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Abstract

One-dimensional cut-and-project point sets obtained from the square lattice in the plane are considered from a unifying point of view and in the perspective of aperiodic wavelet constructions. We successively examine their geometrical aspects, combinatorial properties from the point of view of the theory of languages, and self-similarity with algebraic scaling factor θ. We explain the relation of the cut-and-project sets to non-standard numeration systems based on θ. We finally examine the substitutivity, a weakened version of substitution invariance, which provides us with an algorithm for symbolic generation of cut-and-project sequences.

Classification

52C23, 42C40, 68R15, 11Z05

Keywords

Multiresolution, wavelet, Pisot number, cut-and-project set, quasicrystal, self-similarity, substitution, combinatorics on words

1 Introduction

Initially introduced by Y. Meyer [48, 49] in the context of Harmonic Analysis and more specifically of *harmonious sets*, the cut-and-project sets or model sets have become during

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the two last decades a kind of geometrical paradigm in quasicrystalline studies. Quasicrystals are those alloys whose first sample was discovered in 1982 by Shechtman, Blech, Gratias, and Cahn [61], namely the alloy $Al_{0.86}Mn_{0.14}$, characterized by

i) a diffraction pattern like a dense constellation of more-or-less bright spots, which is an indication of a long-range order,

ii) a spatial organisation of those Bragg peaks obeying five- or ten-fold symmetries, at least locally, which indicates a sort of icosahedral organisation in real space with five-fold symmetries,

iii) a spatial organisation of those Bragg peaks obeying specific scale invariance, more precisely invariance under dilations by a factor equal to some power of the golden mean $\tau = \frac{1+\sqrt{5}}{2}$ and manifestly consistent with the five-fold symmetry since $\tau = 2\cos\frac{2\pi}{10}$.

One-dimensional examples which are usually presented as toy geometrical models of quasicrystals [40] appertain to the so-called Fibonacci chain family. They are discrete quasiperiodic subsets of the real line and are often presented as an illustration of the cut-and-project method, mainly developed in this context by [23, 33]. Consider a semi-open band $\mathcal{B}$ obtained by translating the unit square through the square lattice $\mathbb{Z}^2$ along the straight line D_1 of slope ε. D_1 is referred to as a "cut" or "parallel" space or "physical space". Then project on D_1 and along a straight line D_2 the lattice points lying in $\mathcal{B}$. Note that the latter points belong to a unique path made of horizontal segments (A) and vertical segments (B). The resulting sequence of points lying in D_1 are the nodes of a specific Fibonacci chain if $\varepsilon = \frac{1}{\tau}$ and $D_2 = D_1^{\perp}$. Let us denote this set of nodes by $\mathcal{F}$. The chain itself is made of the projected paths and reads $\ldots ABAABABAABAA\ldots$. Note that a short link B is never adjacent to another B whereas two adjacent long links A can occur. D_2 is called the "internal" space, and $\mathcal{B}\cap D_2$ is the "window" or "acceptance zone", or also "atomic surface".

The set of Fibonacci nodes is equivalently obtained through a purely algebraic filtering procedure. Let us first consider the so-called extension ring of the algebraic integer τ:

$$\mathbb{Z}[\tau] = \{x = m + n\tau \mid m, n \in \mathbb{Z}\} = \mathbb{Z} + \mathbb{Z}\tau.$$

It can be obtained as the projection onto D_1 and along D_2 of the whole square lattice $\mathbb{Z}^2$. There exists in this type of a ring an algebraic conjugation, called Galois automorphism, and defined by:

$$x = m + n\tau \quad \mapsto \quad x' = m + n\tau',$$

where $\tau' = -\frac{1}{\tau} = \frac{1-\sqrt{5}}{2}$ is the other root of the golden mean equation $x^2 = x + 1$. Then define the point set $\Sigma(\Omega)$ using an internal sieving rule in the ring $\mathbb{Z}[\tau]$ itself [52]:

$$\Sigma(\Omega) = \left\{ x = m + n\tau \in \mathbb{Z}[\tau] \;\middle|\; x' = m - n\frac{1}{\tau} \in \Omega \right\} = (\mathbb{Z}[\tau] \cap \Omega)'.$$

The Fibonacci point set $\mathcal{F}$ in the above, with link lengths $A = \tau^2$, $B = \tau$, is precisely that set $\Sigma(\Omega)$ with $\Omega = [0, 1)$.

As was previously mentioned, self-similarity plays a fundamental structural role in the existence of quasicrystals. That property is perfectly illustrated by the Fibonacci point set $\mathcal{F} = \Sigma[0, 1)$ since we check from the algebraic definition that $\tau^2\,\Sigma[0, 1) = \Sigma[0, 1/\tau^2) \subset \Sigma[0, 1)$ and so $\tau^2\,\mathcal{F} \subset \mathcal{F}$: this particular Fibonacci point set is self-similar with scaling factor equal to τ^2. Immediately we get the infinite nested sequence

$$\cdots \subset \mathcal{F}/\tau^{2j-2} \subset \mathcal{F}_j := \mathcal{F}/\tau^{2j} \subset \mathcal{F}/\tau^{2j+2} \subset \cdots, \tag{1.1}$$

as increasing aperiodic discretizations of $\mathbb{R}$. Since the distance between two adjacent points of $\mathcal{F}_j$ is equal to $1/\tau^{2j}$ or $1/\tau^{2j+1}$, it is clear that the inductive limit $\mathcal{F}_\infty := \lim\limits_{j\to\infty} \mathcal{F}_j$ densely fills the real line. It is precisely this property which led us to examine in recent works (see [29, 5, 4] and references therein) the problem of constructing wavelets by following discretization schemes of the real line like (1.1). In [4], the construction was based on multiresolution analysis with spline wavelets earlier elaborated by Lemarié-Rieusset [39] and Bernuau [11, 12] (more details will be given below). Our aim was to eventually apply these wavelets to the analysis of aperiodic structures, like diffraction spectra of Fibonacci chain or some other related spectral problem, and to compare our results with more standard wavelet analysis (e.g. dyadic wavelets) [6].

Let us recall the main features of wavelet analysis in the framework of the Hilbert space $L^2(\mathbb{R})$. More complete information can be found in comprehensive textbooks, like [43]. We shall just outline here the essential of that field whose development since the beginning of the eighties amazingly parallels that one of quasicrystals. Under the name *wavelet* is commonly understood a function $\psi(x) \in L^2(\mathbb{R})$ such that the family of functions $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k)$ for $j, k \in \mathbb{Z}$ forms an orthonormal (in a restrictive sense) or at least a *Riesz basis* for $L^2(\mathbb{R})$. A family of vectors (v_n) in a separable Hilbert space V is a Riesz basis if and only if each $v \in V$ can be expressed uniquely as $v = \sum_n a_n v_n$ and there exist positive constants K_2 and K_2, $0 < K_1 \leq K_2$, such that

$$K_1 \sum_n |a_n|^2 \leq \Big\| \sum_n a_n v_n \Big\|^2 \leq K_2 \sum_n |a_n|^2$$

for all sequence of scalars a_n. We can say that the v_n's are strongly linearly independent and, if $K_1 = 1 = K_2$, then the basis is orthonormal. A function (or a set of functions) generating through dilations and translations an orthonormal basis for $L^2(\mathbb{R})$ can be found using a *multiresolution analysis* of $L^2(\mathbb{R})$ (shortly MRA), a method settled by S. Mallat [44] and precisely based on an increasing sequence of periodic discretizations of $\mathbb{R}$. The genuine MRA ingredients are:

(i) one *scaling* function $\varphi(x) \in L^2(\mathbb{R})$ such that $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal system,

(ii) the Hilbert subspace V_0 which is the linear span of $\{\varphi(x-k)\,|\,k \in \mathbb{Z}\}$ and which corresponds to the "central" element of the sequence of discretizations of $\mathbb{R}$,

(iii) the increasing sequence of nested Hilbert subspaces $\cdots V_{j-1} \subset V_j \subset V_{j+1} \cdots$ which are defined by $f(x) \in V_0 \Leftrightarrow f(2^j x) \in V_j$ and are such that $\bigcap_j V_j = \{0\}$ and $\bigcup_j V_j$ is dense in $L^2(\mathbb{R})$,

(iv) one wavelet, i.e. a function $\psi(x)$ such that $\{\psi(x-k)\,|\,k \in \mathbb{Z}\}$ spans the orthogonal complement W_0 of V_0 in $V_1 = V_0 \oplus W_0$.

Note that the orthonormality of the basis $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ of the subspace V_0 is a strong constraint. This condition is usually weakened by just imposing that the system $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ be a Riesz basis of V_0.

We thus note that the dilatation factor is genuinely $\theta = 2$. Indeed, the construction of a wavelet basis within the MRA framework relies on the fact that the lattices $2^{-j}\mathbb{Z}$ are increasing for the inclusion. This property is preserved only when θ is an integer. Then, what about choosing another number θ as a scaling factor? Auscher [8] considered the following problem: given a real number $\theta > 1$, does there exist a finite set $\{\psi_1, \psi_2, \ldots, \psi_\ell\}$ of functions in $L^2(\mathbb{R})$ such that the family $\theta^{j/2}\psi_i(\theta^j x - k)$, $j, k \in \mathbb{Z}$, $1 \le i \le \ell$, is an orthonormal basis for $L^2(\mathbb{R})$? Then he proved that a basis of this type exists if θ is a rational number. More precisely, for $\theta = p/q > 1$, p and q being relatively prime integers, there exists a set of $p-q$ wavelet functions satisfying the previous condition. Now, what about an irrational scaling factor? We already mentioned the works [11, 12] and [4], the latter being mainly devoted to the cases in which θ is encountered in quasicrystallography, like $\tau = (1+\sqrt{5})/2$ or $\tau^2 = (3+\sqrt{5})/2$. Other "quasicrystallographic" numbers have been observed: $1+\sqrt{2}$, $2+\sqrt{3}$. All of them belong to the class of quadratic Pisot-Vijayaraghavan units. To such numbers are associated discretization sequences like in (1.1),

$$\cdots \ \subset \ \Lambda_{j-1} \ \subset \ \Lambda_j := \Lambda/\theta^j \ \subset \ \Lambda_{j+1} \ \subset \ \cdots , \tag{1.2}$$

where $\Lambda = \Lambda_0$ is a selfsimilar Delone set with scaling factor θ, $\theta\,\Lambda \subset \Lambda$ and is such that the inductive limit $\Lambda_\infty := \lim_{j\to\infty} \Lambda_j$ densely fills the real line. Recall that by Delone set we mean that Λ is *uniformly discrete* (the distances between any pair of points in Λ are greater than a fixed $r > 0$) and *relatively dense* (there exists $R > 0$ such that $\mathbb{R}$ is covered by intervals of length $2R$ centered at points of Λ).

In 1992 Buhmann and Micchelli [18] proposed a construction of a wavelet spline basis corresponding to non-uniform and non-self-similar knot sequences, which are actually nested sequences of Delone sets

$$\cdots \subset \Lambda_{j-1} \subset \Lambda_j \subset \Lambda_{j+1} \subset \cdots . \tag{1.3}$$

They consider only two successive elements, say Λ_0 and $\Lambda_1 \supset \Lambda_0$ for their purpose of proving the existence of what they call prewavelets, with minimal support, which span the orthogonal complement W_0 of V_0 in $V_1 = V_0 \oplus W_0$. Here, V_0 and V_1 are the spaces of linear combinations of *B-splines* on Λ_0 and Λ_1 respectively. They first suppose that the "refining" of the "coarse" knot sequence Λ_0 leading to the "finer" Λ_1 consists in adding a new knot between each two adjacent knots of Λ_0. The case of multiple insertions is eventually examined. Let us now give some insight on these spline functions, so intimately linked to the notion of a Delone set.

Any Delone set Λ determines a space of splines of order s, $s \geq 2$, in the following way.

Definition 1.1 *Let $s \geq 2$. Then $V_0^{(s)}(\Lambda)$ is the closed subspace of $L^2(\mathbb{R})$ defined by*

$$V_0^{(s)}(\Lambda) := \left\{ f(x) \in L^2(\mathbb{R}) \,\middle|\, \frac{\mathrm{d}^s}{\mathrm{d}x^s} f(x) = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \right\}.$$

An equivalent definition is given in terms of the restriction of functions to intervals determined by consecutive elements of Λ. Suppose $\Lambda = \{\lambda_n \mid n \in \mathbb{Z}\}$, where $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{Z}$. There results from Definition 1.1 that

$$V_0^{(s)}(\Lambda) = \left\{ f \in C^{s-2} \cap L^2(\mathbb{R}) \,\middle|\, f_{|[\lambda_n, \lambda_{n+1}]} \text{ is a polynomial of degree } \leq s-1 \right\}.$$

Therefore, $V_0^{(s)}(\Lambda)$ is the space of splines of order s with nodes in Λ. Let us now give a classical result about the existence of a Riesz basis for $V_0^{(s)}(\Lambda)$ [60].

Theorem 1.2 *For all Delone sets $\Lambda = \{\lambda_n \mid n \in \mathbb{Z}\} \subset \mathbb{R}$ and for all $s \geq 2$, there exists a Riesz basis $\{B_\lambda^{(s)} \mid \lambda \in \Lambda\}$ of $V_0^{(s)}(\Lambda)$. The function $B_\lambda^{(s)}$ (called B-spline) is the unique function in $V_0^{(s)}(\Lambda)$ satisfying the following conditions:*

(i) $\operatorname{supp} B_\lambda^{(s)} = [\lambda, \lambda']$*, where* $\lambda' \in \Lambda$.

(ii) The interval (λ, λ') contains exactly $s-1$ points of Λ.

(iii) $\displaystyle\int_{\mathbb{R}} B_\lambda^{(s)} = \frac{\lambda' - \lambda}{s}$.

See [60] for proof. Note that (i) and (ii) give precise information on the (compact) support of $B_\lambda^{(s)}$ whilst (iii) is a normalization condition.

The construction of $B_\lambda^{(s)}(x)$ can be carried out in various ways, by recurrence, by using the condition of minimal support, or by inverse Fourier transform. In the latter case, one can prove that the Fourier transform of $B_{\lambda_n}^{(s)}(x + \lambda_n)$ depends on the s-tuple $(\lambda_{n+1} - \lambda_n, \ldots, \lambda_{n+s} - \lambda_n)$ only. Now suppose that the Delone set Λ is of *finite local complexity*, which means [38, 36] that, for all $R > 0$, the point set

$$\bigcup_{\lambda \in \Lambda} \left\{ (\Lambda - \lambda) \cap (-R, R) \right\}$$

is finite, i.e. local environments of points in Λ are not different in infinite fashions. Typically, such sets Λ are mathematical models for one-dimensional structures having a long-range order, like quasicrystals. We can then assert the following:

Proposition 1.3 *Let $\Lambda \subset \mathbb{R}$ be a Delone set of finite local complexity. Then the B-splines of order s based on Λ are of the form $B_\lambda^{(s)}(x) = \phi_\lambda(x-\lambda)$, $\lambda \in \Lambda$, where the set $\{\phi_\lambda(x) \mid \lambda \in \Lambda\}$ is a finite set of functions with compact support.*

Therefore, in the finite local complexity case, it is possible to partition the indexing set $\mathbb{Z}$ for Λ into a finite set of equivalence classes $\bar{0}, \bar{1}, \ldots, \bar{q}$, where the equivalence between k and n is given by

$$B_{\lambda_k}^{(s)}(x+\lambda_k) = B_{\lambda_n}^{(s)}(x+\lambda_n)\,, \qquad \text{for all } x\,.$$

Correspondingly, for a given s, the point set Λ is partitioned into $\Lambda = \bigcup_{\bar{n}=\bar{0}}^{\bar{q}} \Lambda_{\bar{n}}$ with $\Lambda_{\bar{n}} = \{\lambda_k \in \Lambda \,|\, k \in \bar{n}\}$. The equivalence between k and n means that λ_k and λ_n are left-hand ends of identical s-letter words if we identify each interval $(\lambda_k, \lambda_{k+1})$ with a letter of the allowed alphabet. To each class $\bar{n}$ is biunivocally associated the function $\phi_{\bar{n}}(x) \equiv \phi_{\lambda_k}(x) = B_{\lambda_k}^{(s)}(x+\lambda_k)$, $k \in \bar{n}$. In this way, the space $V_0^{(s)}(\Lambda)$ decomposes into the direct sum

$$V_0^{(s)}(\Lambda) = \bigoplus_{\bar{n}=\bar{0}}^{\bar{q}} V_{0,\bar{n}},$$

where $V_{0,\bar{n}}$ is the closure of the linear span of the functions $\phi_{\bar{n}}(x-\lambda_k)$, $k \in \bar{n}$.

Let us now go back to the case in which there is self-similarity (like in (1.2)) and finite local complexity and let us see which issue holds in term of MRA and existence and properties of wavelets. More concretely, let Λ be a Delone set of finite local complexity and self-similar with inflation factor $\theta > 1$, $\theta\Lambda \subset \Lambda$. Changing the scale allows us to define subspaces $V_j^{(s)}(\Lambda)$, $j \in \mathbb{Z}$, as

$$V_j^{(s)}(\Lambda) = \left\{ f(x) \in L^2(\mathbb{R}) \;\mid\; \frac{\mathrm{d}^s}{\mathrm{d}x^s} f(x) = \sum_{\lambda \in \Lambda} a_\lambda \delta_{\theta^{-j}\lambda} \right\}.$$

Therefore, $V_j^{(s)}(\Lambda)$ is the space of splines of order s with nodes in the jth scaled version of Λ.

We now have at our disposal an inductive chain of spaces allowing analysis at any scale. More precisely, with the above notations, we have the following statement.

Proposition 1.4 *The sequence of subspaces $(V_j^{(s)}(\Lambda))_{j\in\mathbb{Z}}$ is a θ-multiresolution analysis of $L^2(\mathbb{R})$, i.e.*

(i) for any $j \in \mathbb{Z}$, $V_j^{(s)}(\Lambda)$ is a closed subspace of $L^2(\mathbb{R})$,

(ii) $\cdots \subset V_{-1}^{(s)}(\Lambda) \subset V_0^{(s)}(\Lambda) \subset V_1^{(s)}(\Lambda) \subset \cdots$,

(iii) $\bigcup_{j\in\mathbb{Z}} V_j^{(s)}(\Lambda)$ is dense in $L^2(\mathbb{R})$,

(iv) $\bigcap_{j\in\mathbb{Z}} V_j^{(s)}(\Lambda) = \{0\}$,

(v) $f(x) \in V_j^{(s)}(\Lambda)$ if and only if $f(\theta^{-j}x) \in V_0^{(s)}(\Lambda)$,

(vi) there exists a finite number of functions $\phi_{\bar{n}}(x) \in V_0^{(s)}$, called scaling functions such that $\{\phi_{\bar{n}}(x - \lambda_k)\}_{k\in\bar{n},\, 0\le n\le q}$ is a Riesz basis in $V_0^{(s)}$.

The proof is straightforward from definitions and Proposition 1.3.

As a consequence of the the above statements, we have the important result obtained by Bernuau [11, 12]:

Theorem 1.5 *Let $\Lambda = \{\lambda_n \mid n \in \mathbb{Z}\} \subset \mathbb{R}$ be a Delone set of finite local complexity, self-similar with factor $\theta > 1$. Let us denote the elements of $\theta^{-1}\Lambda$ by $\kappa_n = \theta^{-1}\lambda_n$, $n \in \mathbb{Z}$. Then for all $s > 1$ there exists a Riesz basis of $L^2(\mathbb{R})$ of the form:*

$$\left\{\theta^{j/2}\psi_{\kappa_n}^{(s)}(\theta^j x - \kappa_n),\ \kappa_n \in \theta^{-1}\Lambda,\ \kappa_{n+1} \notin \Lambda,\ j \in \mathbb{Z}\right\},$$

where $\left\{\psi_{\kappa_n}^{(s)}\right\}$ is a finite set of compactly supported functions of order C^{s-2}.

Such a result offers the possibility of constructing explicit (spline) wavelet basis for a large class of self-similar Delone set of finite local complexity, like in particular those ones obtained through the cut-and-project method. It was done for some very specific cases in [4]. Nevertheless, we think that a systematic study of the properties of (not necessarily self-similar) cut-and-project sets is still lacking.

The aim of the present paper is to fill this gap. Its content is devoted to the study of one-dimensional cut-and-project sets built in a rather generic way, without supposing a priori any algebraic nature for the respective slopes of D_1 and D_2 and by using the freedom of making the window Ω vary in a continuous way. Hence, our results can be viewed as paving the way to further investigations concerning explicit wavelet constructions for arbitrary cut-and-project sets. Indeed, what we learn from previous works is the importance of knowing in a precise way the environment of each point in the Delone set, i.e. its complexity, in order to build the scaling functions and the associated wavelets. Moreover, we should not underestimate the structural importance of the nested sequence (1.3) in five of its features, namely

(i) the way in which the new points (i.e. the *details*) intertwine the old ones at each step $\Lambda_j \subset \Lambda_{j+1}$ of the increasing sequence (1.3), or, equivalently, the characteristics of the detail sets $\Lambda_{j+1} \setminus \Lambda_j$,

(ii) the way in which the sequence behaves at the limit $j \to \infty$, i.e. the structure of its inductive limit which should be dense in $\mathbb{R}$,

(iii) the specific advantages brought by the self-similarity hypothesis,

(iv) related to (i) and (iii), the specific advantages brought by the *substitivity* hypothesis, a notion which is introduced in (6.2).

