Updating our beliefs about inconsistency: The Monty-Hall case

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1   Introduction

Since the sixties, various researchers have come to the conclusion that participants in

psychological studies revise their probability judgments differently from the way

experimenters do using Bayes' rule. People seem unable to revise their degrees of belief in a

Bayesian manner (for a review see Gilovich *et al*. 2002). They are thus thought to be

inconsistent. However this conclusion appears debatable when one sheds light on the notion

of Bayesian consistency. In such light, the methodology underlying the above-mentioned

studies does not seem concerned by individual consistency. It rather amounts to an analysis of

accuracy. The experimenter plays the role of an expert who forms "correct predictions"

(Baratgin & Politzer 2006). The Monty-Hall Puzzle constitutes a much-studied paradigm

emblematic of this literature. The standard version originates from a popular television game

show "*Let's Make A Deal*" programmed in 1963 in North America. It involves only three

numbered doors and equal prior probabilities (from now on we will refer to it as standard

MHP)[1]. This paper will focus on a generalized probabilistic version of the Monty-Hall Puzzle

(that we will note MHP_{nn-2}). It corresponds to a four stages story that reads as follows[2]:

Stage (i)   A TV host shows you n numbered doors No.1, ...No.n, one hiding a car and the

other n-1 hiding goats. Let $_{car}No.i$ be the event "the car is behind door No.i". Let

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[1] It was phrased as a problem of choice where the host after picking door No.3 to reveal a goat, asks participant: "should you switch your original choice No.1 to No.2?". Despite a choice is required in this version, participants must first of all make a probabilistic judgment. An older version called the Three Prisoners Problem is cast in a probabilistic format (Gardner 1959). This paper focuses on the mere probabilistic judgment disregarding the further question whether to switch. We will indifferently make references to the Three Prisoners Problem or to the Monty Hall Puzzle under the same notation. The question of choosing whether to switch door can be treated independently from the probability judgment question. Chun (2003) for example analyses standard MHP as a decision problem. Conversely numerous experimental studies focus on the sole question of choice whether to switch door (see for example Gilovich *et al*. 1995; Granberg & Brown 1995; Friedman 1998).

[2] This version is an adaptation of standard MHP (see for example Gillman 1992).

$$P(_{car}No.i) = \frac{a_i}{A}, i = 1, ..., i = n, \text{ with } A = \sum_{s=1}^{s=n} a_s$$ be the corresponding prior probabilities given at this stage. The version with equal prior probabilities will be noted standard MHP_{nn-2}.

Stage (ii)   You get to pick a door, winning whatever is behind it. You choose door No.1, say. What is the probability that the car is behind door No.1?

Stage (iii)   The host, who knows where the car is, tells you "I will show you n-2 doors (out of the n-1 other doors) that hide goats".

Stage (iv)   Then the host opens doors No.3, ... to No.n to reveal goats (a message that we note '$_{goat}$No.3...n' with single quotation marks), and asks you what is now the probability that the car is behind door No.1?

Solving MHP_{nn-2} requires comparing the probabilities that *the car is respectively behind doors No.1 or No.2* given that, the host *(*who knows where the car is*) has chosen to open doors No.3... to No.n* (out of the n-1 other possibilities) revealing goats behind.

In the various experiments on the Monty Hall Puzzle, participants' modal probabilistic response is $\frac{a_1}{a_1 + a_2}$ (noted from now on MR). It departs from experimenters' "Bayesian solution" called "EBS", which general form in MHP_{nn-2} establishes to $\frac{a_1}{a_1 + (n-1)a_2}$ (see for a review of studies on standard versions (with equal prior probabilities) Krauss & Wang (2003); and for experiments that propose a version with unequal prior probabilities see: Ichikawa (1989); Ichikawa & Takeichi (1990); Johnson-Laird, Legrenzi, Girotto & Sonino-Legrenzi (1999); Granberg (1996); Yamagishi (2003)).

Most psychologists, economists and lawyers consider these studies as mounting evidence of the inconsistency of human beings (Piattelli-Palmerini 1994; Friedman 1998; Caplan 2000; Risinger & Loop 2002; Kluger & Wyat 2004). The purpose of this paper is to

analyze the rationale for such a conclusion. The aim here is not to conduct a thorough review of the literature testing the experimental robustness of MR but rather to discuss the underlying idea of participants' inconsistency. In order to validate this often drawn conclusion, experimenters should make sure that participants share their representation of MHP_{nn-2}. Alternatively, the response usually made by participants may be seen as consistent with respect to a different representation of the puzzle[3]. In this view, *specific normative[4] solutions can be determined for each particular representation (experimenters' and participants' ones).*

This paper is organized in three sections. First, experimenters' representation of MHP_{nn-2} is presented. "EBS" is the correct solution in a traditional situation of probability revision where the message '$_{goat}$No.3...n' *focuses* attention on a given subset of the original set of hypotheses. Second we explain that MR corresponds to a correct solution in an *updating* interpretation of the situation of revision where the message modifies the structure of the original problem. We show by paying attention to MHP_{nn-2}'s statement that pragmatics reasons can explain the biased interpretation of the revision message. In a third part, we show that the different explanations proposed in the literature to descriptively account for participants' responses can be derived from an updating framework.

2   Experimenters' representation of MHP_{nn-2}

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[3] The hypothesis that participants' response could be consistent with a representation of the experimental paradigm different from what experimenters expect is rarely envisaged in the literature on probabilistic revision. However a significant time ago various authors in the field of formal logic retained this idea to study the possible interpretations of the conditional. Indeed when participants interpret a conditional as a biconditional (*If p then q* leads one to infer that *If not-p then not-q*), they correctly proceed to an Affirmation of the Consequent although this affirmation would be considered as an error by experimenters (Geis & Zwicky 1971; Ducrot 1972; Fillenbaum 1976).

[4] The term "normative" is taken here in the sense of a theoretical model considered as a referential norm for "rational" judgment. This is the case, for example, of the Subjective Expected Utility Theory in the framework of decision theory, or of the Bayesian model in the context of probabilistic judgment. The question asked within this methodology is whether man is "consistent" in his judgments and decisions with respect to the norm.

2.1 Basics postulates

Any empirical study is based on experimenters' belief that participants accept with certainty the data provided in the statement of the problem. In MHP_{nn-2}, participants are supposed to distinguish the four stages (i), (ii), (iii) and (iii) and to accept the explicit and implicit rules related to these stages. The explicit rules are[5]:

Rule 1: The host has a distribution of preference ($P(_{car}No.i) = \dfrac{a_i}{A}$, with $A = \sum_{s=1}^{s=n} a_s$) when he initially places the car behind one door.

Rule 2: The host cannot open the door chosen by participants (and conventionally chosen as No.1).

Rule 3: The host never opens a door (among the n-2 doors he or she will open) that hides the car. This rule implies:

    Rule 3.1: If the car is behind door No.i (with $i = 2$, ..., $i = n$), the host is bound to show goats behind the n-2 doors No.2, ...,No.i-1, No.i+1, ..., No.n.

    Rule 3.2: if the car is behind door No.1, the host is bound to show goats behind n-2 doors chosen among the n-1 doors No.2..., No.n. In accordance with the experimental literature, we consider the additional following rule:

        Rule 3.2.1: In the specific case where the car is behind door No.1, to show goats, the host has no preference among the n-1 doors No.2, ..., No.n to choose n-2 doors.

Hence experimenters in MHP_{nn-2} form the following first postulate.

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[5] These rules are immediate translations of the rules applied to MHP and standard MHP_{nn-2} by all experimenters. In Appendix 1 we discuss the difficulty of transposing the rule 3.2 in a general MHP_{nk} (untreated by psychologists) where k doors with goats behind are opened ($k \leq$ n-2).

Postulate 1. Participants consider the four stages and accept the implicit and explicit rules 1, 2, 3, 3.1 and 3.2 implied by MHP_{nn-2}'s statement.

The second set of experimenters' beliefs, largely accepted in the great majority of experimental paradigms on probabilities revision (see Baratgin & Politzer 2006), seems a natural consequence of Postulate 1. Experimenters think that participants adhere to their two-fold representation of MHP_{nn-2}. To the mental construction of the various stages of MHP_{nn-2} (fixed system) based on the three first stages on the one hand and to the correct comprehension of the message of revision given in stage (iv) on the other hand.

Postulate 2. Participants share experimenters' representation of MHP_{nn-2} derived from by postulate 1. They specifically have the same structure of believe (representation of system) and the same interpretation of the message of revision.

Let's next analyze experimenters' double representation.

2.2   A bi-probabilistic level

The semantics of possible worlds as used in Economics and Artificial Intelligence helps to model the belief structure experimenters expect from participants surrounded by uncertainty (Walliser & Zwirn 1997; Billot & Walliser 1999; Fagin *et al*. 1999; Walliser & Zwirn forthcoming). Due to uncertainty, participants envisage a plurality of worlds derived from the alternative situations proposed by the experimental paradigm that they consider as possibly real. In most experiments on probability revision, participants are requested to form several layers of uncertain beliefs related to the various properties and meta-properties characterizing the object of prediction (for a review, see Baratgin & Politzer 2007a)[6]. After

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[6] For instance, in the well-known "medical diagnosis" problem (Hammerton 1973), participants (cast as physicians) are requested to form a belief about a patient. They consider the different symptoms (mammography positive, negative), the possibly related diseases (breast cancer or not) and the belief whether the patient suffers from the disease. Various other paradigms are designed in the same way,

the original participant's choice of door No.1, MHP_{nn-2} can be modeled as a "two-level hierarchy belief structure" (Walliser & Zwirn 1997, forthcoming).

For experimenters the participant in MHP_{nn-2} has to form uncertain beliefs on the n-2 doors to be opened by the host which depend on his (or her) uncertain beliefs on the door that he (or she) believes to hide the car. In experimenters' view, participants after to have picked the door No.1 construct three layers of beliefs:

- A layer 0 illustrates the basic properties of the door (whether to be opened by the host). It comprises the n-1 possible combinations offered to the host at stage (iii) when choosing n-2 doors out of n-1. It corresponds to the following n-1 worlds $\{'_{goat}No.2.4...n'$, $'_{goat}No.2.3.5.6...n'$, ..., $'_{goat}No.2...n-1'$, $'_{goat}No.3...n'\}$,

- A layer 1 accounts for the secondary properties of the door (whether to hide the car). It comprises the n possibilities offered to the participant at stage (ii). It corresponds to the following n worlds $\{_{car}No.1, ..., _{car}No.n\}$,

- A layer 2 defines the door of prediction (the door originally picked by the participant). It corresponds to the world $\{No.1\}$.

By Postulate 1, participants know that the host's choice to open n-2 doors that hide goats at stage (iii) depends on their choice at stage (ii). Hence, participants define a causal relation between the two layers 0 and 1. Similarly, participants know that the host's choice to initially place the car is limited to n doors. Hence, participants define a populational relation between layers 1 and 2.

Experimenters assume that participants translate these relations into two objective distributions of probability elaborated from the explicit and implicit rules of MHP_{nn-2}. The

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see for example the "cab problem" (Kahneman & Tversky 1972), and the "engineer-lawyer problem" (Kahneman & Tversky 1973).

distribution of probability from layer 2 to 1 is explicitly defined by rule 1. It is represented by the prior probabilities $P(_{car}No.i)$ (with $i = 1, ..., i = n$). The distribution of probability (called likelihoods) of layer 1 to 0 is precisely defined by rules 2, 3, 3.1 and 3.2. If the car is behind door No.1, the host uses an indifference strategy and the n-1 worlds of layer 0 have the same probability of 1/(n-1) of being opened to show a goat. If the car is behind door No.i (with $i >$ 1), the host can open only the n-2 doors No.3...i-1.i+1...n.

We thus establish the first property of experimenters' representation of MHP_{nn-2}.

Property 1. Experimenters' structure of belief of MHP_{nn-2} is a bi-probabilistic structure (see Figure 1a)[7] where the distributions of probabilities are consistent with rules 1, 2, 3, 3.1 and 3.2.

Insert Figure 1 about here

2.3   A focusing situation of revision

We can note that the bi-probabilistic structure of MHP_{nn-2} can be collapsed into an equivalent probabilistic structure of one level with the 2(n-1) possible combined worlds (see Figure 1b): $\{_{car}No.1\wedge'_{goat}No.2.4…n', _{car}No.1\wedge'_{goat}No.2.3.5.6…n', ..., _{car}No.1\wedge'_{goat}No.2…n-1',$ $_{car}No.1\wedge'_{goat}No.3…n', _{car}No.2\wedge'_{goat}No.3…n', _{car}No.3\wedge'_{goat}No.2.4…n', ..., _{car}No.n\wedge'_{goat}No.2…n-1'\}$. Hence we have the following relations between the two structures: (1)

$P(_{car}No.1) = P(_{car}No.1\wedge'_{goat}No.2.4…n') + P(_{car}No.1\wedge'_{goat}No.2.3.5…n') + ...$

$+ P(_{car}No.1\wedge'_{goat}No.2…n-1') + P(_{car}No.1\wedge'_{goat}No.3…n')$

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[7] As in Walliser and Zwirn (forthcoming) a physical world is represented here by a square and a psychical world is represented by a circle. Here experimenters have a subjective belief that participants' beliefs match this objective structure and in level 2, the 2-world is represented by a circle.

$P(_{car}No.2) = P(_{car}No.2 \wedge \text{`}_{goat}No.3…n\text{'})$

...

