilon(2) \otimes \epsilon(2)$ is associated with the coupling of two spin angular momenta $j_1 = 1/2$ and $j_2 = 1/2$ or two qubits (in the vector $u \otimes v$, u and v correspond to j_1 and j_2, respectively). Four of the five MUBs for $d = 4$ can be constructed from the direct products

$$|ab : \alpha\beta\rangle = |a\alpha; 0\rangle \otimes |b\beta; 0\rangle \tag{181}$$

which are eigenvectors of the operators

$$w_{ab} = v_{0a} \otimes v_{0b} \tag{182}$$

(the operators v_{0a} and v_{0b} refer to the two spaces $\epsilon(2)$, the vectors of type $|a\alpha; 0\rangle$ and $|b\beta; 0\rangle$ are given by the master formula (72) for $d = 2$). Obviously, the set

$$B_{0a0b} = \{|ab : \alpha\beta\rangle : \alpha, \beta = 0, 1\} \tag{183}$$

is an orthonormal basis in $\mathbb{C}^4$. It is evident that B_{0000} and B_{0101} are two unbiased bases since the modulus of the inner product of $|00 : \alpha\beta\rangle$ by $|11 : \alpha'\beta'\rangle$ is

$$|\langle 00 : \alpha\beta|11 : \alpha'\beta'\rangle| = |\langle 0\alpha; 0|1\alpha'; 0\rangle\langle 0\beta; 0|1\beta'; 0\rangle| = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{4}} \tag{184}$$

A similar result holds for the two bases B_{0001} and B_{0100}. However, the four bases B_{0000}, B_{0101}, B_{0001} and B_{0100} are not mutually unbiased. A possible way to overcome this no-go result is to keep the bases B_{0000} and B_{0101} intact and to re-organize the vectors inside the bases B_{0001} and B_{0100} in order to obtain four MUBs. We are thus left with four bases

$$W_{00} \equiv B_{0000}, \quad W_{11} \equiv B_{0101}, \quad W_{01}, \quad W_{10} \tag{185}$$

which together with the computational basis B_4 give five MUBs. Specifically, we have

$$\begin{aligned}
W_{00} &= \{|00 : \alpha\beta\rangle : \alpha, \beta = 0, 1\} && (186)\\
W_{11} &= \{|11 : \alpha\beta\rangle : \alpha, \beta = 0, 1\} && (187)\\
W_{01} &= \{\lambda|01 : \alpha\beta\rangle + \mu|01 : \alpha \oplus 1\beta \oplus 1\rangle : \alpha, \beta = 0, 1\} && (188)\\
W_{10} &= \{\lambda|10 : \alpha\beta\rangle + \mu|10 : \alpha \oplus 1\beta \oplus 1\rangle : \alpha, \beta = 0, 1\} && (189)
\end{aligned}$$

where

$$\lambda = \frac{1-i}{2}, \quad \mu = \frac{1+i}{2} \tag{190}$$

As a résumé, only two formulas are necessary for obtaining the $d^2 = 16$ vectors for the bases W_{ab}, namely

$$\begin{aligned}
W_{00}, W_{11} &: \quad |aa : \alpha\beta\rangle && (191)\\
W_{01}, W_{10} &: \quad \lambda|aa \oplus 1 : \alpha\beta\rangle + \mu|aa \oplus 1 : \alpha \oplus 1\beta \oplus 1\rangle && (192)
\end{aligned}$$

for all a, α and β in $\mathbb{Z}/2\mathbb{Z}$. The five MUBs are listed below as state vectors and column vectors with

$$|0\rangle \equiv |j = 1/2, m = 1/2\rangle \text{ or } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle \equiv |j = 1/2, m = -1/2\rangle \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{193}$$

The canonical basis:

$$|0\rangle \otimes |0\rangle, \quad |0\rangle \otimes |1\rangle, \quad |1\rangle \otimes |0\rangle, \quad |1\rangle \otimes |1\rangle$$

or in column vectors

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{194}$$

The W_{00} basis:

$$|00:00\rangle = +\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |00:01\rangle = -\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$
$$|00:10\rangle = -\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |00:11\rangle = +\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

or in developed form

$$\begin{aligned}
|00:00\rangle &= +\frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)\\
|00:01\rangle &= -\frac{1}{2}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)\\
|00:10\rangle &= -\frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)\\
|00:11\rangle &= +\frac{1}{2}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)
\end{aligned}$$

or in column vectors

$$\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad -\frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \quad -\frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} \tag{195}$$