(v) the relevance of a specific choice of a nested sequence with regard to the domain of application of the corresponding wavelet analysis.

Note the crucial importance of the first point in connection with the so-called *scaling* or *refinement* equations which couple with the inclusion $V_0 \subset V_1 = V_0 \oplus W_0$. Suppose there exist spline bases $(\varphi_{0,k})_{k\in\mathcal{I}_0\subset\mathbb{Z}}$, $(\varphi_{1,k})_{k\in\mathcal{I}_1\subset\mathbb{Z}}$ for subspaces V_0 and V_1 respectively, and a wavelet basis $(\psi_{0,k})_{k\in\mathcal{J}_0\subset\mathbb{Z}}$ for W_0. Then the following hilbertian decomposition should hold:

$$\varphi_{1,j}(x) = \sum_{k\in\mathcal{I}_0} c_{1,jk}\varphi_{0,k}(x) + \sum_{k\in\mathcal{J}_0} d_{1,jk}\psi_{0,k}(x). \tag{1.4}$$

The way the *tendency* coefficients $c_{1,jk}$ and the *detail* coefficients $d_{1,jk}$ behave for large k is a crucial question in wavelet analysis, and so the way this question depends on the set inclusion $\Lambda_0 \subset \Lambda_1$ deserves special attention.

In consequence, we have organized the paper by following the hierarchy of questions (i)-(v), and furthermore including in the scheme other aspects of possible interest, like some considerations on numeration systems related to cut-and-project scheme. Section 2 is devoted to the geometrical aspects of those point sets in the line issued from the square lattice in the plane through "cut" and "projection": definition, study of distances between adjacent points, properties of invariance or covariance under the group $SL(2,\mathbb{Z}) \times \{-1,1\}$ acting on the square lattice. The material presented there is not specifically new. However it represents an original overview, in which one focuses on the universality of many features of these cut-and-project sets, independently of specific algebraic or substitutional characteristics.

In Section 3 we examine the combinatorial aspects of the cut-and-project sequences from the point of view of the theory of languages: subword complexity, Rauzy graphs, and occurrence of specific classes of finite words (or *factors*) in the bidirectional word biunivocally associated to cut-and-project sequences. One can find there original results (Propositions 3.8, 3.11, and 3.16). The important case of sturmian words is also considered and we establish three properties, 3.18, 3.19, 3.20, describing their factors.

Self-similar cut-and-project sets are the object of Section 4 in which we solve (Theorem 4.1) the precise relation between self-similarity and algebraic properties of the irrational scaling factor(s) on one hand and the irrational numbers involved in the cut-and-project scheme on the other hand.

In Section 5 we revisit the important notion of β-integers, i.e. those real numbers which do not have "β-fractional" part when expanded in "basis" $\beta > 1$, in the light of their possible or not relation with cut and projection, and we give a necessary and sufficient condition for β (Proposition 5.2) under which the positive β-integers coincide with the positive part of a cut-and-project set.

Another original part of the paper is found in Section 6. It is well known that infinite words associated to many cut-and-project sets present the so-called substitution invariance, and this property can be crucial for understanding or even for creating the relation $\Lambda_j \to \Lambda_{j+1}$ in a multiresolution sequence of sets and the scaling equations (1.4) issued from the companion inclusion $V_j \subset V_{j+1}$. Now, even though the substitution invariance is absent for a given bidirectional infinite word $u = \cdots u_{-2}u_{-1}|u_0u_1u_2\cdots$, where u_k belongs to some alphabet $\mathcal{A}$, there exist cases in which it could be "hidden" behind the weaker notion of *substitutivity*, which means that there exists another infinite word $v = \cdots v_{-2}v_{-1}|v_0v_1v_2\cdots$ over an alphabet $\mathcal{B}$ which has substitution invariance and a letter projection $\psi : \mathcal{B} \to \mathcal{A}$ such that $u = \cdots u_{-2}u_{-1}|u_0u_1u_2\cdots = \cdots\psi(v_{-2})\psi(v_{-1})|\psi(v_0)\psi(v_1)\psi(v_2)\cdots$.

With regards to this property, we shall give in Section 6 an algorithm (Theorem 6.6) allowing to "pull back", a given word u pertaining to the algebraic cut-and-project scheme to the word v mentioned in the above. In order to illustrate this result, the algorithm is carried out on an example of cut-and-project set defined by the algebraic (Sturm) number $1/\sqrt{2}$.

Eventually, we shall give in the conclusion some hints about possible applications of our results, mainly in direction of wavelet constructions, of mathematical diffraction, and of design of aperiodic pseudo-random number generators.

2 Cut-and-project sequences

In this section we define cut-and-project sequences arising by a projection of a 2-dimensional lattice. We also describe their basic properties, including the invariance under certain transformations. We further show that cut-and-project sequences are geometric representations of a three or two interval exchange. Codings of two interval exchanges are in one-to-one correspondence with mechanical words $\underline{s}_{\alpha,\beta}$, $\overline{s}_{\alpha,\beta}$, (see definition by (2.16) and (2.17)), which are in fact sturmian words (Definition 3.2).

2.1 Definition and properties

The construction of a cut-and-project sequence starts with a choice of a 2-dimensional lattice L and two straight lines D_1, D_2. One of the lines plays the role of the space onto which the lattice L is projected, the other line determines the direction of the projection. If A is an arbitrary non-singular linear map on $\mathbb{R}^2$, then the cut-and-project sequence constructed using a lattice AL and straight lines AD_1, AD_2 is the same as the cut-and-project sequence constructed using a lattice L and straight lines D_1, D_2. Therefore it is not necessary to consider general L and D_1, D_2. Some authors allow arbitrary lattice L and for D_1, D_2 take mutually orthogonal straight lines. Others, including us, prefer to fix the lattice $\mathbb{Z}^2$ and consider arbitrary straight lines D_1, D_2.

Let us take two distinct irrational numbers ε, η and let us consider straight lines $D_1 : y = \varepsilon x$, $D_2 : y = \eta x$. If we choose vectors

$$\vec{x}_1 = \frac{1}{\varepsilon - \eta}(1, \varepsilon) \quad \text{and} \quad \vec{x}_2 = \frac{1}{\eta - \varepsilon}(1, \eta)$$

in the subspaces D_1, D_2 of $\mathbb{R}^2$, then for every lattice point $(a, b) \in \mathbb{Z}^2$ we have

$$(a, b) = (b - a\eta)\vec{x}_1 + (b - a\varepsilon)\vec{x}_2\,.$$

Obviously, the projection of $\mathbb{Z}^2$ on D_1 along D_2 is the set

$$\mathbb{Z}[\eta]\vec{x}_1\,,$$

where $\mathbb{Z}[\eta]$ is the abelian group

$$\mathbb{Z}[\eta] := \{a + b\varepsilon \mid a, b \in \mathbb{Z}\}\,.$$

Similarly, the projection of $\mathbb{Z}^2$ on D_2 along D_1 is the set $\mathbb{Z}[\varepsilon]\vec{x}_2$, Since numbers ε, η are irrational, the mappings $(a, b) \mapsto a + b\eta$, $(a, b) \mapsto a + b\varepsilon$ are bijections between $\mathbb{Z}^2$ and $\mathbb{Z}[\eta]$, resp. $\mathbb{Z}^2$ and $\mathbb{Z}[\varepsilon]$. Therefore there exists also a bijection

$$\star : \mathbb{Z}[\eta] \to \mathbb{Z}[\varepsilon]$$

defined by the prescription

$$x = a + b\eta \quad \mapsto \quad x^\star = a + b\varepsilon\,,$$

which is called the star map. Directly from its definition we obtain

$$(x + y)^\star = x^\star + y^\star \qquad \text{for every} \;\; x, y \in \mathbb{Z}[\eta]\,.$$

Let us now introduce the definition of cut-and-project sets. It is easy to observe that, as a consequence of irrationality of ε and η, the sets $\mathbb{Z}[\varepsilon]$, $\mathbb{Z}[\eta]$ are dense in $\mathbb{R}$. However, if instead of all the lattice $\mathbb{Z}^2$, we project only those points in $\mathbb{Z}^2$ that belong to a chosen strip parallel to D_1, the resulting set in D_1 has no limit points. The width and position of the projected strip is determined by an interval in D_2. Formally, we have the following definition.

Definition 2.1 *Let ε, η be distinct irrational numbers and let Ω be a bounded interval. The set*

$$\Sigma_{\varepsilon,\eta}(\Omega) = \{a + b\eta \mid a, b \in \mathbb{Z},\ a + b\varepsilon \in \Omega\} = \{x \in \mathbb{Z}[\eta] \mid x^\star \in \Omega\}$$

is called a cut-and-project sets, or C&P set. The interval Ω is called the acceptance window of $\Sigma_{\varepsilon,\eta}(\Omega)$.

The above definition is a special case of the very general 'model sets'. Important contributions to the study of model sets as mathematical models of quasicrystals are due to [36, 38, 49, 50, 52, 51].

Let us mention some of the properties of cut-and-project sequences that follow directly from the definition or were derived by cited authors.

Remark 2.2

1. *Trivially from the definition we have*

$$\Sigma_{\varepsilon,\eta}(\Omega_1) \subset \Sigma_{\varepsilon,\eta}(\Omega_2) \quad \textit{for} \quad \Omega_1 \subset \Omega_2\,.$$

   More generally, if $(\Omega_i)_{i\in\mathbb{Z}}$ is a sequence of nested bounded intervals such that

$$\cdots \subset \Omega_{i-1} \subset \Omega_i \subset \Omega_{i+1} \subset \cdots \qquad \textit{and} \qquad \bigcup_{i\in\mathbb{Z}} \Omega_i = \mathbb{R}\,,$$

   then

$$\cdots \subset \Sigma_{\varepsilon,\eta}(\Omega_{i-1}) \subset \Sigma_{\varepsilon,\eta}(\Omega_i) \subset \Sigma_{\varepsilon,\eta}(\Omega_{i+1}) \subset \cdots$$

   and

$$\bigcup_{i\in\mathbb{Z}} \Sigma_{\varepsilon,\eta}(\Omega_i) = \mathbb{Z}[\eta]\,.$$

2. *Since $\mathbb{Z}[\eta]$ and $\mathbb{Z}[\varepsilon]$ are additive groups, the C&P sequence satisfies*

$$x + \Sigma_{\varepsilon,\eta}(\Omega) = \Sigma_{\varepsilon,\eta}(\Omega + x^\star) \qquad \textit{for every} \ \ x \in \mathbb{Z}[\eta].$$

   This property further implies that $\Sigma_{\varepsilon,\eta}(\Omega)$ is not invariant under any translation, i.e. is aperiodic.

3. *Any model set is Delone, see [50]. In our one-dimensional case it implies that there exists a sequence $(x_n)_{n\in\mathbb{Z}}$, and two positive numbers r_1, r_2 such that $r_1 < x_{n+1} - x_n < r_2$, for all $n \in \mathbb{Z}$, and $\Sigma_{\varepsilon,\eta}(\Omega) = \{x_n \mid n \in \mathbb{Z}\}$.*

4. *The density of points of $\Sigma_{\varepsilon,\eta}(\Omega)$, defined as*

$$\varrho\left(\Sigma_{\varepsilon,\eta}(\Omega)\right) := \lim_{N\to+\infty} \frac{\#\ \big([-N,N] \cap \Sigma_{\varepsilon,\eta}(\Omega)\big)}{2N+1}$$

   is proportional to the length of the interval Ω, see [51].

5. *Since Ω is an interval, it is easy to see that there exists a finite set $F^\star \subset \mathbb{Z}[\varepsilon]$ such that $\Omega - \Omega \subset \Omega + F^\star$. Hence there also exists a finite set F such that*

$$\Sigma_{\varepsilon,\eta}(\Omega) - \Sigma_{\varepsilon,\eta}(\Omega) \subset \Sigma_{\varepsilon,\eta}(\Omega) + F\,.$$

   Thus $\Sigma_{\varepsilon,\eta}(\Omega)$ satisfies the so-called Meyer property. In fact, every model set $\Lambda \subset \mathbb{R}^n$ is a Meyer set, i.e. is Delone and satisfies $\Lambda - \Lambda \subset \Lambda + F$ for a finite set F, cf. [50]. Note that in this sense a cut-and-project set is a generalization of a lattice, because a lattice satisfies the above property with $F = \{0\}$.

6. *Since $\Sigma_{\varepsilon,\eta}(\Omega)$ is a Meyer set, it is of finite local complexity, i.e. it has only a finite number of local configurations of a fixed size [36]. More precisely, for $\varrho > 0$ we define the ϱ-neighbourhood of a point $x \in \Sigma_{\varepsilon,\eta}(\Omega)$ as*

$$N_\varrho(x) = \{y \in \Sigma_{\varepsilon,\eta}(\Omega) \mid |x - y| < \varrho\}\,.$$

   The family of ϱ-neighbourhoods $\{N_\varrho(x) - x \mid x \in \Sigma_{\varepsilon,\eta}(\Omega)\}$ is finite for any positive ϱ. In particular, there is only finitely many distances between adjacent points of any C&P sequence, i.e. the set $\{x_{n+1} - x_n \mid n \in \mathbb{Z}\}$ is finite.

7. *The boundary of the acceptance interval Ω influences the structure of the C&P sequence only trivially. The sets $\Sigma_{\varepsilon,\eta}[c, c+\ell)$, $\Sigma_{\varepsilon,\eta}[c, c+\ell]$, $\Sigma_{\varepsilon,\eta}(c, c+\ell)$, and $\Sigma_{\varepsilon,\eta}(c, c+\ell]$ differ at most in two points. If $c, c + \ell \notin \mathbb{Z}[\varepsilon]$, then all these sets coincide.*

8. *If the acceptance window Ω is chosen to be a semi-closed interval, then the number and shape of ϱ-neighbourhoods of a C&P sequence does not depend on the position Ω, but only on its length $|\Omega|$.*

9. *The set $\Sigma_{\varepsilon,\eta}(\Omega)$, where $\Omega = [c, c+\ell)$ or $\Omega = (c, c+\ell]$ contains every finite configuration infinitely many times. More precisely, for every $\varrho > 0$ and every $x \in \Sigma_{\varepsilon,\eta}(\Omega)$ there exists infinitely many points $y \in \Sigma_{\varepsilon,\eta}(\Omega)$, such that*

$$N_\varrho(x) - x = N_\varrho(y) - y\,.$$

   We say that such C&P sequences are repetitive.

2.2 Distances

As it was mentioned in (6) of Remark 2.2, any cut-and-project sequence has only a finite number of distances between adjacent points. It turns out that the number of distances does not exceed 3. We quote the result of [30], which is a generalization of the famous 3-distance theorem, and provide algorithms for determining the distances for any particular acceptance interval.

Theorem 2.3 *Let Ω be a semi-closed interval. For every $\Sigma_{\varepsilon,\eta}(\Omega)$ there exist positive numbers Δ_1, $\Delta_2 \in \mathbb{Z}[\eta]$ such that the distances between adjacent points in $\Sigma_{\varepsilon,\eta}(\Omega)$ take values in $\{\Delta_1, \Delta_2, \Delta_1 + \Delta_2\}$. The numbers Δ_1, Δ_2 depend only on the parameters ε, η and on the length $|\Omega|$ of the interval Ω. They are linearly independent over $\mathbb{Q}$ and satisfy $\Delta_1^\star > 0$, $\Delta_2^\star < 0$, and $\Delta_1^\star - \Delta_2^\star \geq |\Omega|$.*

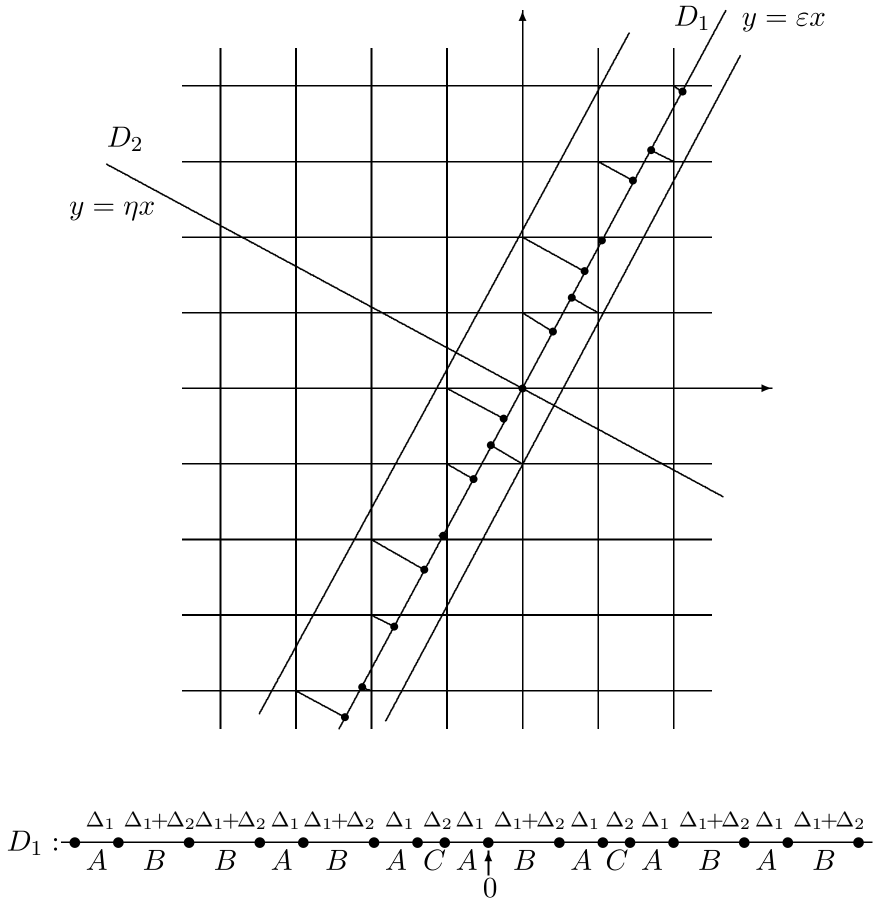


Figure 1: Construction of a cut-and-project sequence and assignment of the infinite word which codes the order of distances in the cut-and-project sequence.

More precisely, every C&P sequence $\Sigma_{\varepsilon,\eta}(\Omega) = \{x_n \mid n \in \mathbb{Z}\}$ has always two or three type of distances between adjacent points, namely

$$\{x_{n+1} - x_n \mid n \in \mathbb{Z}\} = \begin{cases} \{\Delta_1, \Delta_2, \Delta_1 + \Delta_2\} & \text{if } \Delta_1^\star - \Delta_2^\star > |\Omega|\,, \\ \{\Delta_1, \Delta_2\} & \text{if } \Delta_1^\star - \Delta_2^\star = |\Omega|\,. \end{cases} \tag{2.5}$$

Therefore one can naturally assign to it a binary or ternary bidirectional infinite word $u_{\varepsilon,\eta}(\Omega) = (u_n)_{n\in\mathbb{Z}}$, for example in the alphabet $\{A, B, C\}$, by

$$u_n = \begin{cases} A & \text{if } \; x_{n+1} - x_n = \Delta_1\,, \\ B & \text{if } \; x_{n+1} - x_n = \Delta_1 + \Delta_2\,, \\ C & \text{if } \; x_{n+1} - x_n = \Delta_2\,. \end{cases} \tag{2.6}$$

An example of construction of a cut-and-project sequence together with the assignment of the infinite word is shown in Figure 1.

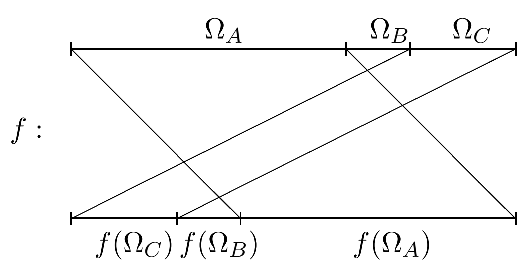
The successor of a point x in the C&P sequence is determined using its star-map image $x^\star$. If $\Omega = [c, c+\ell)$, then the inequality $\Delta_1^\star - \Delta_2^\star \geq \ell$ from Theorem 2.5 ensures that the nearest right neighbour of the point x in $\Sigma_{\varepsilon,\eta}(\Omega)$ is equal to $x + \Delta_1$ if $x^\star \in [c, c+\ell-\Delta_1^\star)$, to $x+\Delta_2$ if $x^\star \in [c-\Delta_2^\star, c+\ell)$, or to $x+\Delta_1+\Delta_2$ if $x^\star \in [c+\ell-\Delta_1^\star, c-\Delta_2^\star)$. Thus we can define a piecewise linear map $f : [c, c+\ell) \to [c, c+\ell)$ which satisfies $f(x^\star) = y^\star$ if y is the nearest right neighbour of x. This mapping plays an important role in our considerations.

Definition 2.4 *Let $c \in \mathbb{R}$, $\ell > 0$. The stepping function of the interval $[c, c+\ell)$ is a mapping $f : [c, c+\ell) \to [c, c+\ell)$ defined by*

$$f(y) = \begin{cases} y + \Delta_1^\star & \text{if } \; y \in [c, c+\ell-\Delta_1^\star)\,, \\ y + \Delta_1^\star + \Delta_2^\star & \text{if } \; y \in [c+\ell-\Delta_1^\star, c-\Delta_2^\star)\,, \\ y + \Delta_2^\star & \text{if } \; y \in [c-\Delta_2^\star, c+\ell)\,. \end{cases}$$

The graph of the map f is illustrated on Figure 2.