$P(_{car}No.n) = P(_{car}No.n \wedge \text{`}_{goat}No.2…n-1\text{'})$.

In the last stage (iv) of MHP_{nn-2} participants learn that the host has opened doors No.3, ... to No.n showing goats behind following the protocol described in stage (iii) (among the n-1 possible doors No.2, ...and No.n). In experimenters' opinion, this message indicates that the subsystem where the world '$_{goat}$No.3...n' is true should be extracted from the initial system (bi-probabilistic structure of Figure 1). Participants should form their beliefs about this extracted subsystem according to message '$_{goat}$No.3…n' by keeping the same distribution of probabilities P. They are expected to *focus* their attention on such subset. Practically, this subset comprises the two worlds $\{_{car}No.1 \wedge \text{'}_{goat}No.3…n\text{'}, _{car}No.2 \wedge \text{'}_{goat}No.3…n\text{'}\}$, the other n-2 following worlds being temporally left out. Thus participants should arrive to a one level belief hierarchy corresponding to a simpler probabilistic structure where the basic worlds are combined over layers 0 and 1 of the initial structure (see Figure 1c). This operation, called the synthetic rule by Walliser and Zwirn (forthcoming) exactly corresponds to the Bayesian conditioning rule[8] under the one probabilistic level structure (see Figure 1d)[9]. It operates a change of reference class but is not a true revision process (de Finetti 1949, 1972). It is appropriate in the focusing type situation of revision (Dubois & Prade 1992; Dubois *et al.*

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[8] The two traditional equivalent specifications of Bayes' rule are:

Form 1, "Bayes' identity": $P(_{car}No.1 |\text{'}_{goat}No.3...n\text{'}) = \dfrac{P(_{car}No.1) \times P(\text{'}_{goat}No.3...n\text{'}|_{car}No.1)}{P(\text{'}_{goat}No.3...n\text{'})}$. This specification can naturally be deduced from the bi-probabilistic structure of figure 1.

Form 2, "conditional probability": $P(_{car}No.1 |\text{'}_{goat}No.3...n\text{'}) = \dfrac{P(_{car}No.1 \wedge \text{'}_{goat}No.3...n\text{'})}{P(\text{'}_{goat}No.3...n\text{'})}$. This specification can naturally be deduced from the one-probabilistic structure of figure 1b.

with $P(\text{'}_{goat}No.3...n\text{'}) = \sum\limits_{i=1}^{i=n} P(_{car}No.i)P(\text{'}_{goat}No.3...n\text{'}|_{car}No.i)$

$= P(_{car}No.1)P(\text{'}_{goat}No.3...n\text{'}|_{car}No.1) + P(_{car}No.2)P(\text{'}_{goat}No.3...n\text{'}|_{car}No.2)$

1996; Dubois & Prade 1997; Walliser & Zwirn forthcoming). In such focusing situation the message concerns an object drawn at random from a population of objects (fixed universe). For instance a clinician who knows about the distribution of the diseases and the links between these diseases and the symptoms in a class of population and who collects the symptoms on a patient, changes his (or her) reference class in order to focus on the cases that share the same symptoms as the ones he (or she) has collected.

Consequently we thereby establish our second property of experimenters' representation of MHP_{nn-2}.

Property 2. Experimenters' representation of revision in MHP_{nn-2} is a focusing situation for which the Bayes' rule of conditioning applies.

MHP_{nn-2} can be illustrated in a focusing framework as a problem of balls, urns and meta-urns in which the message concerns an information on a ball that has been extracted (Walliser & Zwirn forthcoming). Such problem is advantageously worded in a populational format making explicit the bi-probabilistic level hierarchy and the situation of revision (see below the section 4.2.4).

The urns and balls problem[10]**:** There is a meta urn No.1 that contains A urns. There are: a_1 urns labeled $_{car}$No.1, ..., a_i urns labeled $_{car}$No.i, ... and a_n urns labeled $_{car}$No.n (with $A = \sum_{s=1}^{s=n} a_s$). Each a_1 urns labeled $_{car}$No.1 contains n-1 balls labeled from 2 to n. Among the A- a_1 other urns, each urn labeled $_{car}$No.i contains n-1 balls labeled i. Each a_2 urn labeled $_{car}$No.2 contains n-1 balls labeled 2, ... each a_i urn labeled $_{car}$No.i contains n-1 balls labeled i, ... and each a_n urn labeled $_{car}$No.n contains n-1 balls labeled n. One urn is randomly extracted from the meta urn and you must assess the probability that it is labeled urn $_{car}$No.1. Next one ball is extracted at random from this urn and you learn the message (at layer 0) "this ball is

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[9] For a demonstration see Appendix 2 in Walliser & Zwirn (forthcoming)

not labeled 3, ... to n" (corresponding to '$_{goat}$No.3...n'). What is now your belief that the selected urn is $_{car}$No.1 ($P(_{car}$No.1 | '$_{goat}$No.3...n'))?

In the urns and balls problem, the message "a urn is selected and a ball is extracted, which is not labeled 3, ... to n" is without ambiguity a focusing message. It implies that the selected urn is not any of the urns $_{car}$No.3 , ... to $_{car}$No.n and that the probability that it is labeled $_{car}$No.1 is $a_1/a_2(n-1)$ times the probability that it is labeled $_{car}$No.2.

2.4   Experimenters' Bayesian solution ("EBS") for MHP$_{nn-2}$

Bayesian conditioning amounts to a re-standardization of the probabilities of the remaining worlds compatible with the focusing message '$_{goat}$No.3...n' (see Figure 1e). We have:

$$P(_{car}\text{No.1}|'_{goat}\text{No.3...n'}) = \frac{P(_{car}\text{No.1}\wedge'_{goat}\text{No.3...n'})}{P(_{car}\text{No.1}\wedge'_{goat}\text{No.3...n'}) + P(_{car}\text{No.2}\wedge'_{goat}\text{No.3...n'})} = \frac{a_1}{a_1 + (n-1)a_2} \quad (2)$$

$$P(_{car}\text{No.2}|'_{goat}\text{No.3...n'}) = \frac{P(_{car}\text{No.2}\wedge'_{goat}\text{No.3...n'})}{P(_{car}\text{No.1}\wedge_{goat}\text{No.3...n'}) + P(_{car}\text{No.2}\wedge'_{goat}\text{No.3...n'})} = \frac{(n-1)a_2}{a_1 + (n-1)a_2} \quad (3)$$

and hence the relations: $\dfrac{P(_{car}\text{No.1}|''_{goat}\text{No.3...n''})}{P(_{car}\text{No.2}|''_{goat}\text{No.3...n''})} = \dfrac{a_1}{(n-1)a_2}$ $\qquad\qquad\qquad$ (4)

We get theorem 1:

Theorem 1: (i) For a_2 strictly superior to the mean of set of values a_i (with $i > 2$),

$$P(_{car}\text{No.1}|'_{goat}\text{No.3...n'}) < P(_{car}\text{No.1}) \; = \; \frac{a_1}{A} \quad \text{and}$$

(ii) in the specific case where a_2 corresponds to the mean of set of values a_i (with $i > 2$), $P(_{car}\text{No.1}|('_{goat}\text{No.3...n'}) = P(_{car}\text{No.1})$

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[10] This problem may be worded as the generalization of the traditional problem of "the three small boxes" (Bertrand 1889).

Proof. By elementary calculation:

$$P(_{car}No.1\,|'_{goat}No.3...n') \le P(_{car}No.1) \iff a_1 + (n-1)a_2 \ge \sum_{i=1}^{i=n} a_i \iff a_2 \ge \frac{\sum_{i=3}^{i=n} a_i}{(n-2)}$$

In particular, for standard MHP_{nn-2}, we have[11]: $P(_{car}No.1\,|'_{goat}No.3...n') = \dfrac{1}{n}$ \hfill (5)

and $P(_{car}No.2\,|'_{goat}No.3...n') = 1 - \dfrac{1}{n}$ \hfill (6)

3 Participants' point of view

3.1 The situation of revision brings about a new hypothesis to account for MR

Although MR is commonly analyzed as evidence for the "inconsistency" of participants, some authors have yet sketched two possible explanations for MR as a consistent response. First, from a Bayesian point view, rules 1 and 3 are questionable because participants' MR could be consistent if based on subjective prior probabilities and likelihoods different from experimenters' beliefs[12]. For example, if participants believe that when the car is behind door No.1, the host always opens doors No.3 ... to No.n ($P('_{goat}No.3...n'|_{car}No.1) = 1$), then MR would correspond to a Bayesian solution[13] albeit different from experimenters'.

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[11] In standard MHP, we have: $P(_{car}No.1|'_{goat}No.3') = P(_{car}No.1) = 1/3$ and $P(_{car}No.2|'_{goat}No.3') = 2/3$.

[12] For a general discussion of this point in the experimental literature on probability judgment see Baratgin & Politzer (2006), for a general Bayesian solution to standard MHP taking into account the possible host's strategies see for example Wechsler, Esteve, Simonis, and Peixoto (2005). For a discussion on the hypothesis that despite answering "MR" to standard MHP participants may actually have the same representation of the task as experimenters, but assume a different host's strategy than those given by implicit and explicit "rules" see for example Nickerson (1996).

[13] In standard MHP and standard MHP_{nn-2}, for example the rule 3.2.1 seems intuitive because if the host has originally no preference for a specific door when he/she initially places the car, he/she is expected to have no preference when choosing to open n-2 doors with goats behind. Yet, it seems more debatable in MHP and MHP_{nn-2} versions because initially the host has a preference on the doors (unequal prior probabilities).

Second, one can question Postulate 1 that considers message '$_{goat}$No.3...n' as certain[14] and Property 1 that forces to stick to the additivity framework[15]. However these two explanations actually appear limited, as participants seem to adhere to experimenters' Property 1 in the numerous experiments on standard MHP and MHP versions. When participants are indeed explicitly required to state their prior probabilities, they actually share experimenters' priors (Ichikawa & Takeichi 1990; Baratgin & Politzer 2003; Franco-Watkins *et al.* 2003). Besides when experimenters explicitly specify the strategy of the host, participants nevertheless give MR (Shimojo & Ichikawa 1989; Ichikawa & Takeichi 1990; Granberg & Brown 1995; Krauss & Wang 2003; Burns & Wieth 2004)[16]. Consequently an explanation for MR still needs to be found.

Let's next analyze whether participants adhere to experimenters' Property 2. The relevant literature shows that experimenters who regard MR as erroneous generally fail to consider that participants' interpretation of the host's message might not conform to the only situation they envisage, namely a focusing situation of revision. Yet participants in MHP$_{nn-2}$ can infer pieces of information that differ from those meant by the experimenter. This approach is new to psychological research because, other situations of revision than focusing have scarcely ever been studied (for review, see Baratgin & Politzer 2007a). In few studies, participants learn a message that specifies or invalidates an initial belief regarding a universe also considered as fixed. In this situation, called "revising", Bayes' rule remains the adequate rule (among other possible ones) for an individual to change his (or her) degrees of belief.

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[14] In the situation where the message would be dubious, Jeffrey's rule would be adequate (for a solution to standard MHP in this situation see Loschi *et al.* 2006).

[15] This postulate is traditional in the field of studies on probability judgment. It assumes that participants' degrees of belief are quantitative and obey to Kolmogorov' s axioms. However the plausibility of this hypothesis is debatable (Baratgin & Politzer 2006). Some authors have proposed a solution for standard MHP in a formalism different from traditional additive probability (see Appendix 2).

[16] However no experimental study analyzes whether participants have a precise distribution of probability between layers 1 and 0. In this case, participants' representation of MHP would be a "distribution of events" (see Appendix 2 on this question).

For instance in the urns and balls problem the message "there are no balls labeled 3, ... to n in the extracted urn" is a revising message that changes our belief into the certitude that it is urn $_{car}$No.2 that has been extracted. This operation amounts to eliminating the possible urns with balls labeled 3, ... to n and changes our beliefs on the remaining urns (here the only urn $_{car}$No.2) according to Bayes' rule. Such process appropriate to a bi-probabilistic structure is called "maximal rule" (Walliser & Zwirn forthcoming). In contrast to the focusing and revising situations, individuals may also have to revise their degrees of belief when the initial universe - now considered as dynamic - has changed according to the information conveyed by the new message. Such situation, called "updating", has a clear definition supported by both a set of axioms for belief revision (Katsuno & Mendelzon 1992) and for probability revision (Walliser & Zwirn 2002). In the updating situation of revision, the message specifies a change of the universe, which is considered as time evolving. The updating message is easy to illustrate when it stems from an action or an intervention that modifies the initial knowledge of the original universe. Some possibilities may be removed by an exterior action. But it is noteworthy that updating may occur without an explicit intervention. This would be the case with the outbreak of a new disease that transforms the knowledge base of physicians, or with new statistical data that radically alter a demographic model. In the urns and balls problem, an updating message at layer 0 would correspond to the message "the balls that are labeled 3, ... to n have been removed from all urns" (in MHP_{nn-2} this message is equivalent to "the doors numbered 3, .. to n (with goats behind) has been removed" and noted $_{car}$No.3...n). The updating message indicates that the original problem has evolved.

