And yet it rocks!

Fluctuations and growth in Ragnar Frisch's rocking horse model

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December 5, 2020

Abstract

Ragnar Frisch's famous "rocking horse" model has been the object of much praise and even controversy since its publication in 1933. In this paper, we propose a new simulation of the trajectories of the model to clarify those controversies and show that there exists cyclical solutions for a large set of parameters. By building an analytical solution that takes the same form as Frisch's original solution, we are also able to provide new insights into the ideas that he encapsulated in his model. In particular, we show that the author tried to construct a model that would combine both cycles and growth. Finally, the exploration of Frisch's formal construction of the model leads us to link his statistical work on the decomposition of time series with the construction of the 1933 model.

*I wish to thank Michaël Assous, for his help and advice on this paper, Pedro Garcia Duarte, Antonin Pottier, Ariane Dupont-Kieffer and Samuel Demeulemeester for their comments on early drafts, as well as the participants to the workshop of PhD students in economics at Triangle for their questions and comments.

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1 Introduction

In 1969, Ragnar Frisch received the first "Nobel Prize" in economics (shared with Jan Tinbergen), "for having developed and applied dynamic models for the analysis of economic processes."[1] One of his pioneering contributions in this respect was his 1933 article "Propagation Problems and Impulse Problems in Dynamic Economics" (Frisch, 1933). Today, this article is still remembered and celebrated for his approach to macroeconomic phenomena, especially by the New Classicals who have regularly insisted on the necessity to find propagation mechanisms and to differentiate them from the impulses (see for instance the references to Frisch in Robert Lucas and Thomas Sargent [1979] or Edward Prescott [2006]).[2] The idea of separating a propagation mechanism, explaining the form of the return to equilibrium, and an impulse mechanism, explaining the persistence of business cycles when the propagation is in the form of a damped cycle, has been one of the most enduring contributions of Frisch's article and his work.

How did Frisch come to this idea in 1933? His debate with John M. Clark in the *Journal of Political Economy* in 1931-1933 has been rightly underlined as one of the main impetus to write this article and clarify the importance of a determinate system.[3] Another important source of inspiration was his correspondance with Joseph Schumpeter and the role of the pendulum metaphor, that has been particularly discussed by Franscisco Louçã (2001), who used an original correspondence between the two economists to show their different opinions on the validity of the metaphor. The influence of other Scandinavian economists on Frisch, and particularly the Swedish school, has also been the subject of several detailed studies, for instance from Mauro Boianovsky and Hans-Michael Trautwein

[1] See the "facts" page for Frisch on the website of the Nobel Prize (`https://www.nobelprize.org/prizes/economic-sciences/1969/frisch/facts/`, consulted in september 2020).

[2] On the definition of shocks by Frisch, and the role he played into establishing a modern understanding of them, see Duarte and Kevin Hoover (2012), who examine how the notion of shocks was transformed afterward, and "rediscovered" by the New Classicals. Muriel Dal Pont Legrand and Harald Hagemann (2019) argue that the role of propagation mechanisms progressively lost importance afterward.

[3] On the origins of Frisch's model and for a detailed anaysis of the hypotheses of his model, see Ariane Dupont-Kieffer (2012a). Olav Bjerkholt (2007) lists many of the influences of Frisch during the end of the 1920s and early 1930s. On Frisch himself, particularly during the interwar period, see the works of Jens C. Andvig (1981); Bjerkholt (1995); Bjerkholt and Dupont-Kieffer (2010). Dupont-Kieffer (2003) showed in particular in her thesis the opposition of approaches between Frisch and Wesley C. Mitchell.

(2007), who showed that Frisch introduced his distinction between impulse and propagation in part as an answer to a debate with Johan Åkerman, after Frisch assisted to the defense of the latter's PhD thesis in 1928. Dupont-Kieffer (2003, 2012b) underlined the importance of the collection of statistics and national accounts, in particular in connection with Erik Lindahl's work at the same time. And of course, it is well-known that Frisch attributed to Knut Wicksell the image of a rocking horse periodically hit that he used to illustrate the combination of impulses and propagation. The influence of the soviet statisticians and mathematicians, and especially of Eugen Slutzky, was also very important, particularly on the idea that shocks could revive the cycle.[4] Finally, one of the most important source of inspiration of Frisch was his own methodological approach to model-building, that resonated with a small - but growing - group of economists who gathered to create the Econometric Society and the new field of econometrics.[5] The Society fostered important discussions from its first European meeting in Lausanne, where Jan Tinbergen presented several models based on difference-differential equations. It is important to note that both Frisch and Michał Kalecki also used mixed difference-differential equations in the models they presented in another Econometric Society meeting, two years later in Leyden.

In this article, we would like to insist on the evolution of Frisch from his ideas on how to analyze time series to his model published in 1933. In our opinion, it is essential to have in mind the particular approach to harmonic analysis that was developed by Frisch to understand the vision he tried to encapsulate in his 1933 model. In particular, we believe that the formal structure of the model (a mixed difference-differential equation) can only be understood as a consequence of Frisch's early programme of statistical exploration.

To understand the link between his empirical programme and the 1933 paper, we present new simulations of his rocking horse model. These simulations are based on an original analytical solution that takes a form similar to the one in Frisch's article,

[4]See in particular Vincent Barnett (2006), who discusses the different interpretations of Slutzky's ideas, published in 1927 in Russian but already well-known before their translation in English ten years later in *Econometrica* (Slutzky, 1937).

[5]See Mary Morgan (1990) and Louçã (2007) for detailed studies of how econometric ideas came into existence and spread among economists.

allowing us to recover the point on which he originally wanted to insist. They also clarify several controversial results that have been published in the past decades. In particular, we explain where Frisch's methodology led him to a wrong set of parameters in 1933, which explains why Stefano Zambelli was rightly led to conclude that the model, with the original parameters, did not oscillate toward the equilibrium (Zambelli, 1992, 2007). Unlike Zambelli however, we show that the model can fluctuate unambiguously after only a slight change in the parameters. By giving a solution in a similar form to that of Frisch but derived with mathematical and computing tools that he did not have access to in the 1930s, we are able to find parameters of his model for which the generated trajectories are similar to what he explains in his 1933 paper.

2 From the decomposition of time series to their explanation

While Frisch's distinction between impulse and propagation is well-known, we feel that this has obscured other more peculiar characteristics of his model. These aspects can only be understood in the context of what was being done before the econometricians began building macrodynamic models in the 1930s. During the 1920s, the study of the business cycle was essentially empirical, and was driven by several methodologies dealing with the explosion in availability of economic data. One of the big ideas of this decade was that the business cycle could be disaggregated into several component cycles and that it was possible to analyze the structure of lags, relative importance and frequencies of those components to make predictions on the future phase of the business cycle. This idea stemmed largely from the discovery during the second part of the 19th century of cycles of different lengths, and it was built upon and applied in particular by Warren Persons, who established the famous tool of the "economic barometer" and the ABC curves.[6]

While Persons' method relied on the elimination of seasonal variation and trend, and

[6]See in particular chapter 2 of Morgan (1990) on the developments of the empirical approach to the business cycle.

the computations of correlations between time series, another method relied on the frequency domain decomposition of economic time series into components of varying phase, amplitude and period.[7] The techniques employed date back from the groundbreaking work of Joseph Fourier in the early 19th century, who explained how almost any curve could be synthesized as the sum of simple periodic curves (sinusoidal curves), giving birth to Fourier analysis and a host of related tools, which form a central branch of modern mathematics and physics. The periodogram analysis was one of these tools, developed at the end of the 19th century in particular by Arthur Schuster, a british physicist. It was used to identify in a time series which particular frequencies were most important, that is, which component cycles (sinusoids) of a Fourier decomposition of a time series had the most energy (the largest amplitude).