The stepping function f of Figure 2 has been studied in the field of dynamical systems under the name of three interval exchange (in case that f has two discontinuity points) or two interval exchange (if f has only one discontinuity point). In the former situation, the two discontinuity points divide the acceptance interval Ω into three disjoint intervals, say Ω_A, Ω_B, Ω_C, from left to right. The image $f(\Omega) = \Omega$ is again divided into three disjoint intervals $f(\Omega_C)$, $f(\Omega_B)$, $f(\Omega_A)$, in the order from left to right. Therefore one sometimes uses the graphical notation shown in the following scheme.



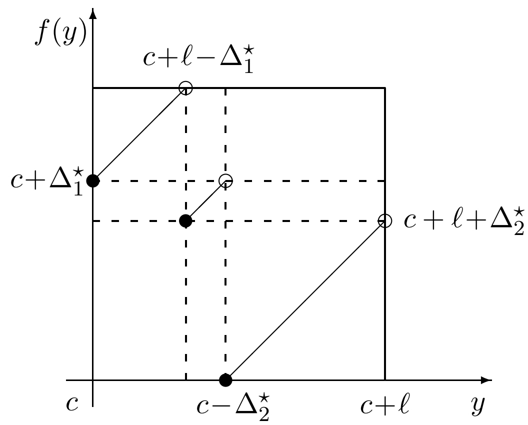


Figure 2: Stepping function f associated to $\Sigma_{\varepsilon,\eta}[c, c+\ell)$. If the distances between neighbours in the C&P set $\Sigma_{\varepsilon,\eta}[c, c+\ell)$ take only two values Δ_1, Δ_2, i.e. $\ell = \Delta_1^\star - \Delta_2^\star$, then the discontinuity points $c+\ell-\Delta_1^\star$ and $c-\Delta_2^\star$ coincide.

The relation of three interval exchange to simultaneous approximation of a pair of irrational numbers is treated in [1, 26, 57]. In the theory of symbolic dynamical systems one studies the orbit of a point $x \in \Omega$, under the mapping f, i.e. the sequence $(f^{(n)})_{n\in\mathbb{N}_0}$[1]. Therefore C&P sequences can be viewed as geometric representations of three interval exchange transformations.

Changing continuously the length ℓ of the acceptance interval $\Omega = [c, c+\ell)$ causes discrete changes of the triplet of distances $(\Delta_1, \Delta_2, \Delta_1+\Delta_2)$. Recall that the triplet does not depend on c, therefore we consider $c = 0$. Let $\Sigma_{\varepsilon,\eta}[0,\ell)$ be a C&P sequence with three distances between its neighbours, and let $\Delta_1^\star > 0$, $\Delta_2^\star < 0$, $\Delta_1^\star+\Delta_2^\star$, be the star map images of these distances. According to (2.5), we must have $\ell < \Delta_1^\star - \Delta_2^\star$. Growing ℓ up to the value $\Delta_1^\star - \Delta_2^\star$ causes appearance of new points in the C&P sequence, which split the large distance $\Delta_1+\Delta_2$ into two distances Δ_1 and Δ_2. When ℓ reaches the value $\Delta_1^\star - \Delta_2^\star$, the large distance $\Delta_1+\Delta_2$ disappears completely.

On the other hand, diminishing the length ℓ of the acceptance interval causes that the frequency of the distance $\Delta_1+\Delta_2$ grows to the detriment of occurrences of the distances Δ_1 and Δ_2. This happens until ℓ reaches a certain limit value for which one of the distances Δ_1 or Δ_2 disappears.

Starting from a given initial value ℓ_0, for which the set $\Sigma_{\varepsilon,\eta}[0,\ell_0)$ has two distances between adjacent points, we can determine by recurrence the increasing sequence of lengths ℓ_n, $n \in \mathbb{Z}$, of the acceptance windows for which $\Sigma_{\varepsilon,\eta}[0,\ell_n)$ has only two distances. The initial value ℓ_0 is determined below (Remark 2.6).

[1]Note that we use the notation $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Let $\Delta^\star_{n1} > 0$ and $\Delta^\star_{n2} < 0$ be the star images of distances occurring in the sequence $\Sigma_{\varepsilon,\eta}[0,\ell_n)$, i.e., according to (2.5), $\ell_n = \Delta^\star_{n1} - \Delta^\star_{n2}$.

$$\begin{aligned}
&\text{If } \Delta^\star_{n1} + \Delta^\star_{n2} > 0 \text{ then} \\
&\qquad \ell_{n-1} := \Delta^\star_{n1}, \quad \Delta^\star_{(n-1)1} := \Delta^\star_{n1} + \Delta^\star_{n2}, \quad \Delta^\star_{(n-1)2} := \Delta^\star_{n2}. \\
&\text{If } \Delta^\star_{n1} + \Delta^\star_{n2} < 0 \text{ then} \\
&\qquad \ell_{n-1} := -\Delta^\star_{n2}, \quad \Delta^\star_{(n-1)1} := \Delta^\star_{n1}, \quad \Delta^\star_{(n-1)2} := \Delta^\star_{n1} + \Delta^\star_{n2}.
\end{aligned} \tag{2.7}$$

Similarly, the algorithm which determines the triple $\ell_{n+1}, \Delta_{(n+1)1}, \Delta_{(n+1)2}$ from the triple $\ell_n, \Delta_{n1}, \Delta_{n2}$ has the inverse form

$$\begin{aligned}
&\text{If } \Delta_{n1} > \Delta_{n2} \text{ then} \\
&\quad \ell_{n+1} := \Delta^\star_{n1} - 2\Delta^\star_{n2}, \quad \Delta^\star_{(n+1)1} := \Delta^\star_{n1} - \Delta^\star_{n2}, \quad \Delta^\star_{(n+1)2} := \Delta^\star_{n2}. \\
&\text{If } \Delta_{n1} < \Delta_{n2} \text{ then} \\
&\quad \ell_{n+1} := 2\Delta^\star_{n1} - \Delta^\star_{n2}, \quad \Delta^\star_{(n+1)1} := \Delta^\star_{n1}, \quad \Delta^\star_{(n+1)2} := \Delta^\star_{n2} - \Delta^\star_{n1}.
\end{aligned} \tag{2.8}$$

From this algorithm it can be seen that the C&P sequences $\Sigma_{\varepsilon,\eta}[0,\ell_n)$ and $\Sigma_{\varepsilon,\eta}[0,\ell_{n-1})$ have exactly one type of distances in common. It is the shorter one among $\Delta_{(n-1)1}$, $\Delta_{(n-1)2}$. Moreover, the distances in $\Sigma_{\varepsilon,\eta}[0,\ell)$, for $\ell_{n-1} < \ell < \ell_n$, are three, and they are given by the union of the sets of distances for $\Sigma_{\varepsilon,\eta}[0,\ell_n)$ and $\Sigma_{\varepsilon,\eta}[0,\ell_{n-1})$.

2.3 Transformations

We would like to identify those parameters $\varepsilon, \eta, \Omega$ which provide essentially the same cut-and-project sequences. For example, we have

$$a + b\eta + \Sigma_{\varepsilon,\eta}(\Omega) = \Sigma_{\varepsilon,\eta}(\Omega + a + b\varepsilon)\,, \qquad \text{for } \ a, b \in \mathbb{Z}\,. \tag{2.9}$$

Such translation of the C&P sequence corresponds to a translation of the lattice $\mathbb{Z}^2$. The group of all linear transformations of the lattice $\mathbb{Z}^2$ onto itself is

$$G = \left\{ \mathbb{A} \in M_2(\mathbb{Z}) \;\middle|\; \det \mathbb{A} = \pm 1 \right\}.$$

Consider the matrix $\mathbb{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For arbitrary irrational numbers ε, η and arbitrary interval Ω it holds that

$$
\begin{aligned}
\Sigma_{\varepsilon,\eta}(\Omega) &= \left\{p + q\eta \;\middle|\; p,q \in \mathbb{Z},\, p + q\varepsilon \in \Omega\right\} = \\
&= \left\{(1,\eta)\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \;\middle|\; p,q \in \mathbb{Z},\, (1,\varepsilon)\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \in \Omega\right\} = \\
&= \left\{(1,\eta)\mathbb{A}\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \;\middle|\; p,q \in \mathbb{Z},\, (1,\varepsilon)\mathbb{A}\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \in \Omega\right\} = \\
&= \left\{(a + c\eta, b + d\eta)\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \;\middle|\; p,q \in \mathbb{Z},\, (a + c\varepsilon, b + d\varepsilon)\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \in \Omega\right\} = \\
&= (a + c\eta)\left\{\left(1, \tfrac{b+d\eta}{a+c\eta}\right)\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \;\middle|\; p,q \in \mathbb{Z},\, \left(1, \tfrac{b+d\varepsilon}{a+c\varepsilon}\right)\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right) \in \tfrac{1}{a+c\varepsilon}\Omega\right\} = \\
&= (a + c\eta)\Sigma_{\frac{b+d\varepsilon}{a+c\varepsilon},\frac{b+d\eta}{a+c\eta}}\left(\tfrac{1}{a+c\varepsilon}\Omega\right).
\end{aligned}
$$

Let us study the consequences of the above relation if we choose for the matrix $\mathbb{A}$ one of the three generators $\mathbb{A}_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\mathbb{A}_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\mathbb{A}_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of the group G,

$$
\begin{aligned}
\Sigma_{\varepsilon,\eta}(\Omega) &= \Sigma_{1+\varepsilon,\ 1+\eta}(\Omega)\,, && (2.10)\\
\Sigma_{\varepsilon,\eta}(\Omega) &= \Sigma_{-\varepsilon,-\eta}(-\Omega)\,, && (2.11)\\
\Sigma_{\varepsilon,\eta}(\Omega) &= \eta\ \Sigma_{\frac{1}{\varepsilon},\frac{1}{\eta}}(\tfrac{1}{\varepsilon}\Omega)\,. && (2.12)
\end{aligned}
$$

The mentioned transformations were used in [30] for the proof of the following theorem.

Theorem 2.5 *For every irrational numbers ε,η, $\varepsilon \neq \eta$ and every bounded interval Ω, there exist $\tilde\varepsilon \in (-1,0)$, $\tilde\eta > 0$ and an interval $\tilde\Omega$, satisfying $\max(1+\tilde\varepsilon, -\tilde\varepsilon) < |\tilde\Omega| \leq 1$, such that*

$$\Sigma_{\varepsilon,\eta}(\Omega) = s\Sigma_{\tilde\varepsilon,\tilde\eta}(\tilde\Omega)\,, \qquad \textit{for some} \quad s \in \mathbb{R}\,.$$

Moreover, if $|\tilde\Omega| \neq 1$, then the distances between adjacent points in $\Sigma_{\tilde\varepsilon,\tilde\eta}(\tilde\Omega)$ are $\tilde\eta$, $1+\tilde\eta$, and $1+2\tilde\eta$. The distances take only two values $\tilde\eta$, $1+\tilde\eta$, only if $|\tilde\Omega| = 1$.

According to the above theorem, every C&P sequence is geometrically similar to another C&P sequence whose parameters satisfy certain restricted conditions. In particular, without loss of generality we can consider $\varepsilon \in (-1,0)$, $\eta > 0$ and the length of the acceptance interval Ω in the range $(\max(1+\varepsilon,-\varepsilon),1]$. If moreover we are interested only in the ordering of the distances in the C&P sequence and not on their actual lengths, i.e. we consider only the infinite word $u_{\varepsilon,\eta}(\Omega)$ we can choose any fixed $\eta > 0$. The words $u_{\varepsilon,\eta_1}(\Omega)$, $u_{\varepsilon,\eta_2}(\Omega)$ for $\eta_1 \neq \eta_2$ coincide. Therefore choosing $\eta = -\frac{1}{\varepsilon}$, which corresponds to a cut-and-project scheme with orthogonal projection, causes no loss of generality when studying only combinatorial properties of C&P sequences. The choice of η however influences geometry of the sequences, such as existence of self-similarity factor, etc. (cf. Section 4).

Remark 2.6 *Note that according to Theorem 2.5, the length $|\Omega|$ of the interval Ω being equal to 1 is the only case among $\big(\max(1+\tilde{\varepsilon},-\tilde{\varepsilon}),1\big]$ for which the C&P set has only two distances between neighbours. We shall thus take it as the initial case for the algorithm given in* (2.7), (2.8). *We have*

$$\ell_0 = 1, \quad \Delta^\star_{01} = 1+\varepsilon, \quad \Delta^\star_{02} = \varepsilon\,. \tag{2.13}$$

Example 2.7 *As an example, let us study the case $|\Omega| = 1$, which gives a C&P sequence with two distances between adjacent points. Set $\alpha = -\varepsilon \in (0,1)$ and put $\Omega = (\beta-1,\beta]$ for some $\beta \in \mathbb{R}$ as the acceptance window. Since the condition $a + b\varepsilon \in \Omega$ rewrites as $\beta - 1 < a - b\alpha \le \beta$, we obtain $a = \lfloor b\alpha + \beta\rfloor$ and the C&P sequence is of the form*

$$\Sigma_{-\alpha,\eta}(\beta-1,\beta] = \{\lfloor b\alpha+\beta\rfloor + b\eta \mid b \in \mathbb{Z}\}\,. \tag{2.14}$$

Since $\alpha,\eta > 0$, the sequence $x_n := \lfloor n\alpha+\beta\rfloor + n\eta$ is strictly increasing and thus the distances between adjacent points of the C&P set $\Sigma_{-\alpha,\eta}(\beta-1,\beta]$ are of the form

$$x_{n+1} - x_n = \eta + \lfloor (n+1)\alpha+\beta\rfloor - \lfloor n\alpha+\beta\rfloor = \begin{cases} \eta+1\,, \\ \eta\,. \end{cases} \tag{2.15}$$

From this expression it can be seen that the distances in the C&P sequence are arranged in the same order as 0's and 1's in the so-called lower and upper mechanical word. Recall that the lower mechanical word $\underline{s}_{\alpha,\beta} : \mathbb{Z} \to \{0,1\}$ is defined by the prescription

$$\underline{s}_{\alpha,\beta}(n) = \lfloor (n+1)\alpha+\beta\rfloor - \lfloor n\alpha+\beta\rfloor\,, \tag{2.16}$$

where α is called the slope and β the intercept of the word $\underline{s}_{\alpha,\beta}$. Similarly, upper mechanical word $\overline{s}_{\alpha,\beta} : \mathbb{Z} \to \{0,1\}$ is defined by the prescription

$$\overline{s}_{\alpha,\beta}(n) = \lceil (n+1)\alpha+\beta\rceil - \lceil n\alpha+\beta\rceil\,. \tag{2.17}$$

The infinite word $u_{\varepsilon,\eta}(\beta-1,\beta]$ with parameters $\eta > 0$, $\varepsilon = -\alpha$ is in fact the lower mechanical word $\underline{s}_{\alpha,\beta}$. Similarly, the choice $[\beta,\beta+1)$ for the acceptance window provides the upper mechanical word $\overline{s}_{\alpha,\beta}$. The mechanical words are in fact related to the well-known sturmian words, see Definition 3.2 and Remark 3.3.

3 Combinatorial properties of C&P sequences

Ordering of the distances in the C&P sequence $\Sigma_{\varepsilon,\eta}(\Omega)$ on the real line defines naturally an infinite binary or ternary word $u_{\varepsilon,\eta}(\Omega)$ (cf. equation (2.6)). In this section we describe some combinatorial properties of these infinite words. Some of the results derived here can be found in [27]. Nevertheless, the geometric approach to three interval exchange makes the proof simpler.

Obviously, geometrically similar C&P sequences correspond to the same infinite words. Therefore according to Theorem 2.5 we can consider only

$$\begin{aligned} \varepsilon \in (-1,0),\ \eta > 0 \quad &\text{and} \quad \Omega = [c, c+\ell), \\ \text{where} \quad \max(1+\varepsilon, -\varepsilon)\ &<\ \ell\ \leq\ 1. \end{aligned} \tag{3.18}$$

In this case the stepping function has the form

$$f(y) = \begin{cases} y+1+\varepsilon & \text{if } \ y\ \in\ [c, c+\ell-1-\varepsilon) & =:\ \Omega_A\,, \\ y+1+2\varepsilon & \text{if } \ y\ \in\ [c+\ell-1-\varepsilon, c-\varepsilon) & =:\ \Omega_B\,, \\ y+\varepsilon & \text{if } \ y\ \in\ [c-\varepsilon, c+\ell) & =:\ \Omega_C\,. \end{cases} \tag{3.19}$$

For simplicity, we denote the discontinuity points of the stepping function

$$\delta_1 := c+\ell-1-\varepsilon\,, \qquad \delta_2 := c-\varepsilon\,.$$

As was already mentioned, the infinite word $u_{\varepsilon,\eta}(\Omega)$ is defined over a binary alphabet if and only if the length of the acceptance window is $\ell = 1$, because in that case $\delta_1 = \delta_2$ and thus $\Omega_B = \emptyset$. Otherwise the alphabet of $u_{\varepsilon,\eta}(\Omega)$ has three letters.

Let us recall some basic notions of combinatorics on words. An alphabet $\mathcal{A}$ is a finite set of symbols - letters. A finite concatenation w of letters is called a finite word. The set of all finite words (including the empty word ϵ) over the alphabet $\mathcal{A}$ is denoted by $\mathcal{A}^*$. The concatenation of n letters a is denoted by a^n. The length of a word w is the number of letters concatenated in w, it is denoted by $|w|$. One considers also one-directional infinite words

$$u = u_0u_1u_2u_3\cdots$$

and bidirectional infinite words

$$u = \cdots u_{-2}u_{-1}u_0u_1u_2\cdots\,.$$

In relation to C&P sequences, mainly bidirectional infinite words are important. We denote the set of such words by $\mathcal{A}^{\mathbb{Z}}$. A word $w = w_0w_1\cdots w_{k-1}$ is called a *factor* of a word $u \in \mathcal{A}^{\mathbb{Z}}$ if $w = u_iu_{i+1}\cdots u_{i+k-1}$ for some i. Note that such i is called the *occurrence* of w in u. The set of factors of a word $u \in \mathcal{A}^{\mathbb{Z}}$ with the length n is denoted by

$$\mathcal{L}_n = \{u_iu_{i+1}\cdots u_{i+n-1} \mid i \in \mathbb{Z}\}\,.$$

The set of all factors of the word u (the *language* of u) is denoted by

$$\mathcal{L} = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n\,.$$

The number of different n-tuples that appear in the infinite word is given by the so-called complexity function, see for example [3].

Definition 3.1 *The complexity of a word $u \in \mathcal{A}^{\mathbb{Z}}$ is a mapping $\mathcal{C} : \mathbb{N} \to \mathbb{N}$ such that*

$$\mathcal{C}(n) = \#\{u_i u_{i+1} \cdots u_{i+n-1} \mid i \in \mathbb{Z}\} = \#\mathcal{L}_n \,.$$

Obviously, if u is an infinite word over a k-letter alphabet, then its complexity satisfies

$$1 \leq \mathcal{C}(n) \leq k^n \,, \qquad \text{for every } n \in \mathbb{N} \,.$$

It is known [53] that if there exists an $n \in \mathbb{N}$ such that $\mathcal{C}(n) \leq n$, then the word u is periodic, i.e. of the form $u = \cdots wwww \cdots$ for a finite word w. An aperiodic word of minimal complexity thus satisfies $\mathcal{C}(n) = n + 1$ for all $n \in \mathbb{N}$. An example of such a word is the word $\cdots 0001000 \cdots$ on the alphabet $\{0, 1\}$. The structure of such words is little interesting, since the occurrence of the letter 1 is singular. Obviously, they cannot be obtained by a cut-and-project scheme. In order to avoid such strange phenomena, we consider only those words which have reasonable density of their letters. The density of a letter a in the infinite word $u = \cdots u_{-2} u_{-1} u_0 u_1 u_2 \cdots$ is defined by

$$\varrho_a := \lim_{k \to \infty} \frac{\#\big\{ i \in \mathbb{Z} \cap [-k, k] \;\big|\; u_i = a \big\}}{2k + 1} \,,$$

if the limit exists.

Definition 3.2 *An infinite word $u \in \mathcal{A}^{\mathbb{Z}}$ is called sturmian if $\mathcal{C}(n) = n + 1$ for all $n \in \mathbb{N}$ and the densities of its letters are irrational.*

Such words have been extensively studied. We shall focus on them later in this section. Let us mention that the condition $\mathcal{C}(n) = n + 1$ for all $n \in \mathbb{N}$ in the case of one-directional infinite words already implies irrationality of the densities of letters. Our notion of sturmian words follows [54], however, sturmian words are often considered only as one-directional. A survey of properties of one-directional sturmian words can be found in [41].