You are now faced with a meta urn No.1 that contains that contains A urns. There are: a_1 urns labeled $_{car}$No.1, ..., a_i urns labeled $_{car}$No.i, ... and a_n urns labeled $_{car}$No.n (with i = 1, ... to n and $A = \sum_{s=1}^{s=n} a_s$). The a_1 urns labeled $_{car}$No.1 contain only one ball labeled 2, the a_2 urns $_{car}$No.2 always contain n-1 balls labeled 2 and each other urns $_{car}$No.i (i = 3, ... to n) are now

empty. It implies that the selected urn is not any of the urns $_{car}$No.3 , ... to $_{car}$No.n and that the probability that it is labeled $_{car}$No.1 is a_1/a_2 times the probability that it is labeled $_{car}$No.2. This result corresponds to MR in MHP$_{nn-2}$. The operation amounts to removing balls labeled 3 to n from the possible urns and to changing their composition accordingly by Bayes' rule. The updating rule appropriate in a bi-probabilistic structure is called "minimal rule (Walliser & Zwirn forthcoming) (see section 3.3.2). A correspondence exists between the situations of revision at different layers (see for a generalized analysis Walliser & Zwirn, forthcoming). In the urns and balls problem, after the updating message at layer 0 "the balls that are labeled 3, ... to n have been removed from all urns" , induces the focusing message at layer 1 (urns level): "the extracted urn from the large urn is not a urn of the type $_{car}$No.3, ... to $_{car}$No.n" (that is equivalent in MHP$_{nn-2}$ to message "the doors No.3, ... to No.n hide goats" noted $_{goat}$No.3...n)[17]. Psychologists have never explicitly considered the situation of updating in their numerous investigations. Studying this situation however appears necessary for a better understanding of human probability revision (see for a review Baratgin & Politzer 2006). Even though experimenters assume or instruct participants that the universe is fixed, it may be the case that participants yet have a dynamic interpretation of it. So, when assessing the coherence of participants' judgment, experimenters must be aware of this possibility and ready to use the appropriate formalism. A wrongful participants' response given in a focusing situation (for which Bayes' rule is the only adequate measure) may actually reveal correct if participants interpret the situation in an updating framework (for which Bayes' rule does not apply) and the response is consistent with the adequate revision rule for updating. We argue that MR may be seen as consistent if *participants typically infer the focusing message '$_{goat}$No.3...n' as the updating message $_{car}$~~No.3...n~~.*

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17 This situation is isomorphic to a problem with two sorts of urns; the urns $_{car}$No.1 that contain only one ball labeled 2 and the urns $_{car}$No.2 that contain n-1 balls labeled 2 Thus the updating message is also equivalent to an updating message at layer 1 (urns level): "the urns No.3, ..., No.n have been removed from the meta urn".

3.2   A pragmatic explanation for the updating representation

The hypothesis that participants interpret MHP_{nn-2} as an updating situation of revision can be supported by pragmatic analysis[18].

In the two first stages (i) and (ii) of MHP_{nn-2}, participants are required to pick a door that hides a car among n doors presented in front of them. They make their initial choice of a door (here No.1) without any special information related to this object. Because of uncertainty participants attach a prior probability to each door (here explicitly given by experimenters). In the framework of relevance theory (Sperber & Wilson 1995), the communicative principle posits that any utterance conveys a presumption of its own (optimal) relevance. In other words, participants assume that if the experimenter sends a message, it deserves to be treated. Thus for participants any new information given by the experimenter carries a presumption of importance in relation with the aim of the problem (to know where the car is). In fact, participants can expect the new information to be highly important (relevant) as they have initially chosen the door randomly and this is mutual knowledge shared with the experimenter. In addition according to relevance theory, participants are more inclined to pay attention to messages with *important contextual effect* and *low cognitive effort* of treatment. The *focusing* message '$_{car}$No.3...n' corresponds to stages (iii) and (iv). At stage (iii), the host warns that he will open n-2 doors (out of the n-1 remaining doors) that hide goats. This message actually comprises two compacted pieces of information. The first element is implicit. It corresponds to the host's strategies to open n-2 doors among doors No.2, ... to No.n that hide goats. It constitutes the key to construct the representation of the puzzle (Figure 1a). The second element refers to the host's explicit declaration to open n-2 doors among doors No.2, ... to No.n that hide goats. The comprehension of the first element

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[18] Some studies on probability judgment have supported the hypothesis along which participants infer a different representation of the task than experimenters do for various pragmatic reasons (see for

requires a sizeable cognitive effort because participants must work out the n-2 complementary possibilities that the host may show goats behind doors (No.2, No.4, ... and No.n), ..., or (No.2, ... and No.n-1) then keep this in memory. The second element of the message requires low a cognitive effort of treatment and is contextually powerful and expected (so would be the case of the n-2 other possible messages). It carries a strong presumption of importance in relation with the aim of the problem (to know where the car is). Participants induce that "n-2 of the n-1 doors (No.2, ..., No.n) will be eliminated". This should lead them to favor the two remaining doors (n-2 doors are removed as well as the possibilities to have a car behind). Participants nevertheless take into account the host's implicit strategies and expect following the host's declaration to be promptly confronted with a "new simplified" problem with two doors (No.1 and an other door) to which they may attach a new distribution of probability. The hypothesis that participants are in expectation of an evolving problem was tested in Baratgin (1999; 2004) with a revised version of standard MHP limited to three stages. After the first two stages (i) and (ii), the host delivers at stage (iii) an uninformative message (that is the already known message of stage (i)): "at least one door (out of No.2 and No.3) has a goat behind". A redundancy effect was evidenced as a sizeable number of participants nevertheless modified their initial degrees of belief and give in a large majority MR. In other words although a message may actually be uninformative, it still may prompt participants to infer information from it and to revise their perception of the problem into a simplified problem with two doors (No.1 and an other door). Now, when participants receive the message (resulting from the host's action) "the car is not behind doors No.3, ...to No.n". They consider it as the answer to the expectation to see n-2 possible doors removed. The message benefits from an overwhelming contextual effect and amounts in participants' view to concretely eliminating doors No.3, ..., No.n from the set of possible doors ($_{car}$~~No.3...n~~).

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instance Dulany & Hilton 1988; Krosnick *et al*. 1990; Politzer & Noveck 1991; Macchi 1995; Politzer

Participants are then left with two doors (door No.1 and door No.2) they nevertheless adhere to Property 1 (the bi-probabilistic level structure). Consequently, participants are strongly invited to increase their beliefs related to doors No.1 and No.2[19]. From an experimental point of view the analysis of the verbal responses of the majority of participants in standard MHP is consistent with this pragmatic explanation: "*we have shifted from a 3-doors problem to a 2-doors problem after the host's action* " (Baratgin & Politzer 2003).

Overall we conclude that participants interpret standard MHP as an updating problem of revision and that "EBS" violates expectations resulting from ostensive communication. The focusing situation *does not really correspond to a dynamic process of probability revision* because in a focusing situation there are no truly invalidated worlds. On receiving the message '$_{goat}$No.3...n', participants should change the reference of the whole initial set of possible worlds (see section 2.3) but stick to the same probability distribution P. This is explicit in the urns and balls problem but on the contrary, the statement of MHP$_{nn-2}$ suggests an updating situation of revision. At time t_0 participants are invited to work out their prior distribution on n doors (layer 1); at time t_1 they integrate the new message resulting from the host's action (layer 0). This action modifies perceptibly the problem (two doors). Then the experimenter asks participants a judgment of probability that can be interpreted as a true revision of their initial probability distribution P leading to a updating revision. This hint contained in the statement deters participants from envisaging the focusing situation.

3.3   MHP$_{nn-2}$ in an updating interpretation

3.3.1   The general Imaging rule

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[19] Hence "EBS" contradicts the principle of relevance in numerous cases (see Theorem 1). Notably in all experiments where experimenters have proposed a version of MHP where $a_2 > a_3$ or a standard version of MHP$_{nn\_2}$. "EBS" neither increases (nor changes for standard MHP$_{nn-2}$) participants' initial degrees of belief attributed to $_{car}$No 1. The message '$_{car}$No.3...n' would only lead participants to increase their initial degrees of

In $\text{MHP}_{nn\text{-}2}$ if the message '$_{goat}$No.3...n' is interpreted (albeit mistakenly) as the updating message "the doors No.3, ...to No.n have been removed by the host" ; it translates a physical operation on the system, modifying its initial structure. In this situation, the adequate method of revision for a one probabilistic level is different and Lewis' rule (or Imaging) (Lewis 1976; Gärdenfors 1988; Lepage 1997) reveals the appropriate operation of revision (see the Imaging generalized rule Walliser & Zwirn 2002). . Generally Bayesian conditioning rule and imaging yield different solutions[20]. Imaging causes a revision of the prior probabilities on the possible worlds in such a way that the posterior probabilities are obtained by shifting the initial probabilities of invalidated worlds to newly validated worlds. Each invalidated world sees its probability attributed to the closest set of worlds that remain. One difficulty of Imaging lies in defining the notion of "closest" world. Hence the few authors who have envisaged an imaging solution to standard MHP have proposed two different possible distances (Dubois & Prade 1992; Cross 2000). Yet these authors have considered an Imaging operation with regards to the compacted probabilistic level of Figure 1b. Under this compacted representation there is no clue to choose a particular distance. Let $P^{+}_{'goat No.3...n'}(_{car}No.1)$ and $P^{+}_{'goat No.3...n'}(_{car}No.2)$ be the probabilities obtained by Imaging after the

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belief attached to $_{car}$No.2. These results can be interpreted as a violation of logic (see Dubois & Prade 1992, for a discussion in standard MHP).

[20] It is easy to see that the difference with Bayes' rule lies in the redistribution of the invalidated world ($_{car}$No.1∧'$_{goat}$No.2'). Indeed Bayes' rule can also be seen as a redistribution of invalidated worlds proportionally to prior probability of remaining worlds. Consequently "BS" can be formulated as follows:

$$P(_{car}No.1 |'_{goat}No.3...n') = \frac{P(_{car}No.1 \wedge '_{goat}No.3...n')}{P(_{car}No.1 \wedge '_{goat}No.3...n') + P(_{car}No.2 \wedge '_{goat}No.3...n')}$$

$$= P(_{car}No.1 \wedge '_{goat}No.3...n')$$

$$+ \frac{P(_{car}No.1 \wedge '_{goat}No.3...n')}{P(_{car}No.1 \wedge '_{goat}No.3...n') + P(_{car}No.2 \wedge '_{goat}No.3...n')}\left(1 - P(_{car}No.1 \wedge '_{goat}No.3...n') - P(_{car}No.2 \wedge '_{goat}No.3...n')\right)$$

$$= P(_{car}No.1 \wedge '_{goat}No.3...n')$$

$$+ \frac{P(_{car}No.1 \wedge '_{goat}No.3...n')}{P(_{car}No.1 \wedge '_{goat}No.3...n') + P(_{car}No.2 \wedge '_{goat}No.3...n')}\left(P(_{car}No.1 \wedge '_{goat}No.2.4...n') + ... + P(_{car}No.3 \wedge '_{goat}No.2.4...n')\right)$$

For a detailed axiomatic analysis of this difference see Walliser & Zwirn (2002)

updating message '$_{goat}$No.3...n'. $P^+_{'_{goat}No.3...n'}(_{car}No.1)$ is called the *image* of $P(_{car}No.1)$ on worlds where '$_{goat}$No.3...n' is true. When the updating message '$_{goat}$No.3...n' is learnt, the probabilities concerning the 2(n-2) invalidated worlds $\{_{car}No.1\wedge'_{goat}No.2.4...n', _{car}No.1\wedge'_{goat}No.2.3.5.6...n',$ ..., $_{car}No.1\wedge'_{goat}No.2...n-1', _{car}No.3\wedge'_{goat}No.2.4...n', ..., _{car}No.n\wedge'_{goat}No.2...n-1'\}$ are redistributed over the other possible worlds that participants consider as the closest. The n-2 worlds $\{_{car}No.1\wedge'_{goat}No.2.4...n', _{car}No.1\wedge'_{goat}No.2.3.5.6...n', ..., ..., _{car}No.1\wedge'_{goat}No.2...n-1'\}$ are incompatible with the world $_{car}No.2$ but perfectly compatible with the world $_{car}No.1$. Thus it is natural to assume that the closest world to these n-2 worlds is $_{car}No.1$ and that the weight $P(_{car}No.1\wedge'_{goat}No.2.4...n') + P(_{car}No.1\wedge'_{goat}No.2.3.5.6...n') +...+ P(_{car}No.1\wedge'_{goat}No.2...n-1')$ is reallocated over the single world $_{car}No.1$ (Dubois & Prade 1992; Cross 2000). As far as the invalided worlds $\{_{car}No.3\wedge'_{goat}No.2.4...n', ..., _{car}No.n\wedge'_{goat}No.2...n-1'\}$ corresponding to n-2 worlds $_{car}No.i$ (with i = 3, ..., i = n, cf. relation 1) are concerned, there are different possibilities to define their "closest" world. In a general Imaging (Gärdenfors 1988) the weight of the invalidated worlds are redistributed over several remaining worlds. In MHP_{nn-2} each weight of the invalidated worlds, $P(_{car}No.i)$ (with i = 3, ..., i = n) can be redistributed on the two remaining worlds $_{car}No.1$ and $_{car}No.2$.