The W_{11} basis:

$$|11:00\rangle = -\frac{|0\rangle - i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - i|1\rangle}{\sqrt{2}}, \quad |11:01\rangle = +\frac{|0\rangle - i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$
$$|11:10\rangle = +\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - i|1\rangle}{\sqrt{2}}, \quad |11:11\rangle = -\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

or in developed form

$$\begin{aligned}
|11:00\rangle &= -\frac{1}{2}(|0\rangle \otimes |0\rangle - i|0\rangle \otimes |1\rangle - i|1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)\\
|11:01\rangle &= +\frac{1}{2}(|0\rangle \otimes |0\rangle + i|0\rangle \otimes |1\rangle - i|1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)\\
|11:10\rangle &= +\frac{1}{2}(|0\rangle \otimes |0\rangle - i|0\rangle \otimes |1\rangle + i|1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)\\
|11:11\rangle &= -\frac{1}{2}(|0\rangle \otimes |0\rangle + i|0\rangle \otimes |1\rangle + i|1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)
\end{aligned}$$

or in column vectors

$$-\frac{1}{2}\begin{pmatrix}1\\-i\\-i\\-1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\i\\-i\\1\end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix}1\\-i\\i\\1\end{pmatrix}, \quad -\frac{1}{2}\begin{pmatrix}1\\i\\i\\-1\end{pmatrix} \tag{196}$$

The W_{01} basis:

$$\begin{aligned}
\lambda|01:00\rangle + \mu|01:11\rangle &= +\mu\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - i|1\rangle}{\sqrt{2}} - \lambda\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + i|1\rangle}{\sqrt{2}}\\
\mu|01:00\rangle + \lambda|01:11\rangle &= -\lambda\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - i|1\rangle}{\sqrt{2}} + \mu\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + i|1\rangle}{\sqrt{2}}\\
\lambda|01:01\rangle + \mu|01:10\rangle &= -\mu\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + i|1\rangle}{\sqrt{2}} + \lambda\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - i|1\rangle}{\sqrt{2}}\\
\mu|01:01\rangle + \lambda|01:10\rangle &= +\lambda\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + i|1\rangle}{\sqrt{2}} - \mu\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - i|1\rangle}{\sqrt{2}}
\end{aligned}$$

or in developed form

$$\begin{aligned}
\lambda|01:00\rangle + \mu|01:11\rangle &= +\frac{i}{2}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle - i|1\rangle \otimes |0\rangle - i|1\rangle \otimes |1\rangle)\\
\mu|01:00\rangle + \lambda|01:11\rangle &= +\frac{i}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + i|1\rangle \otimes |0\rangle - i|1\rangle \otimes |1\rangle)\\
\lambda|01:01\rangle + \mu|01:10\rangle &= -\frac{i}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle - i|1\rangle \otimes |0\rangle + i|1\rangle \otimes |1\rangle)\\
\mu|01:01\rangle + \lambda|01:10\rangle &= -\frac{i}{2}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + i|1\rangle \otimes |0\rangle + i|1\rangle \otimes |1\rangle)
\end{aligned}$$

or in column vectors

$$\frac{i}{2}\begin{pmatrix}1\\-1\\-i\\-i\end{pmatrix}, \quad \frac{i}{2}\begin{pmatrix}1\\1\\i\\-i\end{pmatrix}, \quad -\frac{i}{2}\begin{pmatrix}1\\1\\-i\\i\end{pmatrix}, \quad -\frac{i}{2}\begin{pmatrix}1\\-1\\i\\i\end{pmatrix}, \tag{197}$$

The W_{10} basis:

$$\begin{aligned}
\lambda|10:00\rangle + \mu|10:11\rangle &= +\mu\frac{|0\rangle - i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} - \lambda\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\\
\mu|10:00\rangle + \lambda|10:11\rangle &= -\lambda\frac{|0\rangle - i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \mu\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\\
\lambda|10:01\rangle + \mu|10:10\rangle &= -\mu\frac{|0\rangle - i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} + \lambda\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}\\
\mu|10:01\rangle + \lambda|10:10\rangle &= +\lambda\frac{|0\rangle - i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} - \mu\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}
\end{aligned}$$

or in developed form

$$\lambda|10:00\rangle + \mu|10:11\rangle = +\frac{i}{2}(|0\rangle \otimes |0\rangle - i|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle - i|1\rangle \otimes |1\rangle)$$
$$\mu|10:00\rangle + \lambda|10:11\rangle = +\frac{i}{2}(|0\rangle \otimes |0\rangle + i|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle - i|1\rangle \otimes |1\rangle)$$
$$\lambda|10:01\rangle + \mu|10:10\rangle = -\frac{i}{2}(|0\rangle \otimes |0\rangle + i|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle + i|1\rangle \otimes |1\rangle)$$
$$\mu|10:01\rangle + \lambda|10:10\rangle = -\frac{i}{2}(|0\rangle \otimes |0\rangle - i|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + i|1\rangle \otimes |1\rangle)$$

or in column vectors

$$\frac{i}{2}\begin{pmatrix} 1 \\ -i \\ -1 \\ -i \end{pmatrix}, \quad \frac{i}{2}\begin{pmatrix} 1 \\ i \\ 1 \\ -i \end{pmatrix}, \quad -\frac{i}{2}\begin{pmatrix} 1 \\ i \\ -1 \\ i \end{pmatrix}, \quad -\frac{i}{2}\begin{pmatrix} 1 \\ -i \\ 1 \\ i \end{pmatrix} \tag{198}$$