3.1 Complexity

For the determination of the factors in the infinite bidirectional word $u_{\varepsilon,\eta}(\Omega)$ associated with the C&P sequence $\Sigma_{\varepsilon,\eta}(\Omega)$ it is essential to study the stepping function f. Its properties imply that the word $w = w_0 w_1 \cdots w_{k-1}$ in the alphabet $\mathcal{A} = \{A, B, C\}$ is a factor of $u_{\varepsilon,\eta}(\Omega)$ if and only if there exists an $x \in \Sigma_{\varepsilon,\eta}(\Omega)$ such that

$$x^\star \in \Omega_{w_0} \,, \quad f(x^\star) \in \Omega_{w_1} \,, \quad \ldots \,, \quad f^{k-1}(x^\star) \in \Omega_{w_{k-1}} \,,$$

where Ω_A, Ω_B, Ω_C are defined in (3.19). This means that

$$w = w_0 w_1 \cdots w_{k-1} \in \mathcal{L}_k \quad \Longleftrightarrow \quad \Omega_{w_0} \cap f^{-1}(\Omega_{w_1}) \cap \cdots \cap f^{-(k-1)}(\Omega_{w_{k-1}}) \neq \emptyset \,.$$

In case that $w = w_0 w_1 \cdots w_{k-1} \in \mathcal{L}_k$, we denote

$$\Omega_w := \Omega_{w_0} \cap f^{-1}(\Omega_{w_1}) \cap \cdots \cap f^{-(k-1)}(\Omega_{w_{k-1}})\,.$$

Properties of the stepping function f imply that Ω_w is an interval, closed from the left, open from the right. Obviously, we have

$$\Omega = \bigcup_{w \in \mathcal{L}_k} \Omega_w\,,$$

where the union is disjoint. In order that points $x^\star, y^\star \in \Omega$ belong to different intervals $x^\star \in \Omega_{w^{(1)}}$, $y^\star \in \Omega_{w^{(2)}}$, where $w^{(1)} \neq w^{(2)}$, $w^{(1)}, w^{(2)} \in \mathcal{L}_k$, there must exist $i = 0, 1, \dots, k-1$ such that at least one discontinuity point of the function f lies between $f^i(x^\star)$ and $f^i(y^\star)$. Thus boundaries between intervals Ω_w for $w \in \mathcal{L}_k$ are all points z such that $f^i(z)$ is a discontinuity point of the function f, i.e. $f^i(z) \in \{\delta_1, \delta_2\}$. This implies that the number of different factors of the word $u_{\varepsilon,\eta}(\Omega)$ of length k is equal to the number of elements

$$\#\mathcal{L}_k = \#\Big\{c, \delta_1, f^{-1}(\delta_1), \dots, f^{-k+1}(\delta_1), \delta_2, f^{-1}(\delta_2), \dots, f^{-k+1}(\delta_2)\Big\}\,. \tag{3.20}$$

For determination of the cardinality of the set $\mathcal{L}_k$, i.e. complexity of the infinite word, we need to use two properties of the stepping function f, which follow from the irrationality of ε.

$$\begin{aligned} &1)\ \text{Let } x \in \Omega. \quad \text{Then } f^i(x) \neq x \text{ for all } i \in \mathbb{Z},\ i \neq 0. \\ &2)\ \text{Let } x, y \in \Omega. \quad \text{Then } \exists\, i \in \mathbb{Z}, \text{ such that } f^i(x) = y \text{ iff } x - y \in \mathbb{Z}[\varepsilon]. \end{aligned} \tag{3.21}$$

Since $c = f(\delta_2)$, the property 1) implies that $c, \delta_2, f^{-1}(\delta_2), \dots, f^{-k+1}(\delta_2)$ are distinct. Thus

$$\mathcal{C}(k) = \#\mathcal{L}_k \geq k+1\,, \qquad \text{for all} \quad k \in \mathbb{N}\,.$$

Equality holds only in the case that $\delta_1 = \delta_2$. Therefore for C&P sequences $u_{\varepsilon,\eta}(\Omega)$ with parameters $\varepsilon, \eta, \Omega$ satisfying (3.18) it holds that

$$\mathcal{C}(k) = k+1 \qquad \Longleftrightarrow \qquad |\Omega| = 1\,.$$

We have thus derived the following well known fact.

Remark 3.3 *Every mechanical word* (2.16) *or* (2.17) *is a sturmian word. The opposite is also true [22, 54].*

The results about the complexity function of all C&P sequences are summarized in the following theorem.

Theorem 3.4 ([30]) *Let $\mathcal{C}$ be the complexity function of the infinite word $u_{\varepsilon,\eta}(\Omega)$ with $\Omega = [c, c+\ell)$, and let f be the corresponding stepping function.*

- *If $\ell \notin \mathbb{Z}[\varepsilon]$, then*
$$\mathcal{C}(n) = 2n+1\,, \qquad \textit{for}\ \ n \in \mathbb{N}\,.$$
- *If $\ell \in \mathbb{Z}[\varepsilon]$, then there exists a unique $n_0 \in \mathbb{N}_0$ such that*
$$\mathcal{C}(n) = \begin{cases} 2n+1 & \textit{for}\ \ n \le n_0\,, \\ n+n_0+1 & \textit{for}\ \ n > n_0\,. \end{cases}$$

Obviously, generic cut-and-project sequences have complexity $2n+1$. In case that the length of the acceptance window is in $\mathbb{Z}[\varepsilon]$, the cut-and-project sequence has a specific property which is explained in the following remark.

Remark 3.5 *Theorem 3.4 says that infinite words $u_{\varepsilon,\eta}(\Omega)$ with $|\Omega| \in \mathbb{Z}[\varepsilon]$ have complexity $\mathcal{C}(n) = n + const.$ for sufficiently large n. One-directional words with such complexity are called quasisturmian by Cassaigne in [21]. This author shows that such words have a sturmian structure, i.e. up to a finite prefix they are images under a morphism of a one-direction sturmian word. Following the same ideas, one can show that the bidirectional infinite word $u_{\varepsilon,\eta}(\Omega)$ with $|\Omega| \in \mathbb{Z}[\varepsilon]$ corresponding to a C&P sequence satisfies the following: there exists a sturmian word $v = \cdots v_{-2}v_{-1}|v_0v_1v_2\cdots \in \{0,1\}^{\mathbb{Z}}$ and finite words $W_0, W_1 \in \{A,B,C\}^*$ such that*
$$u_{\varepsilon,\eta}(\Omega) = \cdots W_{v_{-2}}W_{v_{-1}}|W_{v_0}W_{v_1}W_{v_2}\cdots\,,$$
i.e. $u_{\varepsilon,\eta}(\Omega)$ can be obtained by concatenation of words W_0, W_1 in the order of 0's and 1's in the sturmian word v.

Example 3.6 *Consider $\varepsilon = -\frac{1}{\tau}$ and $\eta = \tau$, where $\tau = \frac{1}{2}(1+\sqrt{5})$, (see Introduction). For the acceptance window choose $\Omega_1 = \big[-\frac{7}{\tau}+4, -\frac{17}{\tau}+11\big)$, $\Omega_2 = \Omega_1 + 2 - \frac{3}{\tau}$. Since the length of the acceptance windows $\ell = |\Omega_1| = |\Omega_2|$ satisfies $\ell = 7 - \frac{10}{\tau} \in \big(\max(1+\varepsilon,\varepsilon), 1\big]$, according to Theorem 2.5, the distances between adjacent points in the C&P sequences $\Sigma_{\varepsilon,\eta}(\Omega_1)$, $\Sigma_{\varepsilon,\eta}(\Omega_2)$ are $1+\tau$, coded by the letter A; $1+2\tau$, coded by the letter B; and τ, coded by the letter C.*

Using (2.9) the sequences $\Sigma_{\varepsilon,\eta}(\Omega_1)$, $\Sigma_{\varepsilon,\eta}(\Omega_2)$ are the same, up to a shift by $2+3\tau$. Both of them contain 0, since $0 \in \Omega_1$, $0 \in \Omega_2$. Figure 3 shows a segment of the infinite words coding these sequences, where we mark the point 0 in both of them.

It can be shown that the word $u_{\varepsilon,\eta}(\Omega_1)$ can be obtained from the upper mechanical sequence $\overline{s}_{\frac{1}{\tau},-\frac{1}{\tau^2}}$ by substituting for 0 the word $w_0 = BACAACA$ and for 1 the word $w_1 = BABACA$. Similarly, the infinite word $u_{\varepsilon,\eta}(\Omega_2)$ can be obtained from the upper mechanical sequence $\overline{s}_{\frac{1}{\tau},-\frac{1}{\tau}}$ by substituting for 0 the word $w_0 = BACABA$ and for 1 the word $w_1 = CAACA$.

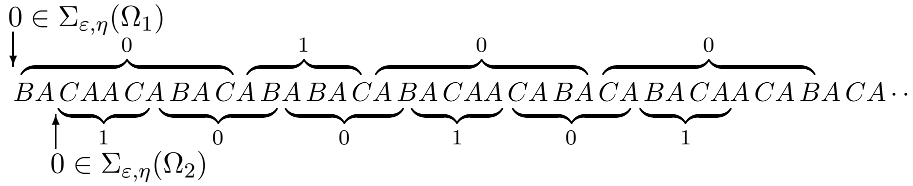


Figure 3: Block structure of quasisturmian words.

3.2 Properties of the language

Let us study the language of the infinite word $u_{\varepsilon,\eta}(\Omega)$. Since the stepping function f_Ω corresponding to the C&P sequence $\Sigma_{\varepsilon,\eta}(\Omega)$ and the stepping function $f_{\Omega+z}$ corresponding to the C&P sequence $\Sigma_{\varepsilon,\eta}(\Omega+z)$ satisfy the relation

$$f_{\Omega+z}(x) = z + f_\Omega(x-z)\,,$$

the language $\mathcal{L}$ and the complexity $\mathcal{C}$ of a C&P sequence depend only on the length of the acceptance interval Ω and not on its position.

First we determine the density of a given factor. Recall that the density of a particular factor w in the infinite bidirectional word $u = \cdots u_{-2}u_{-1}u_0u_1u_2\cdots$ is defined by

$$\varrho_w := \lim_{k\to\infty} \frac{\#\{i \in \mathbb{Z}\cap[-k,k] \mid u_iu_{i+1}\ldots u_{i+n-1} = w\}}{2k+1}\,, \tag{3.22}$$

if the limit exists. Elements $x \in \Sigma_{\varepsilon,\eta}(\Omega)$ such that the word corresponding to the k-tuple of the right neighbours of x is $w = w_0w_1\cdots w_{k-1} \in \mathcal{L}_k$ satisfy $x^\star \in \Omega_w$. Therefore the occurrences of the factor w in the infinite word $u_{\varepsilon,\eta}(\Omega)$ are given by the set

$$\{x \in \mathbb{Z}[\eta] \mid x^\star \in \Omega_w\}\,.$$

Since Ω_w is a semi-closed interval $\Omega_w \subset \Omega$, it is a C&P set, and its density is proportional to the length of the acceptance window (see fact 4 of Remark 2.2). This implies that the density of a factor $w \in \mathcal{L}$ in $u_{\varepsilon,\eta}(\Omega)$ is given by

$$\varrho_w = \frac{|\Omega_w|}{|\Omega|}\,.$$

Another important property of the language of the infinite word $u_{\varepsilon,\eta}(\Omega)$ is given by the following proposition.

Proposition 3.7 *The language $\mathcal{L}$ of the infinite word $u_{\varepsilon,\eta}[c,c+\ell)$ is stable under mirror image, i.e.*

$$w = w_0w_1\cdots w_{n-1} \in \mathcal{L} \quad\Longrightarrow\quad \overline{w} = w_{n-1}w_{n-2}\cdots w_0 \in \mathcal{L}\,.$$

Moreover, the densities of the factors w and $\overline{w}$ coincide, $\varrho_w = \varrho_{\overline{w}}$.

ProofSince the language of the infinite word $u_{\varepsilon,\eta}[c, c+\ell)$ depends only on the length ℓ of the acceptance interval and not on its position, it suffices to show the statement for the infinite word $u_{\varepsilon,\eta}[-\frac{\ell}{2}, \frac{\ell}{2})$, which codes the C&P sequence $\Sigma_{\varepsilon,\eta}[-\frac{\ell}{2}, \frac{\ell}{2})$. If $-\frac{\ell}{2} \notin \mathbb{Z}[\varepsilon]$, then

$$\Sigma_{\varepsilon,\eta}[-\tfrac{\ell}{2}, \tfrac{\ell}{2}) = \Sigma_{\varepsilon,\eta}(-\tfrac{\ell}{2}, \tfrac{\ell}{2})$$

and thus it is a centrally symmetric set and the statement of the proposition is obvious. If $-\frac{\ell}{2} = a + b\varepsilon \in \mathbb{Z}[\varepsilon]$, then for the proof it suffices to realize that the central symmetry of the set $\Sigma_{\varepsilon,\eta}[-\frac{\ell}{2}, \frac{\ell}{2})$ is broken by a unique point, namely $a + b\eta$. Since every factor $w \in \mathcal{L}$ occurs in $u_{\varepsilon,\eta}[-\frac{\ell}{2}, \frac{\ell}{2})$ infinitely many times, we can still use the same argument to justify the proposition. ■

For sturmian words the above property is well known, its proof can be found in [41].

For different lengths $\ell_1, \ell_2 \in \big(\max(-\varepsilon, 1+\varepsilon), 1\big]$ the languages of the infinite words $u_{\varepsilon,\eta}[c, c+\ell_1)$, $u_{\varepsilon,\eta}[c, c+\ell_2)$ are different. However, if we are interested only in factors of a given length n, the sets $\mathcal{L}_n$ can coincide even for infinite words corresponding to acceptance intervals of different lengths. Let $u_{\varepsilon,\eta}(\Omega)$ be an infinite word with the length of the acceptance window $|\Omega| = \ell$. We denote $\mathcal{L}_n(\ell)$ its set of factors of length n. For example $\mathcal{L}_1(\ell)$ is equal to the alphabet $\{A, B, C\}$ for every length $\max(-\varepsilon, 1+\varepsilon) < \ell < 1$. Let us now see how much we can change the length ℓ of the acceptance interval Ω without changing the set $\mathcal{L}_n(\ell)$.

Proposition 3.8 *Let $n \in \mathbb{N}$ be fixed. Denote by $\mathcal{C}_\ell$ the complexity function of the infinite word $u_{\varepsilon,\eta}(\Omega)$ with $\Omega = [c, c+\ell)$. Define*

$$\mathcal{D}_n = \left\{ \ell \;\middle|\; \max(-\varepsilon, 1+\varepsilon) < \ell \leq 1,\ \mathcal{C}_\ell(n) < 2n+1 \right\}.$$

Then elements of $\mathcal{D}_n$ divide the interval $(\max(-\varepsilon, 1+\varepsilon), 1]$ into a finite disjoint union of sub-intervals, such that $\mathcal{L}_n(\ell)$ is constant on the interior of each of these intervals.

The proof for special case $\varepsilon = -\frac{1}{\tau}$, $\eta = \tau$, can be found in [47]. The demonstration of the general statement follows analogous ideas.

Example 3.9 *Consider again the parameters $\varepsilon = -\frac{1}{\tau}$, $\eta = \tau$. For the sake of illustration of the previous proposition, let us choose $n = 4$ and find the division of the interval $\big(\max(-\varepsilon, 1+\varepsilon), 1\big] = (\frac{1}{\tau}, 1]$ into intervals such that the set $\mathcal{L}_4(\ell)$ is constant on the interior of these intervals. For that, we need to find ℓ so that $\mathcal{C}_\ell(n) < 2n+1$ for $n = 4$. Using* (3.20) *this happens if*

$$f^{(k)}(\delta_1) = \delta_2 \quad \text{or} \quad f^{(k)}(\delta_2) = \delta_1\,, \qquad \text{for some } k = 0, 1, 2, 3\,.$$

For solving these equations, one has to realize that not only the discontinuity points $\delta_1 = c + \ell - 1 - \varepsilon$, $\delta_2 = c - \varepsilon$ depend on ℓ, but also the prescription for the function f depends *on it. However, since every iteration of f is piecewise linear, the above equations can be easily solved. We find that*

$$\mathcal{D}_4 = \{4 - 2\tau, -4 + 3\tau, 1\}\,.$$

The division of the interval $(\frac{1}{\tau}, 1]$ by the elements of $\mathcal{D}_4$ is illustrated in Figure 4. The figure also shows the set of factors of length 4 for each of the subintervals and for the singular lengths $\ell \in \mathcal{D}_4$. Note that the set of factors corresponding to the interior of a subinterval is a union of sets of factors corresponding to the boundary points of the subinterval, for example

$$\mathcal{L}_4(\ell) = \mathcal{L}_4(4-2\tau) \cup \mathcal{L}_4(-4+3\tau)\,, \qquad \textit{for all} \;\; \ell \in (4-2\tau, -4+3\tau)\,.$$

$\frac{1}{\tau}$ $4-2\tau$ $-4+3\tau$ 1

$\mathcal{L}_4(\ell)$:						
	$ABAB$	$ABAB$	$AACA$	$AACA$	$AACA$	$AACA$
	$ABAC$	$ABAC$	$ABAB$	$ABAC$	$ABAC$	$ACAA$
	$ABBA$	$ACAB$	$ABAC$	$ACAA$	$ACAA$	$ACAC$
	$ACAB$	$BABA$	$ACAA$	$ACAB$	$ACAB$	$CAAC$
	$BABA$	$BACA$	$ACAB$	$BACA$	$ACAC$	$CACA$
	$BABB$	$CABA$	$BABA$	$CAAC$	$BACA$	
	$BACA$		$BACA$	$CABA$	$CAAC$	
	$BBAB$		$CAAC$		$CABA$	
	$CABA$		$CABA$		$CACA$	

Figure 4: The appearance of factors of length 4 in $\Sigma_{-\frac{1}{\tau},\tau}(-\Omega)$ in function of the length of the acceptance window.

3.3 Special factors

Let us introduce some important notions which help us understand the structure of factors in the language $\mathcal{L}$. The notions have been introduced in [20]. Consider arbitrary bidirectional infinite word v in an alphabet $\mathcal{A}$,

$$v = \cdots v_{-2}v_{-1}v_0v_1v_2\cdots$$

For every factor $w \in \mathcal{L}$ of v there exists at least one letter $a \in \mathcal{A}$ such that $aw \in \mathcal{L}$. Such letter a is called a left extension of the factor w. The set of left extensions of the factor w is denoted by $\mathrm{Lext}(w) \subset \mathcal{A}$.

Remark 3.10 *If $\tilde{w}$ is a prefix of the factor w, then*

$$\mathrm{Lext}(\tilde{w}) \supseteq \mathrm{Lext}(w)\,.$$

Since for every factor $\tilde{w} \in \mathcal{L}_{n+1}$ we have $\tilde{w} = aw$ for some $w \in \mathcal{L}_n$ and a letter $a \in \mathrm{Lext}(w)$, the increment of the complexity function can be computed as

$$\Delta\mathcal{C}(n) = \mathcal{C}(n+1) - \mathcal{C}(n) = \#\mathcal{L}_{n+1} - \#\mathcal{L}_n = \sum_{w \in \mathcal{L}_n} \big(\#\mathrm{Lext}(w) - 1\big)\,. \tag{3.23}$$

Similarly one can define the notion of right extension of a factor and obtain analogical relation

$$\Delta\mathcal{C}(n) = \sum_{w \in \mathcal{L}_n} \big(\#\mathrm{Rext}(w) - 1\big)\,. \tag{3.24}$$

Obviously, for determining the increment of complexity, only such factors w are interesting that have $\#\mathrm{Lext}(w) \geq 2$ or $\#\mathrm{Rext}(w) \geq 2$. Such factors are called left (resp. right) special factor.

Let us study these notions for infinite words corresponding to C&P sequences. Proposition 3.7 implies

$$w \text{ is a left special factor of } u_{\varepsilon,\eta}(\Omega)$$
$$\Updownarrow$$
$$\overline{w} \text{ is a right special factor of } u_{\varepsilon,\eta}(\Omega)\,.$$

Therefore we can limit our considerations to the study of left special factors. Theorem 3.4 implies that

$$1 \leq \Delta\mathcal{C}(n) \leq 2\,.$$

Thus for every $n \in \mathbb{N}$ there exists at least one and at most two left special factors of length n. Let us explain how one can decide whether a given factor $w \in \mathcal{L}_n$ is a left special factor or not. Recall from the beginning of Section 3.1 that every factor $w \in \mathcal{L}_n$ is linked with an interval $\Omega_w \subset \Omega$ of the form $\Omega_w = [a,b)$, where

$$a, b \in \Big\{c, \delta_1, f^{-1}(\delta_1), \dots, f^{-k+1}(\delta_1), \delta_2, f^{-1}(\delta_2), \dots, f^{-k+1}(\delta_2)\Big\}\,,$$

and δ_1, δ_2 are the discontinuity points of the stepping function f.

For every $x \in \Sigma_{\varepsilon,\eta}(\Omega)$ such that $x^\star \in \Omega_w$ the n-tuple of distances in the right neighbourhood of the point x corresponds to the word w. The nearest left neighbour of the point x is determined by $f^{-1}(x^\star)$. In order that the word w is a left special factor, the interior Ω_w° must contain at least one discontinuity point of f^{-1}. If the discontinuity point of f^{-1} lies only on the boundary of Ω_w, then w has only one left extension and thus is not a left special factor. The discontinuity points of the function f^{-1} are

$$c + \ell + \varepsilon = f(\delta_1) \qquad \text{and} \qquad c + 1 + \varepsilon = f(c) = f^2(\delta_2)\,.$$

Properties (3.21) of the stepping function imply that if $\ell \notin \mathbb{Z}[\varepsilon]$, then the discontinuity points of f^{-1} do not belong to the set

$$\Big\{c, \delta_1, f^{-1}(\delta_1), \dots, f^{-n+1}(\delta_1), \delta_2, f^{-1}(\delta_2), \dots, f^{-n+1}(\delta_2)\Big\}$$

for any $n \in \mathbb{N}$, and, therefore, if a discontinuity point of f^{-1} lies in Ω_w, then it lies in its interior. We can therefore conclude with the following proposition, which is proved in a different way in [27].