One of the first advocate of using the periodogram in economics was Henry L. Moore.[8] However, Moore's work was discredited by the fact that he explained the persistent oscillation of the business cycle with the movements of Venus. Frisch himself worked on this idea in the late 1920s, and tried to devise a new method to compute the coefficients of components with varying amplitude and period (Frisch, 1928).[9] Frisch illustrated his problem with a chain of pendula: if we have the information on the movement of the last pendulum of the chain, and the lengths of each pendulum, we can obtain the movements of each intermediate pendulum in the chain with a harmonic decomposition of the movement observed at the end of the chain. However if the lengths of the pendula or the intensity of the gravitational field are changing over time, a Fourier decomposition of the movement will not be able to decompose correctly the intermediate movements because it supposes that parameters are held constants. This is a well-known problem in signal

[7] On the development of spectral methods see in particular Harold Davis (1941, chapter 1), and Thomas Cargill (1974).

[8] A detailed presentation of Moore's ideas on this subject can be found in Morgan (1990, chapter 1). On the transfer of the periodogram from physics to economics by Moore, see also Philippe Le Gall (1999). For a recent evaluation of Moore's work and his approach to periodogram analysis, see Paul Turner and Justine Wood (2020), who argue that Moore's idea of a cycle driven by Venus should not hide the originality of his approach.

[9] Morgan (1990: 83-90) has given one of the most thorough discussions of Frisch's work in time series analysis. See also Bjerkholt (2007) for a review of the publications and communications of Frisch during this period. Boianovsky and Trautwein (2007) also discuss the competing views of Frisch and Åkerman on the decomposition of statistical time series.

processing and one would use the short-time Fourier transform to obtain "snapshots" of the components at different moments in time. This seems to be what Frisch explains in his own particular language (see Morgan [1990: 88-89] on this communication problem). What is more interesting is that Frisch used a metaphor of pendula, something that took a central place in the 1933 model. If his pendula were damped, as they are in real life, it would mean that the solution of his decomposition would involve a sum of damped sinusoids changing over time, a rather complex system.

His ideas on this subject, while acknowledged by Davis, did not have an important impact on the development of spectral analysis, in part because other advances were made at the same time, and also because the interest of economists was already shifting under the critiques of Schumpeter and Mitchell. Morgan also argues convincingly that it was abandoned because it could not represent in any way the *relationships* between the variables, something that Frisch was very much conscious of (Morgan, 1990: 90).

In fact, one can argue that the identification of those relationships formed another part of the program of empirical exploration that we can find laid out for instance in a communication made during the 1930 meeting of the American Statistical Association (Frisch, 1931). In this paper, Frisch identified four groups of problems in the analysis of time series: their decomposition (the subject of his communication and of his previous work published in 1928), the comparison of different time series, the forecasting problem, and the "explanation problem" (Frisch, 1931: 74). It is useful to quote Frisch directly on this issue:

> When we have found that a given series contains certain components, we ask the further question: *How did these things come into the series?* In a sense, this is the crucial question of time series analysis. ... But answering such a question means working out a whole rational explanation of the phenomenon at hand. This is not a question of time series technique any more, but a question of the whole content of the *theory* of the particular phenomenon at hand. (*ibid.*)

This quote underlines the fact that the statistical exploration of time series and the increasingly complex schemes developed by Frisch to decompose them were only one side of his analysis of business cycles. The other side, answering the "explanation problem",

seems to have been increasingly on the mind of the Norwegian economist in the early 1930s. The output of his reflexion on this matter was obviously encapsulated in his 1933 paper, which provided an answer to the "explanation problem". The pendula metaphor offers us a crucial link between his earlier statistical work and his new macrodynamic work. In 1928, the pendula were used to illustrate his process of decomposition, as a simplification of a time series. What the 1933 paper amounted to was to find the equations of motion of the system of pendula, rather than taking as a given the trajectory of the last one of the chain. This continuity can also be seen if we look at the formal structure of the model build in 1933, and compare it to the way in which Frisch suggested to decompose time series: it is a striking feature of his 1933 model that the solution can be expressed as a sum of components, that can take the form of sinusoids (cycles) or trends, in the same way that the harmonic decomposition of a time series will be a sum of sinusoids with different amplitudes.

This continuity between the decomposition of time series and the components of his model has also been noted by Marcel Boumans, who argues that Frisch "assumed that each of the economic variables were composed of a number of component cycles or waves" (1995: 133). It is in fact a general characteristic of mixed difference-differential equations that they are infinite dimensional, and can be expressed as an infinite series of sinusoidal functions.[10] This feature was known to Frisch because Tinbergen had already started to study those equations in the early 1930s, and had presented his preliminary results in the 1931 Lausanne meeting of the Econometric Society. Thus one is led to conclude that Frisch selected this specific form for his model because he knew that he would obtain a solution made of a sum of component cycles.

Indeed, Frisch took advantage of the fact that models mixing difference and differential equations gave rise to an infinity of components to decompose the solution of his models into a trend component and cyclical components featuring different periods, dampings,

[10]In an article published in *Econometrica* in 1938, James and Belz (1938) expressed explicitly the solution of a mixed difference-differential equation with the help of a Fourier transform. James had been working with Frisch the years before and their article was part of a larger production of econometricians on those type of equations. See Guido Erreygers (2019) for more background on James and Belz.

phases and amplitudes. This analytical approach mirrored the statistical approach of the 1920s: instead of estimating the cycles' features directly from the data, a theoretical model was built and (loosely) calibrated with the data, and it was solved such that the different components of the general solution would be apparent, and could be compared to the cycles periodicity already well studied. This meant that the model also took a life of its own: Frisch underlined in his article that in addition to the well-known cycles of 8.5 and 3.5 years, he also found another cycle with a period of about two years, and he went a step further by hypothesizing that "if the various statistical production or monetary series that are now usually studied in connection with business cycles are scrutinized more thoroughly, ... *then we shall probably discover evidence also of the tertiary cycle, i.e. a cycle of a little more than two years.*" (Frisch, 1933: 20).