The five preceding bases are of central importance in quantum information for expressing any ququart or quartic (corresponding to $d = 4$) in terms of qudits (corresponding to $d = 2$). It is to be noted that the vectors of the W_{00} and W_{11} bases are not intricated (i.e., each vector is the direct product of two vectors) while the vectors of the W_{01} and W_{10} bases are intricated (i.e., each vector is not the direct product of two vectors). To be more precise, the degree of intrication of the state vectors for the bases W_{00}, W_{11}, W_{01} and W_{10} can be determined in the following way. In arbitrary dimension d, let

$$|\Phi\rangle = \sum_{k=0}^{d-1}\sum_{l=0}^{d-1} a_{kl}|k\rangle \otimes |l\rangle \tag{199}$$

be a double qudit state vector. Then, it can be shown that the determinant of the $d \times d$ matrix $A = (a_{kl})$ satisfies

$$0 \leq |\det A| \leq \frac{1}{\sqrt{d^d}} \tag{200}$$

as discussed in (Albouy, 2009). The case $\det A = 0$ corresponds to the absence of *global* intrication while the case

$$|\det A| = \frac{1}{\sqrt{d^d}} \tag{201}$$

corresponds to a maximal intrication. As an illustration, we obtain that all the state vectors for W_{00} and W_{11} are not intricated and that all the state vectors for W_{01} and W_{10} are maximally intricated.

Generalization of (191) and (192) can be obtained in more complicated situations (two qupits, three qubits, ...). The generalization of (191) is immediate. The generalization of (192) can be achieved by taking linear combinations of vectors such that each linear combination is made of vectors corresponding to the same eigenvalue of the relevant tensor product of operators of type v_{0a}.

4.3 Mutually unbiased bases and Lie agebras

4.3.1 Generalized Pauli matrices

We now examine the interest for quantum information of the Weyl pair $(\mathbf{X}, \mathbf{Z})$ introduced in section 3.1.4. The linear operators corresponding to the matrices $\mathbf{X}$ and $\mathbf{Z}$ are known in quantum information and quantum computing as shift and clock operators, respectively. (Note however that for each of the operators x and z, the *shift* or *clock* character depends on which state the operator acts. The qualification adopted in quantum information and quantum computing corresponds to the action of x and z on the computational basis B_d.) For d arbitrary, they are at the root of the Pauli group P_d, a finite subgroup of $U(d)$ (see section 3.1.4). The normaliser of P_d in $U(d)$ is a Clifford-type group in d dimensions noted Cl_d. More precisely, Cl_d is the set $\{\mathbf{U} \in U(d) | \mathbf{U} P_d \mathbf{U}^\dagger = P_d\}$ endowed with matrix multiplication (the elements of P_d being expressed in terms of the matrices $\mathbf{X}$ and $\mathbf{Z}$). The Pauli group P_d, as well as any other invariant subgroup of Cl_d, is of considerable importance for describing quantum errors and quantum fault tolerance in quantum computing (see (Havlíček & Saniga, 2008; Planat, 2010; Planat & Kibler, 2010) and references therein for recent geometrical approaches to the Pauli group). These concepts are very important in the case of n-qubit systems (corresponding to $d = 2^n$).

The Weyl pair $(\mathbf{X}, \mathbf{Z})$ turns out to be an integrity basis for generating the set $\{\mathbf{X}^a \mathbf{Z}^b : a, b \in \mathbb{Z}/d\mathbb{Z}\}$ of d^2 generalized Pauli matrices in d dimensions (see for instance (Bandyopadhyay et al., 2002; Gottesman et al., 2001; Kibler, 2008; Lawrence et al., 2002; Pittenger & Rubin, 2004) in the context of MUBs and (Balian & Itzykson, 1986; Patera & Zassenhaus, 1988; Šťovíček & Tolar, 1984) in group-theoretical contexts). As seen in section 3.1.4, the latter set constitutes a basis for the Lie algebra of the linear group $GL(d, \mathbb{C})$ (or its unitary restriction $U(d)$) with respect to the commutator law. Let us give two examples of these important generalized Pauli matrices.