Proposition 3.11 *Let $\ell \notin \mathbb{Z}[\varepsilon]$. Consider the one-directional infinite word $u^{(i)} = u_0^{(i)} u_1^{(i)} u_2^{(i)} u_3^{(i)} \cdots$, $i = 1, 2$, coding the orbits $\{f^n(c + \ell + \varepsilon) \mid n \in \mathbb{N}_0\}$ and $\{f^n(c + 1 + \varepsilon) \mid n \in \mathbb{N}_0\}$. Then a finite word w is a left special factor of $u_{\varepsilon,\eta}(\Omega)$ if and only if it is a prefix of $u^{(1)}$ or $u^{(2)}$.*

Remark 3.12 *Since $c + \ell + \varepsilon$ is the image of δ_1, which is on the boundary between intervals Ω_B, Ω_C, then every prefix of the infinite word $u^{(1)}$ has $\{B, C\}$ as its left extension. Similarly, every prefix of $u^{(2)}$ has in its left extension letters A, B. It can happen that a word w is a prefix of both $u^{(1)}$ and $u^{(2)}$. Then $\mathrm{Lext}(w) = \{A, B, C\}$. However, since the infinite words $u^{(1)}$, $u^{(2)}$ are different, starting from a certain length of the factor w we have $\#\mathrm{Lext}(w) = 2$.*

3.4 Rauzy graphs

Another important tool for the study of combinatorial properties of infinite words are the so-called Rauzy graphs, [56, 7].

Definition 3.13 *Let u be an infinite word in the alphabet $\mathcal{A}$ and let $\mathcal{L}_n$ be the set of its factors of length n, $n \in \mathbb{N}$. Rauzy graph Γ_n is a directed graph whose set of vertices is $\mathcal{L}_n$ and set of directed edges is $\mathcal{L}_{n+1}$. The edge $e \in \mathcal{L}_{n+1}$ starts at a vertex $x \in \mathcal{L}_n$ and ends at a vertex $y \in \mathcal{L}_n$, if x is a prefix of e and y is its suffix, i.e.*

$$\underset{x = w_0 w_1 \cdots w_{n-1}}{\bullet} \xrightarrow{\ e = w_0 w_1 \cdots w_{n-1} w_n\ } \underset{y = w_1 \cdots w_{n-1} w_n}{\bullet}$$

The number of edges starting at a vertex x is called the outdegree of x and denoted by $d^+(x)$, the number of edges ending at x is called the indegree of x and denoted by $d^-(x)$.

From the definition of a Rauzy graph, we have

$$d^+(w) = \#\mathrm{Rext}(w) \qquad \text{and} \qquad d^-(w) = \#\mathrm{Lext}(w) \tag{3.25}$$

Remark 3.10 implies that if Γ_{n+1} contains a vertex with outdegree K, then the graph Γ_n contains a vertex with outdegree $\geq K$. Similar statement holds also for indegrees. Therefore

$$\max_{w \in \mathcal{L}_n} d^-(w) \ \geq \ \max_{w \in \mathcal{L}_{n+1}} d^-(w) \qquad \text{and} \qquad \max_{w \in \mathcal{L}_n} d^+(w) \ \geq \ \max_{w \in \mathcal{L}_{n+1}} d^+(w)\,. \tag{3.26}$$

Example 3.14 *Let us consider the lower mechanical word*

$$u_n = \left\lfloor \frac{n+1}{\tau} \right\rfloor - \left\lfloor \frac{n}{\tau} \right\rfloor , \qquad n \in \mathbb{Z}, \qquad \textit{where } \tau = \frac{1+\sqrt{5}}{2} .$$

Using Example 2.7, this infinite word in the alphabet $\{0,1\}$ *is a coding of the C&P sequence* $\Sigma_{-\frac{1}{\tau},\eta}(\beta-1,\beta]$. *According to Remark 3.3, it is a sturmian word, i.e. of complexity* $\mathcal{C}(n) = n+1$. *It can be easily computed that*

$$\begin{aligned} \mathcal{L}_3 &= \{010, 011, 101, 110\} , \\ \mathcal{L}_4 &= \{0101, 0110, 1010, 1011, 1101\} , \\ \mathcal{L}_5 &= \{01011, 01101, 10101, 10110, 11010, 11011\} . \end{aligned}$$

The Rauzy graphs Γ_3, Γ_4 *are illustrated on Figure 5.*

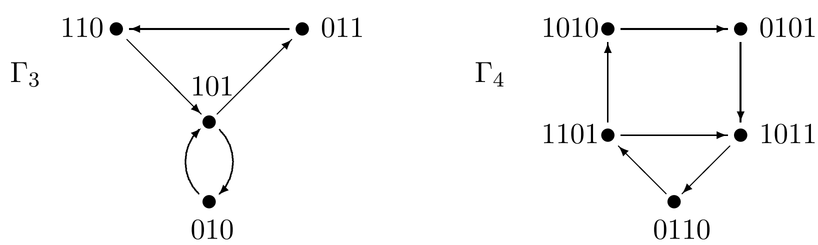


Figure 5: Rauzy graphs for the Fibonacci word.

Let us list some of the properties of the Rauzy graph Γ_n of the infinite word $u_{\varepsilon,\eta}[c, c+\ell)$ corresponding to a C&P sequence.

1. The graph Γ_n is strongly connected for every $n \in \mathbb{N}$. It means that for every pair of vertices x, y of the graph, there exists a directed path starting at x ending at y. This is a consequence of the repetitivity of the infinite word $u_{\varepsilon,\eta}[c, c+\ell)$, see (9) of Remark 2.2.

2. For the length of the acceptance window $\ell = 1$, the infinite word $u_{\varepsilon,\eta}[c, c+1)$ is sturmian and thus $\Delta\mathcal{C}(n) = 1$ for every $n \in \mathbb{N}$. Using (3.23), (3.24) and (3.25) for every n the graph Γ_n contains exactly one vertex $x \in \mathcal{L}_n$ with outdegree 2 and exactly one vertex $y \in \mathcal{L}_n$ with indegree 2. These vertices may or may not coincide, as we have seen in Example 3.14.

   For $\ell \in \mathbb{Z}[\varepsilon]$ the Theorem 3.4 implies $\Delta\mathcal{C}(n) = 1$ for sufficiently large n. Therefore the graphs have the same indegrees and outdegrees as in the sturmian case.

3. If $\ell \notin \mathbb{Z}[\varepsilon]$, then using Theorem 3.4 we have $\Delta\mathcal{C}(n) = 2$ for all $n \in \mathbb{N}$. Since the language of the infinite word $u_{\varepsilon,\eta}[c, c+\ell)$ is stable under mirror image (Proposition 3.7), relations (3.23) and (3.24) imply that in the graph Γ_n there is either one vertex with indegree 3 and one with outdegree 3, or there are two vertices with indegree 2 and two with outdegree 2. Remark 3.12 states that a vertex with out or indegree 3 can occur only in a graph Γ_n for small n.

4. Let us denote by $\bar{\Gamma}_n$ the graph created from Γ_n by the change of the orientation of the edges. Then $\bar{\Gamma}_n$ and Γ_n are isomorphic graphs, i.e. there exists a bijection π between the vertices of $\bar{\Gamma}_n$ and Γ_n, such that for every two vertices x, y of $\bar{\Gamma}_n$ there is a directed edge from x to y if and only there is a directed edge in the graph Γ_n from $\pi(x)$ to $\pi(y)$. This property follows from Proposition 3.7.

The last mentioned property can be stated in an even stronger version, if we consider the densities of factors in $\mathcal{L}_{n+1}$ as labels of the edges in the graph Γ_n.

Definition 3.15 *If the densities of all factors of the infinite word u are well defined, every edge e in the Rauzy graph Γ_n can be assigned a non-negative number, namely the density ϱ_e of the factor e. The resulting graph is called a weighted Rauzy graph.*

For every vertex x of the weighted Rauzy graph Γ_n we have obviously a ‘conservation law’,

$$\sum_{\text{edge } e \text{ ending in } x} \varrho_e \quad = \quad \sum_{\text{edge } f \text{ starting in } x} \varrho_f \,. \tag{3.27}$$

With the mentioned properties we can prove that the factors in a C&P word take at most 5 values.

Proposition 3.16 *Let $\mathcal{L}_n$ be the set of factors of length n of the infinite word $u_{\varepsilon,\eta}[c, c+\ell)$, $\ell \notin \mathbb{Z}[\varepsilon]$. The densities of factors in $\mathcal{L}_n$ take at most 5 values, i.e.*

$$\#\{\varrho_w \mid w \in \mathcal{L}_n\} \leq 5\,.$$

ProofConsider the weighted Rauzy graph Γ_n of $u_{\varepsilon,\eta}[c, c+\ell)$. If for every vertex x of Γ_n we have $d^+(x) = d^-(x) = 1$, then the relation (3.27) implies that the density of the edge e ending at x and of the edge f starting at x coincide. Since the graph is strongly connected, these edges are different, $e \neq f$. Denote by y the starting vertex of the edge e and by z the ending vertex of the edge f. From the graph Γ_n we remove the vertex x and edges e, f and replace it by a new edge starting at y and ending at z. We assign the new edge with the weight $\varrho_e = \varrho_f$. This reduction of the graph is illustrated of Figure 6.

The reduction of the Rauzy graph Γ_n is repeated until there are no vertices with both outdegree and indegree 1. The resulting graph is called the reduced weighted Rauzy graph $R\Gamma_n$. The construction implies that also $R\Gamma_n$ is a strongly connected graph, the weights of

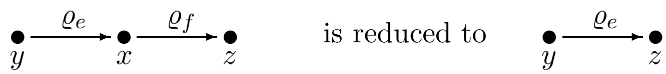


Figure 6: Reduction of the weighted Rauzy graph.

its edges satisfy the conservation law and the set of weights of the graph $R\Gamma_n$ is the same as the set of weight of the graph Γ_n. Moreover, the graph $\overline{R\Gamma_n}$ created by reversing the direction of edges in $R\Gamma_n$ is isomorphic to $R\Gamma_n$.

Using the property 3 of the Rauzy graph Γ_n for sufficiently large n there are two vertices with outdegree 2, the outdegree of the remaining vertices is 1. Similarly, there are two vertices with indegree 2 and the indegree of other vertices is 1. It may happen that a vertex with outdegree 2 coincides with a vertex with indegree 2. This implies that the reduced Rauzy graph has 2, 3 or 4 vertices.

Let us discuss the case that $R\Gamma_n$ has 4 vertices, i.e. the case when none of the vertices has in the same time indegree and outdegree 2. It can be easily derived that the reduced weighted Rauzy graph has one of the forms illustrated on Figure 7. Since all the possible reduced graphs have six edges, the original weighted Rauzy graph has at most six different densities. We can eliminate the sixth value in graphs G_1, G_2 and G_3 using the conservation law. In the graph G_1 we have $\varrho_1 = \varrho_4 + \varrho_6 = \varrho_3$. Similarly in the graph G_2 we have $\varrho_1 = \varrho_2 + \varrho_5 = \varrho_3$. In the graph G_3 we have $\varrho_1 = \varrho_4 - \varrho_6 = \varrho_3$.

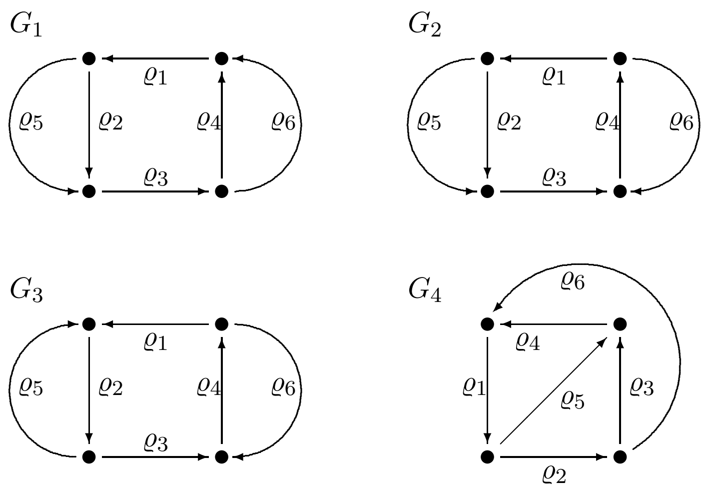


Figure 7: Possible reduces weighted Rauzy graphs.

The conservation law is not sufficient for reducing the number of densities in the graph G_4. Here we use the property 4 of the Rauzy graph of a C&P, namely that by changing the direction of the edges in G_4 we obtain an isomorphic graph $\bar{G}_4$. The graphs are illustrated on Figure 8. The only permutation π of the vertices which realizes the isomorphism of the graphs G_4 and $\bar{G}_4$ is the permutation $\pi(x) = v$, $\pi(y) = z$, $\pi(z) = y$, $\pi(v) = x$. The isomorphism preserves the densities, thus $\varrho_2 = \varrho_4$, $\varrho_5 = \varrho_6$.

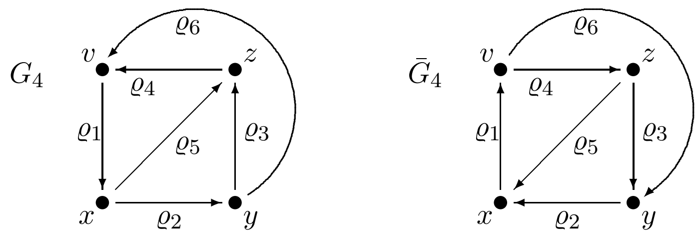


Figure 8: Isomorphic reduced graphs G_4 and $\bar{G}_4$.

We have thus solved the case that the reduced weighted graph $R\Gamma_n$ has 4 vertices. If $R\Gamma_n$ has 2 or 3 vertices, then such a graph has at most 5 edges. Therefore there are at most 5 values of densities.

For small values of n it can happen that the graph Γ_n has one vertex with outdegree 3 and one vertex with indegree 3, the other vertices having both outdegree and indegree 1. In this case the reduced Rauzy graph $R\Gamma_n$ has 1 or 2 vertices and at most 4 edges, thus the number of different values of densities is less or equal to 4. ∎

Remark 3.17 *In case that $\ell \in \mathbb{Z}[\varepsilon]$, then the densities of factors of length n of the infinite word $u_{\varepsilon,\eta}[c, c+\ell)$ take at most 3 values for sufficiently large n , because the resulting word is either sturmian or quasisturmian and the number of densities can be read from the corresponding reduced Rauzy graph, which has always at most three edges. Let us mention that the fact that the densities of factors in sturmian words take at most three values has been stated in [14], in fact, it can be deduced already from [62].*

3.5 Sturmian words

As we have seen, sturmian words can be defined in several different equivalent ways, namely as

- bidirectional infinite words with complexity $\mathcal{C}(n) = n + 1$and irrational densities of letters,
- mechanical words $\underline{s}_{\alpha,\beta}$, $\overline{s}_{\alpha,\beta}$ with irrational slope α,
- codings of cut-and-project sequences $u_{\varepsilon,\eta}(\Omega)$, where Ω is a semi-closed interval of unit length, and ε, η are irrational numbers satisfying $\varepsilon \in (-1, 0)$, $\eta > 0$.

There exist other equivalent definitions, for example using the number of palindromes of given length or using the so-called return words. For a nice overview of these definitions see [13, 41].

The arithmetical definition of mechanical words allows one to easily derive further combinatorial properties of sturmian words. In Example 2.7 we have shown that the upper mechanical word $\overline{s}_{\alpha,\beta}$ corresponds to a cut-and-project sequence with an acceptance window which is closed from the left and open from the right. The upper mechanical word $\underline{s}_{\alpha,\beta}$ corresponds to a cut-and-project sequence with acceptance interval of opposite type. Since $\Sigma_{\varepsilon,\eta}(-\Omega) = -\Sigma_{\varepsilon,\eta}(\Omega)$ and since the language of a sturmian sequence is closed under reversal, for the study of the properties of the language we can limit our considerations to upper mechanical words $\overline{s}_{\alpha,\beta}$, see (2.17).

Let us now prove three properties which have been used in [5] for the construction of aperiodic wavelets. Note that Property 3.19 can be found already in [21].

Property 3.18 *The number of letters 1 in a factor of length n of the mechanical word $\overline{s}_{\alpha,\beta}$ is equal to $\lfloor n\alpha \rfloor$ or $\lceil n\alpha \rceil$.*

Proof Consider a factor w of length n, $w = \overline{s}_{\alpha,\beta}(i)\overline{s}_{\alpha,\beta}(i+1)\cdots\overline{s}_{\alpha,\beta}(i+n-1)$. Since the alphabet of the mechanical word is $\{0,1\}$, the number of letters 1 in w is equal to

$$\sum_{j=0}^{n-1}\overline{s}_{\alpha,\beta}(i+j) = \lceil (i+n)\alpha+\beta \rceil - \lceil i\alpha+\beta \rceil =$$

$$= \lceil n\alpha + \underbrace{i\alpha+\beta-\lceil i\alpha+\beta\rceil}_{\in(-1,0)} \rceil = \begin{cases} \lfloor n\alpha \rfloor, \\ \lceil n\alpha \rceil. \end{cases}$$

∎

Property 3.19 *All $n+1$ factors of length n of the mechanical word $\overline{s}_{\alpha,\beta}$ appear in the factor w of length $2n$ of the mechanical word $\overline{s}_{\alpha,-\alpha}$, given by*

$$w = \overline{s}_{\alpha,-\alpha}(-n+1)\overline{s}_{\alpha,-\alpha}(-n+2)\cdots\overline{s}_{\alpha,-\alpha}(0)\cdots\overline{s}_{\alpha,-\alpha}(n)\,.$$

Proof Example 2.7 says that $\overline{s}_{\alpha,\beta}$ codes the distances in the cut-and-project sequence $\Sigma_{-\alpha,\eta}[\beta,\beta+1)$ for arbitrary $\eta > 0$. Since the language of a cut-and-project sequence does not change by translation of the acceptance interval, we can study without loss of generality the language of the mechanical word $\overline{s}_{\alpha,-\alpha}$, i.e. of the cut-and-project sequence $\Sigma_{-\alpha,\eta}[-\alpha,1-\alpha)$. The stepping function has a unique discontinuity point, namely $\delta_1 = 0$. The same considerations as for determining the complexity in Section 3.1 lead to the fact that the acceptance window $\Omega = [-\alpha, 1-\alpha)$ is divided by n points δ_1, $f^{-1}(\delta_1)$, $\ldots$, $f^{-n+1}(\delta_1)$ into $n+1$ disjoint subintervals closed from the left and open from the right, say $\Omega_{w^{(1)}}$, $\Omega_{w^{(2)}}$, $\ldots$, $\Omega_{w^{(n+1)}}$, with the following property: if x, y are elements of

$\Sigma_{-\alpha,\eta}[-\alpha, 1-\alpha)$, then the n-tuples of distances starting from x and from y coincide if and only if $x^\star$, $y^\star$ belong to the same interval $\Omega_{w^{(i)}}$ for some $1 \le i \le n+1$. Since the left boundary points of all the intervals belong to $\mathbb{Z}[\alpha]$, these boundary points are star map images of points of $\Sigma_{-\alpha,\eta}[-\alpha, 1-\alpha)$. The boundary points of the intervals $w^{(i)}$ are explicitly given by:

$$-\alpha = f(\delta_1), 0 = \delta_1, f^{-1}(\delta), \cdots, f^{-n+1}(\delta_1)\,.$$

Therefore it suffices to consider all n-tuples of distances in $\Sigma_{-\alpha,\eta}[-\alpha, 1-\alpha)$, starting at point 0, at its right neighbour, and at its $n-1$ left neighbours. From Example 2.7 we know that every element $x_k \in \Sigma_{-\alpha,\eta}[-\alpha, 1-\alpha)$ has the form $x_k = \lceil k\alpha - \alpha \rceil + k\eta$ for $k \in \mathbb{Z}$. Thus $x_0 = 0$ and we must study the n-tuples of distances between points x_{-n+1}, x_{-n+2}, $\ldots$, x_0, x_1, $\ldots$, x_{n+1}. Since $\overline{s}_{\alpha,-\alpha}(k)$ codes the distance between x_k and x_{k+1}, the proof is finished. ■

Property 3.20 *The number of factors of length n in the mechanical word $\overline{s}_{\alpha,\beta}$ prefixed by 1 is equal to $\lceil n\alpha \rceil$.*

ProofProperty 3.19 implies that, for the description of the first letter of all $n+1$ different factors of length n, it suffices to focus on letters $\overline{s}_{\alpha,-\alpha}(-n+1)$, $\overline{s}_{\alpha,-\alpha}(-n+2)$, $\ldots$, $\overline{s}_{\alpha,-\alpha}(0)$, $\overline{s}_{\alpha,-\alpha}(1)$. The number of letters 1 among them is

$$\sum_{k=-n+1}^{1} \overline{s}_{\alpha,-\alpha}(k) = \lceil \alpha \rceil - \lceil -n\alpha \rceil = 1 - \lceil -n\alpha \rceil = \lceil n\alpha \rceil\,.$$

■

Let us mention an interesting consequence of Property 3.18. If w and w' are factors of $\overline{s}_{\alpha,\beta}$ of the same length, then the numbers of letters 1 in w and in w' differ at most by 1. This obviously implies that also numbers of letters 0 in w and w' differ at most by 1. Infinite words with this property are called balanced. Sturmian words are balanced. On the other hand, every aperiodic balanced infinite word is sturmian. We have thus obtained another equivalent definition of sturmian words. The above implies other properties:

- Either 00 or 11 is not a factor of a sturmian word.
- If 00 is not a factor of $\overline{s}_{\alpha,\beta}$ or $\underline{s}_{\alpha,\beta}$, and if 01^x0 is a factor, then $x = b$ or $x = b+1$, where $b = [\frac{\alpha}{1-\alpha}]$.