Let β_i be the proportion of weight $P(_{car}No.i)$ reallocated to world $_{car}No.1$ (for i = 3, ..., i = n, $\beta_i \in [0, 1]$). The general Imaging rule on considering the one probabilistic level gives:

$$P^+_{'_{goat}No.3...n'}(_{car}No.1) = P(_{car}No.1\wedge'_{goat}No.3...n') + P(_{car}No.1\wedge'_{goat}No.2.4...n') +$$

$$... + P(_{car}No.1\wedge'_{goat}No.2...n-1') + \sum_{i=3}^{i=n}\beta_i P(_{car}No.i)$$

$$P^+_{'_{goat}No.3...n'}(_{car}No.1) = P(_{car}No.1) + \sum_{i=3}^{i=n}\beta_i P(_{car}No.i) \tag{7}$$

and similarly $P^+_{'_{goat}No.3...n'}(_{car}No.2) = P(_{car}No.2) + \sum_{i=3}^{i=n}(1-\beta_i)P(_{car}No.i) \tag{8}$

We thus establish theorem 2 for general imaging:

Theorem 2. (i) If there is at least one i with $\beta_i \neq 0$ then $P^+_{'goat\ No.3...n'}(_{car}No.1) > P(_{car}No.1)$

and (ii) if $\beta_i = 0$, for $i = 3, ..., i = n$, then $P^+_{'goat\ No.3...n'}(_{car}No.1) = P(_{car}No.1)$.

However, as we have seen in section 2.2, the belief representation of MHP_{nn-2} corresponds to the bi-probabilistic level of Figure 1a and the one-probabilistic level of Figure 1b is only a consequence of this bi-probabilistic level. On considering the bi-probabilistic level it is now possible to detail the appropriate redistributions of the weight of invalidated worlds $_{car}No.i$ (with $i = 3, ..., i = n$) and thus define (β_i), $i = 3, ..., i = n$.

3.3.2 The "proportional Imaging rule" gives MR in MHP_{nn-2}

In a general analysis of the revision rules for bi-probabilistic level, Walliser & Zwirn (1997; forthcoming) propose the "minimal rule" as the appropriate updating rule. It consists in revising in a Bayesian way the distributions of level 1 that are compatible with the message, the other distributions being abandoned with a necessary homothetic adjustment of the distribution at level 2 (see Figure 2a).

Insert Figure 1 about here

This rule is obvious in the urns and balls problem (see section 3.1). We call this rule the "proportional Imaging rule " because it corresponds (in the probabilistic structure of one level) to the reallocation of the weight of incompatible worlds over the remaining worlds proportionally to their prior probabilities (see Figure 2b). In MHP_{nn-2}, the weight $P(_{car}No.i)$ (for $i = 3, ..., i = n$) must be reallocated over the worlds $_{car}No.1$ and $_{car}No.2$ proportionally to their prior probabilities. In this case $\beta_i = \dfrac{P(_{car}No.1)}{P(_{car}No.1) + P(_{car}No.2)}$ and is independent from i.

The solution of MHP_{nn-2} by this proportional Imaging rule (from now on "PIS") leads to the same result as the revision by Bayes' rule after the simpler focusing message $_{goat}No.3...n$ (see also the urns and balls problem in section 3.1). Hence we have the following theorem 3.

Theorem 3. The proportional Imaging rule is mathematically isomorphic to a Bayesian conditioning on the "naive set" $\{_{car}No.1, ..., _{car}No.n\}$[21].

Proof.

$$P^+_{'goat\,No.3...n'}(_{car}No.1) = P(_{car}No.1) + \frac{P(_{car}No.1)}{P(_{car}No.1) + P(_{car}No.2)} \times \sum_{i=3}^{i=n} P(_{car}No.i)$$

$$= P(_{car}No.1)\left(1 + \frac{\sum_{i=3}^{i=n} P(_{car}No.i)}{P(_{car}No.1) + P(_{car}No.2)}\right)$$

$$P^+_{'goat\,No.3...n'}(_{car}No.1) = \frac{P(_{car}No.1)}{P(_{car}No.1) + P(_{car}No.2)} = P(_{car}No.1|_{goat}No.3...n) = \frac{a_1}{a_1 + a_2} \tag{9}$$

Similarly: $P^+_{'goat\,No.3...n'}(_{car}No.2) = \dfrac{P(_{car}No.2)}{P(_{car}No.1) + P(_{car}No.2)} = P(_{car}No.2|_{goat}No.3...n) = \dfrac{a_2}{a_1 + a_2}$ $\qquad$ (10)

and we have the relation: $\dfrac{P^+_{'goat\,No.3...n'}(_{car}No.1)}{P^+_{'goat\,No.3...n'}(_{car}No.2)} = \dfrac{P(_{car}No.1)}{P(_{car}No.2)} = \dfrac{a_1}{a_2}$ $\qquad$ (11)

In particular, for standard MHP_{nn-2}, we have: $P^+_{'goat\,No.3...n'}(_{car}No.1) = P^+_{'goat\,No.3...n'}(_{car}No.2) = \dfrac{1}{2}$ $\qquad$ (12)

It is therefore obvious that: $P^+_{'goat\,No.3...n'}(_{car}No.1) > P(_{car}No.1)$ and $P^+_{'goat\,No.3...n'}(_{car}No.2) > P(_{car}No.2)$.

Solutions (9) and (10) exactly correspond to MR in MHP[22] (Ichikawa 1989; Ichikawa & Takeichi 1990; Granberg 1996; Yamagishi 2003). Proportional Imaging verifies theorem 2

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[21] We make a parallel here with the remark of some theorists who have interpreted MR as the result of a Bayesian conditioning on a "naive set" while "EBS" is the result of a Bayesian conditioning on a more "sophisticated set" (Jeffrey 1988; Grünwald & Halpern 2003). In Appendix 3, we show that the proportional Imaging rule is equivalent to Pearl (2000)' s Bayesian conditioning after one *intervention* (designed by Pearl as the "do" operator).

[22] In standard MHP, we have: $P_{'goat\,No.3'}(_{car}No.1) = P_{'goat\,No.3'}(_{car}No.2) = \dfrac{1}{2}$.

(i) and supports the pragmatic argument of section 3.2. It makes a good candidate to experimentally describe and theoretically gauge *participants' judgment* (see also section 4). Finally "PIS" verifies the below theorem 4.

Theorem 4. For any $n > 2$, $P^+_{'goat No.3...n'}(_{car}No.1) > P(_{car}No.1 |'_{goat}No.3...n')$ and

$P^+_{'goat No.3...n'}(_{car}No.2) < P(_{car}No.2 |'_{goat}No.3...n')$.

Proof. It is obvious because for $n > 2$; $\dfrac{a_1}{a_1 + a_2} > \dfrac{a_1}{a_1 + (n-1)a_2}$ and $\dfrac{(n-1)a_2}{a_1 + (n-1)a_2} > \dfrac{a_2}{a_1 + a_2}$

3.3.3   The "egalitarian Imaging rule"

It is noteworthy that when a_1 is equal to a_2, the proportional Imaging rule is isomorphic to the e*galitarian Imaging rule*[23], which redistributes on the worlds $_{car}No.1$ and $_{car}No.2$ half of the weight of the cancelled worlds $_{car}No.i$ (with $i = 3, ..., i = n$). The equidistance of the two worlds $_{car}No.1$ and $_{car}No.2$ to worlds $_{car}No.i$ has been suggested by Dubois and Prade (1992) in standard MHP, and may be seen as the result of the n possible worlds being initially equally likely. Let $P'^+_{'goat No.3...n'}(_{car}No.1)$ be the revision of $P(_{car}No.1)$ with this "e*galitarian Imaging rule*" (we use P' to differentiate from proportional Imaging rule). The general solution by egalitarian Imaging rule in MHP_{nn-2} (from now on "EIS") gives:

$$P'^+_{'goat No.3...n'}(_{car}No.1) = \frac{a_1}{A} + \frac{1}{2} \times \frac{A - a_1 - a_2}{A} = \frac{A + a_1 - a_2}{2A} = \frac{1}{2}\left(1 + P(_{car}No.1) - P(_{car}No.2)\right) \qquad (13)$$

and $P'^+_{'goat No.3...n'}(_{car}No.2) = \dfrac{A - a_1 + a_2}{2A} = \dfrac{1}{2}\left(1 + P(_{car}No.2) - P(_{car}No.1)\right)$ \qquad (14)

Thus we have the relation: $\dfrac{P'^+_{'goat No.3...n'}(_{car}No.1)}{P'^+_{'goat No.3...n'}(_{car}No.2)} = \dfrac{1 + P(_{car}No.1) - P(_{car}No.2)}{1 + P(_{car}No.2) - P(_{car}No.1)} = \dfrac{1 + a_1 - a_2}{1 + a_2 - a_1}$ \qquad (19)

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[23] The term *Egalitarian Imaging rule* is borrowed from Walliser & Zwirn (2002). This rule is also analyzed in Lepage (1997).

In the case of standard MHP_{nn-2} the solution gives MR:

$$P'^{+}_{'goat No.3...n'} (_{car}No.1) = P'^{+}_{'goat No.3...n'} (_{car}No.2) = \frac{1}{n} + \frac{1}{2} \times \frac{(n-2)}{n} = \frac{1}{2} \qquad (20)$$

3.3.4   The "simple Imaging rule "

As suggested by Cross (2000) in standard MHP a second distance can be assumed in the probabilistic structure of one level. Such distance considers that the worlds $_{car}$No.i (with i > 1) are closer to one another rather than to $_{car}$No.1. In this case, the weights $P(_{car}$No.i) are reallocated on the sole world $_{car}$No.2. Intuitively this distance can be understood as if the weight of the worlds $_{car}$No.2$v...v_{car}$No.n should not change after the updating message '$_{goat}$No.3 ... .n'. This distance seems artificial with the bi-probabilistic level of Figure 1a but can be analyzed as the appropriate rule in a different representation (with a supplementary level) where the different doors No.i (with i >1) are gathered under a similar probability. For instance the door No.1 is white and all others doors are black (see Figure 2c). In a focusing framework this tri-probability level structure is isomorphic to the original bi-probabilistic level but in an updating framework the minimal rule corresponds to the simple Imaging rule.

Let $P''^{+}_{'goat No.3...n'} (_{car}No.1)$ be the revision of $P(_{car}$No.1) with this "*simple Imaging rule* " (we use P' to differentiate from proportional Imaging rule).

$$P''^{+}_{'goat No.3...n'} (_{car}No.1) = P(_{car}No.1) = \frac{a_1}{A} \qquad (21)$$

$$P''^{+}_{'goat No.3...n'} (_{car}No.2) = P(_{car}No.2) + \sum_{i=3}^{i=n} P(_{car}No.i) = \sum_{i=2}^{i=n} P(_{car}No.i) = \frac{A - a_1}{A} \qquad (22)$$

Thus we have the relation: $\dfrac{P''^{+}_{'goat No.3...n'} (_{car}No.1)}{P''^{+}_{'goat No.3...n'} (_{car}No.2)} = \dfrac{P(_{car}No.1)}{\sum\limits_{i=2}^{i=n} P(_{car}No.i)} = \dfrac{a_1}{A - a_1} \qquad (23)$

The solution (from now on "SIS") applied to MHP_{nn-2} verifies theorem 2 (ii).

More generally, we have the following theorem 4 (derived from theorem 1):

Theorem 5. 'SIS" corresponds to "EBS" in $\text{MHP}_{nn\text{-}2}$ in the specific case where a_2 corresponds to the mean of the set of values a_i (with i = 3, ..., = n)

In standard $\text{MHP}_{nn\text{-}2}$, "SIS" corresponds to "EBS":

$$P''^{+}_{'goat\,No.3...n'}(_{car}No.1) = P(_{car}No.1\,|'_{goat}No.3...n') = \frac{1}{n} \text{ and} \tag{24}$$

$$P''^{+}_{'goat\,No.3...n'}(_{car}No.2) = P(_{car}No.2\,|'_{goat}No.3...n') = \frac{1}{n} + \frac{n-2}{n} = \frac{n-1}{n} \tag{25}$$

4 Experimental hindsight

4.1 *The updating interpretation underlies cognitive explanations for MR*

Several cognitive explanations to account for MR have been put forward in psychological experimental studies conducted on MHP and standard MHP as well as standard $\text{MHP}_{nn\text{-}2}$. They do not reveal *independent* one from the others and appear to be actually under lied by the updating interpretation of '$_{goat}$No.3...n'. In this section we will review three main families of such cognitive arguments. We will adopt the notation D_B as apposed to P to represent participants' degree of belief in order to stress that we focus here on cognitive processes rather than on theoretical relationships.

4.1.1 The heuristics explanations

The first set of explanations proposes that participants use "simple heuristics" or "subjective heuristics" to form their judgments of probabilities (Ichikawa 1989; Shimojo & Ichikawa 1989; Ichikawa & Takeichi 1990; Falk 1992; Yamagishi 2003).

4.1.1.1 The heuristics explanations for MR

According to these authors, participants who give MR in standard MHP use the two following heuristics:

"The number of cases heuristic" (NC): When the number of possible alternatives is N, the probability of each alternative is 1/N.

"The constant ratio heuristic" (CR): When one alternative is eliminated, the ratio of probabilities for the remaining alternatives is equal to the ratio of respective prior probabilities.