Example 9: The $d = 2$ case. For $d = 2 \Leftrightarrow j = 1/2$ ($\Rightarrow q = -1$), the matrices of the four operators u_{ab} with $a, b = 0, 1$ are

$$\mathbf{I}_2 = \mathbf{X}^0 \mathbf{Z}^0, \quad \mathbf{X} = \mathbf{X}^1 \mathbf{Z}^0, \quad \mathbf{Y} = \mathbf{X}^1 \mathbf{Z}^1, \quad \mathbf{Z} = \mathbf{X}^0 \mathbf{Z}^1 \tag{202}$$

or in terms of the matrices $\mathbf{V_{0a}}$

$$\mathbf{I}_2 = (\mathbf{V_{00}})^2, \quad \mathbf{X} = \mathbf{V_{00}}, \quad \mathbf{Y} = \mathbf{V_{01}}, \quad \mathbf{Z} = (\mathbf{V_{00}})^\dagger \mathbf{V_{01}} \tag{203}$$

In detail, we get

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{204}$$

Alternatively, we have

$$\mathbf{I}_2 = \sigma_0, \quad \mathbf{X} = \sigma_x, \quad \mathbf{Y} = -i\sigma_y, \quad \mathbf{Z} = \sigma_z \tag{205}$$

in terms of the usual (Hermitian and unitary) Pauli matrices σ_0, σ_x, σ_y and σ_z

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{206}$$

The approach developed here leads to generalized Pauli matrices in dimension 2 that differ from the usual Pauli matrices. This is the price one has to pay in order to get a systematic

generalization of Pauli matrices in arbitrary dimension. It should be observed that the commutation and anti-commutation relations given by (133) and (134) with $d = 2$ correspond to the well-known commutation and anti-commutation relations for the usual Pauli matrices transcribed in the normalization $\mathbf{X}^1\mathbf{Z}^0 = \sigma_x$, $\mathbf{X}^1\mathbf{Z}^1 = -i\sigma_y$, $\mathbf{X}^0\mathbf{Z}^1 = \sigma_z$.
From a group-theoretical point of view, the matrices $\mathbf{I}_2$, $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ can be considered as generators of the group $U(2)$. On the other hand, the Pauli group P_2 contains eight elements; due to the factor $-i$ in $\mathbf{Y} = -i\sigma_y$, the group P_2 is isomorphic to the group of hyperbolic quaternions rather than to the group of ordinary quaternions.
In terms of column vectors, the vectors of the bases B_{00}, B_{01} and B_2 (see (177)) are eigenvevtors of σ_x, σ_y and σ_z, respectively (for each matrix the eigenvalues are 1 and -1).
Example 10: The $d = 3$ case. For $d = 3 \Leftrightarrow j = 1$ ($\Rightarrow q = e^{i2\pi/3}$), the matrices of the nine operators u_{ab} with $a, b = 0, 1, 2$, viz.,

$$\begin{aligned} &\mathbf{X}^0\mathbf{Z}^0 = \mathbf{I}_3, \quad \mathbf{X}^1\mathbf{Z}^0 = \mathbf{X}, \quad \mathbf{X}^2\mathbf{Z}^0 = \mathbf{X}^2 \\ &\mathbf{X}^0\mathbf{Z}^1 = \mathbf{Z}, \quad \mathbf{X}^0\mathbf{Z}^2 = \mathbf{Z}^2, \quad \mathbf{X}^1\mathbf{Z}^1 = \mathbf{X}\mathbf{Z} \\ &\mathbf{X}^2\mathbf{Z}^2, \quad \mathbf{X}^2\mathbf{Z}^1 = \mathbf{X}^2\mathbf{Z}, \quad \mathbf{X}^1\mathbf{Z}^2 = \mathbf{X}\mathbf{Z}^2 \end{aligned} \tag{207}$$

are

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{X}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \quad \mathbf{Z}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad \mathbf{X}\mathbf{Z} = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & q^2 \\ 1 & 0 & 0 \end{pmatrix} \tag{208}$$

$$\mathbf{X}^2\mathbf{Z}^2 = \begin{pmatrix} 0 & 0 & q \\ 1 & 0 & 0 \\ 0 & q^2 & 0 \end{pmatrix}, \quad \mathbf{X}^2\mathbf{Z} = \begin{pmatrix} 0 & 0 & q^2 \\ 1 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad \mathbf{X}\mathbf{Z}^2 = \begin{pmatrix} 0 & q^2 & 0 \\ 0 & 0 & q \\ 1 & 0 & 0 \end{pmatrix}$$

The generalized Pauli matrices (208) differ from the Gell-Mann matrices used in elementary particle physics. They constitute another extension of the Pauli matrices in dimension $d = 3$ of interest for the Lie group $U(3)$ and the Pauli group P_3.
In terms of column vectors, the vectors of the bases B_{00}, B_{01}, B_{02} and B_3 (see (178)) are eigenvectors of $\mathbf{X}$, $\mathbf{XZ}$, $\mathbf{XZ}^2$ and $\mathbf{Z}$, respectively (for each matrix the eigenvalues are 1, q and q^2).