In this context, Frisch's perception of the output of his model becomes a lot more understandable. He viewed this output as symmetric to what was done on the decomposition of time series, and as we will see he pushed this idea a bit too far when he came to consider that the components he obtained could depend on different initial conditions. After Frisch, the idea of using the components of the model never really gained any traction. He himself evolved toward the problems posed by economic policy, especially in the framework of national accounting (Dupont-Kieffer, 2012b). Tinbergen and Kalecki, using simpler versions of difference-differential equations, kept only the first component and looked for parameters that would make it oscillate. They were (mostly) justified in doing so, because of the simpler form of their equations. After them, Roy Allen (1959: 302) called these higher terms "spurious", "arising because of the rigid and unrealistic assumption of a fixed time-delay", and, reflecting on this period, Paul Samuelson acknowledged that he was stumped by the meaning of these components: "As a young student, what I found mystifying was the meaning of the infinite number of sinusoidal components of Frisch's more transcendental mixed difference-differential equation" (Samuelson, 1974: 9). James and Belz (1938) proposed to interpret them as the overtone or harmonics of a fundamental note, drawing a justified analogy with physical processes. But while we can interpret the different solutions arising, for instance, from a system of springs and

masses (via the concept of normal modes), such an interpretation has never been given in economics, and the idea was largely abandoned.

It remains that, in 1933, Frisch saw in mixed difference and differential equations a way to represent both growth and fluctuations, because in the infinite series a monotonous component could appear alongside the cyclical components. Only by going back to this idea of a sum of components can we get a deeper understanding of Frisch's model, and where it could fail.

3 New insights from the simulation of the model

Frisch's model is made of three equations:[11]

$$\dot{x}_t = c - \lambda(r \cdot x_t + s \cdot z_t) \tag{1}$$

$$y_t = m \cdot x_t + \mu \cdot \dot{x}_t \tag{2}$$

$$z_t = \frac{1}{\epsilon} \int_{t-\epsilon}^{t} y(\tau) d\tau \tag{3}$$

The first one is an equation giving the variation of consumption, x. Frisch relates this equation to Walras's *encaisse désirée*: r and s are coefficients of the demand for cash to buy consumption goods and capital goods respectively. When this *encaisse désirée* increases because consumption or capital goods production has increased, it yields increasing tensions on the money supply and can eventually lead to a fall in consumption. c is a constant level of increase in consumption, while λ is a parameter that determines the (negative) influence of the *encaisse désirée* on consumption. From an economic point of view, this may be the weaker part of Frisch's argument, where his model did not anticipate the developments of macroeconomics during the 1930s.

The second equation is an accelerator relationship explaining the level of investment as a function of consumption and changes in consumption. m is a coefficient for the total

[11] We begin directly with Frisch's second model, the first one being an illustration of a system that does not give rise to oscillations.

depreciation of the capital stock, while μ governs the changes in investment needed to accomodate changes in consumption.

Finally, the third equation gives the level of activity going on at an instant t, as a function of the production started during the period between $t - \epsilon$ and t.

To solve his model, Frisch did not have a lot of possibilities available. The theory of delay differential equations (DDE), as they are generally called today, was not even begun, and the behaviour of such equations is often much more complex than that of ordinary differential equations or difference equations.[12] Frisch remarked that the solution of his equation could be expressed as a sum of exponentials, which is one way to express a Fourier series. In the case of the latter the arguments of the exponentials will be complex numbers with a multiple of 2π in the imaginary part and a real part equal to zero, so that every component of the series is cyclical and has a constant amplitude. Frisch did not pose such restrictions on the inputs of the exponential functions of his series, which means that they could very well be real numbers, yielding a monotonous trajectory, or complex numbers with a nonzero real part, generating either a damped (real part inferior to zero), or explosive (real part superior to zero) trajectory.

To find a solution, Frisch still had to obtain from the parameters of the model and a set of initial conditions the frequencies and damping of the components of the infinite series, as well as their coefficients (the latter being the customary part of Fourier analysis). This is on this point that we can give a more elegant answer than Frisch's to the computation of the components, while respecting his original process of finding a solution made of a sum of damped sinusoids. This is possible because of the work done by Richard Bellman, Kenneth Cooke and others during the 1940s and 1950s, which resulted in the publication by Bellman and Cooke of an important monograph on DDEs in 1963 (Bellman and Cooke, 1963). It should also be noted that this is the logical extension of Frisch's approach to the problem. During the 1930s, he was not the only economist working with DDEs, as Tinbergen, Kalecki, Robert James and Maurice Belz were also publishing models based on

[12] In fact DDEs belong to a special class of functional differential equation and are closer to partial differential equations (PDE), than to ordinary differential equations. It is not a coincidence that Fourier-type analysis was developed to solve PDEs and is well-suited for DDEs.

those equations and their solutions in *Econometrica*, and their pioneering works are cited by Bellman and Cooke, who even spent a section on the solutions of Kalecki's 1933-35 model.

While Bellman and Cooke develop several approaches to deal with DDEs, one in particular allows us to retrieve the same infinite sum that Frisch used in his article: the Laplace transform and its inversion. This analytical tool, a staple of modern engineering, allows one to solve very efficiently and simply a wide range of differential equations. Its true power is unleashed when applied more formally to intricate equations such as those involving differential or integral operators as well as lags and delays. At its core, the transformation transports us into a new domain where differentiation and integration are transformed into operations of multiplication and division, making it much easier to find a solution. The difficulty then lies in going back to the original domain, the temporal domain where the solutions can describe their trajectories as a function of time. This transform is close to the Fourier transform, but more adapted for the study of difference-differential equations. In particular, once applied to this type of equations, it will give an infinite sum of components, that can be expressed as a function of the parameters of the system, the roots of a characteristic equation, and the initial conditions determined by the model-builder.

To apply the Laplace transform to Frisch's system of three equations, we start by reducing it (by substitution) to one integro-differential equation with one unknown variable, the consumption x (the complete process is described in Appendix I). Doing this, we obtain the following integro-differential equation with a lag in the state variable and the integral:[13]

$$\dot{x}(t) + \lambda(r + \frac{s\mu}{\epsilon}) \cdot x(t) - \frac{\lambda s\mu}{\epsilon} \cdot x(t-\epsilon) + \frac{\lambda sm}{\epsilon} \int_{t-\epsilon}^{t} x(\tau)d\tau = c \tag{4}$$

From this equation, the appendix explains the computations needed to obtain, with the Laplace transform and its inverse, the following solution for x, a sum of components

[13]This equation can also be found in a differentiated form in Zambelli (1992: 32), and Boumans (1999: 80).

similar to the one in Frisch (1933):

$$x(t-\epsilon) = \overbrace{\frac{c}{\lambda(r+sm)}}^{\textbf{(i)}} + \overbrace{k_1 e^{r_1(t-\epsilon)}}^{\textbf{(ii)}} + \overbrace{\sum_{i=2}^{\infty} A_i e^{\alpha_i(t-\epsilon)} cos(\beta_i(t-\epsilon)+\phi_i)}^{\textbf{(iii)}} \tag{5}$$

There are three distinct terms on the right hand side of this equation. (i) On the left, an equilibrium level, determined as a function only of the parameters of the system. (ii) In the middle, we find an exponential function with a real argument, that will be stable or unstable according to the parameters of the system, and that was dubbed by Frisch a "secular trend" because it generates a monotonous trajectory. (iii) On the right, there is an infinite sum of sinusoidal solutions, each with its own frequency and damping exponent, that are ordered from the longest period for the lowest positive index to the smallest period for higher indexes. Both the trend and each of the cycles have their own amplitude determined by the initial conditions, and each cycle has its own phase, determined by the initial conditions as well. Appendix I presents our methodology to obtain the roots that give us the frequency and damping of each cycle, and it is with this equation that we are able to rebuild Frisch's solution and discuss the different trajectories of his model.

