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A SOLUTION TO THE TWO-PERSON IMPLEMENTATION PROBLEM*

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Abstract

We propose strike mechanisms as a solution to the classical problem of Hurwicz and Schmeidler [1978] and Maskin [1999] according to which, in two-person societies, no Pareto efficient rule is Nash-implementable. A strike mechanism specifies the number of alternatives that each player vetoes. Each player simultaneously casts these vetoes and the mechanism selects randomly one alternative among the unvetoed ones. For strict preferences over alternatives and under a very weak condition for extending preferences over lotteries, these mechanisms are deterministic-in-equilibrium. They Nash implement a class of Pareto efficient social choice rules called *Pareto-and-veto* rules. Moreover, under mild richness conditions on the domain of preferences over lotteries, any Pareto efficient Nash-implementable rule is a Pareto-and-veto rule and hence is implementable through a strike mechanism.

Keywords: Nash implementation, Two players, Pareto efficiency.

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1 Introduction

Can one design some protocol that ensures that two players reach a Pareto efficient agreement in equilibrium? The theorems of Hurwicz and Schmeidler [1978] and Maskin [1999], at the outset of implementation theory, provide a negative answer to this question: no deterministic mechanism, except dictatorship, can guarantee that every Nash equilibrium is Pareto efficient. In fact, there is a tension between the conditions for the existence of an equilibrium at every preference profile and those which ensure that each outcome is Pareto efficient. We refer to this impossibility as the *two-person implementation problem*.

We propose a solution to this problem based on "strike" mechanisms. A strike mechanism endows each player i with v_i vetoes to be distributed among the alternatives, with $v_1 + v_2$ being equal to the number of alternatives minus one, so that at least one alternative remains unvetoed. The game is simultaneous and the outcome is the uniform lottery over the unvetoed alternatives.

By allowing lotteries, we introduce a modification of the mechanisms used in general for implementation but, as we shall prove, lotteries do not materialize at equilibrium; they only act as off-equilibrium threats. From a mechanism design perspective, we therefore consider Nash implementation through deterministic-in-equilibrium mechanisms or simply **DE** mechanisms.

The idea of introducing off-equilibrium threats is already present in the implementation literature.[1] More precisely, Sanver [2006] allows for off-equilibrium awards, Bochet [2007] considers lotteries whereas Benoît and Ok [2008] consider mechanisms with awards and mechanisms with lotteries off-equilibrium. But these papers consider three players or more while we consider the two-person case. This aspect is important since the characterizations of Moore and Repullo [1990] and Dutta and Sen [1991] jointly with the mentioned impossibility results suggest that, with two players, "exact implementation is very demanding, at least in the absence of domain restrictions" as Abreu and Sen [1991] puts it, whereas implementable rules

[1] Randomization off-equilibrium is used in other branches of economic theory. See for instance Ederer et al. [2018] for recent work in the theory of incentives where similar techniques are used as a strategy to combat gaming by better informed agents.

of interest exist with three or more players. Another difference is that our work, rather than relying on integer games,[2] builds games –the strike mechanisms– whose rules are simple and explicitly based on vetos.

Since we deal with lotteries, the notion of Pareto efficiency needs some qualification (see Bogomolnaia and Moulin [2001] for a discussion). Two classical definitions are ex-ante and ex-post Pareto efficiency. A lottery is ex-ante Pareto efficient if no other lottery Pareto dominates it, whereas it is ex-post Pareto efficient if no alternative that can be selected by the lottery is Pareto dominated by some other alternative. While we show that the possibility of ex-ante Pareto efficient implementation cannot be hoped for, we establish that ex-post Pareto efficient implementation is possible, by **DE** mechanisms, as soon as preferences over alternatives are strict.[3]

Our main result is that a SCR is Pareto efficient and Nash-implementable by a **DE** mechanism if and only if it is a *Pareto-and-veto* rule: for some pair of integers $v = (v_1, v_2)$, with $v_1 + v_2 + 1$ being the number of alternatives, it selects all Pareto efficient alternatives that are not among the v_i worst alternatives for each player i. The Pareto-and-veto rule with vector v is denoted $\textsc{pv}_v$.[4] We show that the strike mechanism with vetoes v_1 and v_2 Nash implements $\textsc{pv}_v$.

The study of the strike mechanism is made possible by the fact that the best-response reasoning is straightforward in this game. Given the vetoes cast by her opponent, a player can induce any unvetoed alternative as the outcome by adequately casting her vetoes. Thus, her best response amounts to select her best element among the unvetoed alternatives.