If participants apply NC or CR to standard MHP_{nn-2}, they will cut down the number of alternatives from n doors to two when learning the message '$_{goat}$No.3...n'. They will thus conclude that: $D_B(_{car}No.2|'_{goat}No.3...n') = D_B(_{car}No.1|'_{goat}No.3...n') = 1/2$. It proves difficult to know what strategy participants actually implement to answer 1/2 in standard MHP_{nn-2} since NC and CR are undistinguishable. However, when Shimojo and Ichikawa (1989) and Ichikawa and Takeichi (1990) propose MHP to their participants a large majority of participants actually use CR. This leads us to theorem 6.

Theorem 6. "EIS" given by the egalitarian Imaging rule is isomorphic to the solution for standard MHP_{nn-2} given by NC. More generally "PIS" given by the proportional Imaging rule is isomorphic to the solution for MHP_{nn-2} based on CR.

Proof. We know by the Rule 1 that $D_B(_{car}No.i) = P(_{car}No.i)$ for i = 1, ..., i = n. CR definition corresponds to relation (11): $\dfrac{D_B(_{car}No.1|'_{goat}No.3...n')}{D_B(_{car}No.2|'_{goat}No.3...n')} = \dfrac{P(_{car}No.1)}{P(_{car}No.2)} = \dfrac{a_1}{a_2}$, and by the complementarity constraint of additive probability (Property 1) we have:

$D_B(_{car}No.1|'_{goat}No.3...n') + D_B(_{car}No.2|'_{goat}No.3...n') = 1$, thus:

$\left(\dfrac{P(_{car}No.1)}{P(_{car}No.2)} + 1\right) D_B(_{car}No.2|'_{goat}No.3...n') = 1$ and we find the relations (9) and (10):

$$D_B(_{car}No.1 \,|'_{goat}No.3...n') = \frac{P(_{car}No.1)}{P(_{car}No.1) + P(_{car}No.2)} = P^+_{'goat No.3...n'}(_{car}No.1) \text{ and}$$

$$D_B(_{car}No.2 \,|'_{goat}No.3...n') = \frac{P(_{car}No.2)}{P(_{car}No.1) + P(_{car}No.2)} = P^+_{'goat No.3...n'}(_{car}No.2)$$

4.1.1.2   The heuristics explanations for "EBS" in standard MHP

In parallel Shimojo and Ichikawa (1989, p. 6-7) propose a third heuristic for the thin minority of participants in standard MHP, who give "EBS" (see also Falk, 1992). These authors show that these participants do not actually use Bayes' rule to give "EBS" but use rather the following heuristic:

"The Irrelevant, Therefore Invariant heuristic" (ITI): "If it is certain that at least one of the several alternatives $(A_1, A_2, ..., A_k)$ will be eliminated. When the information specifying what alternatives to be eliminated is given, it does not change the probability of the other alternatives $(A_{k+1}, A_{k+2}, ..., A_n)$".

Participants who use this heuristic do not modify their prior probability $P(_{car}No.1)$ after the message '$_{goat}No.3...n$' just as "SIS".

Theorem 7. "SIS" given by the simple Imaging rule is isomorphic to the solution for MHP_{nn-2} given by ITI.

Proof. Contrary to CR and NC, the definition of ITI does not predict how the probabilities of the remaining alternatives must change once some have been removed. Nevertheless, using the basic complementarity constraint of the axiom of additivity of probabilities (Property 1), we can suppose that:

$$D_B(_{car}No.2 \,|'_{goat}No.3...n') = 1 - D_B(_{car}No.1 \,|'_{goat}No.3...n') = 1 - P(_{car}No.1) = \sum_{i=2}^{i=n} P(_{car}No.i) \text{ and we}$$

find again relations (21) and (22) of "SIS".

4.1.1.3   The "Subjective Superior heuristic" (SS):

Finally Shimojo and Ichikawa (1989) assume that the answers based on the three above mentioned heuristics actually result from a "Subjective Superior heuristic" defined as participants' following belief.

The "Subjective Superior heuristic": "The posterior probabilities of the remaining alternatives should never be less than their priors when one alternative is removed".

This belief is incompatible with theorem 1 and more generally with the process of conditioning following Bayes' rule (see for a demonstration Ichikawa 1989). However it is compatible with theorem 2 corresponding to an updating situation.

4.1.2   The Mental Models Theory

The second main explanation for MR was proposed by Johnson-Laird et al. (1999) based on their Mental Models Theory (from now on noted MMT) (see also Girotto & Gonzalez 2000). According to these authors, participants are not able to construct the complete set of mental models the puzzle entails because they have a limited working memory. MMT in probability situation can be described by four principles.

 (i) The **"truth principle":** Participants first represent the probability problem by constructing the sets of "mental models" (representations of the different possibilities), which probabilities are strictly superior to 0, (ii) the **"equiprobability principle":** Participants construct a diagram of equiprobable alternatives (iii) The **"proportionality principle":** Participants assess the probability of an event E proportionally to the number of mental models (n_E) in which event E occurs, that is to say n_E/n with n the number of mental models and (iv) the **subset principle**: Granted equiprobability, a conditional probability, $P(E|B)$, depends on the subset of B that is E, and applying the proportionality principle to E and B yields the numerical value.

We notice that the principles (i), (ii) and (iii) actually make up an algorithm to construct the experimenter's representation of Figure 1b in a populational format (see section 4.2.4). Meanwhile the subset principle (iv) corresponds to a natural translation of a focusing procedure[24]. Hence MMT can be interpreted as an alternative normative method to describe MHP_{nn-2} (mental models corresponding to possible worlds).

For instance, the description of standard MHP in terms of mental models involves 6 models: 2 models when the car is behind the chosen door (door No.1) because the host then either exhibits a goat behind door No.2 (first model) or behind door No.3 (second model).

Hence we have the following models $\begin{Bmatrix} _{car}No.1 & '_{goat}No.2' \\ _{car}No.1 & '_{goat}No.3' \end{Bmatrix}$. The *equiprobability* principle, assumes that 2 identical models exist in the case where the car is behind door No.i (i > 1) and the host shows that door No.j (with j > 1 and j ≠ i) hides a goat. Hence we have the additional

4 models: $\begin{Bmatrix} _{car}No.2 & '_{goat}No.3' \\ _{car}No.2 & '_{goat}No.3' \\ _{car}No.3 & '_{goat}No.2' \\ _{car}No.3 & '_{goat}No.2' \end{Bmatrix}$.

Now when the host shows that door No.3 hides a goat, the "subset principle" retains the 3 models that contain '$_{goat}$No.3': $\begin{Bmatrix} _{car}No.1 & '_{goat}No.3' \\ _{car}No.2 & '_{goat}No.3' \\ _{car}No.2 & '_{goat}No.3' \end{Bmatrix}$ and by the "*proportionality* principle" we arrive at "EBS".

However, Johnson-Laird et al. (1999), outline that when the car is behind door No.1, for working memory limitation, participants simply think: "the host chooses another door". Similarly, participants fail to construct the identical models in each case when the car is not

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[24] These principles actually rediscover Laplace's (1986, p. 38-39) traditional first two principles: (i) "The probability is the ratio of the number of favorable cases to the number of all possible cases"; (ii) but this holds under the assumption that the various cases are equally possible; if they are not…. the probability will be the sum of the possibilities of each possible case" (Author's translation).

behind door No.1. They consequently built only the 3 following mental models:

$$\begin{cases} _{car}No.1 & other\ door \\ _{car}No.2 & other\ door \\ _{car}No.3 & other\ door \end{cases}$$. When the host shows that door No.3 hides a goat, the use of the

"subset principle" implies that participants consider only the 2 mental models compatible

with this message: $\begin{cases} _{car}No.1 & other\ door \\ _{car}No.2 & other\ door \end{cases}$ and by the proportionality principle it gives MR.

It is obvious that manipulating mental models requires significant a memorial

cognitive effort ($(n-1)\sum_{i=1}^{i=n} a_i$ models for MHP_{nn-2}). TMM's explanation would hold if

participants effectively use mental models to form their probability judgments. However this

hypothesis is not widely spread (see for example Brase 2002). In MHP_{nn-2}, participants seem

to accept Property 1 of experimenters' representation (see section 3.1) thus confirming a full

representation of the puzzle. Participants can also use a representation different from mental

models to understand MHP_{nn-2}. For instance, the formal bi-probabilistic structure described in

section 2.2 (see Figure 1) requires less effort of manipulation in memory than a description in

terms of mental models. Nevertheless, if one accepts the principles of TMM, an alternative

explanation for MR can be again formulated in terms of the updating interpretation of revision

situation. If participants interpret the message '$_{goat}$No.3...n' in an updating framework, the

"subset principle" (iv) no longer holds. The problem has been modified and participants must

again (as if answering to a new problem) construct a new set of mental models. In standard

MHP, the "truth principle" induces a new set of 2 mental models: $\begin{cases} _{car}No.1 & _{goat}No.2 \\ _{car}No.2 & _{goat}No.1 \end{cases}$.

Participants by the "equiprobability principle" find MR. This process corresponds to the

proportional Imaging rule in a MHP_{nn-2} formulated in a populational format.

4.1.3   The proportional Imaging rule explains the partition-edit-count strategy

A third explanation for MR was proposed by Fox and Levay (2004) (see also Fox & Rottenstreich 2003; Fox & Clemen 2005).

Participants evaluate conditional probabilities in (i) **subjectively partitioning** the sample space into a set of n interchangeable events suggested by the statement of the problem, (ii) **editing** out events that can be eliminated by the new message and (iii) **counting** the number of remaining events and reporting the ratio.

However these authors think that partitioning correctly the sample space in standard MHP proves difficult because participants only focus on the sample space most obviously suggested by the statement. In this view, participants in MHP_{nn-2} do not envisage the correct experimenters' set (i.e. 2(n-1) possible worlds {$_{car}$No.1∧'$_{goat}$No.2.4…n', $_{car}$No.1∧'$_{goat}$No.2.3.4.6…n', ..., $_{car}$No.1∧'$_{goat}$No.2…n-1', $_{car}$No.1∧'$_{goat}$No.3…n', $_{car}$No.2∧'$_{goat}$No.3…n', $_{car}$No.3∧'$_{goat}$No.2.4…n', ..., $_{car}$No.n∧'$_{goat}$No.2…n-1'} of Figure 1b) but the limited sample space of n possible worlds {$_{car}$No.1, ..., $_{car}$No.n}. Hence, applying (ii) and (iii) on this naïve space gives MR. As we have seen in relations (9) and (10) this cognitive explanation supports further our hypothesis of an updating interpretation of the message of revision. Indeed the updating interpretation of message '$_{goat}$No.3...n' forces participants to consider only the simpler probabilistic level corresponding to this naive set (see section 3.2.1).

4.2 "EBS" is facilitated when the format of MHP_{nn-2} enhances the focusing interpretation

In the experimental literature some studies show that specific formats of MHP_{nn-2}' statement seem to prompt a higher rate of "EBS" in participants' response. These results have triggered some psychological debates on the underlying cognitive processes at play (Barbey & Sloman Forthcoming). However our binary view of MHP_{nn-2}, in terms of experimenter's "correct" focusing and participants' "misleading" updating interpretation of the situation of

revision, invites to a new reading of these experimental results. It leads us to conclude that the different formats that facilitate "EBS" actually enhance the focusing understanding of the situation of revision.

4.2.1 Natural frequencies format

Some authors argue that the human mind would have developed in its evolution a module designed to process natural frequencies acquired by natural sampling (Cosmides & Tooby 1996; Gigerenzer & Selten 2001). Various studies prove that probability judgments appear more Bayesian when experimental paradigms refer to natural frequencies (as opposed to single-event probabilities) (Gigerenzer & Hoffrage 1995; Cosmides & Tooby 1996; Brase *et al.* 1998; Hoffrage *et al.* 2002). This result is confirmed in problems written in a natural frequencies format isomorphic (from a mathematical point of view) to standard MHP (Tubau & Alonso 2003; Yamagishi 2003). For instance the following problem used in Yamagishi (2003)

The Gemstone problem (natural frequencies format): A factory manufactures 1200 artificial gemstones daily. Among the 1200, 400 gemstones are blurred, 400 are cracked, and 400 contain neither. An inspection machine removes all cracked gemstones, and retains all clear gemstones. However, the machine removes half of the blurred gemstones. How many gemstones pass the inspection, and how many among them are blurred?

In this natural frequencies format, the bi-probabilistic structure illustrated by Figure 1 in section 2.2 is made explicit and unambiguous. To draw the parallel with standard MHP, we respectively have blurred, cracked, and neither that correspond to $_{car}No.1$, $_{car}No.3$ and $_{car}No.2$ while '$_{goat}No.3$' corresponds to the inspection process. However this format is different from a pragmatic point of view. The three pragmatic arguments inherent to MHP_{nn-2}'s three stages (see section 3.2) to explain the updating interpretation, no longer exist. Participants know that

the situation of revision is static and that they must stick to their original distribution P (explicit in the statement). The message not cracked equivalent to '$_{goat}$No.3' does not require cognitive effort. The process of inspection is explicitly presented as non-symmetrical with respect to blurred and clear gemstones. Moreover the question worded in frequentist terms modifies the perception of the situation of revision and sets up the situation of focusing (Dubois & Prade 1992; Dubois *et al*. 1996). Participants must simply focus on the reference class of gemstone on a subset to modify P(gemstone) into P(gemstone│inspection) by Bayesian conditioning[25] . The natural frequencies format improves Participants' "EBS" rate as it encourages the comprehension of the focusing situation. In MHP$_{nn-2}$, if participants envisage the bi-probabilistic structure, the focusing situation is clearly identifiable in this context where participants are first asked the frequency question "Has the car appeared behind door No.1 in numerous repetitions?" and then P($_{car}$No.1│'$_{goat}$No.3...n') to the second frequency question: "Has the car appeared behind door No.1 in numerous repetitions when the host who knows where the car is, had opened doors No.3; …; No.n to reveal goats? Similarly some other manipulations may enhance (i) the explicit bi-probabilistic structure and (ii) help participants to interpret the situation of revision in a focusing framework through a specific wording of the question. The two theories that have proposed arguments alternative to frequentist module arguments (the "question form" and the "nested sets") can be analyzed in this way.