Of course this is not the only approach to solve this model. However, it allows us to obtain this sum of components, that was in the original article of Frisch. Another approach would be to simulate the whole solution with a numerical integration of either (4) or the system of three equations. Frisch gave some elements in this direction, but due to the lack of computing powers in 1933 he was obviously limited in this regard. In addition, while this type of solution is useful to check the validity of our analytical solution and the level of approximation coming from the necessary truncation of the infinite series, it does not inform us in any way on the behaviour of the individual components. This was however the approach adopted by Zambelli in 1992 and 2007. From his two articles, we can gather that he discretized the model of three equations, that is, he transformed a continuous time model into a discrete time model to compute the solution step by step.

The following summarizes the two systems that result from this operation (on the left the continuous system, on the right the discrete one).

$$\dot{x}(t) = c - \lambda(rx(t) + sz(t)) \qquad x(t+h) = x(t) + h(c - \lambda(rx(t) + sz(t))$$

$$y(t) = mx(t) + \mu\dot{x}(t) \qquad y(t) = mx(t) + \mu\frac{x(t+h) - x(t)}{h}$$

$$z(t) = \frac{1}{\epsilon}\int_{t-\epsilon}^{t} y(\tau)d\tau \qquad z(t) = \frac{h}{\epsilon}\left[\sum_{i=1}^{\frac{\epsilon-h}{h}} y(t-ih) + \frac{y(t) + y(t-\epsilon)}{2}\right]$$

In the discrete system, the integral has been approximated using the trapezoid rule, dividing the interval between $t - \epsilon$ and t into smaller intervals, where h is an arbitrary step size. We also extracted the first and last bounds of the new sum to make apparent the dependence of $z(t)$ on the current value of another variable, $y(t)$. Neither in 1992 nor in 2007 does Zambelli explain how he solved this second system, and the solution is not "straightforward" (Zambelli, 2007: 162, Zambelli 1992: 42) since it is a rather complex system of simultaneous difference equations with several lags. The problem is simple: at each point in time, if we look at the first equation we see that to compute $x(t+h)$, one needs the value of $z(t)$, and the second equation tells us that $y(t)$ is necessary to compute $z(t)$; but in order to compute $y(t)$, one also needs $x(t+h)$! This is by definition a system of simultaneous equations and one has to solve all of them at the same time, and cannot just start to compute $x(t+h)$ or another variable to compute the others. Nowhere in his articles does Zambelli signals that he has taken this problem into account, and in fact it is highly likely that he did not: we extracted $y(t)$ from the sum in order to make it apparent, but more importantly, if we just use one variable to determine the others we will obtain Zambelli's result of a monotonous return to equilibrium, while a computation using the simultaneous solution of the discrete model gives us the same results as with our analytical sum of components.

We don't need to solve this system of equations for all three variables, and with careful substitutions we can obtain an expression of $x(t+h)$ as a function of only $x(t)$. In addition to being rather cumbersome to carry out, this process does not offer us any insights into

how Frisch solved his model and how he perceived its relevance for the economy and business cycles. But it does confirm that our analytical solution holds true thus we present in Appendix II the expression that we obtain when we correctly apply Zambelli's methodology to Frisch's model.

4 Cycles and Growth in the rocking horse model

If we assume, as does Zambelli, that the model is initially in a state of equilibrium and that there is a 10% increase of x, the consumption, we can trace the return to equilibrium with our equation.[14] To obtain this trajectory, we compute the sum of the first one thousand components of equation (5).[15] Focusing on the first four components, we can give a decomposition that shows precisely why the cycle is not apparent once we aggregate the different components: the first component (component (ii) in equation (5)), that is, the "trend" component that takes the form of a monotonic return to equilibrium, dominates largely the cycles, both in amplitude and in damping.

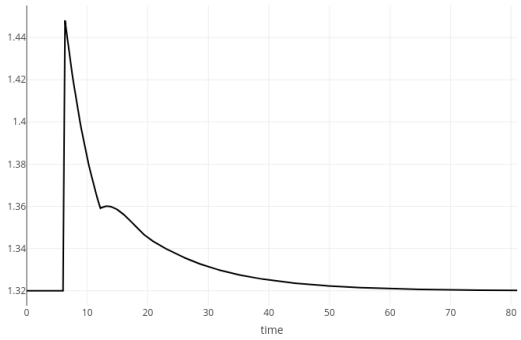


Figure 1: Solution of x(t) with the original parameters and 1000 components

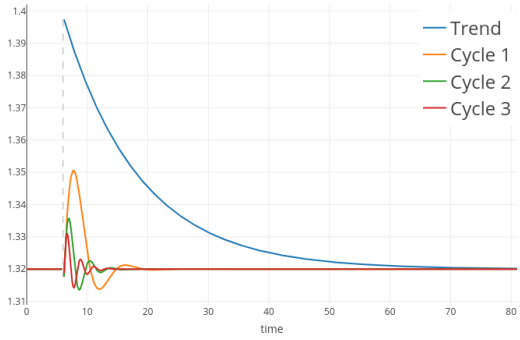


Figure 2: Decomposition with components 1 to 4 of equation (5)

[14]The code used to make the following figures is available at `https://gist.github.com/vcarret/2c0832e815dcf1f7918fbfd140d57ba5`. At the end of the script, we also show that the correct expression from Zambelli's discrete-time system gives the same result as with our Laplace transform. An application simulating our solution by components is also available at `https://cbheem.shinyapps.io/Frisch/`.

[15]Using more components does not change significatively the solution; in fact using only one hundred components would have given us essentially the same picture, because their amplitude goes very quickly to zero. We can check easily that the result is the same with the equation correctly derived from the discrete system in Appendix II.

This confirms that, for the parameters chosen by Frisch, we obtain a monotonic return to equilibrium: the trend component largely dominates in amplitude the cyclical components, and once they are aggregated, it produces the result in figure 2. But why did Frisch overlooked this in 1933? In the appendix, we compute a solution for x as a function of the initial conditions, the parameters of the system, and the roots of a characteristic polynomial. Doing so, we see clearly (cf in particular equation (18)) that all the components will depend on the same initial conditions. But Frisch poses two different initial conditions: one for the "trend": "x_0 shall be unity at origin" (Frisch, 1933: 18), and one for all the cycles "[w]e may, for instance, require that $x_1(0) = 0$ and $\dot{x}_1(0) = \frac{1}{2}$." (Frisch, 1933: 20).[16] Hence it appears that Frisch did as if the first component (the "trend") was independent of the other components (the cycles), and that he could impose different initial conditions on them, thereby artificially lowering the amplitude of the trend and making the cycles much more apparent. He was not allowed to do that and it prevented him to see that for the parameters he chose, the general solution would not present a cyclical appearance. To Zambelli's credit, he was right on this point.