We prove that, when preferences are strict, the equilibria of this game are pure and strict. Then, the nice feature of best responses has three consequences for equi-

[2]Jackson [2001] summarizes some views on the limits of these games.

[3]The current results do not extend to the setting where the players are indifferent among several alternatives. Indeed, as proved by Sanver [2006], no selection of Pareto set is (Maskin) monotonic and hence can be implemented.

[4]It is not the first time that Pareto-and-veto rules are found to be of interest in the literature. Moulin [1983] defines $\textsc{pv}$ under the name "veto core" (Chapter 6.5). Abreu and Sen [1991] (pp. 1016-17) present this class of rules as the main example that is virtually implementable but fails to be Nash-implementable. In a setting where monetary transfers are allowed, Sanver [2018] designs a direct veto mechanism that implements alternatives which are Pareto efficient and preferable to some disagreement outcome by both players.

librium behavior. First, each veto mechanism is **DE**, with a unique alternative remains unvetoed in equilibrium. Second, any equilibrium outcome is Pareto efficient. Third, the equilibrium strategies have a natural shape: if x is the implemented alternative and v_i is the number of vetoes, player i vetoes all alternatives preferred to x by her opponent (say k alternatives) and she vetoes also $v_i - k$ among the alternatives less preferred than x by her opponent. If both strategies veto disjoint sets of alternatives, this forces each player to accept her opponent's strategy. In any equilibrium, this is case: the players veto disjoint sets of alternatives and only one alternative, the implemented one, remains unvetoed.

The simple shape of the best responses also has consequences for out-of equilibrium behavior. As we shall prove, in the considered game, iterated best responses converge to the equilibrium, and they do that quickly: in less than n steps.

All these results hold under the standard von Neumann and Morgenstern expected utility framework and are even more general than that. Indeed, they remain true under a mild condition that we term "best-element bias": for any set of alternatives, a player prefers the (sure) lottery that consists of her most preferred element in the set to any lottery with support in the same set. One can formally define the output of the mechanism as a set of alternatives rather than a lottery, then results equivalent to the presented ones are achieved, as will be shown, under mild hypothesis on how preferences over alternatives extend to preferences over sets.

At this point, we have described a solution to the two-person implementation problem: a mechanism that implements a Paretian SCR. We then show that, in some sense, there cannot be a different solution. This necessity part is more involved. Here, the key concept is the veto power generated by a mechanism: a mechanism μ endows player i with veto power over some set X of alternatives if and only if player i has some strategy that prevents any alternative in X to be selected with positive probability whatever her opponent plays. As we show, any mechanism μ that ensures Pareto efficient outcomes must endow each player i with veto power over every set of alternatives whose cardinality does not exceed some integer v_i^μ with $v_1^\mu + v_2^\mu + 1$ being the number of alternatives. This is a strong result which almost directly entails that only sub-correspondences of PV_v are Nash-implementable. The necessity is established on a domain of preference extensions over lotteries that is rich enough to include specific extended preferences that we label "priority" extensions. In words, a "priority" extended preference is defined by the property that whenever all the elements of a set X are preferred to all elements outside X, any lottery that put some weight (however small) on some element of X is preferred to any lottery that puts no weight on X. For instance, the domain of vNM preferences satisfies this requirement.

We also identify a set of conditions that characterize the class of Pareto-and-veto rules which, thus, turn to be necessary and sufficient for two-person Nash implementability with **DE** mechanisms. These conditions are Pareto efficiency, Maskin monotonicity, neutrality-on-its-vetoes (a weakening of neutrality) and the intersection property which is the key distinction between two-player and many-player implementation models. These conditions are independent as shown in the Appendix. As such, our conditions are weaker than the necessary and sufficient conditions identified by Moore and Repullo [1990] and Dutta and Sen [1991] for two-person Nash implementability without **DE** mechanisms, as their conditions coincided with dictatorship over the full domain of preferences.

The structure of the paper is as follows: Section 2 introduces the basic notions and Section 3 presents the strike mechanisms. Section 4 presents the outcomes of these mechanisms while Section 5 tackles the necessity issue. Section 6 provides impossibility results that highlight the limitations of ex-ante implementation through **DE** mechanisms. Section 7 shows how the current results are related to the classical characterization of implementable social choice rules with two players. Section 8 presents a review of the various other ideas that have emerged in the literature to bypass the Hurwicz-Schmeildler impossibility of Paretian implementation and makes some concluding remarks.