4.2.2   The question form

Girotto & Gonzalez (2001) argue that participants are able to make correct probability judgment for single-event probabilities when the distributions of probabilities are formulated in term of chance. Their studies were not concerned with MHP$_{nn-2}$ so we will stick to the

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[25] On can noted that Dubois & Prade (1992) use a frequentist framework to explain the focusing situation

previous gemstone problem and propose the following statement (see however the studies cited in Girotto & Gonzalez 2000):

The Gemstone problem (question form): A factory manufactures artificial gemstones daily. A gemstone that is produced has 2 chances out of 6 to be blurred, 2 chances out of 6 to be cracked, and 2 chances out of 6 to be neither. It is then tested by the inspection machine. A blurred gemstone has 1 chance out of 2 to be removed by the inspection machine, a cracked gemstone has 2 chances out of 2 to be removed by the inspection machine and a clear gemstone has no chance (out of 2) to be removed by the inspection machine. Imagine that a gemstone is tested. Out of the total of 6 chances, this gemstone has __ chances of passing the inspection. What are besides the chances that it has passed the inspection and that it is blurred?

With this formulation the bi-probabilistic hierarchical structure is again explicit (three layers in the same way of natural frequencies format and the two distributions of probability reflected by the given respective proportions). It is also clear that here the "question format" conducts participants to directly use Laplace's principle (ratio of proportions) which exactly corresponds to the focusing process.

4.2.3 Nested set format

Sloman, Over, Slovak and Stibel (2003) propose a general explanation for natural frequencies effect: framing problems in terms of instances rather than properties actually clarifies critical set relations. In this line Yamagishi (2003) claims that participants in MHP tend to answer MR because they cannot comprehend "nested sets" namely, subsets relative to larger sets, in the task structure. A solution is to propose a schematic representation of standard MHP (equivalent to Figure 1) (Ichikawa 1989; Cheng & Pitt 2003; Yamagishi 2003). For instance the use of a roulette-wheel diagram (see Figure 3) favors "EBS" in MHP

(Ichikawa 1989; Yamagishi 2003). This diagram is a schematic representation of the bi-probabilistic level of belief described in Figure 1a. The outside circle corresponds to the set of worlds of layer 0, the inner circle corresponds to the set of worlds of layer 1 and the central point represents the single set of layer 2. The different areas represent the two distributions of probabilities related to layers 1 to 0 and to layers 2 to 1. Now if a small ball runs along the edge of the inner disk and stops at random like a roulette-wheel when participants receive the message '$_{goat}$No.3', it is certain that the ball stops in the dotted area, that is, Area $_{car}$No.2 or 1/2 of Area $_{car}$No.1. This schematic representation, leads participants to correctly interpret '$_{goat}$No.3' as a focusing message.

Insert Figure 3 about here

4.2.4   Populational format a natural translation of possible worlds semantics

More generally, a populational format of presentation helps participants to induce experimenters' representation of MHP$_{nn-2}$. This format relies on a finite number of discrete objects (balls) and discrete meta objects (urns) representing worlds and meta-worlds. These objects (and meta objects) are liable to receive individualizing properties (and meta properties) on which participants will form beliefs. The traditional hierarchy of balls, urns and meta-urn (corresponding respectively to worlds of layer 0, layer 1 and layer 2) is the most convenient introductive set. The focusing situation being here illustrated by the message of the type " a ball is extracted of an urn which had been previously extracted, this ball has a specific characteristic to be named". The most elementary translation in a populational format of MHP$_{nn-2}$ is the urns and balls problem described in section 2.3. Such a representation indeed facilitates the distinction between the focusing message "the ball is not labeled 3, ... to

n" and the updating message "the balls labeled 3, ... to n have been removed" (Baratgin & Politzer 2007b). The three former formats that improve the interpretation of the focusing situation can be grouped under this populational representation. For instance Gigerenzer's natural frequency format actually corresponds to a general translation (among various others) of this populational representation.

5   Conclusion

MHP_{nn-2} appears in the literature as the typical paradigm that concludes that human probability judgment fails to implement basic revision rule such as Bayesianism conditioning. However a careful analysis of the experimental statement in the light of the Relevance Theory and a thorough review of the broad Psychological literature on the subject leads us to reconsider such conclusion. The heuristics put forward to explain participants' "non-Bayesian" response as well as the experimental clues and formats that prompt a higher rate of Bayesian response can be seen as evidence that participants actually interpret the puzzle in an updating framework in which the universe undergoes a significant change. Such updating situation as opposed to the focusing interpretation expected by experimenters requires a different revision rule which explains the departure from the Bayesian Solution.

MHP_{nn-2} actually illustrates experimenters' difficult choice of a methodology in the studies on probability revision. Before conducting any experimental study, experimenters should define the consistency principles they retain and outline the implications of the reference model they use for their empirical work. If an experimenter is to consider the Bayesian model as the reference to study revision of probabilistic judgment, he or she should first of all conduct a pragmatic analysis of the representation of the task by participants. More generally, psychologists should check that participants' representation of the task coincides with the type of revision situation they intend (Baratgin & Politzer 2006).

In this line MHP$_{nn-2}$ is the typical example where such a protocol is not respected. First in the numerous experiments on the puzzle experimenters limit Bayesianism to the simple use of Bayes' rule. Participants are judged inconsistent because their responses depart from the result Bayes' rule yields on the data given in the statement. Second, experimenters lose sight aware that Bayes' rule can be a normative rule in the sole respect of a focusing situation of revision. Consequently experimenters overlook the fact that participants may not interpret the puzzle in a focusing framework.

Finally from a psychological point of view, experimenters would advantageously consider the fairly general Imaging' rules as a valuable reference to study human probability judgment. It may actually reveal to have a sizeable descriptive power and robustness as participants seem to naturally opt for an updating framework. The updating situation of revision was hardly addressed in the literature. Nevertheless it has been indirectly studied in three particular research fields. It has been considered explicitly in a *deductive* framework, in connection with the problem of belief revision for knowledge bases (Elio & Pelletier 1997; Politzer & Carles 2001), and implicitly in the field of counterfactual reasoning: a counterfactual statement indeed expresses a modification of the universe, albeit virtual. Finally, the situation called "intervention" in the field of causal reasoning can be considered as a special case of updating (see Appendix 3). Some experimental studies on causal reasoning have shown that this situation of intervention seems natural to participants as they are competent to predict the consequences of an intervention (Sloman & Lagnado 2005; Waldmann & Hagmayer 2005). This new approach to causal reasoning offers additional reasons to take a serious view at the updating situation in probability judgment revision.

Appendix

1 "EBS$_k$" is not unique in general MHP$_{nk}$

Let MHP$_{nk}$ be the version of the Monty-Hall puzzle with n doors No.i (i = 1, ..., n) and k the number of doors opened by the host to show goats when participants' original choice is No.1 (k ≤ n-2). For parsimony and convenience sake we will order the doors following the choice of the host and we will assume that the host opens the k last doors (message '$_{goat}$No.n-k+1...n' with k ∈ [1, ..., n-2]). Rules 1, 2, 3, are straightforwardly transposed into MHP$_{nk}$ while rules 3.2 and 3.2.1 are easily generalized. On the contrary rule 3.1 can have various generalizations. With respect to rules 3.1, 3.2 and 3.2.1, we assume that the probabilities P('$_{goat}$No.n-k+1...n' $|$ $_{car}$No.i) (for i = 1, ..., i = n) are independent from prior probabilities but may depend on the remaining number of closed doors (n-k). The likelihoods P('$_{goat}$No.n-k+1...n' $|$ $_{car}$No.i) (i = 1, ..., i = n) are thus function of n and k. Let α_i (i = 1, ..., i = n) be the likelihoods P('$_{goat}$No.n-k+1...n' $|$ $_{car}$No.i). Let's next consider the generalized rules:

Rule 3.2g: If the car is behind door No.1, the host is bound to show goats behind k doors chosen among doors No.2..., No.n.

Also the rule 3.2.1 (no preference between the n-1 doors if the car is behind door No.1) becomes:

Rule 3.2.1$_g$: In the specific case where the car is behind door No.1, the host chooses to show k goats behind doors selected among n-1 doors. The host has thus C_{n-1}^k possibilities and

$$\alpha_1 = \frac{1}{C_{n-1}^k}.$$

The general Bayesian solution to MHP$_{nk}$,"EBS$_k$" becomes:

$$P(_{car}No.i \,|'_{goat}No.n\text{-}k+1...n') = \frac{a_i\alpha_i}{a_1\alpha_1 + \sum\limits_{j=2}^{j=n\text{-}k}\alpha_j a_j}, \text{ with } 0 < i < n\text{-}k+1 \qquad (26)$$

Before going further an assumption needs to be made on the host's strategies when the car is behind door No.i (with i>1). We propose first the natural host's strategy 1.

*1.1   **Strategy 1**: For any i the host adopts the same rule whatever door No.i that hides the*

    car

In this case the host has no preference between the n-2 doors No.2, ...No.i-1; No.i+1, ...No.n when the car is behind door No.i. Hence the implicit rule 3.1 is no longer ambiguous and can be generalized by the following rule.

Rule 3.1$_{g1}$: If the car is behind door No.i (with i > 1), the host chooses to show goats behind k doors selected among n-2 doors (n doors except doors No.1 and No.i). The host has thus C_{n-2}^k possibilities and $\alpha_i = \dfrac{1}{C_{n-2}^k}$.

In this case (26) becomes "EBS$_{k1}$":

$$P(_{car}No.1\,|'_{goat}No.n\text{-}k+1...n') = \frac{a_1\dfrac{(n-1-k)!k!}{(n-1)!}}{a_1\dfrac{(n-1-k)!k!}{(n-1)!} + \dfrac{(n-2-k)!k!}{(n-2)!}\sum\limits_{j=2}^{j=n\text{-}k}a_j}$$

$$= \frac{a_1(n-1-k)}{a_1(n-1-k) + (n-1)\sum\limits_{j=2}^{j=n\text{-}k}a_j} \qquad (27)$$

$$P(_{car}No.i\,|'_{goat}No.n\text{-}k+1...n') = \frac{a_i(n-1)}{a_1(n-1-k)+(n-1)\sum\limits_{j=2}^{j=n\text{-}k}a_j}, \text{ for } i = 2, ..., i = n \qquad (28)$$

and hence the relation: $\dfrac{P(_{car}No.1\,|'_{goat}No.n\text{-}k+1...n')}{P(_{car}No.i\,|'_{goat}No.n\text{-}k+1...n')} = \dfrac{a_1(n-1-k)}{a_i(n-1)}$ \qquad (29)

Theorem 8: When the mean of the set of values a_i (with $i = 2, ... i = n-k$) is strictly superior to the mean of the set of values a_i (with $i = n-k+1, ..., i = n$),

$P(_{car}No.1 |'_{goat}No.n - k +1...n') < P(_{car}No.1)$ and in the specific case where the mean of the set of values a_i (with $i = 3, ... i = n-k$) is equal to the mean of the set of values a_i (with $i = n-k+1, ... i = n$) $P(_{car}No.1 |'_{goat}No.n - k +1...n') = P(_{car}No.1)$.

Proof. By simple elementary calculation

$$P(_{car}No.1 |'_{goat}No.n - k +1...n') < P(_{car}No.1) \Leftrightarrow \frac{a_1(n-1-k)}{a_1(n-1-k) + (n-1)\sum_{j=2}^{j=n-k} a_j} < \frac{a_1}{a_1 + \sum_{j=2}^{j=n} a_j}$$

$$\Leftrightarrow \left(\frac{(n-1)}{(n-1-k)} - 1\right)\sum_{i=2}^{i=n-k} a_i > \sum_{i=n-k+1}^{i=n} a_i \Leftrightarrow \frac{\sum_{i=2}^{i=n-k} a_i}{(n-1-k)} > \frac{\sum_{i=n-k+1}^{i=n} a_i}{k}$$

In particular when the a_i (with $i > 1$) are equal to a_2 for any k, "EBS_{k2}" coincides with "EBS".

$$P(_{car}No.1 |'_{goat}No.n - k +1...n') = P(_{car}No.1 |'_{goat}No.3...n') = P(_{car}No.1) \quad (30)$$

Proof. By simply applying relations (27) and (28), we have:

$$P(_{car}No.1 |'_{goat}No.n - k +1...n') = \frac{a_1}{a_1 + (n-1)a_2} = P(_{car}No.1) \quad (31)$$

$$P(_{car}No.i |'_{goat}No.n - k +1...n') = \frac{a_2(n-1)}{a_1(n-1-k) + a_2(n-1-k)(n-1)}$$

$$= \frac{a_2(n-1)}{(n-1-k)(a_1 + a_2(n-1))} = \frac{a_2}{a_1 + a_2(n-1)} + \frac{ka_2}{(n-1-k)(a_1 + a_2(n-1))}$$

$$= P(_{car}No.i) + \frac{1}{(n-k-1)} \sum_{j=n-k+1}^{n} P(_{car}No.j) \quad (32)$$

The relations (31) and (32) correspond to the solution by simple Imaging rule "SIS_k" for MHP_{nk} for the k worlds invalidated by the updating message '$_{goat}No.n-k+1...n$' are redistributed proportionally on the (n-k-1) worlds $(_{car}No.i)$ (with $i = 2, ..., i = n-k$).