But does that mean that the model can never fluctuate for economically relevant parameters as Zambelli also argues? Such an equation is unlikely to show only a monotonous return to equilibrium for a wide range of parameters. We can in fact easily find parameters that make the cycle clearly apparent on the aggregated solution, either because the amplitude of the "trend" component is smaller than one or more of the cycles' amplitudes, or because this "trend" is more damped than one or more of the cycles.

One such combination of parameters, that remains within the bounds given by Zambelli in his 2007 appendix, can be obtained just by increasing λ, for instance to 0.3 (to avoid cluttering the figures we will only represent the "trend" component and the first cycle, which has the longest period and the highest amplitude and is thus more likely to impact the final form of the equation).

[16] He then repeats the same condition for x_2, thus implying that he used it for all cycles.

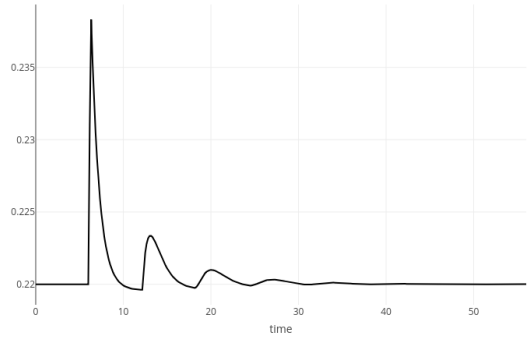


Figure 3: Solution of x(t) with $\lambda = 0.3$, 1000 components

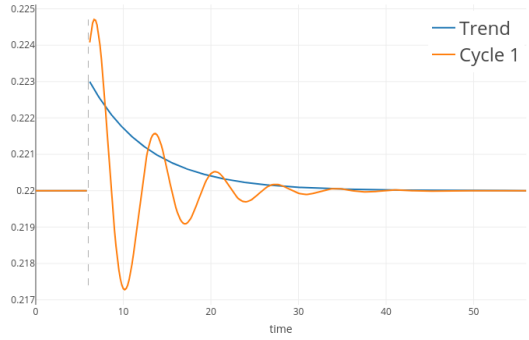


Figure 4: Components 1 and 2 (trend and major cycle)

In figure 3 and 4, we see that the amplitude of the first cycle has grown relatively to the "trend" component, such that the return to equilibrium is clearly of a cyclical form, bouncing above and below the equilibrium level, although it bounces farther above than below. Still within the bounds given by Zambelli, we can change more parameters to obtain an even clearer cycle, where the damping out of the "trend" is quicker than the disappearance of the first cycle; for instance, with $m = 1$, $\lambda = 0.3$, $r = 1$, $s = 2$ and $\mu = 15$, we obtain the following figures.

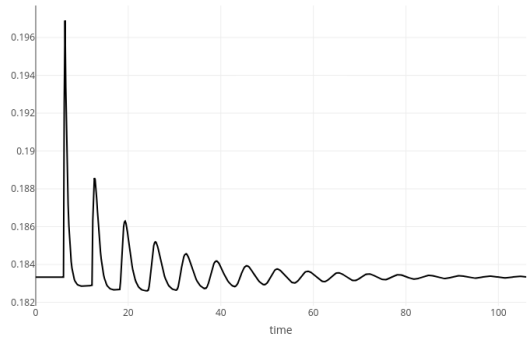


Figure 5: Solution of x(t), see parameters in the text, 1000 components

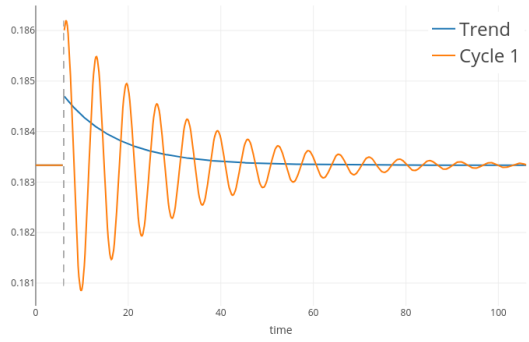


Figure 6: Components 1 and 2 (trend and major cycle)

This result is very important, because it means that Frisch's model can generate a trend component which will leave room for some cycles. This is only possible because of

the complexity of equation (4), and this result is hardly possible in a simpler difference-differential model such as that of Kalecki (1935) or Tinbergen (1959 [1931]).[17]

As a final example, to underline Frisch's original vision of trend and cycles, we can give a solution showing the type of "transitory growth" that was present in his original model. Indeed if we look at the equations 23(a),(b),(c) and (d) on page 22 of Frisch (1933), we see that the equation for x_0 is a positive trend, similar to the blue (monotonous) curve in figure 8 below. To obtain this component, we abandoned the constraint of supposing that the economy starts at the equilibrium, and that a shock displaces it away from it, that was used to observe the shape of the return to equilibrium. Instead, we suppose that the economy was artificially maintained at a lower level than its equilibrium level for some time (at least a time equal to ϵ). In the following two figures, we kept the same parameters used in figures 5 and 6 above:

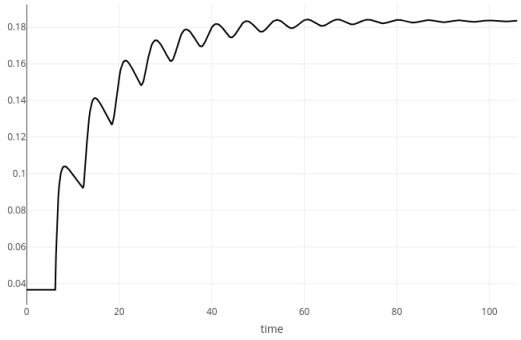


Figure 7: Solution of x(t), see parameters in the text, 1000 components

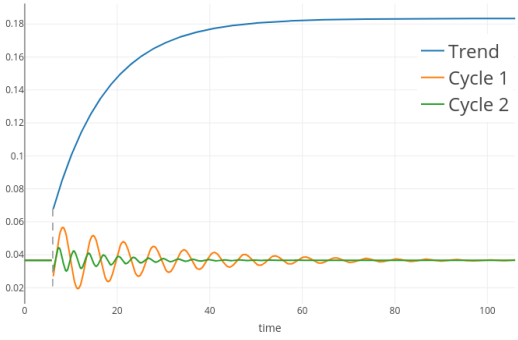


Figure 8: Components 1, 2 and 3 (trend and two cycles)

Figures 7 and 8 are the closest ones to the components originally presented by Frisch in his article: there is a "trend" component that brings us toward the equilibrium level of the economy from the level where the economy was stuck, with about the same speed as in Frisch's original article. Then we have a primary cycle with a period of about 6.5 years, a

[17] The shared visions as well as differences of those models is detailed in the forthcoming book (Assous and Carret, 2021), where we apply the type of analysis made here to the models built in the 1930s and 1940s by Ludwig Hamburger, Tinbergen and others in Europe, as well as Samuelson and Oskar Lange in the United States (see also Assous and Carret (2020) on the latter). An earlier article by Assous et al. (2017) has already shown that Kalecki's 1933-35 model could not generate both cycles and growth.

secondary cycle with a period of about 3.2 years, and a third cycle (not represented) with a period of about 2.1 years, all values very close to those in Frisch's article. The primary cycle is clearly seen when it is superposed to the trend component: there are several periods before the new equilibrium level is reached, it has a relatively high magnitude (compared to the trend and other cyclical components), and finally it is less damped than the trend component.