Other interesting properties of sturmian words concern substitution invariance. This is the topic of the following section.

Remark 3.21 *Generic cut-and-project sequences with three distances between adjacent points do not have explicit formula for determining the n-th letter, which exists for sturmian words. Therefore the study of properties analogous to that mentioned in this subsection is significantly more difficult [27].*

4 Selfsimilarity of C&P sequences

We now turn our attention to cut-and-project sequences with self-similarity. We say that a set $\Lambda \subset \mathbb{R}$ is self-similar if there exists a factor $\gamma > 1$ such that

$$\gamma\Lambda \subset \Lambda\,.$$

An infinite bidirectional word corresponding to a self-similar C&P set may have many interesting properties, namely under some very general condition it is a fixed point of a nontrivial morphism, or it is an image of such a fixed point. These properties are studied in Section 6.

In this section we describe the conditions on the parameters ε, η, and interval Ω, under which the C&P set $\Sigma_{\varepsilon,\eta}(\Omega)$ is self-similar. For that we need to recall some basic number theoretical notions that will be useful also in studying the invariance of C&P sets under morphisms. It turns out that ε and η are different roots of one quadratic equation with integer coefficients. Therefore, we restrict ourselves to notions connected to quadratic numbers.

For an irrational number α we denote by $\mathbb{Q}(\alpha)$ the minimal number field containing $\mathbb{Q}$ and α. If α is a quadratic number, i.e. an irrational solution of a quadratic equation with integer coefficients, then

$$\mathbb{Q}(\alpha) = \{a + b\alpha \mid a, b \in \mathbb{Q}\}\,.$$

The other root α' of the quadratic equation is the algebraic conjugate of α and obviously we have $\alpha' \in \mathbb{Q}(\alpha)$. On $\mathbb{Q}(\alpha)$ one defines the mapping

$$x = a + b\alpha \in \mathbb{Q}(\alpha) \qquad \mapsto \qquad x' = a + b\alpha' \in \alpha \in \mathbb{Q}(\alpha)\,,$$

which is (the so-called Galois) automorphism on $\mathbb{Q}(\alpha)$. This means that it satisfies $(x+y)' = x' + y'$ and $(xy)' = x'y'$ for all $x, y \in \mathbb{Q}(\alpha)$.

A root of a monic quadratic polynomial with integer coefficients is called a quadratic integer. A quadratic integer γ is a quadratic Pisot number, if $\gamma > 1$ and its algebraic conjugate γ' satisfies $|\gamma'| < 1$. The following result may be found also in [10].

Theorem 4.1

1. *The C&P sequence $\Sigma_{\varepsilon,\eta}(\Omega)$ is self-similar if and only if ε is a quadratic number, $\eta = \varepsilon'$ is its algebraic conjugate, and the closure $\overline{\Omega}$ of the acceptance Ω contains the origin. In that case*

$$\Sigma_{\varepsilon,\eta}(\Omega) = \Sigma_{\varepsilon,\varepsilon'}(\Omega) = \Sigma_{\eta',\eta}(\Omega) = \{x \in \mathbb{Z}[\eta] \mid x' \in \Omega\}\,.$$

2. *If γ is the self-similarity factor of $\Sigma_{\eta',\eta}(\Omega)$, then γ is a quadratic Pisot number in $\mathbb{Q}[\eta]$.*

ProofFirst let us show that if ε and η are mutually conjugated quadratic numbers and $0 \in \overline{\Omega}$ then $\Sigma_{\varepsilon,\eta}(\Omega)$ is self-similar. For that we have to find a self-similarity factor γ, such that $\gamma\Sigma_{\eta',\eta}(\Omega) \subset \Sigma_{\eta',\eta}(\Omega)$.

Let ε, η be the roots of the equation $Mx^2 = Kx + L$ for some integers K, L, M. We look for γ in the form $\gamma = a + Mb\eta$ for some integers a, b. Such γ satisfies $\gamma \in \mathbb{Z}[\eta]$ and $\gamma\eta = a\eta + Mb\eta^2 = Lb + \eta(a + Kb) \in \mathbb{Z}[\eta]$. Therefore $\gamma\mathbb{Z}[\eta] \subset \mathbb{Z}[\eta]$.

Since $\gamma \in \mathbb{Z}[\eta] \subset \mathbb{Q}(\eta)$, we determine the image of γ under the Galois automorphism $\gamma' = a + Mb\eta'$. Clearly, $\gamma\gamma' \in \mathbb{Z}$. Since η' is irrational, the set $\mathbb{Z}[\eta'] = \mathbb{Z} + \mathbb{Z}\eta'$ is dense in $\mathbb{R}$ and thus there are infinitely many choices of $a, b \in \mathbb{Z}$ so that $\gamma' \in (0, 1)$. Together with the fact $0 \in \overline{\Omega}$ it follows that $\gamma'\Omega \subset \Omega$. We use the above to obtain

$$\begin{aligned}\gamma\Sigma_{\eta',\eta}(\Omega) &= \gamma\{x \in \mathbb{Z}[\eta] \mid x' \in \Omega\} = \{\gamma x \in \gamma\mathbb{Z}[\eta] \mid \gamma' x' \in \gamma'\Omega\} \subset \\ &\subset \{\gamma x \in \mathbb{Z}[\eta] \mid \gamma' x' \in \Omega\} \subseteq \{y \in \mathbb{Z}[\eta] \mid y' \in \Omega\} = \Sigma_{\eta',\eta}(\Omega)\,.\end{aligned}$$

Since $\gamma' \in (0, 1)$ and $\gamma\gamma' \in \mathbb{Z}$, we have $|\gamma| > 1$. If $\gamma > 1$, it is the desired self-similarity factor, in the opposite case we choose γ^2 for the self-similarity factor.

Let us prove the necessary condition for the self-similarity of a C&P set. Let $\gamma > 1$ satisfy $\gamma\Sigma_{\varepsilon,\eta}(\Omega) \subset \Sigma_{\varepsilon,\eta}(\Omega)$. For a chosen point $x = a + b\eta \in \Sigma_{\varepsilon,\eta}(\Omega)$ we have $\gamma x \in \Sigma_{\varepsilon,\eta}(\Omega) \subset \mathbb{Z}[\eta]$. Therefore there must exist integers $\tilde{a}$, $\tilde{b}$ such that $\gamma x = \gamma(a + b\eta) = \tilde{a} + \tilde{b}\eta$. This implies

$$\gamma = \frac{\tilde{a} + \tilde{b}\eta}{a + b\eta} \qquad \text{and} \qquad \eta = \frac{-\tilde{a} + a\gamma}{\tilde{b} - b\gamma}\,.$$

Therefore $\mathbb{Q}(\gamma) = \mathbb{Q}(\eta)$.

Let $(x_n)_{n\in\mathbb{Z}}$ be the strictly increasing sequence such that $\Sigma_{\varepsilon,\eta}(\Omega) = \{x_n \mid n \in \mathbb{Z}\}$. Recall that the distances between neighbouring points of $\Sigma_{\varepsilon,\eta}(\Omega)$ take values $x_{n+1} - x_n \in \{\Delta_1, \Delta_2, \Delta_1 + \Delta_2\}$, where Δ_1, Δ_2 are positive numbers in $\mathbb{Z}[\eta]$ linearly independent over $\mathbb{Q}$. Take an index n such that $x_{n+1} - x_n = \Delta_1$. Since $\Sigma_{\varepsilon,\eta}(\Omega)$ is self-similar with the factor γ, both γx_n and γx_{n+1} belong to $\Sigma_{\varepsilon,\eta}(\Omega)$. Therefore the gap between the two points is filled by distances Δ_1, Δ_2 and $\Delta_1 + \Delta_2$. It follows that the distance $\gamma x_{n+1} - \gamma x_n$ is an integer combination of Δ_1, Δ_2 with positive coefficients,

$$\gamma\Delta_1 = \gamma x_{n+1} - \gamma x_n = k_{11}\Delta_1 + k_{12}\Delta_2$$

for some non-negative integers k_{11}, k_{12}. Analogously we obtain

$$\gamma\Delta_2 = k_{21}\Delta_1 + k_{22}\Delta_2\,, \qquad k_{21}, k_{22} \in \mathbb{N}_0\,.$$

We denote by $\mathbb{K}$ the 2×2 matrix $\mathbb{K} = (k_{ij})$ and write the above as

$$\mathbb{K}\begin{pmatrix}\Delta_1\\ \Delta_2\end{pmatrix} = \gamma\begin{pmatrix}\Delta_1\\ \Delta_2\end{pmatrix}.$$

This means that γ is an eigenvalue of the integer-valued 2×2 matrix $\mathbb{K}$ and as such is a root of a monic quadratic polynomial with integer coefficients. Since $\mathbb{Q}(\eta) = \mathbb{Q}(\gamma)$ and η is irrational, γ is a quadratic integer.

The eigenvector corresponding to γ is $\binom{\Delta_1}{\Delta_2}$. As Δ_1, Δ_2 belong to the quadratic field $\mathbb{Q}(\eta) = \mathbb{Q}(\gamma)$, we can apply the Galois automorphism to obtain the other eigenvector and eigenvalue of the matrix $\mathbb{K}$,

$$\mathbb{K}\begin{pmatrix} \Delta_1' \\ \Delta_2' \end{pmatrix} = \gamma' \begin{pmatrix} \Delta_1' \\ \Delta_2' \end{pmatrix}.$$

The Perron-Frobenius theorem for positive integer matrices implies that $|\gamma'| < \gamma$.

We now explain the relation between the Galois automorphism and the star map in the cut-and-project scheme. Take any pair of points $x, y \in \Sigma_{\varepsilon,\eta}(\Omega)$, $x < y$. Their distance $y - x$ belongs to $\Sigma_{\varepsilon,\eta}(\Omega) - \Sigma_{\varepsilon,\eta}(\Omega) = \Sigma_{\varepsilon,\eta}(\Omega - \Omega)$. From the definition of the C&P set, we have $(y - x)^\star \in \Omega - \Omega$. Now let $x = \gamma^m x_n$, $y = \gamma^m x_{n+1}$ for any integer power m and for some n such that $x_{n+1} - x_n = \Delta_1$, or $x_{n+1} - x_n = \Delta_2$ respectively. From the self-similarity of $\Sigma_{\varepsilon,\eta}(\Omega)$, the points x, y belong to $\Sigma_{\varepsilon,\eta}(\Omega)$, and hence $(\gamma^m \Delta_1)^\star$, $(\gamma^m \Delta_2)^\star \in \Omega - \Omega$. Therefore the sequence of vectors

$$\mathbb{K}^m \begin{pmatrix} \Delta_1^\star \\ \Delta_2^\star \end{pmatrix} = \left(\mathbb{K}^m \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \right)^\star = \left(\gamma^m \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \right)^\star = \begin{pmatrix} (\gamma^m \Delta_1)^\star \\ (\gamma^m \Delta_2)^\star \end{pmatrix} \tag{4.28}$$

is bounded with $m \to \infty$. In the above we have used the property of the star map $(kx)^\star = kx^\star$ for $x \in \mathbb{Z}[\eta]$ and any integer k. Since the eigenvectors of the matrix $\mathbb{K}$ form a basis of $\mathbb{R}^2$, we can write

$$\begin{pmatrix} \Delta_1^\star \\ \Delta_2^\star \end{pmatrix} = \alpha_1 \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} + \alpha_2 \begin{pmatrix} \Delta_1' \\ \Delta_2' \end{pmatrix}$$

for some real coefficients α_1, α_2. Substituting into (4.28) we derive that the sequence of vectors

$$\alpha_1 \gamma^m \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} + \alpha_2 \gamma'^m \begin{pmatrix} \Delta_1' \\ \Delta_2' \end{pmatrix}$$

is bounded. Since $\gamma > 1$, we have $\alpha_1 = 0$ and $|\gamma'| < 1$. We can conclude that γ is a quadratic Pisot number and $\binom{\Delta_1^\star}{\Delta_2^\star} = \alpha_2 \binom{\Delta_1'}{\Delta_2'}$. The lengths Δ_1, Δ_2 belong to $\mathbb{Z}[\eta]$ and hence can be written in the form $\Delta_1 = a_1 + b_1\eta$, $\Delta_2 = a_2 + b_2\eta$ for some integers a_1, a_2, b_1, b_2. We have

$$\alpha_2 = \frac{a_1 + b_1\varepsilon}{a_2 + b_2\varepsilon} = \frac{a_1 + b_1\eta'}{a_2 + b_2\eta'},$$

which implies $(a_1b_2 - a_2b_1)(\varepsilon - \eta') = 0$. Since Δ_1, Δ_2 are linearly independent over $\mathbb{Q}$, we have $a_1b_2 - a_2b_1 \neq 0$ and thus $\varepsilon = \eta'$ as the theorem claims. The star map in such a cut-and-project scheme coincides with the Galois automorphism on the quadratic field $\mathbb{Q}(\eta) = \mathbb{Q}(\varepsilon) = \mathbb{Q}(\gamma)$.

The last to be verified is that $\overline{\Omega}$ contains the origin. Since $\gamma\Sigma_{\varepsilon,\eta}(\Omega) \subset \Sigma_{\varepsilon,\eta}(\Omega)$, it follows easily that $\gamma'\Omega \subset \Omega$. This implies $0 \in \overline{\Omega}$. ■

From the proof of the above theorem it follows that if ε, η are mutually conjugated quadratic numbers and $0 \in \overline{\Omega}$, (i.e. $\Sigma_{\varepsilon,\eta}(\Omega)$ is self-similar) there exists infinitely many factors γ such that $\gamma\Sigma_{\varepsilon,\eta}(\Omega) \subset \Sigma_{\varepsilon,\eta}(\Omega)$. It can be shown that all of these factors are quadratic Pisot numbers in $\mathbb{Z}[\eta]$. The exact description of all self-similarity factors of a given C&P set is straightforward, but rather technical. For a generalisation of self-similarity studied on the most common example $\eta = \tau$ we refer to [45].

Finally, let us mention that first results about self-similar Delone sets with Meyer property (which include C&P sets) have been obtained by Meyer in [49]. He shows that the self-similarity factor of such sets must be a Pisot or Salem number, i.e. an algebraic integer > 1 with all conjugates in the unit disc. In a even more general setting, self-similarity of Delone sets has been studied in [34, 35].

5 Non-standard numeration systems and C&P sequences

Another example of self-similar sets are sequences formed by β-integers. We show how they are related to C&P sequences. For the definition of β-integers we introduce the notion of β-expansion, which has been first given by Rényi [58]. The β-expansions are studied from the arithmetical point of view for example in [59, 55, 28, 16].

Let β be a real number greater than 1. For a non-negative $x \in \mathbb{R}$ we find a unique k such that $\beta^k \le x < \beta^{k+1}$ and put

$$x_k := \left[\frac{x}{\beta^k}\right], \qquad r_k := x - x_k\beta^k .$$

The coefficients x_i, $i \in \mathbb{Z}$, $i \le k-1$ we define recursively

$$x_i := [\beta r_{i+1}] , \qquad r_i := \beta r_{i+1} - x_i .$$

The described procedure is called the greedy algorithm. It ensures that

$$x = \sum_{i=-\infty}^{k} x_i\beta^i .$$

The above expression of x using an infinite series is called the β-expansion of x. For $\beta = 2$ or $\beta = 10$ we obtain the usual binary or decimal expansion of x. The real numbers x for which the coefficients x_{-1}, x_{-2}, x_{-3}, … in the β-expansion of $|x|$ vanish, are called β-integers. They form the set denoted by $\mathbb{Z}_\beta$,

$$\mathbb{Z}_\beta = \left\{ \pm\sum_{i=0}^{k} x_i\beta^i \;\middle|\; \sum_{i=0}^{k} x_i\beta^i \text{ is a } \beta\text{-expansion of an } x \ge 0 \right\} .$$

The greedy algorithm implies that if $x = \sum_{i=-\infty}^{k} x_i\beta^i$ is the β-expansion of a number x, then $\sum_{i=-\infty}^{k+1} x_{i-1}\beta^i$ is the β-expansion of βx. Therefore we trivially have for β-integers

$$\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta\,.$$

Let us mention that $\mathbb{Z}_\beta$ has also many other self-similarity factors.

If β is an integer greater than 1, the set of β-integers coincides with rational integers, $\mathbb{Z}_\beta = \mathbb{Z}$. Drawn on the real line, the distances between adjacent points of $\mathbb{Z}_\beta$ are all 1, and all integers > 1 are self-similarity factors of $\mathbb{Z}_\beta$. In this case the coefficients (digits) x_i in a β-expansion take values 0, 1, ..., $\beta - 1$, and every finite sequence formed by these digits is a β-expansion of some number x.

The situation is very different if $\beta \notin \mathbb{Z}$. As a consequence of the greedy algorithm, the digits in a β-expansion take values 0, 1, ..., $\lceil\beta\rceil - 1$. However, not all strings of these digits correspond to a number x as its β-expansion. Which sequences of digits are permissible in β-expansions and what are the distances between adjacent points in $\mathbb{Z}_\beta$ depends on the so-called Rényi development of 1. We define a mapping

$$T_\beta(x) = \beta x - [\beta x]\,, \qquad \text{for} \ \ x \in [0,1]\,.$$

Put $t_i := \left[\beta T_\beta^{i-1}(1)\right]$ for $i = 1, 2, 3, \ldots$. The sequence

$$d_\beta(1) = t_1t_2t_3\cdots$$

is called the Rényi development of 1.

In order to decide, whether a series $\sum_{i=0}^{n} x_i\beta^i$ is a β-expansion, we use the condition of Parry [55].

Proposition 5.1 *Let $\beta > 1$. Then $\sum_{i=0}^{n} x_i\beta^i$ is a β-expansion of a number x if and only if the word $x_ix_{i-1}\cdots x_0$ is lexicographically strictly smaller than $t_1t_2t_3\cdots$ for all $0 \le i \le n$.*

In [63] it is shown that the distances between neighbouring points in the set $\mathbb{Z}_\beta$ are of the form

$$\sum_{k=1}^{\infty} \frac{t_{i+k}}{\beta^k}\,, \qquad \text{for} \ \ i = 0, 1, 2, \ldots\,.$$

A necessary condition in order that $\mathbb{Z}_\beta$ has only finitely many distances between neighbouring points is that the Rényi development of 1 is eventually periodic. The construction of $d_\beta(1)$ implies that

$$1 = \sum_{i=1}^{\infty} \frac{t_i}{\beta^i}\,. \tag{5.29}$$

If moreover $d_\beta(1)$ is eventually periodic, β is a root of a monic polynomial with integer coefficients. Such β is an algebraic integer.

Our aim is to describe which parameters have to be chosen in order that the sets $\Sigma_{\varepsilon,\eta}(\Omega)$ and $\mathbb{Z}_\beta$ coincide on the positive half-axis[2], i.e. when

$$\Sigma_{\varepsilon,\eta}(\Omega) \ \cap \ \mathbb{R}_0^+ \quad = \quad \mathbb{Z}_\beta \ \cap \ \mathbb{R}_0^+ \,. \tag{5.30}$$

In [19] an example of such a relation is given together with the parameters $\varepsilon, \eta, \Omega$. In particular, the authors study the case of quadratic Pisot units. All quadratic Pisot units can be expressed as the positive roots of a quadratic equation

$$\beta^2 = m\beta + 1\,, \quad \text{for } m \geq 1 \qquad \text{or} \qquad \beta^2 = m\beta - 1\,, \quad \text{for } m \geq 3\,.$$

It is shown that

$$\Sigma_{\beta',\beta}\big[-1, -\frac{1}{\beta'}\big) \ \cap \ \mathbb{R}_0^+ \quad = \quad \mathbb{Z}_\beta \cap \mathbb{R}_0^+ \quad \text{for} \ \ \beta^2 = m\beta + 1, \ m \geq 1, \tag{5.31}$$

$$\Sigma_{\beta',\beta}\big[\,0, \frac{1}{\beta'}\big) \ \cap \ \mathbb{R}_0^+ \quad = \quad \mathbb{Z}_\beta \cap \mathbb{R}_0^+ \quad \text{for} \ \ \beta^2 = m\beta - 1, \ m \geq 3. \tag{5.32}$$

The above equalities imply that the C&P sequences with given windows have two distances only between adjacent points. Thus from Theorem 2.5 they are geometrically similar to C&P sequences with unit acceptance interval, and therefore the infinite binary words corresponding to β-integers are sturmian words.

In the following proposition we prove that a quadratic Pisot unit β is the only example of a basis for which positive β-integers coincide with the restriction of a cut-and-project set to its positive part.

Proposition 5.2 *Positive part of the set $\mathbb{Z}_\beta$ coincides with the positive part of a cut-and-project set $\Sigma_{\varepsilon,\eta}(\Omega)$ if and only if β is a quadratic Pisot unit.*

ProofOne implication is obvious from (5.31) and (5.32). Let us prove $\Rightarrow$. Since $\mathbb{Z}_\beta$ is a self-similar set, we impose the requirement of self-similarity also on the C&P sets, which implies that ε, η are mutually conjugated quadratic numbers, i.e. $\varepsilon = \eta'$ and $0 \in \overline{\Omega}$. According to Theorem 4.1 the self-similarity factor β is a quadratic Pisot number. Thus it remains to show that β is a unit.