4.3.2 MUBs and the special linear group

In the case where d is a prime integer or a power of a prime integer, it is known that the set $\{\mathbf{X}^a\mathbf{Z}^b : a, b = 0, 1, \ldots, d-1\}$ of cardinality d^2 can be partitioned into $d+1$ subsets containing each $d-1$ commuting matrices (cf. (Bandyopadhyay et al., 2002)). Let us give an example before going to the case where d is an arbitrary prime number.
Example 11: The $d = 5$ case. For $d = 5$, we have the six following sets of four commuting matrices

$$\begin{aligned} &\mathcal{V}_0 = \{01, 02, 03, 04\}, \quad \mathcal{V}_1 = \{10, 20, 30, 40\} \\ &\mathcal{V}_2 = \{11, 22, 33, 44\}, \quad \mathcal{V}_3 = \{12, 24, 31, 43\} \\ &\mathcal{V}_4 = \{13, 21, 34, 42\}, \quad \mathcal{V}_5 = \{14, 23, 32, 41\} \end{aligned} \tag{209}$$

where ab is used as an abbreviation of $\mathbf{X}^a\mathbf{Z}^b$.

Proposition 16. *For $d = p$ with p a prime integer, the $p+1$ sets of $p-1$ commuting matrices are easily seen to be*

$$\begin{aligned}
\mathcal{V}_0 &= \{\mathbf{X}^0\mathbf{Z}^a : a = 1,2,\ldots,p-1\} \\
\mathcal{V}_1 &= \{\mathbf{X}^a\mathbf{Z}^0 : a = 1,2,\ldots,p-1\} \\
\mathcal{V}_2 &= \{\mathbf{X}^a\mathbf{Z}^a : a = 1,2,\ldots,p-1\} \\
\mathcal{V}_3 &= \{\mathbf{X}^a\mathbf{Z}^{2a} : a = 1,2,\ldots,p-1\} \\
&\vdots \\
\mathcal{V}_{p-1} &= \{\mathbf{X}^a\mathbf{Z}^{(p-2)a} : a = 1,2,\ldots,p-1\} \\
\mathcal{V}_p &= \{\mathbf{X}^a\mathbf{Z}^{(p-1)a} : a = 1,2,\ldots,p-1\}
\end{aligned} \tag{210}$$

Each of the $p+1$ sets $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_p$ can be put in a one-to-one correspondance with one basis of the complete set of $p+1$ MUBs. In fact, $\mathcal{V}_0$ is associated with the computational basis B_p; furthermore, in view of

$$\mathbf{V_{0a}} \in \mathcal{V}_{a+1}, \quad a = 0,1,\ldots,p-1 \tag{211}$$

it follows that $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_p$ are associated with the p remaining MUBs B_{00}, B_{01}, ..., B_{0p-1}, respectively.

Keeping into account the fact that the set $\{\mathbf{X}^a\mathbf{Z}^b : a,b = 0,1,\ldots,p-1\} \setminus \{\mathbf{X}^0\mathbf{Z}^0\}$ spans the Lie algebra of the special linear group $SL(p,\mathbb{C})$, we have the next theorem.

Theorem 4. *For $d = p$ with p a prime integer, the Lie algebra $sl(p,\mathbb{C})$ of the group $SL(p,\mathbb{C})$ can be decomposed into a sum (vector space sum indicated by $\uplus$) of $p+1$ abelian subalgebras each of dimension $p-1$, i.e.,*

$$sl(p,\mathbb{C}) \simeq v_0 \uplus v_1 \uplus \ldots \uplus v_p \tag{212}$$

where the $p+1$ subalgebras $v_0, v_1, \ldots, v_p$ are Cartan subalgebras generated respectively by the sets $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_p$ containing each $p-1$ commuting matrices.

The latter result can be extended when $d = p^e$ with p a prime integer and e an integer ($e \geq 2$): there exists a decomposition of $sl(p^e,\mathbb{C})$ into p^e+1 abelian subalgebras of dimension p^e-1 (cf. (Boykin et al., 2007; Kibler, 2009; Patera & Zassenhaus, 1988)).

5. Conclusion

The quadratic discrete Fourier transform studied in this chapter can be considered as a two-parameter extension, with a quadratic term, of the usual discrete Fourier transform. In the case where the two parameters are taken to be equal to zero, the quadratic discrete Fourier transform is nothing but the usual discrete Fourier transform. The quantum quadratic discrete Fourier transform plays an important role in the field of quantum information. In particular, such a transformation in prime dimension can be used for obtaining a complete set of mutually unbiased bases. It is to be mentioned that the quantum quadratic discrete Fourier transform also arises in the determination of phase operators for the groups $SU(2)$ and $SU(1,1)$ in connection with the representations of a generalized oscillator algebra (Atakishiyev et al., 2010; Daoud & Kibler, 2010). As an open question, it should be worth investigating the relation between the quadratic discrete Fourier transform and the Fourier transform on a finite ring or a finite field.

Acknowledgements

The author is greatly indebted to M. Daoud, M. Planat, O. Albouy, M. Saniga, N.M. Atakishiyev and K.B. Wolf for collaboration on various facets of the material contained in this chapter.