There is however one caveat compared to Frisch's original article:[18] in order to obtain apparent cycles at the aggregate level, we had to decrease the damping of the system. In fact the return to equilibrium is much longer than in Frisch's original article (compare the return to equilibrium of the first components in figure 2 and in figure 8). Is this a problem? We don't think that it necessarily is so: the propagation mechanism was only one part of the whole model, the second part being the impulse mechanism, which was used to explain the persistence of otherwise damped cycles. The fact that the propagation mechanism itself can explain a larger part of the persistence of cycles appears to be in line with Frisch's original objective of explaining the phenomena of sustained fluctuations in the business cycle. It is also true that we merely presented some examples of fluctuations, and that others could be find with different combinations of parameter, maybe quicker to return to equilibrium.

One thing we can say is that throughout all the parameters we tested, we were unable to find a set of economically relevant parameters that would make the system unstable. We believe that this sheds an interesting light on a well-known fact about Frisch's approach to business cycles: that they are the manifestation of a stable process that would eventually reach a stationary level if left undisturbed.

[18] We thank Pedro Garcia Duarte for calling our attention on this point.

Appendix I - Solving the model with a shifted Laplace transform

The equations of Frisch's model are the following:[19]

$$\dot{x}_t = c - \lambda(r \cdot x_t + s \cdot z_t) \tag{6}$$

$$y_t = m \cdot x_t + \mu \cdot \dot{x}_t \tag{7}$$

$$z_t = \frac{1}{\epsilon} \int_{t-\epsilon}^{t} y(\tau) d\tau \tag{8}$$

As a preliminary, we can compute the equilibrium of the system, which will help us verify later on that our computations are right. After at least ϵ time at the equilibrium, we know that the rate of change will be zero, and the past values will be the same as the equilibrium values. We have thus three equations to determine the three unknown equilibrium values:

$$c = \lambda(r \cdot \bar{x} + s\bar{z}) \tag{9}$$

$$\bar{y} = m \cdot \bar{x} \tag{10}$$

$$\bar{z} = \bar{y} \tag{11}$$

Solving for the equilibrium values, we get that

$$\bar{x} = \frac{c}{\lambda(r + sm)} \tag{12}$$

$$\bar{y} = \bar{z} = \frac{mc}{\lambda(r + sm)} \tag{13}$$

To find solutions satisfying this system, we reduce it to one equation by successive replacements: first, we replace y in z_t and we obtain $z_t = \frac{1}{\epsilon} \int_{t-\epsilon}^{t} m \cdot x(\tau) + \mu \cdot \dot{x}(\tau) d\tau$. This can be simplified as $z_t = \frac{m}{\epsilon} \int_{t-\epsilon}^{t} x(\tau) d\tau + \frac{\mu}{\epsilon} \cdot (x(t) - x(t-\epsilon))$.

[19] Frisch's notation of z_t is rather confusing. This form makes clear that we want the integral of y over the interval $t - \epsilon$ to t, and computations yield the same result as Frisch's.

Putting this into (1), we obtain $\dot{x}_t = c - \lambda \left[r \cdot x_t + s \cdot \left(\frac{m}{\epsilon} \int_{t-\epsilon}^{t} x(\tau) d\tau + \frac{\mu}{\epsilon} \cdot (x(t) - x(t-\epsilon)) \right) \right]$

Combining and rearranging terms, we obtain the integro-differential equation that was given in the text as equation (4):

$$\dot{x}_t + \lambda(r + \frac{s\mu}{\epsilon}) \cdot x_t - \frac{\lambda s \mu}{\epsilon} \cdot x(t-\epsilon) + \frac{\lambda s m}{\epsilon} \int_{t-\epsilon}^{t} x(\tau) d\tau = c$$

We can easily check that the equilibrium value of this equation is the same as x in the system above. If we manage to find a solution for this equation, it will then just be a matter of differentiating it and replacing into the system (1)-(3) to obtain the other variables.

To simplify computations, we pose that $a = \lambda(r + \frac{s\mu}{\epsilon})$, $b = -\frac{\lambda s \mu}{\epsilon}$ and $d = \frac{\lambda s m}{\epsilon}$.

We will now use the Laplace transform to solve this equation. Its definition is $\mathscr{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt = F(s)$, and in order to have a solution to our equation in terms of the initial conditions from 0 to ϵ, we shift this definition by ϵ to obtain $\int_\epsilon^\infty f(t) e^{-st} dt$ (this follows Bellman and Cooke (1963) approach and avoids expressing our solution as a function of a negative time from $-\epsilon$ to 0). To simplify the application of our transform, we separate in two parts the integral in equation (4). The first part goes from 0 to t and the second from $t - \epsilon$ to 0, we then invert the second part and change variables to obtain:

$$\dot{x}(t) + a \cdot x(t) + b \cdot x(t-\epsilon) + d \int_0^t x(\tau) d\tau - d \int_\epsilon^t x(\tau - \epsilon) d\tau = c \tag{14}$$

The Laplace transform is linear and applying it to this equation term by term is a rather straightforward, if unwieldy, computation. $\dot{x}(t)$ will give us an initial condition on $x(\epsilon)$ as can be expected. For $b \cdot x(t-\epsilon)$ we use a change of variable to make apparent the initial condition:

$$b \int_\epsilon^\infty x(t-\epsilon) e^{-st} dt = b e^{-s\epsilon} \int_0^\infty x(t) e^{-st} dt = b e^{-s\epsilon} \left[\int_0^\epsilon x(t) e^{-st} dt + \int_\epsilon^\infty x(t) e^{-st} dt \right]$$

The first integral in the brackets involves our initial condition on the development of

$x(t)$ during the period 0 to ϵ. The second is the definition of the transform, which we will call $X(s)$.