All quadratic Pisot numbers can be expressed as the positive roots of a quadratic equation

$$\beta^2 = m\beta + n, \ \ \text{for } 1 \leq n \leq m, \qquad \text{or} \qquad \beta^2 = m\beta - n, \ \ \text{for } 1 \leq n \leq m-2.$$

Our considerations can thus be divided into two cases.

[2]Positive half-axis $[0, +\infty)$ is denoted by $\mathbb{R}_0^+$.

1. Let $\beta^2 = m\beta + n$, $m \geq n \geq 1$. Then the conjugated root to β is the number $\beta' \in (-1,0)$. The Rényi development of 1 has the form

$$d_\beta(1) = mn$$

and the distances between β-integers are

$$\sum_{i=1}^{\infty} \frac{t_i}{\beta^i} = 1 \qquad \text{and} \qquad \sum_{i=1}^{\infty} \frac{t_{i+1}}{\beta^i} = \beta - t_1 = \beta - m = \frac{n}{\beta}\,.$$

In this case a series $\sum_{i=0}^{k} x_i\beta^i$ with non-negative integer coefficients x_i is a β-expansion if $x_i x_{i-1}$ is strictly lexicographically smaller than mn, i.e. $x_i x_{i-1} \prec mn$ for all $1 \leq i \leq k$, which means that $x_i \in \{0,1,\ldots,m\}$ and every digit $x_i = m$ in the string $x_k x_{k-1}\cdots x_1 x_0$ is followed by a digit $x_{i-1} \leq n-1$.

Since in a self-similar C&P set the star map and the Galois automorphism coincide, we can find a candidate for the acceptance interval Ω in order that (5.30) be satisfied. In the following estimations we use $\beta' \in (-1,0)$. For $x = \sum_{i=0}^{k} x_i\beta^i \in \mathbb{Z}_\beta$ we have

$$x' = \sum_{i=0}^{k} x_i {\beta'}^i < m + m{\beta'}^2 + m{\beta'}^4 + \cdots = \frac{m}{1-{\beta'}^2}\,.$$

Similarly,

$$x' = \sum_{i=0}^{k} x_i {\beta'}^i > m\beta' + m{\beta'}^3 + m{\beta'}^5 + \cdots = \frac{m\beta'}{1-{\beta'}^2}\,.$$

Clearly, the only candidate for Ω is the interval $\left[\frac{m\beta'}{1-\beta'^2}, \frac{m}{1-\beta'^2}\right)$. It is obvious that for such a window one inclusion of (5.30) is verified,

$$\Sigma_{\eta',\eta}(\Omega) \,\cap\, \mathbb{R}_0^+ \quad \supseteq \quad \mathbb{Z}_\beta \,\cap\, \mathbb{R}_0^+\,.$$

Since $\mathbb{Z}_\beta$ has only two possible distances between neighbouring elements, namely 1 and $\frac{n}{\beta}$, the equality in the above inclusion is reached according to 8. of Remark 2.2 if $|\Omega| = \Delta_1' - \Delta_2'$, i.e. if

$$\frac{m}{1-{\beta'}^2} - \frac{m\beta'}{1-{\beta'}^2} = \frac{m}{1+\beta'} = 1 - \frac{n}{\beta}\,.$$

Using the quadratic equation ${\beta'}^2 = m\beta' + n$ we obtain the condition $(1-n)\beta' = 0$ which implies $n = 1$. Thus β is a unit.

2. Let us study the case $\beta^2 = m\beta - n$, $m-2 \geq n \geq 1$. Here the conjugated root β' belongs to the interval $(0,1)$. The Rényi development of 1 has coefficients $t_1 = m-1$ and $t_i = m-n-1$ for $i \geq 2$, i.e.

$$d_\beta(1) = (m-1)(m-n-1)(m-n-1)\cdots = (m-1)(m-n-1)^\omega\,.$$

In this case a β-expansion of a number x has digits in the set $\{0, 1, 2, \ldots, m-1\}$ and forbidden are the strings of digits equal or lexicographically greater than $(m-1)(m-n-1)^s(m-n)$ for arbitrary non-negative integer s. The distances between neighbouring β-integers are

$$\sum_{i=1}^{\infty} \frac{t_i}{\beta^i} = 1 \qquad \text{and} \qquad \sum_{i=1}^{\infty} \frac{t_{i+1}}{\beta^i} = \beta - t_1 = \beta - (m-1) = 1 - \frac{n}{\beta}\,.$$

Again, using the Galois conjugation of $x = \sum_{i=0}^{k} x_i \beta^i \in \mathbb{Z}_\beta$ we find a candidate on the acceptance interval Ω for (5.30),

$$0 \leq x' = \sum_{i=0}^{k} x_i \beta'^i < m - 1 + (m-2)\beta' + (m-2)\beta'^2 + \cdots = 1 + \frac{m-2}{1-\beta'}\,.$$

Let us therefore set $\Omega = \big[0, 1 + \frac{m-2}{1-\beta'}\big)$. In order that a C&P set with such an acceptance window have only two distances between neighbours, we must have

$$|\Omega| = 1 + \frac{m-2}{1-\beta'} = \Delta_1' - \Delta_2' = 1 - 1 + \frac{n}{\beta'} = \frac{n}{\beta'}\,.$$

After manipulations we obtain $(n-1)\beta' = 0$ which implies $n = 1$. This completes the proof. ∎

6 C&P sequences and substitutions

Construction of an arbitrarily long segment of a C&P sequence directly from the definition of $\Sigma_{\varepsilon,\eta}(\Omega)$ is numerically very demanding, since precise computation with irrational numbers requires a special arithmetics dependent on the form in which the irrational numbers ε, η are given. For a class of self-similar C&P sequences the sequence of distances between adjacent points (i.e. the infinite word $u_{\varepsilon,\eta}(\Omega)$) can be generated effectively using substitution rules. For this purpose we introduce the following notions.

The set $\mathcal{A}^*$ of finite words on an alphabet $\mathcal{A}$ equipped with the empty word ϵ and the operation of concatenation is a free monoid. A morphism on the monoid $\mathcal{A}^*$ is a map $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ satisfying $\varphi(wz) = \varphi(w)\varphi(z)$ for any pair of words $w, z \in \mathcal{A}^*$. Clearly, the morphism φ is determined by $\varphi(a)$ for all $a \in \mathcal{A}$. The action of a morphism φ can be easily extended to one-directional infinite words $u = u_0u_1u_2\cdots$ over $\mathcal{A}$ by the prescription

$$\varphi(u) = \varphi(u_0u_1u_2\cdots) = \varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots$$

Let $u = u_0u_1u_2\cdots$ be an infinite word over an alphabet $\mathcal{A}$ and let φ be a morphism on $\mathcal{A}^*$ satisfying $|\varphi(a)| \geq 1$ for all $a \in \mathcal{A}$ and $|\varphi(u_0)| > 1$. We say that the word u is invariant under the substitution φ if u is its fixed point, i.e. $\varphi(u) = u$.

The φ-invariance of u implies that $\varphi(u_0)$ has the form $\varphi(u_0) = u_0u'$ for some non-empty word $u' \in \mathcal{A}^*$ and that $\varphi^n(u) = u$ for every $n \in \mathbb{N}$. The word $\varphi^n(u_0)$ is a prefix of the fixed point u and its length grows to infinity with n, therefore we can formally write $u = \lim_{n\to\infty} \varphi^n(u_0)$. The substitution under which an infinite word u is invariant allows one to generate u starting from the initial letter u_0 repeating the rewriting rules infinitely many times.

As an example of a substitution invariant C&P sequence let us recall the β-integers, as presented in the previous section. Consider first $\mathbb{Z}_\beta$ for $\beta^2 = m\beta + 1$, where the distances are $\Delta_1 = 1$ and $\Delta_2 = \frac{1}{\beta}$. Associating the letter A to the distance 1 and the letter B to the distance $\frac{1}{\beta}$ we create a one-directional infinite word u in the alphabet $\{A, B\}$. It can be easily seen from the properties of β-expansions and from the Parry condition that, if x, y are neighbours in $\mathbb{Z}_\beta$ such that $y - x = 1$, then, between the points βx and βy, there is m times the distance 1 followed by one distance $\frac{1}{\beta}$. Similarly, if x, y are neighbours in $\mathbb{Z}_\beta$ such that $y - x = \frac{1}{\beta}$, then the points βx, βy are also neighbours and have distance 1. The above considerations imply that the infinite word u is not changed, if every letter A is replaced by the finite word A^mB, and every letter B is replaced by A. We say, that the word u corresponding to $\mathbb{Z}_\beta$ is invariant under the substitution φ given by

$$\varphi(A) = A^mB, \qquad \varphi(B) = A.$$

Similarly we can derive that for β-integers where $\beta^2 = m\beta - 1$, the infinite word u corresponding to $\mathbb{Z}_\beta$ is invariant under the substitution

$$\varphi(A) = A^{m-1}B, \qquad \varphi(B) = A^{m-2}B.$$

We have presented the substitutions only for those β-integers that correspond to C&P sequences. However in general, every $\mathbb{Z}_\beta$ which has a finite number of distances between neighbours is invariant under a non-trivial substitution [25].

To every substitution φ on the alphabet $\mathcal{A} = \{a_1, \ldots, a_k\}$ one associates naturally the substitution matrix $M \in M_k(\mathbb{N}_0)$, where

$$M_{ij} = \text{the number of letters } a_j \text{ in the word } \varphi(a_i)\,.$$

If all letters a_i of the alphabet $\mathcal{A}$ have a well defined density ϱ_i in the infinite word u invariant under the substitution φ, then the vector $(\varrho_1, \varrho_2, \ldots, \varrho_k)$ is a left eigenvector of the matrix M. If the matrix M is primitive, i.e. it has a positive power, then the substitution is called *primitive*. A fixed point of a primitive substitution can be represented geometrically as a self-similar sequence in the following way.

According to the Perron-Frobenius theorem, the matrix M has an up to a scalar factor unique positive right eigenvector $(y_1, y_2, \ldots, y_k)^T$ corresponding to the dominant eigenvalue, say λ. To the infinite word $u = u_0u_1u_2\cdots$ we associate the sequence $(z_n)_{n\in\mathbb{N}_0}$ such that

$$z_0 = 0 \qquad \text{and} \qquad z_{n+1} - z_n = y_i \quad \text{if } \ u_n = a_i\,. \tag{6.33}$$

The sequence $(z_n)_{n\in\mathbb{N}_0}$ is self-similar, since we have

$$\lambda\{z_n \mid n \in \mathbb{N}_0\} \ \subset\ \{z_n \mid n \in \mathbb{N}_0\}\,.$$

6.1 Substitution invariance

Infinite words corresponding to C&P sequences are bidirectional. From the property 2 of Remark 2.2 it follows that, without loss of generality, we can consider only those C&P sequences which have $0 \in \Omega$, i.e. $0 \in \Sigma_{\varepsilon,\eta}(\Omega)$. We define a pointed bidirectional infinite word $u_{\varepsilon,\eta}(\Omega) = \cdots u_{-2}u_{-1}|u_0u_1u_2\cdots$ such that $u_0u_1u_2\cdots$ corresponds to the order of distances between adjacent points of $\Sigma_{\varepsilon,\eta}(\Omega)$ on the right of 0, and $\cdots u_{-3}u_{-2}u_{-1}$ corresponds to the order of distances between adjacent points of $\Sigma_{\varepsilon,\eta}(\Omega)$ on the left of 0. This word is ternary or binary.

Let $u = \cdots u_{-2}u_{-1}|u_0u_1u_2\cdots$ be a pointed bidirectional infinite word over an alphabet $\mathcal{A}$. Let φ be a morphism on $\mathcal{A}^*$ such that $|\varphi(u_{-1})| > 1$ and $|\varphi(u_0)| > 1$. We say that the word u is invariant under the substitution φ, if it satisfies

$$u = \cdots u_{-2}u_{-1}|u_0u_1u_2\cdots = \cdots\varphi(u_{-2})\varphi(u_{-1})|\varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots = \varphi(u)\,.$$

In this case we formally write

$$u = \lim_{n\to\infty} \varphi^n(u_{-1}) \mid \varphi^n(u_0)\,.$$

Let us mention what is known about the substitution invariance of infinite words associated to C&P sequences. First consider the binary words. As explained in Example 2.7, all such words coincide with lower and upper mechanical words $\underline{s}_{\alpha,\beta}$, $\overline{s}_{\alpha,\beta}$ for irrational $\alpha \in (0,1)$ and any real $\beta \in [0,1)$. The question about substitution invariance of mechanical words has been solved independently by different authors [64, 15, 9]. In order to state the result we need to introduce the notion of a Sturm number.

Definition 6.1 *A quadratic irrational number $\alpha \in (0,1)$ whose algebraic conjugate α' satisfies $\alpha' \notin (0,1)$ is called a Sturm number.*

Let us mention that originally Sturm numbers were defined by a special form of their continued fraction. The characterization presented here is due to [2]. The necessary and sufficient condition for substitution invariance of mechanical words is given by the following theorem [9].

Theorem 6.2 *The mechanical word $\underline{s}_{\alpha,\beta}$, resp. $\overline{s}_{\alpha,\beta}$, for irrational $\alpha \in (0,1)$ and real $\beta \in [0,1)$ is invariant under a substitution if and only if*

(i) α is a Sturm number,

(ii) $\beta \in \mathbb{Q}(\alpha)$,

(iii) $\alpha' \le \beta' \le 1-\alpha'$ *or* $1-\alpha' \le \beta' \le \alpha'$.

If we represent the substitution invariant lower mechanical word $\underline{s}_{\alpha,\beta}$ geometrically, as described in (6.33), we find that this geometrical representation coincides with the C&P sequence $\Sigma_{\varepsilon,\varepsilon'}(\beta-1,\beta]$, where $\alpha = -\varepsilon$. Similar statement is valid for the substitution invariant upper mechanical word $\overline{s}_{\alpha,\beta}$. This implies that substitution invariance of a binary word u associated to a C&P sequence forces existence of a self-similar C&P sequence $\Sigma_{\varepsilon,\varepsilon'}(\Omega)$ such that $u = u_{\varepsilon,\varepsilon'}(\Omega)$.

Substitution invariance of ternary words corresponding to C&P sequences has not yet been solved completely. The authors however conjecture that, even in this case, substitution invariance forces self-similarity of the corresponding C&P sequence.

6.2 Substitutivity

The original aim for studying substitution properties of C&P sequences was the possibility of symbolic generation of $u_{\varepsilon,\eta}(\Omega)$. For our present purpose, it is enough to consider a property weaker than substitution invariance, namely the substitutivity. We take the formulation of Durand [24].

Definition 6.3 *We say that the infinite word u over an alphabet $\mathcal{A}$ is substitutive if there exist an infinite word v over an alphabet $\mathcal{B}$ and a letter projection $\psi : \mathcal{B} \to \mathcal{A}$ such that v is invariant under a substitution φ on $\mathcal{B}^*$ and*

$$\cdots \psi(v_{-2})\psi(v_{-2})|\psi(v_0)\psi(v_1)\psi(v_2)\cdots = \cdots u_{-2}u_{-1}|u_0u_1u_2\cdots$$

If moreover φ is a primitive substitution, then the infinite word u is said to be primitive substitutive.

If an infinite word u is substitutive, it can be constructed in such a way that generating by substitution the word v and using the projection ψ allows us to obtain u.

Using Theorem 2.5 we can without loss of generality consider the ternary words associated to C&P sequences $\Sigma_{\varepsilon,\eta}[c, c+\ell)$, where

$$\varepsilon \in (-1,0), \quad \eta > 0, \quad c \le 0 < c+\ell \quad \text{and} \quad \max(-\varepsilon, 1+\varepsilon) < \ell < 1\,. \tag{6.34}$$

The description of infinite words associated to C&P sequences which are substitutive can be derived from the paper of Adamczewski [1].

Theorem 6.4 *Let $\varepsilon, \eta, c, \ell$ satisfy (6.34). The infinite word $u_{\varepsilon,\eta}[c, c+\ell)$ is primitive substitutive if and only if ε is a quadratic irrational number and $c, \ell \in \mathbb{Q}(\varepsilon)$.*

From a practical point of view it is important to know the procedure which, given a substitutive word u over a ternary alphabet $\{A, B, C\}$, allows to determine an alphabet $\mathcal{B}$, a substitution invariant word v over $\mathcal{B}$ and a projection $\psi : \mathcal{B} \to \mathcal{A}$ such that $\psi(v) = u$. The bidirectional pointed word $v = \cdots v_{-2}v_{-1}|v_0v_1v_2$ is in fact given by the initial letters $v_{-1}|v_0$ and the substitution φ under which it is invariant, since we have $v = \lim_{n\to\infty} \varphi^n(v_{-1})|\varphi^n(v_0)$.

In the rest of this section we describe the algorithm for solving this problem in case that the parameters satisfy besides the necessary conditions (6.34), an additional condition that $-\varepsilon$ is a Sturm number, i.e. the algebraic conjugate ε' of ε satisfies $\varepsilon' < -1$ or $\varepsilon' > 0$. Using the transformations (2.10) and (2.11) we have

$$\Sigma_{\varepsilon,\eta}(\Omega) = \Sigma_{-1-\varepsilon,-1-\eta}(-\Omega)\,,$$

and thus we can without loss of generality consider only $\varepsilon' > 0$. As for the parameter η, we know that, for fixed ε, c, ℓ satisfying (6.34), the words $u_{\varepsilon,\eta}[c, c+\ell)$ coincide for all choices of $\eta > 0$. In case that $\varepsilon' > 0$, it is suitable to put $\eta = \varepsilon'$. In this case $\Sigma_{\varepsilon,\varepsilon'}[c, c+\ell)$ is a self-similar set. This is a crucial property for proving the correctness of the algorithm presented below.

Remark 6.5 *The sequence $\Sigma_{\varepsilon,\varepsilon'}[c, c+\ell)$ with parameters satisfying* (6.34) *has according to Theorem 2.5 three types of distances between adjacent points, namely $\varepsilon', 1+\varepsilon', 1+2\varepsilon'$. According to Theorem 4.1 it is a self-similar set. The stepping function f on the acceptance interval $\Omega = [c, c+\ell)$ has in this case the form*

$$f(x) = \begin{cases} x+1+\varepsilon & \text{for} \quad x \in [c, c+\ell-1-\varepsilon) & =: \Omega_A\,, \\ x+1+2\varepsilon & \text{for} \quad x \in [c+\ell-1-\varepsilon, c-\varepsilon) & =: \Omega_B\,, \\ x+\varepsilon & \text{for} \quad x \in [c-\varepsilon, c+\ell) & =: \Omega_C\,. \end{cases}$$

This function is a bijection on the acceptance interval Ω, i.e. is invertible.

Algorithm:

Input: quadratic $\varepsilon \in (-1, 0)$ with $\varepsilon' > 0$, $c, \ell \in \mathbb{Q}(\varepsilon)$, such that $c \le 0 < c+\ell$, $\max(-\varepsilon, 1+\varepsilon) < \ell \le 1$.

Output: alphabet $\mathcal{B}$, letters $v_{-1}, v_0 \in \mathcal{B}$, morphism φ on $\mathcal{B}^*$, projection $\psi : \mathcal{B} \to \mathcal{A}$.

Step 1 Find a quadratic unit $\gamma \in (0, 1)$ such that $\gamma\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon]$ and its conjugate $\gamma' > 1$. It results in solving a Diophantine equation (more precisely the so-called Pell equation) which has always a solution.

Step 2 For $x \in \Omega$ we define

$$g_\gamma(x) = \frac{1}{\gamma} f^{-\mathrm{ind}(x)}(x), \qquad \text{where} \quad \mathrm{ind}(x) = \min\{i \in \mathbb{N}_0 \mid f^{-i}(x) \in \gamma\Omega\}. \tag{6.35}$$

Find the minimal set $S \subset \Omega$ such that

$$\{c, c+\ell-1-\varepsilon, c-\varepsilon\} \subseteq S \qquad \text{and} \qquad g_\gamma(S) \subseteq S\,.$$

Such a set is finite, let us denote its elements by $S = \{c_0, c_1, \dots, c_k\}$, where $c = c_0 < c_1 < \dots < c_k$, and denote $c_{k+1} := c + \ell$. Note that the elements of the set S divide the acceptance window into small subintervals

$$\Omega = \bigcup_{i=0}^{k} [c_i, c_{i+1})\,.$$

Step 3 Define the alphabet $\mathcal{B} := \{0, 1, \dots, k\}$ and to every letter $i \in \mathcal{B}$ associate the number

$$j_i = \min\{j \in \mathbb{N} \mid f^j(\gamma c_i) \in \gamma\Omega\}$$

and the word $\varphi(i) = w_0^{(i)} w_1^{(i)} \cdots w_{j_i-1}^{(i)}$ by the prescription

$$w_j^{(i)} := m \in \mathcal{B} \qquad \text{if} \quad f^j(\gamma c_i) \in [c_m, c_{m+1})\,.$$

Step 4 Define the initial letters of the infinite word $v = \cdots v_{-2}v_{-1}|v_0v_1v_2\cdots$ over the alphabet $\mathcal{B}$ as

$$\begin{array}{lcllcl} v_0 & = & m \in \mathcal{B} & \text{if} \quad 0 & \in & [c_m, c_{m+1})\,, \\ v_{-1} & = & m \in \mathcal{B} & \text{if} \quad f^{-1}(0) & \in & [c_m, c_{m+1})\,. \end{array}$$

Step 5 Define the projection $\psi : \mathcal{B} \to \mathcal{A} = \{A, B, C\}$ by

$$\psi(i) = \begin{cases} A \text{ if } c_i \in \Omega_A\,, \\ B \text{ if } c_i \in \Omega_B\,, \\ C \text{ if } c_i \in \Omega_C\,. \end{cases}$$

Theorem 6.6 *Let parameters ε, η, c, ℓ satisfy* (6.34). *Let moreover ε be a quadratic irrational number, such that $\varepsilon' > 0$, and let $c, \ell \in \mathbb{Q}(\varepsilon)$. Then the alphabet $\mathcal{B}$, letters $v_{-1}, v_0 \in \mathcal{B}$, morphism φ on $\mathcal{B}^*$, and projection $\psi : \mathcal{B} \to \mathcal{A}$, defined in the above algorithm, satisfy*

$$u_{\varepsilon,\eta}[c, c+\ell) = \psi(v)\,, \qquad \textit{where} \quad v = \lim_{n\to\infty} \varphi^n(v_{-1})|\varphi^n(v_0)\,.$$

The proof of the theorem follows the same ideas as in [46], where the correctness of the algorithm for $\varepsilon = -\frac{1}{\tau}$ is shown. Note that the crucial point in the algorithm is to ensure that the set S of Step 2 is finite.