Appendix: Wigner-Racah algebra of $SU(2)$ in the $\{j^2, x\}$ scheme

In this self-contained Appendix, the bar does not indicate complex conjugation. Here, complex conjugation is denoted with a star.
The Wigner-Racah algebra of the group $SU(2)$ in the $SU(2) \supset U(1)$ or $\{j^2, j_z\}$ scheme is well known. It corresponds to the use of bases of type B_{2j+1} resulting from the simultaneous diagonalization of the Casimir operator j^2 and of the Cartan generator j_z of $SU(2)$. Any change of basis of type

$$|j,\mu\rangle = \sum_{m=-j}^{j} |j,m\rangle\langle j,m|j,\mu\rangle \tag{213}$$

(where for fixed j the elements $\langle j,m|j,\mu\rangle$ define a $(2j+1) \times (2j+1)$ unitary matrix) leads to another acceptable scheme for the Wigner-Racah algebra of $SU(2)$. In this scheme, the matrices of the irreducible representation classes of $SU(2)$ take a new form as well as the coupling coefficients (and the associated $3-jm$ symbols). For instance, the Clebsch-Gordan or coupling coefficients $(j_1 j_2 m_1 m_2|jm)$ are simply replaced by

$$(j_1 j_2 \mu_1 \mu_2|j\mu) = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m=-j}^{j} (j_1 j_2 m_1 m_2|jm) \\ \langle j_1, m_1|j_1, \mu_1\rangle^* \langle j_2, m_2|j_2, \mu_2\rangle^* \langle j, m|j, \mu\rangle \tag{214}$$

when passing from the $\{jm\}$ quantization to the $\{j\mu\}$ quantization while the recoupling coefficients, and the corresponding $3(n-1)-j$ symbols, for the coupling of n ($n \geq 3$) angular momenta remain invariant. The adaptation to the $\{j\mu\}$ quantization scheme afforded by Eq. (213) is transferable to $SU(2)$ irreducible tensor operators. This yields the Wigner-Eckart theorem in the $\{j\mu\}$ scheme.
We give here the basic ingredients for developing the Wigner-Racah algebra of $SU(2)$ in the $\{j^2, v_{00}\}$ or $\{j^2, x\}$ scheme. For such a scheme, the vector $|j,\mu\rangle$ is of the form $|j\alpha;00\rangle$ so that the label μ can be identified with α. Thus, the inter-basis expansion coefficients $\langle j,m|j,\mu\rangle$ are

$$\langle j,m|j\alpha;00\rangle = \frac{1}{\sqrt{2j+1}} q^{(j+m)\alpha} = \frac{1}{\sqrt{2j+1}} \exp\left[\frac{2\pi \mathrm{i}}{2j+1}(j+m)\alpha\right] \tag{215}$$

with $m = j, j-1, \ldots, -j$ and $\alpha = 0, 1, \ldots, 2j$. Equation (215) corresponds to the unitary transformation (45) with $r = a = 0$, that allows to pass from the standard basis B_{2j+1} to the nonstandard basis B_{00}. Then, the Clebsch-Gordan coefficients in the $\{j^2, v_{00}\}$ scheme are

$$(j_1 j_2 \alpha_1 \alpha_2|j_3 \alpha_3) = \frac{1}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}} \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \\ (q_1)^{-(j_1+m_1)\alpha_1} (q_2)^{-(j_2+m_2)\alpha_2} (q_3)^{(j_3+m_3)\alpha_3} (j_1 j_2 m_1 m_2|j_3 m_3) \tag{216}$$

where the various q_k are given in terms of j_k by

$$q_k = \exp\left(\frac{2\pi \mathrm{i}}{2j_k+1}\right), \quad k = 1,2,3 \tag{217}$$

The symmetry properties of the coupling coefficients $(j_1 j_2 \alpha_1 \alpha_2 | j_3 \alpha_3)$ cannot be expressed in a simple way (except the symmetry under the interchange $j_1\alpha_1 \leftrightarrow j_2\alpha_2$). Therefore, it is interesting to introduce the following $\overline{f}$ symbol through

$$\overline{f}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = \frac{1}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}} \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3}$$
$$(q_1)^{-(j_1+m_1)\alpha_1} (q_2)^{-(j_2+m_2)\alpha_2} (q_3)^{-(j_3+m_3)\alpha_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{218}$$

where the $3-jm$ symbol on the right-hand side of (218) is an ordinary Wigner symbol for the $SU(2)$ group in the $\{j^2, j_z\}$ scheme. (The $\overline{f}$ symbol is to the $\{j^2, x\}$ scheme what the $\overline{V}$ symbol of Racah is to the $\{j^2, j_z\}$ scheme.) The $\overline{f}$ symbol exhibits the same symmetry properties under permutations of its columns as the $3-jm$ Wigner symbol (identical to the $\overline{V}$ Racah symbol): Its value is multiplied by $(-1)^{j_1+j_2+j_3}$ under an odd permutation and does not change under an even permutation. In contrast to the $3-jm$ symbol, not all the values of the $\overline{f}$ symbol are real. In this respect, the $\overline{f}$ symbol behaves under complex conjugation as

$$\overline{f}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}^* = (-1)^{j_1+j_2+j_3} (q_1)^{\alpha_1} (q_2)^{\alpha_2} (q_3)^{\alpha_3}\, \overline{f}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \tag{219}$$