Notably, in standard MHP_{nk} for any k,

$$P(_{car}\text{No.1}\,|'_{goat}\text{No.n - k +1...n'}) = P(_{car}\text{No1}) = P''^{+}_{'goat\,No.n-k+1,...,n'}(_{car}\text{No.1}) \qquad (33)$$

$$\text{for } i > 1, \; P(_{car}\text{No.i}\,|'_{goat}\text{No.n - k +1...n'}) = \frac{(n-1)}{(n-1-k)}\frac{1}{n} = \frac{1}{n} + \frac{k}{(n-1-k)n}$$

$$= P(_{car}\text{No.i}) + \frac{1}{(n-1-k)}\sum_{j=n-k+1}^{j=n}P(_{car}\text{No.j}) \; = \; P''^{+}_{'goat\,No.n-k+1,...,n'}(_{car}\text{No.i}) \qquad (34)$$

We can again generalize Theorem 4 into

Theorem 9. For any $n > 2$ and any k, $P_{''goat\,No.n-k+1...n''}(_{car}\text{No.1}) > P(_{car}\text{No.1}\,|'_{goat}\text{No.n - k +1...n'})$

and $P_{'goat\,No.n-k+1...n'}(_{car}\text{No.i}) < P(_{car}\text{No.i}\,|'_{goat}\text{No.n - k +1...n'})$, for $i = 2, \,...,i = n - k$

Proof. "PIS_{k2}" corresponds to $P_{'goat\,No.n-k+1....n'}(_{car}\text{No.i}) = \dfrac{a_i}{\sum\limits_{j=1}^{n-k}a_j}$, for $i = 1, \,...,i = n$ \qquad (35)

It is obvious because for $n > 2$;

$$\frac{a_1}{\sum\limits_{j=1}^{n-k}a_j} > \frac{a_1}{a_1 + \frac{(n-1)}{(n-1-k)}\sum\limits_{j=2}^{n-k}a_j} \; \text{ and } \; \frac{a_i(n-1)}{a_1(n-1-k)+(n-1)\sum\limits_{j=2}^{n-k}a_j} > \frac{a_j}{\sum\limits_{j=1}^{n-k}a_j} \text{ for } i = 1, \,..., \, i = n$$

1.2 **Strategy 2**: *For (any $i > 1$) the host adopts the same strategy whatever door No.i that*

hides the car

This rule is implicit in standard MHP where the host's strategy is the same for No.2 and

No.3. There are multiple possibilities to generalize the rule of MHP: The host opens door

No.2 if No.3 hides a car and No.3 if No.2 hides a car. A possibility is illustrated below:

Rule 3.1$_{g2}$: If the car is behind door No.i (with $1 < i < n-k$), the host chooses to open door

No.i+1...i+1+k. If the car is behind door No.i (with $n-k \le i \le n$), the host chooses to open

doors No.i+1...n,2,...2+n-k-i. Hence when the host opens the k doors No.n-k+1, ..., No.n, α_{n-k} = 1 and $\alpha_i = 0$ otherwise.

In this case (26) becomes "EBS$_{k2}$":

$$P(_{car}No.1 \mid'_{goat} No.n \text{ - } k +1...n') = \frac{\dfrac{a_1}{C_{n\text{-}1}^{k}}}{\dfrac{a_1}{C_{n\text{-}1}^{k}} + a_{n\text{-}k}} = \frac{a_1}{a_1 + a_{n-k}C_{n\text{-}1}^{k}} \tag{36}$$

Similarly $P(_{car}No.n \text{ - } k \mid'_{goat} No.n \text{ - } k +1...n') = \dfrac{a_{n-k}C_{n\text{-}1}^{k}}{a_1 + a_{n-k}C_{n\text{-}1}^{k}}$ (37)

and hence the relation: $\dfrac{P(_{car}No.1 \mid'_{goat} No.n \text{ - } k +1...n')}{P(_{car}No.n \text{ - } k \mid'_{goat} No.n \text{ - } k +1...n')} = \dfrac{a_1}{a_{n-k}C_{n\text{-}1}^{k}}$ (38)

Theorem 10: (i) For $a_{n\text{-}k}$ strictly superior to $\dfrac{\displaystyle\sum_{i=2,\ i\neq n-k}^{i=n} a_i}{(C_{n\text{-}1}^{k} - 1)}$,

$$P(_{car}No.1 \mid'_{goat} No.n \text{ - } k +1...n') < P(_{car}No.1) = \frac{a_1}{A}$$

and (ii) in the specific case where $a_{n\text{-}k}$ corresponds to the mean of set of values a_i (with i > 2, and i ≠ n-k) and k = 1 or k = n-2,

$$P(_{car}No.1 \mid'_{goat} No.n') = P(_{car}No.1 \mid'_{goat} No.3...n') = P(_{car}No.1) = \frac{a_1}{A} \text{ since}$$

$C_{n\text{-}1}^{k} = 2$ when $k = n - 2$ and $k = 1$.

Theorem 11. (i) $P(_{car}No.1 \mid'_{goat} No.n \text{ - } k +1...n') < P(_{car}No.1 \mid'_{goat} No.n \text{ - } k +1') = \dfrac{a_1}{a_1 + (n-1)a_{n\text{-}k}}$

and (ii) In the special case where all a_i are equals, for i > 1, we get from the well know binomial coefficients property $C_{n\text{-}1}^{k} = C_{n\text{-}1}^{n\text{-}1\text{-}k}$ the following result:

For i = 1, ..., i = n: $P(_{car}No.i \mid'_{goat} No.n \text{ - } k +1...n') = P(_{car}No.i \mid'_{goat} No.k +1...No.n')$

We have for $1 < k < n-2$ the following inequalities:

$$P(_{car}No.1 \mid '_{goat}No.n-k+1...n') > P(_{car}No.1 \mid '_{goat}No.3...n')$$

$$P(_{car}No.n-k \mid '_{goat}No.n-k+1...n') < P(_{car}No.n-k \mid '_{goat}No.3...n')$$

Proof. $\dfrac{a_1}{a_1 + a_2 C_{n-1}^k} > \dfrac{a_1}{a_1 + a_2(n-1)}$ because $C_{n-1}^k \leq \dfrac{(n-1)^k}{k!} \leq \dfrac{(n-1)^{n-2}}{(n-2)!} < n-1$:

In standard MHP_{nk}, we have: $P(_{car}No.n-k \mid '_{goat}No.n-k+1...n') = \dfrac{1}{1 + C_{n-1}^k}$ (39)

$$P(_{car}No.n-k \mid '_{goat}No.n-k+1...n') = \dfrac{C_{n-1}^k}{1 + C_{n-1}^k} \tag{40}$$

We can also generalize Theorem 4:

Theorem 12. For any $n > 2$ and any k, $P^+_{'goat No.n-k+1,...,n'}(_{car}No.1) > P(_{car}No.1 \mid '_{goat}No.n-k+1...n')$

and $P^+_{'goat No.n-k+1,...,n'}(_{car}No.n-k) < P(_{car}No.n-k \mid '_{goat}No.n-k+1...n')$.

Proof. "IPS_{k2}" reads: $P^+_{'goat No.n-k+1,...,n'}(_{car}No.i) = \dfrac{a_i}{a_i + a_{n-k}}$ and it is obvious since for $n > 2$;

$$\dfrac{a_1}{a_1 + a_{n-k}} > \dfrac{a_1}{a_1 + C_{n-1}^k a_{n-k}} \quad \text{and} \quad \dfrac{C_{n-1}^k a_{n-k}}{a_1 + C_{n-1}^k a_{n-k}} > \dfrac{a_{n-k}}{a_1 + a_{n-k}} \quad \text{as for } k < n-1, \ C_{n-1}^k \geq \left(\dfrac{n-1}{k}\right)^k > 1.$$

2   Participants' representation of a distribution of events for MHP_k

The possibility though not tested by psychologists, that participants do not have a precise distribution of probability from layer 1 to layer 0 supports further the hypothesis that participants interpret the message in an updating manner. In this case, the hierarchical structure is formed by a probability distributions from layer 2 to layer 1 always represented by the prior probabilities $P(_{car}No.i)$ and by a distribution of events from layer 1 to layer 0. This structure is called a "distribution of events" (Walliser & Zwirn 1997, forthcoming). In this

representation the value of event $_{car}$No.1 is represented by an interval $[\text{Bel}(_{car}\text{No.1}),$

$\text{Pl}(_{car}\text{No.1})]$ with Bel $(_{car}\text{No.1})$ the lower value associated to $_{car}$No.1 (called "belief function")

and $\text{Pl}(_{car}\text{No.1})$ is the upper value associate to $_{car}$No.1 (called "plausibility") with relation

$\text{Pl}(_{car}\text{No.1}) = 1 - \text{Bel}(^-_{car}\text{No.1})$. The two measures $\text{Bel}(_{car}\text{No.1})$ and $\text{Pl}(_{car}\text{No.1})$ are defined

with the mass function m(B) that are the weight of evidence in favor of $_{car}$No.1.

$$\text{Bel}(_{car}\text{No.1}) = \sum_{B:\, B \subset A \neq \varnothing} m(B) = P(_{car}\text{No.1}) \text{ and}$$

$$\text{Pl}(_{car}\text{No.1}) = 1 - \text{Bel}(^-_{car}\text{No.1}) = \sum_{B: A \wedge B \neq \varnothing} m(B) = P(_{car}\text{No.1}).$$

For such distribution of events the adequate rule of revision after an updating message

'$_{goat}$No.n-k+1...n' is the Dempster-Shafer' rule (Dubois & Prade 1993; Walliser & Zwirn

1997, forthcoming) which is expressed by $\text{Bel}_{goat\,No.n-k+1...n'}(_{car}\text{No.i}) = 1 - \text{Pl}_{goat\,No.n-k+1...n'}(^-_{car}\text{No.i})$

and $\text{Pl}_{goat\,No.n-k+1...n'}(_{car}\text{No.i}) = \dfrac{\text{Pl}(_{car}\text{No.i} \wedge '_{goat}\text{No.n - k +1...n'})}{\text{Pl}('_{goat}\text{No.n - k +1...n'})}$.

Numerous authors have shown that the solution of standard MHP by Dempster-

Shafer's rule gives again 1/2 for the two measures $\text{Bel}_{goat\,No.3'}(_{car}\text{No.1})$ and $\text{Pl}_{goat\,No.3'}(_{car}\text{No.1})$

(Diagonis & Zabell 1986; Pearl 1988; Fagin & Halpern 1991; Walley 1991; Dubois & Prade

1992). More generally it is immediate to see that the solution of MHP_{nk} by Dempster-Shafer's

rule (noted "DS_k") is equivalent to "PIS". Thus we have for the two possibilities of the host's

strategies treated in Appendix 1 the following "DS_{k1}" and "DS_{k2}":

$$\text{Bel}_{goat\,No.n-k+1...n'}(_{car}\text{No.i}) = \text{Pl}_{goat\,No.n-k+1...n'}(_{car}\text{No.i}) = \frac{a_i}{\sum_{i=1}^{n-k} a_i} \text{ , for i} = 1, ..., \text{i} = \text{n} \qquad \text{(Strategy 1) (41)}$$

$$\text{Bel}_{goat\,No.n-k+1...n'}(_{car}\text{No.i}) = \text{Pl}_{goat\,No.n-k+1...n'}(_{car}\text{No.i}) = \frac{a_i}{a_i + a_{n-k}} \text{ , for i} = 1, ..., \text{i} = \text{n} \quad \text{(Strategy 2) (42)}$$

As far as focusing situation the solution is not unique. The adequate rule is then the Fagin Halpern Jaffray's rule (Dubois & Prade 1993; Walliser & Zwirn 1997, forthcoming) which defines that:

$$\text{Bel}(_{car}\text{No.i}|'_{goat}\text{No.n-k}+1...n') = \frac{\text{Bel}(_{car}\text{No.i} \wedge'_{goat}\text{No.n-k}+1...n')}{\text{Bel}(_{car}\text{No.i} \wedge'_{goat}\text{No.n-k}+1...n') + \text{Pl}(^{\neg}{}_{car}\text{No.i} \wedge'_{goat}\text{No.n-k}+1...n')}$$

$$\text{Pl}(_{car}\text{No.i}|'_{goat}\text{No.n-k}+1...n') = \frac{\text{Pl}(_{car}\text{No.i} \wedge'_{goat}\text{No.n-k}+1...n')}{\text{Pl}(_{car}\text{No.i} \wedge'_{goat}\text{No.n-k}+1...n') + \text{Bel}(^{\neg}{}_{car}\text{No.i} \wedge'_{goat}\text{No.n-k}+1...n')}$$

In standard MHP, these rules give a solution in the format of the interval $[0, 1/2]$ (Diagonis & Zabell 1986; Pearl 1988; Fagin & Halpern 1991; Walley 1991; Dubois & Prade 1992) and $\left[0, \text{Pl}_{'goat\,No.n-k+1...n'}(_{car}\text{No.i})\right]$ in MHP_{nk}. Indeed following the host's strategies, we have respectively:

$$\text{Pl}_{'goat\,No.n-k+1...n'}(\text{No.i}) = \sum_{i=1}^{i=n-k} \frac{a_i}{A}$$

$$= \text{Pl}(_{car}\text{No.i} \wedge'_{goat}\text{No.n-k}+1...n') + \text{Bel}(_{car}\text{No.i} \wedge'_{goat}\text{No.n-k}+1...n') \qquad \text{(Strategy 1) (43)}$$

$$\text{Pl}_{'goat\,No.n-k+1...n'}(\text{No.i}) = \frac{a_1 + a_{n-k}}{A}$$

$$= \text{Pl}(_{car}\text{No.i} \wedge'_{goat}\text{No.n-k}+1...n') + \text{Bel}(_{car}\text{No.i} \wedge'_{goat}\text{No.n-k}+1...n') \qquad \text{(Strategy 2) (44)}$$

3   A correspondence with Pearl's Bayesian conditioning after intervention

The proportional Imaging rule is equivalent to Pearl (2000)' s Bayesian conditioning after one *intervention* (designed by Pearl as the "do" operator) introduced in the causal reasoning. MHP_{nn-2} can indeed be modeled by a Bayesian network representing dependencies among: the host's initial choice (X1), participants' choice (X2) and the doors open by host

hiding goats (X3) (see Figure 4a below). All variables X1, X2 and X3 can take n values: door No.1, ..., door No.n.