Remark 6.7

1. Given the infinite word $u_{\varepsilon,\eta}(\Omega)$, the substitution φ is not given uniquely. Indeed, the ambiguity is found in the choice of the unit γ in Step 1 of the algorithm. Note that if γ has required properties, then so does any power γ^j, $j \in \mathbb{N}$.

2. The cardinality of the alphabet $\mathcal{B}$ is given by the cardinality of the set S, which depends on the choice of γ. Taking a power of γ as the unit factor may reduce the number of letters in the alphabet.

3. In case that the word $u_{\varepsilon,\eta}(\Omega)$ is not only substitutive but is also a fixed point of a substitution, then suitable choice of γ (sufficiently high power of minimal factor satisfying Step 1) in the algorithm yields the substitution under which $u_{\varepsilon,\eta}(\Omega)$ is invariant.

Let us illustrate the algorithm for finding the substitution on an example.

Example 6.8 *Consider the C&P sequence $\Sigma_{\varepsilon,\eta}[c, c+\ell)$ with parameters*

$$\varepsilon = -\frac{1}{\sqrt{2}}, \qquad \eta = \varepsilon' = \frac{1}{\sqrt{2}}, \qquad c = 0, \qquad \ell = -2 + 2\sqrt{2} = -2 - 4\varepsilon .$$

Such parameters clearly satisfy the assumptions of the algorithm. The distances between adjacent points of $\Sigma_{\varepsilon,\eta}[c, c+\ell)$ are $\Delta_1 = 1 + \eta = 1 + \frac{1}{\sqrt{2}}$, $\Delta_2 = \eta = \frac{1}{\sqrt{2}}$, and $\Delta_1 + \Delta_2 = 1 + 2\eta = 1 + \sqrt{2}$, and therefore the explicit expression of the stepping function in our case is

$$f(x) = \begin{cases} x + 1 + \varepsilon & \text{for} \quad x \in [0, -3 - 5\varepsilon) & =: \Omega_A , \\ x + 1 + 2\varepsilon & \text{for} \quad x \in [-3 - 5\varepsilon, -\varepsilon) & =: \Omega_B , \\ x + \varepsilon & \text{for} \quad x \in [-\varepsilon, -2 - 4\varepsilon) & =: \Omega_C . \end{cases}$$

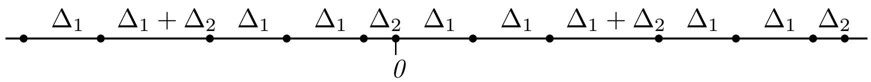
From that we derive the formula for the inverse function

$$f^{-1}(x) = \begin{cases} x - \varepsilon & \text{for} \quad x \in [0, -2 - 3\varepsilon) , \\ x - 1 - 2\varepsilon & \text{for} \quad x \in [-2 - 3\varepsilon, 1 + \varepsilon) , \\ x - 1 - \varepsilon & \text{for} \quad x \in [-1 - \varepsilon, -2 - 4\varepsilon) . \end{cases}$$

We know that $0 \in \Sigma_{\varepsilon,\eta}[c, c+\ell)$. We can generate other elements of the C&P sequences on the right from 0 using the stepping function f. The elements on the left are generated using f^{-1}. We have

$$\begin{aligned} \Sigma_{\varepsilon,\eta}[c, c+\ell) = \{\dots, &-5 - 8\eta, -4 - 6\eta, -3 - 5\eta, -2 - 3\eta, -1 - 2\eta, -\eta, \\ &0,\ 1+\eta,\ 2+2\eta,\ 3+4\eta,\ 4+5\eta,\ 5+6\eta,\ 5+7\eta, \dots\} , \end{aligned}$$

and graphically,



The corresponding bidirectional infinite word $u_{\varepsilon,\eta}[c, c+\ell)$ *is obtained by replacing* Δ_1 *with the letter* A, Δ_2 *with the letter* C *and* $\Delta_1 + \Delta_2$ *with the letter* B, *i.e.*

$$u_{\varepsilon,\eta}[c, c+\ell) = \cdots BABAAC|AABAAC\cdots$$

Let us now use the algorithm described above to derive the substitution generating the word $u_{\varepsilon,\eta}[c, c+\ell)$. *We proceed according to the steps of the algorithm.*

<u>*Step 1*</u> *Put* $\gamma = 3 + 4\varepsilon = 3 - 2\sqrt{2}$. *It is obvious that* γ *is a unit in* $\mathbb{Z}[\varepsilon] \cap (0,1)$ *and its algebraic conjugate* γ' *satisfies* $\gamma' = \frac{1}{\gamma} = 3 - 4\varepsilon = 3 + 2\sqrt{2} > 1$. *We have yet to verify that* $\gamma\mathbb{Z}[\varepsilon] = \mathbb{Z}[\varepsilon]$. *For that it suffices to show that* $\gamma\varepsilon \in \mathbb{Z}[\varepsilon]$ *and* $\gamma^{-1}\varepsilon \in \mathbb{Z}[\varepsilon]$. *We have*

$$\begin{array}{rcrcrcl} \gamma\varepsilon & = & \varepsilon(3+4\varepsilon) & = & 2+3\varepsilon & \in & \mathbb{Z}[\varepsilon]\,, \\ \gamma^{-1}\varepsilon & = & \varepsilon(3-4\varepsilon) & = & -2+3\varepsilon & \in & \mathbb{Z}[\varepsilon]\,, \end{array}$$

where we have used that $\varepsilon^2 = \frac{1}{2}$.

<u>*Step 2*</u> *We need to find the minimal set* $S \subset \Omega$ *closed under the action of* g_γ, *containing the points*

$$c = 0\,, \qquad c + \ell - 1 - \varepsilon = -3 - 5\varepsilon\,, \qquad c - \varepsilon = -\varepsilon\,.$$

Thus we search for values of all iterations $g_\gamma^j(x)$, $j \in \mathbb{N}$, *of the above points. Important for the definition* (6.35) *of the function* g_γ *is the index of a point* x, *i.e. the first exponent* $i \in \mathbb{N}_0$ *such that* $f^{-i}(x)$ *belongs to the interval*

$$\gamma\Omega = (3+4\varepsilon)\big[0, -2-4\varepsilon\big) = \big[0, -14-20\varepsilon\big)\,.$$

Let us find the image under g_γ *of the point* $x = 0$. *Clearly,* $0 \in \gamma\Omega$, *thus* $\mathrm{ind}(0) = 0$. *Therefore* $g_\gamma(0) = \frac{1}{\gamma} f^0(0) = 0$. $g_\gamma^j(0) = 0$ *for all* $j \in \mathbb{N}$.

Let us find the image under g_γ *of the point* $x = -\varepsilon$. *We have*

$$f^{-2}(-\varepsilon) = f^{-1}(-1-2\varepsilon) = -2-3\varepsilon \in \gamma\Omega\,,$$

which implies

$$\mathrm{ind}(-\varepsilon) = 2 \quad \textit{and} \quad g_\gamma(-\varepsilon) = \frac{1}{\gamma} f^{-2}(-\varepsilon) = (3-4\varepsilon)(-2-3\varepsilon) = -\varepsilon\,.$$

Thus $g_\gamma^j(-\varepsilon) = -\varepsilon$ *for all* $j \in \mathbb{N}$.

Let us now find the image under g_γ of the point $x = -3 - 5\varepsilon$. We have

$$\begin{aligned} f^{-4}(-3-5\varepsilon) = f^{-3}(-4-6\varepsilon) = f^{-2}(-5-8\varepsilon) = \\ = f^{-1}(-6-9\varepsilon) = -7-10\varepsilon \in \gamma\Omega\,, \end{aligned}$$

which implies

$$\begin{aligned} \operatorname{ind}(-3-5\varepsilon) = 4 \quad \textit{and} \quad g_\gamma(-\varepsilon) &= \frac{1}{\gamma} f^{-4}(-3-5\varepsilon) = \\ &= (3-4\varepsilon)(-7-10\varepsilon) = -1-2\varepsilon\,. \end{aligned}$$

Therefore the set S must contain the point $-1-2\varepsilon$. In order to find further iterations of g_γ on the point $-3-5\varepsilon$, we determine $g_\gamma(-1-2\varepsilon)$. We have seen that $f^{-1}(-1-2\varepsilon) = -2-3\varepsilon \in \gamma\Omega$. Thus

$$\operatorname{ind}(-1-2\varepsilon) = 1 \quad \textit{and} \quad g_\gamma(-1-2\varepsilon) = \frac{1}{\gamma} f^{-1}(-1-2\varepsilon) = -\varepsilon\,.$$

Altogether, we obtain

$$\begin{array}{rcll} g_\gamma^j(0) & = & 0, & \textit{for all } \ j \in \mathbb{N}\,, \\ g_\gamma^j(-\varepsilon), & = & -\varepsilon, & \textit{for all } \ j \in \mathbb{N}\,, \\ g_\gamma(-3-5\varepsilon) & = & -1-2\varepsilon, & \\ g_\gamma^j(-3-5\varepsilon) & = & -\varepsilon, & \textit{for all } \ j \in \mathbb{N}\,,\ j \geq 2\,. \end{array}$$

We can conclude that S contains four elements,

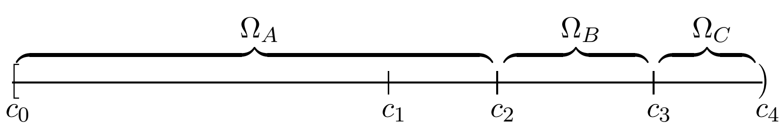
$$c_0 = c = 0\,, \quad c_1 = -1-2\varepsilon\,, \quad c_2 = -3-5\varepsilon\,, \quad c_3 = -\varepsilon\,,$$

and we put $c_4 = -2-4\varepsilon$.

Note that the elements of the set S divide the acceptance window into small subintervals

$$\begin{aligned} \Omega = \bigcup_{i=0}^{3} [c_i, c_{i+1}) &= \\ = [0, -1-2\varepsilon) \cup [-1-2\varepsilon, -3-5\varepsilon) \cup [-3-5\varepsilon, -\varepsilon) \cup [-\varepsilon, -2-4\varepsilon), \end{aligned}$$

as it is illustrated on the following figure.



Step 3 *Since S has four elements, we have the alphabet $\mathcal{B} := \{0,1,2,3\}$ on four letters. In order to define the substitution we compute the iterations $f^j(\gamma c_i)$ and we stop when $f^j(\gamma c_i) \in \gamma\Omega$. First take $i = 0$. We have $\gamma c_0 = 0$ and the iterations*

$$\begin{array}{rclcl} f^0(0) & = & 0 & \in & [c_0, c_1), \\ f^1(0) & = & 1+\varepsilon & \in & [c_0, c_1), \\ f^2(0) & = & 2+2\varepsilon & \in & [c_2, c_3), \\ f^3(0) & = & 3+4\varepsilon & \in & [c_0, c_1), \\ f^4(0) & = & 4+5\varepsilon & \in & [c_1, c_2), \\ f^5(0) & = & 5+6\varepsilon & \in & [c_3, c_4), \\ f^6(0) & = & 5+7\varepsilon & \in & \gamma\Omega, \end{array} \qquad \textit{thus} \quad \begin{array}{l} j_0 = 6 \quad \textit{and} \\ \varphi(0) = 002013. \end{array}$$

Note that the word $\varphi(0) = 002013$ is formed by the indices m of the left-end-points of the intervals $[c_m, c_{m+1})$, read in the column. Similarly for $i = 1$, we have $\gamma c_1 = \gamma(-1-2\varepsilon) = -7-10\varepsilon$. Thus

$$\begin{array}{rclcl} f^0(-7-10\varepsilon) & = & -7-10\varepsilon & \in & [c_0, c_1), \\ f^1(-7-10\varepsilon) & = & -6-9\varepsilon & \in & [c_0, c_1), \\ f^2(-7-10\varepsilon) & = & -5-8\varepsilon & \in & [c_2, c_3), \\ f^3(-7-10\varepsilon) & = & -4-6\varepsilon & \in & [c_0, c_1), \\ f^4(-7-10\varepsilon) & = & -3-5\varepsilon & \in & [c_2, c_3), \\ f^5(-7-10\varepsilon) & = & -2-3\varepsilon & \in & \gamma\Omega, \end{array} \qquad \textit{hence} \quad \begin{array}{l} j_1 = 5 \quad \textit{and} \\ \varphi(1) = 00202. \end{array}$$

For $i = 2$, we have $\gamma c_2 = \gamma(-3-5\varepsilon) = -19-27\varepsilon$. Therefore

$$\begin{array}{rclcl} f^0(-19-27\varepsilon) & = & -19-27\varepsilon & \in & [c_0, c_1), \\ f^1(-19-27\varepsilon) & = & -18-26\varepsilon & \in & [c_0, c_1), \\ f^2(-19-27\varepsilon) & = & -17-25\varepsilon & \in & [c_2, c_3), \\ f^3(-19-27\varepsilon) & = & -16-23\varepsilon & \in & [c_0, c_1), \\ f^4(-19-27\varepsilon) & = & -15-22\varepsilon & \in & [c_2, c_3), \\ f^5(-19-27\varepsilon) & = & -14-20\varepsilon & \in & [c_0, c_1), \\ f^6(-19-27\varepsilon) & = & -13-19\varepsilon & \in & [c_1, c_2), \\ f^7(-19-27\varepsilon) & = & -12-18\varepsilon & \in & [c_3, c_4), \\ f^8(-19-27\varepsilon) & = & -12-17\varepsilon & \in & \gamma\Omega, \end{array} \qquad \begin{array}{l} \textit{hence } j_2 = 8 \quad \textit{and} \\ \varphi(2) = 00202013. \end{array}$$

Last, for $i = 3$, we have $\gamma c_3 = \gamma(-\varepsilon) = -2-3\varepsilon$. Thus

$$\begin{array}{rclcl} f^0(-2-3\varepsilon) & = & -2-3\varepsilon & \in & [c_0, c_1), \\ f^1(-2-3\varepsilon) & = & -1-2\varepsilon & \in & [c_1, c_2), \\ f^2(-2-3\varepsilon) & = & -\varepsilon & \in & [c_3, c_4), \\ f^3(-2-3\varepsilon) & = & 0 & \in & \gamma\Omega, \end{array} \qquad \textit{hence} \quad \begin{array}{l} j_3 = 3 \quad \textit{and} \\ \varphi(3) = 013. \end{array}$$

Altogether, we have the substitution

$$\begin{aligned}
\varphi(0) &= 002013\,,\\
\varphi(1) &= 00202\,,\\
\varphi(2) &= 00202013\,,\\
\varphi(3) &= 013\,.
\end{aligned}$$

<u>*Step 4*</u> *Since* $0 \in [c_0, c_1)$ *and* $f^{-1}(0) = -\varepsilon \in [c_3, c_4)$, *we put as the initial letters* $v_0 = 0$, $v_{-1} = 3$. *Note that 0 is a prefix of* $\varphi(0)$ *and 3 is a suffix of* $\varphi(3)$, *thus the word* $v = \lim_{n\to\infty} \varphi^n(v_{-1})|\varphi^n(v_0) = \lim_{n\to\infty} \varphi^n(3)|\varphi^n(0)$ *is well defined.*

<u>*Step 5*</u> *Since* $c_0, c_1 \subset \Omega_A$, $c_2 \subset \Omega_B$, $c_3 \subset \Omega_C$, *we have the projection* $\psi : \{0,1,2,3\} \to \mathcal{A} = \{A,B,C\}$ *by*

$$\psi(0) = \psi(1) = A\,, \qquad \psi(2) = B\,, \qquad \psi(3) = C\,.$$

Let us write the subsequent iterations of the substitution φ *on the pair of initial letters* $3|0$, *i.e.* $\varphi^n(3)|\varphi^n(0)$. *We have for* $n = 0,1,2$,

$$\begin{array}{r|l}
3 & 0\\
013 & 002013\\
00201300202013 & 00201300201300202013002013002013\\
\vdots &
\end{array}$$

Since each row is a factor of the next one, in the limit we obtain the infinite word

$$v = \cdots 00201300202013|00201300201300202013002013002013\cdots\,.$$

Now we apply the letter projection ψ, *which collapses the letters* 0 *and* 1. *We have*

$$\begin{array}{rl}
v = & \cdots 1\,3\,0\,0\,2\,0\,2\,0\,1\,3|0\,0\,2\,0\,1\,3\,0\,0\,2\,0\,1\,3\,0\,0\,2\,0\,2\,0\,1\,3\,0\,0\cdots\\
 & \;\;\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\;\downarrow\\
u_{\varepsilon,\eta}[c, c+\ell) = & \cdots ACAABABAAC|AABAACAABAACAABABAACAA\cdots
\end{array}$$

Let us study the second iteration of the above substitution φ. *We obtain*

$$\begin{aligned}
\varphi^2(0) &= \varphi(002013) =\\
&= 00201300201300202013002013002013\,,\\
\varphi^2(1) &= \varphi(00202) =\\
&= 00201300201300202013002013002013\,,\\
\varphi^2(2) &= \varphi(00202013) =\\
&= 00201300201300202013002013002020130020130020202013\,,\\
\varphi^2(3) &= \varphi(013) =\\
&= 00201300202013\,.
\end{aligned}$$

Note that $\varphi^2(0) = \varphi^2(1)$. Therefore we can consider the letters 0 and 1 as identic. It enables us to define a new substitution $\tilde{\varphi} : \{A, B, C\}^* \to \{A, B, C\}^*$ by

$$\begin{aligned}
\tilde{\varphi}(A) &= AABAACAABAACAABABAACAABAACAABABAAC\,,\\
\tilde{\varphi}(B) &= AABAACAABAACAABABAACAABAACAABABAAC\circ\\
&\quad \circ AABAACAABABAAC\,,\\
\tilde{\varphi}(C) &= AABAACAABABAAC\,,
\end{aligned}$$

under which the word $u_{\varepsilon,\eta}[c, c+\ell)$ is invariant, we namely have

$$u_{\varepsilon,\eta}[c, c+\ell) = \lim_{n\to\infty} \tilde{\varphi}^n(C)|\tilde{\varphi}^n(A)\,.$$

Note that the symbol $\circ$ in the formula for the substitution stands for concatenation.

7 Conclusions

In this paper, we have attempted to give a unifying view of the one-dimensional cut-and-project point sets obtained from the square lattice in the plane. A part of the work is a review of former results which had to be recalled for the sake of clarity. Let us now indicate some possible continuations or applications of our results.

First, it would be interesting to extend the work [4] to the case of splines of larger regularity for other cut-and-project sets, with quadratic self-similarity, or without self-similarity at all, especially having in view the relation between scaling equations and substitution properties of the considered discretizations of $\mathbb{R}$.

Mathematical diffraction of such aperiodic sets obtained by cut and projection should be also envisaged in a systematic way, in relation with the existence of those multiresolution analysis and related wavelets. Concerning diffraction, one can find in the literature on quasicrystals many works devoted to this important subject, in which substitutional properties, or self-similarity, or cut and projection from higher-dimensional lattices, play a central role in the elaboration of rigorous results (see for instance [17, 32, 42] and [37] for a recent review on these questions). The analysis of diffraction spectra by using adapted wavelets, i.e. wavelets "living" on the diffracting aperiodic structure, is a project which remains to be really developed.

Another nice application of the results of this paper can be envisaged in the construction of a new type of pseudo-random number generators. First step in this direction has been made in [31], where the authors use sturmian sequences to combine classical periodic pseudo-random sequences to produce an aperiodic pseudo-random sequence. These aperiodic pseudo-random number generators (APRNG) have been tested using the DIEHARD test suite and using the Maurer test and it turned out that statistical properties of these APRNG's are significantly better than of the original periodic sequences. Moreover, the authors prove that the APRNG passes the spectral test. It would be very interesting to pursue the study of the APRNG's extending the definition to generic (i.e. ternary) cut-and-project sequences.

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