We transform our two integrals using integration by parts. For the first one, we have:

$$\begin{aligned} d\int_{\epsilon}^{\infty}\int_{0}^{t} x(\tau)d\tau\ e^{-st}dt &= d\left[-\frac{1}{s}e^{-st}\int_{0}^{t}x(\tau)d\tau\Big|_{\epsilon}^{\infty}+\frac{1}{s}\int_{\epsilon}^{\infty}x(t)e^{-st}dt\right] \\ &= \frac{d}{s}\left[e^{-s\epsilon}\int_{0}^{\epsilon}x(\tau)d\tau+\int_{\epsilon}^{\infty}x(t)e^{-st}dt\right] \end{aligned}$$

Where the evaluation on the first line of the left term in the brackets is 0 when $t=\infty$, and the transformation gives us a new form of the initial development of $x(t)$. For the second integral term, we obtain a rather similar expression, with the lag giving a similar form to the initial condition as in the transform of the third term above:

$$\begin{aligned} -d\int_{\epsilon}^{\infty}\int_{\epsilon}^{t} x(\tau-\epsilon)d\tau\ e^{-st}dt &= -d\left[-\frac{1}{s}e^{-st}\int_{\epsilon}^{t}x(\tau-\epsilon)d\tau\Big|_{\epsilon}^{\infty}+\frac{1}{s}\int_{\epsilon}^{\infty}x(t-\epsilon)e^{-st}dt\right] \\ &= -\frac{d}{s}e^{-s\epsilon}\left[\int_{0}^{\epsilon}x(t)e^{-st}dt+\int_{\epsilon}^{\infty}x(t)e^{-st}dt\right] \end{aligned}$$

This time the left term inside the bracket will vanish once evaluated, and we are left with the right term which is similar to the term transformed above, giving us the solution on the second line.

With $X(s)=\int_{\epsilon}^{\infty}x(t)e^{-st}dt$, we can replace our computations in equation (14) to obtain:

$$\begin{aligned} sX(s)-x(\epsilon)e^{-s\epsilon}+aX(s)+be^{-s\epsilon}\left[X(s)+\int_{0}^{\epsilon}x(t)e^{-st}dt\right]+\frac{d}{s}\left[X(s)+e^{-s\epsilon}\int_{0}^{\epsilon}x(\tau)d\tau\right] \\ -\frac{d}{s}e^{-s\epsilon}\left[X(s)+\int_{0}^{\epsilon}x(t)e^{-st}dt\right]=\frac{c}{s}e^{-s\epsilon} \end{aligned}$$

Grouping terms we have that:

$$X(s)\cdot\left[s+a+be^{-s\epsilon}+\frac{d}{s}-\frac{d}{s}e^{-s\epsilon}\right]+\left[b-\frac{d}{s}\right]e^{-s\epsilon}\int_{0}^{\epsilon}x(t)e^{-st}dt+\frac{d}{s}e^{-s\epsilon}\int_{0}^{\epsilon}x(\tau)d\tau-x(\epsilon)e^{-s\epsilon}=\frac{c}{s}e^{-s\epsilon}$$

Which gives us our final expression:

$$X(s) = \frac{e^{-s\epsilon}\left(c + s \cdot x(\epsilon) - (s \cdot b - d)\int_0^\epsilon x(t)e^{-st}dt - d\int_0^\epsilon x(\tau)d\tau\right)}{s^2 + as + bse^{-s\epsilon} + d - de^{-s\epsilon}} \tag{15}$$

where we have multiplied $x(\epsilon)$, $\int_0^\epsilon x(t)e^{-st}dt$ and $\int_0^\epsilon x(\tau)d\tau$ by $\frac{s}{s}$.

From this formula, we see that we have picked up several terms on the numerator that are independent from $X(s)$, and that involve the same initial condition on the development of x between 0 and ϵ. These terms will be very important to determine the amplitude of each component; this is the only error in Frisch's solution, because he chose a different initial condition for the component given by the only nontrivial real root than for the cyclical components.

The formula[20] for the inverse Laplace transform is $f(t) = \frac{1}{2\pi i} lim_{T\to\infty} \int_{\gamma-Ti}^{\gamma+Ti} F(s)e^{st}ds$. Following Bellman and Cooke,[21] we compute this contour by shifting it to the left and taking into account the singularities we meet. We start by defining our contour integral:

$$x(t) = \frac{1}{2\pi i}\int_{(b)} \frac{e^{-s\epsilon}\left(c + s \cdot x(\epsilon) - (s \cdot b - d)\int_0^\epsilon x(t)e^{-st}dt - d\int_0^\epsilon x(\tau)d\tau\right)}{s^2 + as + bse^{-s\epsilon} + d - de^{-s\epsilon}} e^{st}ds \tag{16}$$

where (b) is our vertical contour.

Our contour will be equal to the sum of residue times $2\pi i$, which will cancel the $\frac{1}{2\pi i}$ of the inverse formula, so that we are just left with the sum of residues. Because the last expression can be written in the form $x(t) = \frac{1}{2\pi i}\int_{(b)} \frac{g(s)}{h(s)}ds$, where

$$F(s) = \frac{g(s)}{h(s)} = \frac{\left(c + s \cdot x(\epsilon) - (s \cdot b - d)\int_0^\epsilon x(t)e^{-st}dt - d\int_0^\epsilon x(\tau)d\tau\right)e^{s(t-\epsilon)}}{s^2 + as + bse^{-s\epsilon} + d - de^{-s\epsilon}} \tag{17}$$

and because all poles arising from the denominator will be simple, we have that at a simple pole c, $Res(f, c) = lim_{z\to c}(z - c)F(z) = \frac{g(c)}{h'(c)}$. Thus our sum of residues will rise from this expression, once we are able to compute the zeros of our denominator.

[20] See for instance Boas (1983: 696) for a derivation of this formula, usually called a "Bromwich integral" and written with this γ.

[21] See Chapter 1 in Bellman and Cooke (1963) for a discussion of the inversion algorithm, and Chapter 3, section 7 for a detailed application.

This allows us to give a final expression for $x(t)$ as a sum of components, each one with its own amplitude (and phase for cyclical components) arising from the initial conditions:

$$x(t-\epsilon) = \sum_{i=0}^{\infty} \frac{c + r_i \cdot x(\epsilon) + (d - r_i \cdot b)\int_0^{\epsilon} x(t)e^{-r_i t}dt - d\int_0^{\epsilon} x(\tau)d\tau}{2r_i + a + be^{-r_i\epsilon} - b\epsilon r_i e^{-r_i\epsilon} + d\epsilon e^{-r_i\epsilon}} e^{r_i(t-\epsilon)} = \sum_{i=0}^{\infty} k_i e^{r_i(t-\epsilon)} \tag{18}$$

Where the r_i are zeros of the characteristic polynomial $h(s)$, the denominator of equation (17). We take the sum from 0 to ∞, but the conjugate of each complex root is also a solution, and its coefficient is $\bar{k_i}$.

All that remains to do now is to find a procedure to obtain the zeros of our characteristic polynomial. First of, we can remark that our polynomial always has a trivial root at $r_0 = 0$. With this root it is easy to verify that the numerator is equal to c after the integrals cancel each other, and we will in fact obtain a constant component that is the equilibrium level for this system. The reader will not be surprised to see that this level is equal to $\frac{c}{a+b+d\epsilon} = \frac{c}{\lambda(r+sm)}$, the same equilibrium we previously computed. For the other roots, the integrals are not so accomodating, and we have to assume an initial development for x between 0 and ϵ. We use the fact that we assumed a disturbed equilibrium to pose that $x(t) = \bar{x}$, for $0 \leq t < \epsilon$, which means that our numerator will be simplified to $c + r_i \cdot x(\epsilon) + (d - r_i \cdot b)\bar{x}\frac{1}{r_i}(1 - e^{-r_i\epsilon}) - d\bar{x}\epsilon$. Of course this expression cannot be used for our trivial root.