Now if the external intervention "the n-2 doors No.3, ..., No.n are open" is released $(do(X3) = No.3...n)$, the links $X1 \rightarrow X3$ and $X1 \rightarrow X3$ are deleted and the new mechanism (when X2 is equal to No.1) corresponds to conditioning on the action $do(X3) = No.3...n$ (see Figure 4b). We find the relations (9) and (10) induced by the proportional Imaging rule:

$$P(X1 = No.1 \,|\, (X2 = No.1) \wedge do(X3) = No.3...n) = P(_{car} No.1 |_{goat} No.3...n)$$

$$= P^+_{'goat\,No.3,...,n'} (_{car} No.1) \tag{43}$$

$$P(X1 = No.2 \,|\, (X1 = No.1) \wedge do(X3) = No.3...n) = P(_{car} No.2 |_{goat} No.3...n)$$

$$= P^+_{'goat\,No.3,...,n'} (_{car} No.2) \tag{44}$$

Insert Figure 4 about here

This correspondence supports a new psychological argument in favor of the hypothesis of updating interpretation by participants. It explains the hypothesis introduced by Glymour (2001) and studied by Burns and Wieth (2004) that participants are not able to recognize *the collider principle*. The "collider principle" is depicted in Figure 4a. The two variables X1 (experimenter's original choice) and X2 (participant's original choice) are independent whereas the value of variable X3 (doors hiding goats opened by the host) renders X1 and X2 dependent. Now, in the updating representation of MHP_{nn-2} (Figure 4b), the collider principle is ruled out. Hence participants if have an updating interpretation cannot be detected the collider principle.

Acknowledgements

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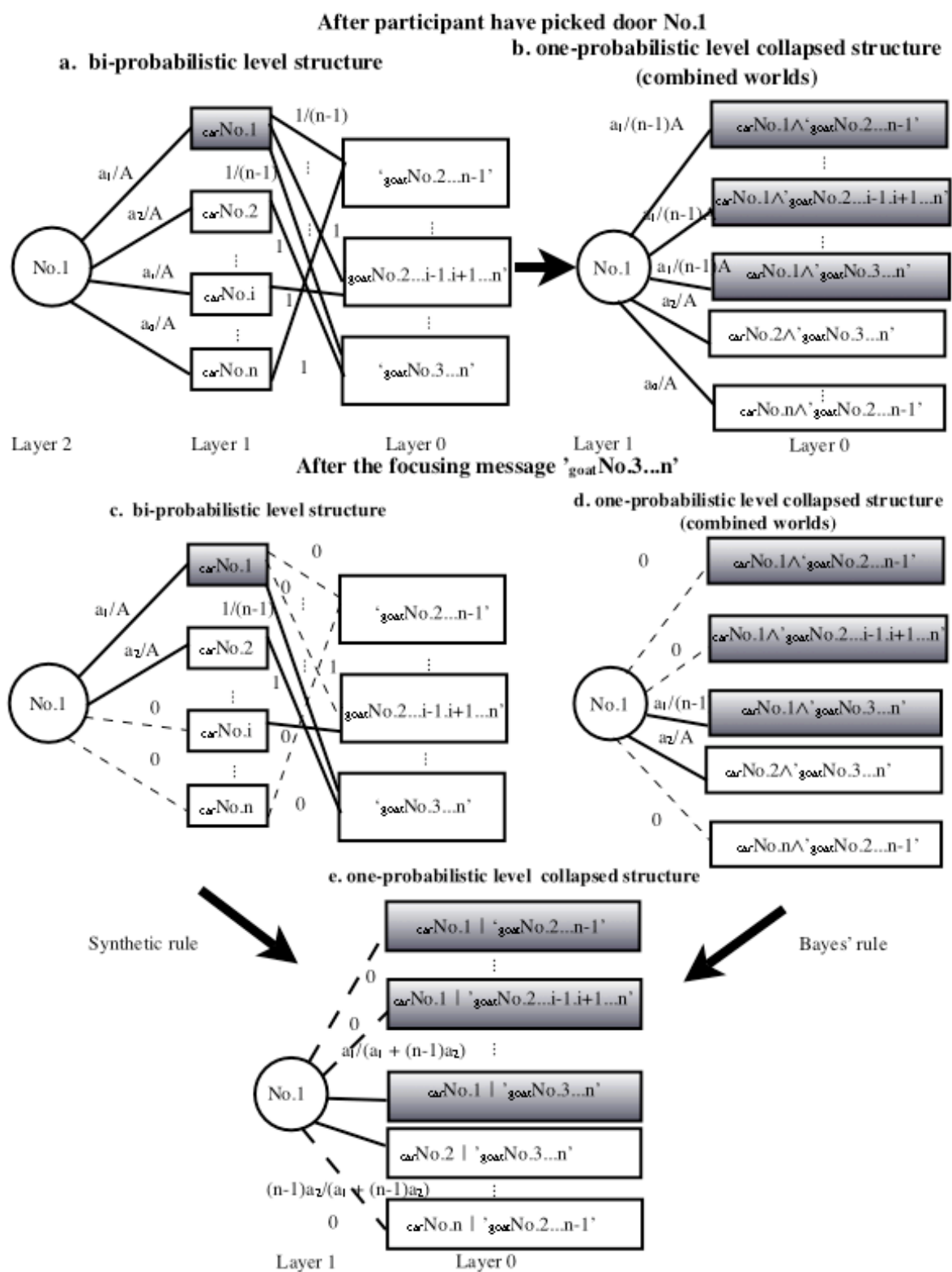**Figure 1. Experimenters' belief structures of MHP$_{nn-2}$**

After participant have picked door No.1

a. bi-probabilistic level structure

b. one-probabilistic level collapsed structure (combined worlds)

After the focusing message '$_{goat}$No.3...n'

c. bi-probabilistic level structure

d. one-probabilistic level collapsed structure (combined worlds)

e. one-probabilistic level collapsed structure

Synthetic rule

Bayes' rule

Figure 2. Participants' belief structures of MHP_{nn-2} after the updating message '$_{goat}No.3...n$'

a. bi-probabilistic level structure

b. one-probabilistic level collapsed structure (combined worlds)

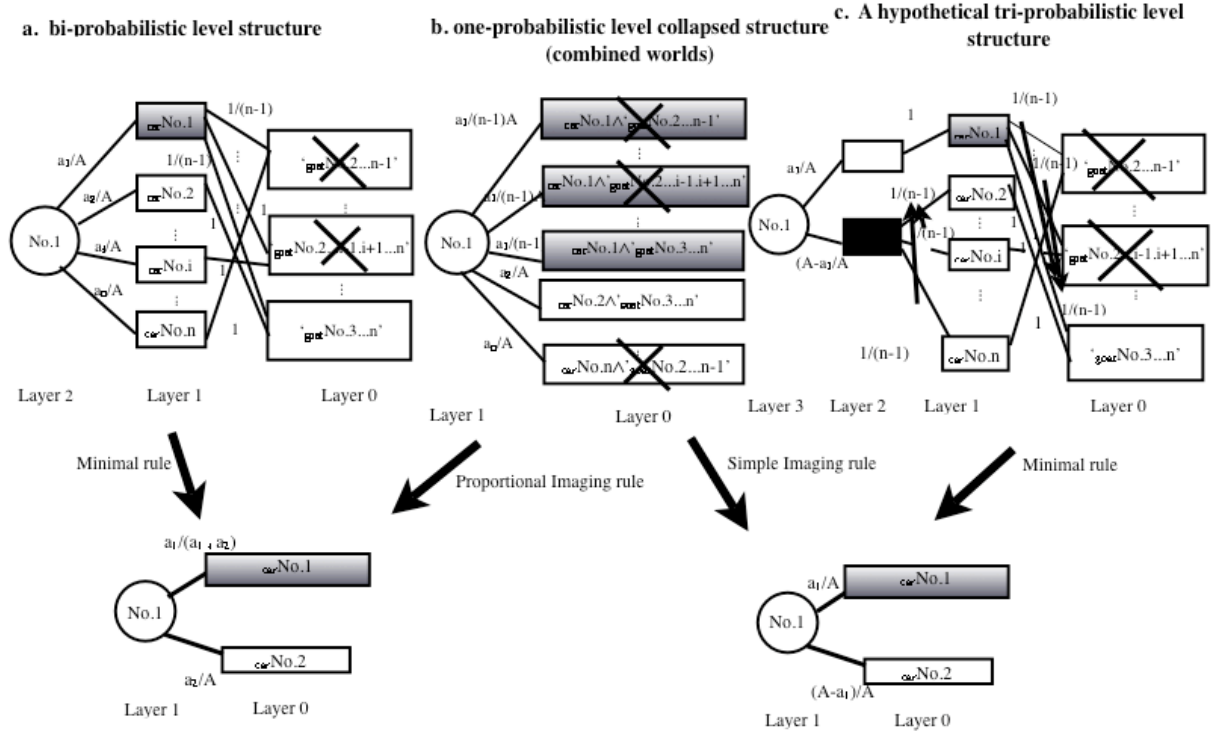c. A hypothetical tri-probabilistic level structure
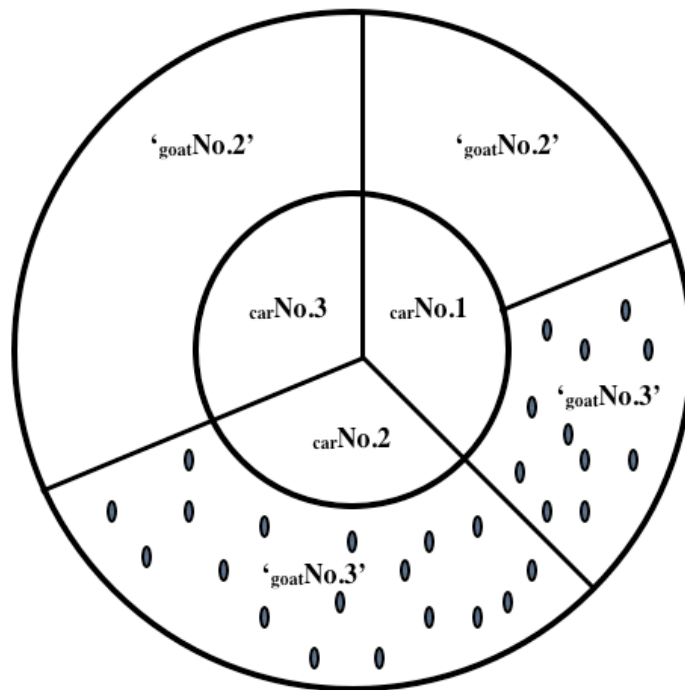
Figure 3. The roulette-wheel diagram

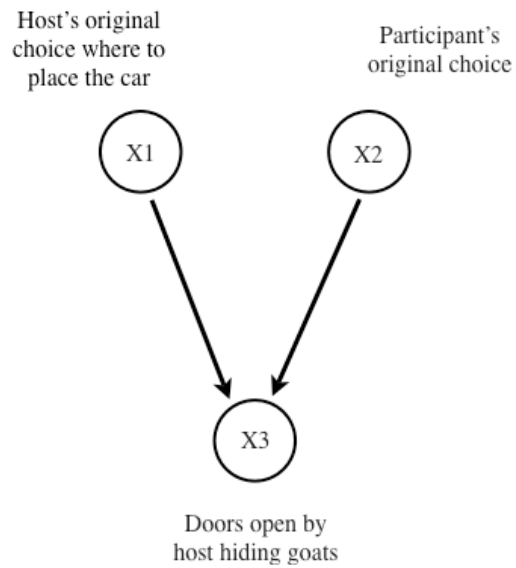## Figure 4. Bayesian network representation

a. of dependencies among three variables in MHP$_{nn-2}$ **b. of the action do(X3)=No.3...n) in MHP$_{nn-2}$**