Other properties (e.g., orthogonality properties, connection with the Clebsch-Gordan coefficients and the Herring-Wigner tensor, etc.) of the $\overline{f}$ symbol and its relations with $3(n-1)-j$ symbols for $n \geq 3$ can be derived along the lines developed in (Kibler, 1968).

6. References

Albouy, O. (2009). Discrete algebra and geometry applied to the Pauli group and mutually unbiased bases in quantum information theory. *Thesis*, (2009), Université de Lyon

Albouy, O. & Kibler, M.R. (2007). SU(2) nonstandard bases: case of mutually unbiased bases. *SIGMA*, 3, (2007) 076

Arik, M. & Coon, D.D. (1976). Hilbert spaces of analytic functions and generalized coherent states. *J. Math. Phys.*, 17, (1976) 524–527

Atakishiyev, N.M.; Kibler, M.R. & Wolf, K.B. (2010). SU(2) and SU(1,1) approaches to phase operators and temporally stable phase states: applications to mutually unbiased bases and discrete Fourier Transforms. *Symmetry*, 2, (2010) 1–24

Balian, R. & Itzykson, C. (1986). Observations sur la mécanique quantique finie. *C. R. Acad. Sci. (Paris)*, 303, (1986) 773–778

Bandyopadhyay, S.; Boykin, P.O.; Roychowdhury, V. & Vatan, F. (2002). A new proof for the existence of mutually unbiased bases. *Algorithmica*, 34, (2002) 512–528

Beckers, J. & Debergh, N. (1990). Parastatistics and supersymmetry in quantum-mechanics. *Nucl. Phys. B*, 340, (1990) 767–776

Bengtsson, I.; Bruzda, W.; Ericsson, Å.; Larsson, J.-Å.; Tadej, W. & Życkowski, K. (2007). Mutually unbiased bases and Hadamard matrices of order six. *J. Math. Phys.*, 48, (2007) 052106

Berndt, B.C.; Evans, R.J. & Williams, K.S. (1998). *Gauss and Jacobi Sums*, Wiley, New York

Boykin, P.O.; Sitharam, M.; Tiep, P.H. & Wocjan, P. (2007). Mutually unbiased bases and orthogonal decompositions of Lie algebras. *Quantum Inf. Comput.*, 7, (2007) 371–382

Brierley, S. & Weigert, S. (2009). Constructing mutually unbiased bases in dimension six. *Phys. Rev. A*, 79, (2009) 052316

Calderbank, A.R.; Cameron, P.J.; Kantor, W.M. & Seidel, J.J. (1997). Z4-Kerdock codes, orthogonal spreads, and extremal Euclidean line-sets. *Proc. London Math. Soc.*, 75, (1997) 436–480

Cerf, N.J.; Bourennane, M.; Karlsson, A. & Gisin, N. (2002). *Phys. Rev. Lett.*, 88, (2002) 127902

Chaichian, M. & Ellinas, D. (1990). On the polar decomposition of the quantum SU(2) algebra. *J. Phys. A: Math. Gen.*, 23, (1990) L291–L296

Champion, J.P.; Pierre, G.; Michelot, F. & Moret-Bailly, J. (1977). Composantes cubiques normales des tenseurs sphériques, *Can. J. Phys.*, 55, (2977) 512–520

Daoud, M.; Hassouni, Y. & Kibler, M. (1998). The k-fermions as objects interpolating between fermions and bosons. In: *Symmetries in Science X*, B. Gruber & M. Ramek (Eds.), 63–77, Plenum Press, New York

Daoud, M. & Kibler, M.R. (2010). Phase operators, temporally stable phase states, mutually unbiased bases and exactly solvable quantum systems. *J. Phys. A: Math. Theor.*, 43, (2010) 115303

Durand, S. (1993). Fractional superspace formulation of generalized mechanics. *Modern Phys. Lett. A*, 8, (1993) 2323–2334

Durt, T.; Englert, B.-G.; Bengtsson, I. & Życkowski, K. (2010). On mutually unbiased bases. *Internat. J. Quantum Info.*, 8, (2010) 535–640

Englert, B.-G. & Aharonov, Y. (2001). The mean king's problem: prime degrees of freedom. *Phys. Lett. A*, 284, (2001) 1–5