To find the other roots of this polynomial, we can remark that $lim_{s\to\infty}h(s) = s^2 + as[1 + \theta_1(s)] + bse^{-\epsilon s}[1 - \theta_2(s)]$, where both θ_1 and θ_2 will tend to zero as s grows to infinity. This means that for large s, the simpler expression $s + a + be^{-\epsilon s} = 0$ will be a good approximation of our roots. Because this is a transcendental equation, we will have an infinity of solutions to this equation, but we can give a closed form solution with the Lambert W function.[22] We first change variables and pose $s = \frac{w}{\epsilon} - a$. Replacing, we have that $\frac{w}{\epsilon} = -be^{-w+a\epsilon}$ and rearranging to have the Lambert form $we^w = -\epsilon be^{a\epsilon}$, which means that $w = W_k(-\epsilon be^{a\epsilon})$ and finally $s = \frac{W_k(-\epsilon be^{a\epsilon})}{\epsilon} - a$, where $k = 0, 1, 2...\infty$ is the branch of the Lambert function giving us an approximation of the value of w for this

[22]See Corless et al. (1996) for the definition of this function and an algorithm to obtain its solutions.

branch. We know that ϵ is positive, b is negative and a and ϵ are real numbers, which means that the expression inside W_k will always be positive. We know that in this case there will always be one nontrivial real root (this is the "trend" identified by Frisch in 1933), and an infinity of complex roots that will give us an infinity of cyclical solutions. The general solution of x will be the sum (superposition) of all these solutions. This gives us an initial guess that is improved with Newton's algorithm, giving us the same results as Frisch (and his assistants) for the roots of the first four components.[23]

Inserting those solutions in equation (18), we obtain the complete solution given in the text as equation (5) (expressed here more formally with the Heaviside step function $u(t-\epsilon)$):

$$x(t-\epsilon)u(t-\epsilon) = \left[\frac{c}{\lambda(r+sm)} + k_1 e^{r_1(t-\epsilon)} + \sum_{i=2}^{\infty} A_i e^{\alpha_i(t-\epsilon)} cos(\beta_i(t-\epsilon)+\phi_i)\right] u(t-\epsilon)$$

Where r_1 is a real root, and the terms in the sum on the right are all sinusoidal functions, with damping and period given by $r_i = \alpha_i + j\beta_i$, $j^2 = -1$. The parameters, roots and initial conditions determine together all the k_i, giving the amplitude of the sinusoidal $A_i = 2 \cdot |k_i|$ and its phase $\phi_i = arg(2 \cdot k_i)$ (we get a factor of two because of the complex conjugate). In the case of the pure exponential, the amplitude is simply k_1.

It is not easy in this equation to prove that the coefficients of the cycles are tending to zero in the limit as the absolute values of the roots tend to infinity. Bellman and Cooke show that to the left of any vertical line in the left-half plane, the residues will always give rise to ever smaller components (in amplitude and damping), at least for retarded and neutral DDE (our equation is a transformation of a neutral DDE). This result does not always hold for more complex DDE, but we can safely believe that this result holds here for at least two reasons: all the computations done by the author have given diminishing

[23]Note that our approximation is working well for cyclical components, but can fail for the lowest component, a real root. In this case we slide down the real line until we find our root. It is also interesting to note that while the complex roots are well-ordered and asymptotically tangent to a vertical line in the complex plane, the nontrivial real root can move to the left of the cycles with longest period (the cycles with the lowest imaginary part). This is crucial in order for cycles to appear in the final solution and is a peculiarity of Frisch's model. This can be seen with a plot of the real and imaginary parts of the characteristic polynomial, which also confirms that we are not forgetting any root with our algorithm.

components for a large range of parameters, and keeping only 100 (even 30 in most cases) components already gives an approximation very close to the numerical integration of the system.

Appendix II - Deriving the correct equation from Zambelli's system II

We repeat Zambelli's "System II" of discretized equations below for the convenience of the reader:

$$\begin{aligned} x(t+h) &= x(t) + h(c - \lambda(rx(t) + sz(t)) \\ y(t) &= mx(t) + \mu\frac{x(t+h) - x(t)}{h} \\ z(t) &= \frac{h}{\epsilon}\left[\sum_{i=1}^{\frac{\epsilon-h}{h}} y(t-ih) + \frac{y(t) + y(t-\epsilon)}{2}\right] \end{aligned}$$

In the third equation for $z(t)$, the important part is $y(t)$. We can extract it, replace it by the equation for $y(t)$ in terms of $x(t)$ and group the rest of $z(t)$ in a new variable:

$$\begin{aligned} z(t) &= \frac{h}{2\epsilon}y(t) + \frac{h}{\epsilon}\left[\sum_{i=1}^{\frac{\epsilon-h}{h}} y(t-ih) + \frac{y(t-\epsilon)}{2}\right] = \frac{h}{2\epsilon}y(t) + \frac{h}{\epsilon}\Gamma \\ &= \frac{h}{2\epsilon}\left(mx(t) + \mu\frac{x(t+h) - x(t)}{h}\right) + \frac{h}{\epsilon}\Gamma \\ &= \frac{\mu}{2\epsilon}x(t+h) + \left(\frac{hm - \mu}{2\epsilon}\right)x(t) + \frac{h}{\epsilon}\Gamma \end{aligned}$$

Where Γ is the expression in brackets on the first line, and depends only on past values of $x(t)$ (before t). We can now insert this equation in $x(t+h)$.

$$x(t+h) = x(t) + hc - h\lambda r x(t) - h\lambda s z(t)$$

$$x(t+h) = hc + (1 - h\lambda r)x(t) - \frac{h\lambda s\mu}{2\epsilon}x(t+h) - h\lambda s\left(\frac{hm-\mu}{2\epsilon}\right)x(t) - \frac{h^2\lambda s}{\epsilon}\Gamma$$

$$\left(1 + \frac{h\lambda s\mu}{2\epsilon}\right)x(t+h) = hc + \left[1 - h\lambda r - h\lambda s\left(\frac{hm-\mu}{2\epsilon}\right)\right]x(t) + \frac{h^2\lambda s}{\epsilon}\Gamma$$

$$x(t+h) = \frac{hc + \left[1 - h\lambda r - h\lambda s\left(\frac{hm-\mu}{2\epsilon}\right)\right]x(t) + \frac{h^2\lambda s}{\epsilon}\Gamma}{1 + \frac{h\lambda s\mu}{2\epsilon}}$$

Note that this equation is the discrete-time counterpart of equation (5). To compute it, we only need to know the evolution of $x(t)$ between 0 and ϵ. Zambelli generally assumes in his articles that x is at equilibrium during this time and that there is a shock at some point after at least ϵ time has elapsed to classify the return to equilibrium. Using this equation, derived from his discretized system, we find the same oscillations as we do using our analytical solution made of components.

The code used to make the figures is available at `https://gist.github.com/vcarret/2c0832e815dcf1f7918fbfd140d57ba5`

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