2 Basic notions and notation

A set $N = \{1, 2\}$ of two players faces a finite set A of $n+1 \geq 3$ alternatives. We write $\mathcal{A} = 2^A$ for the power set of A and $\overline{\mathcal{A}} = \mathcal{A} \setminus \{\varnothing, A\}$. The set of probability distributions (or "lotteries") over A is denoted $\Delta = \{p : A \to [0,1] \mid \sum_{x\in A} p(x) = 1\}$. For each lottery $p \in \Delta$, we let $\textsc{supp}(p) = \{x \in A \mid p(x) > 0\}$ denote the support of p. For each $X \in \mathcal{A}$, $p[X] = \sum_{x\in X} p(x)$ stands for the probability that p selects an alternative in X. Let Δ^{uni} denote the set of all uniform probability distributions over the non-empty subsets of A. Slightly abusing notation, we let $\{x\}$ denote both the singleton set consisting of alternative x and the lottery that selects x with probability one.

The set of linear orders over A is denoted by $\mathcal{L}_A$ and its generic element $\succ_i$ is the preference of $i \in N$.[5] The set of (strict) preference profiles over A is $\mathcal{L}_A^2 = \mathcal{L}_A \times \mathcal{L}_A$ with $\succ = (\succ_1, \succ_2)$ denoting a generic preference profile. We write

$$\textsc{pe}(\succ) = \{x \in A \mid \nexists y \in A : \forall i \in N, y \succ_i x\}$$

for the set of **Pareto efficient** alternatives at $\succ \in \mathcal{L}_A^2$. Let $L(x, \succ_i) = \{y \in A : x \succ_i y\}$ be the (strict) lower contour set and $U(x, \succ_i) = \{y \in A : y \succ_i x\}$ be the (strict) upper contour set of $x \in A$ at $\succ_i \in \mathcal{L}_A$.

A **social choice rule** (SCR) is a mapping $f : \mathcal{L}_A^2 \to \mathcal{A} \setminus \{\emptyset\}$. A SCR is Pareto efficient iff $f(\succ) \subseteq \textsc{pe}(\succ)$ for all $\succ \in \mathcal{L}_A^2$. We say that f is a sub-correspondence of g and write $f \subseteq g$ whenever $f(\succ) \subseteq g(\succ)$ for all $\succ \in \mathcal{L}_A^2$.

In general, a **mechanism** is a function $\mu : \mathcal{M} \to \Delta$ with $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ where $\mathcal{M}_i \neq \varnothing$ is the message space of $i \in N$. In order to properly define the game associated to μ, we do not need to extend preferences over the whole Δ but simply over $\mu(\mathcal{M}) := \{p \in \Delta \mid p = \mu(m) \text{ for some } m \in \mathcal{M}\}$, the range of μ. In this paper, we only consider mechanisms with finite ranges.[6] For example, the set of uniform lotteries over A, denoted $\Delta^{\text{uni}} = \{p \in \Delta \mid p(x) = p(y) \text{ for any } x, y \in \textsc{supp}(p)\}$ is finite. The strike mechanisms, which play the central role in this work, have Δ^{uni} as their range.

We let $\mathcal{P}_{\mu(\mathcal{M})}$ stand for the set of binary relations over $\mu(\mathcal{M})$. A typical element of

[5]More precisely, one of $x \succ_i y$ and $y \succ_i x$ holds for any distinct $x, y \in A$ while $x \succ_i x$ fails for all $x \in A$. Moreover, $x \succ_i y$ and $y \succ_i z$ implies $x \succ_i z$ for all $x, y, z \in A$.

[6]While our results still hold extending over the whole simplex, the richness condition **PREX** becomes harder to satisfy. We would like to thank Bhaskar Dutta for pointing this out.

$\mathcal{P}_{\mu(\mathcal{M})}$ is denoted $\succeq_i^*$ with $\succ_i^*$ being its strict part. We say that $\succeq_i^*$ is an **extension** of $\succ_i$ when $x \succ_i y \Longrightarrow \{x\} \succeq_i^* \{y\}$, $\forall x, y \in A$ with $\{x\}, \{y\} \in \mu(\mathcal{M})$.