Fairlie, D.B.; Fletcher, P.; Zachos, C.K. (1990). Infinite-dimensional algebras and a trigonometric basis for the classical Lie-algebras. *J. Math. Phys.*, 31, (1990) 1088–1094

Gibbons, K.S.; Hoffman, M.J. & Wootters, W.K. (2010). Discrete phase space based on finite fields. *Phys. Rev. A*, 70, (2004) 062101

Gottesman, D.; Kitaev, A. & Preskill, J. (2001). Encoding a qubit in an oscillator. *Phys. Rev. A*, 64, (2001) 012310

Grassl, M. (2005). Tomography of quantum states in small dimensions. *Elec. Notes Discrete Math.*, 20, (2005) 151–164

Havlíček, H. & Saniga, M. (2008). Projective ring line on an arbitrary single qudit. *J. Phys. A: Math. Theor.*, 41, (2008) 015302

Ivanović, I.D. (1981). Geometrical description of quantum state determination. *J. Phys. A: Math. Gen.*, 14, (1981) 3241–3245

Khare, A. (1993). Parasupersymmetry in quantum mechanics. *J. Math. Phys.*, 34, (1993) 1277–1294

Kibler, M. (1968). Ionic and paramagnetic energy levels: algebra. *J. Molec. Spectrosc.*, 26, (1968) 111–130

Kibler, M.R. (2008). Variations on a theme of Heisenberg, Pauli and Weyl. *J. Phys. A: Math. Theor.*, 41, (2008) 375302

Kibler, M.R. (2009). An angular momentum approach to quadratic Fourier transform, Hadamard matrices, Gauss sums, mutually unbiased bases, unitary group and Pauli group. *J. Phys. A: Math. Theor.*, 42, (2009) 353001

Kibler, M.R. & Planat, M. (2006). A SU(2) recipe for mutually unbiased bases. *Internat. J. Modern Phys. B*, 20, (2006) 1802–1807

Klishevich, S. & Plyushchay, T. (1999). Supersymmetry of parafermions. *Modern Phys. Lett. A*, 14, (1999) 2739–2752

Lawrence, J.; Brukner, Č. & Zeilinger, A. (2002). Mutually unbiased binary observable sets on N qubits. *Phys. Rev. A*, 65, (2002) 032320

Lévy-Leblond, J.-M. (1973). Azimuthal quantization of angular momentum. *Rev. Mex. Física*, 22, (1973) 15–23

Mehta, M.L. (1987). Eigenvalues and eigenvectors of the finite Fourier transform. *J. Math. Phys.*, 28, (1987) 781–785

Patera, J. & Winternitz, P. (1976). On bases for irreducible representations of O(3) suitable for systems with an arbitrary finite symmetry group. *J. Chem. Phys.*, 65, (1976) 2725–2731

Patera, J. & Zassenhaus, H. (1988). The Pauli matrices in n dimensions and finest gradings of simple Lie algebras of type A_{n-1}. *J. Math. Phys.*, 29, (1988) 665–673

Pittenger, A.O. & Rubin, M.H. (2004). Mutually unbiased bases, generalized spin matrices and separability. *Linear Algebr. Appl.*, 390, (2004) 255–278

Planat, M. (2010). Pauli graphs when the Hilbert space dimension contains a square: why the Dedekind psi function ? arXiv:1009.3858

Planat, M. & Kibler, M. (2010). Unitary reflection groups for quantum fault tolerance. *J. Comput. Theor. Nanosci.*, 7, (2010) 1–12

Rubakov, V.A. & Spiridonov, V.P. (1988). Parasupersymmetric quantum-mechanics. *Modern Phys. Lett. A*, 3, (1988) 1337–1347

Schwinger, J. (1960). Unitary operator bases. *Proc. Nat. Acad. Sci. USA*, 46, (1960) 570–579

Schwinger, J. (1965). On angular momentum. In: *Quantum theory of angular momemtum*, L.C. Biedenharn & H. van Dam, (Eds.), 55–66, Academic Press, New York

Šťovíček, P. & Tolar, J. (1984). Quantum mechanics in a discrete space-time. *Rep. Math. Phys.*, 20, (1984) 157–170

Tolar, J. & Chadzitaskos, G. (2009). Feynman's path integral and mutually unbiased bases. *J. Phys. A: Math. Theor.*, 42, (2009) 245306

Vourdas, A. (1990). SU(2) and SU(1,1) phase states. *Phys. Rev. A*, 41, (1990) 1653–1661

Vourdas, A. (2004). Quantum systems with finite Hilbert space. *Rep. Prog. Phys.*, 67, (2004) 267–320

Weyl, H. (1931) *The Theory of Groups and Quantum Mechanics*, Dover Publications, New York

Wootters, W.K. & Fields, B.D. (1989). Optimal state-determination by mutually unbiased measurements. *Ann. Phys. (N.Y.)*, 191, (1989) 363–381