For a mechanism $\mu : \mathcal{M} \to \Delta$ and a preference profile over lotteries $\succeq^* = (\succeq_1^*, \succeq_2^*)$, a **Nash equilibrium** is a pair of messages $(m_1, m_2) \in \mathcal{M}$ such that, for all $m_1' \in \mathcal{M}_1$ and all $m_2' \in \mathcal{M}_2$, $\mu(m_1, m_2) \succeq_1^* \mu(m_1', m_2)$ and $\mu(m_1, m_2) \succeq_2^* \mu(m_1, m_2')$. Let $\mathcal{N}^\mu(\succeq^*)$ denote the set of Nash equilibria of the mechanism μ at the profile $\succeq^*$.

We now turn to the question of the domain of preferences to be considered. As already mentioned we work under the assumption that preferences over alternatives are strict, but we are flexible as to the way preferences are extended from alternatives to lotteries. Since there are many ways to do so, we use a notion of admissible extended preferences. Let $\kappa(\succ_i) \subseteq \mathcal{P}_\Delta$ be a set of admissible preferences over lotteries of player i associated with $\succ_i \in \mathcal{L}_A$. Abusing notation, let $\kappa(\succ) \subseteq \mathcal{P}_\Delta^2$ be the set of admissible preference profiles over Δ associated with the preference profile $\succ = (\succ_1, \succ_2)$. Such a correspondence κ that associates to each preference a set of extended preferences (and to each profile of preference a set of profiles of extended preferences) is called a **domain of preference extensions**. Throughout the paper we use the property of Best-element bias: a player with a best-element bias prefers the (sure) lottery that selects her best element in X to any (considered) lottery with support in X.

Best-element bias: Let $\succ_i \in \mathcal{L}_A$ be a strict preference on A, and let $\bar{\Delta} \subseteq \Delta$ be a set of lotteries. An extension $\succeq_i^*$ of $\succ_i$ **exhibits the best element bias in** $\bar{\Delta}$ when for any $X \in \mathcal{A}$ with $\#X > 1$ and any $x \in X$, if $x \succ_i y$ for any $y \in X \setminus \{x\}$, then $\{x\} \succ_i^* p$ for all $p \in \bar{\Delta}$ with $\textsc{supp}(p) \subseteq X$ and $p \neq \{x\}$.

A domain κ is said to satisfy the best element bias (in short: κ satisfies **BEB**) in $\bar{\Delta}$ if, for any strict preference $\succ \in \mathcal{L}_A$, any extension $\succeq_i^* \in \kappa(\succ)$ exhibit the best element bias in $\bar{\Delta}$. Note that **BEB** is satisfied by virtually all domain of preference extensions that are considered in the literature, including the von Neumann and Morgenstern domain.

Given a domain κ, a mechanism μ is **admissible** iff for all $\succ \in \mathcal{L}_A^2$ and all $\succeq^* \in \kappa(\succ)$, $\mathcal{N}^\mu(\succeq^*) \neq \varnothing$. It is deterministic-in-equilibrium (**DE**) iff for all $\succ \in \mathcal{L}_A^2$, all $\succeq^* \in \kappa(\succ)$, and all $m \in \mathcal{N}^{\mu}(\succeq^*)$, $\#\text{supp}(\mu(m)) = 1$. It **Nash-implements** the SCR $f : \mathcal{L}_A^2 \to \mathcal{A}$ iff for all $\succ \in \mathcal{L}_A^2$ and all $\succeq^* \in \kappa(\succ)$, $f(\succ) = \bigcup_{m \in \mathcal{N}^{\mu}(\succeq^*)} \textsc{supp}(\mu(m))$. Note that if μ Nash-implements some SCR f, then μ is admissible.

3 The strike mechanism

3.1 Definition

A strike mechanism endows each player $i \in N$ with a non-negative number v_i of vetoes, with $v_1 + v_2 = n$. The set

$$\mathcal{M}_i = \{X \subseteq A \mid \#X = v_i\}$$

represents the sets of alternatives i can veto, and $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ is the joint message space.

Definition 1. *The **strike mechanism** $\mu_v : \mathcal{M} \to \Delta^{uni}$ associates to each pair of messages $m = (m_1, m_2)$, the lottery $\mu_v(m)$ that is uniform over the set*

$$\textsc{supp}(\mu_v(m)) = A \setminus (m_1 \cup m_2).$$

In other words, an alternative is uniformly drawn from the unvetoed alternatives. Note that, as $v_1 + v_2 = n$, the set $m_1 \cup m_2$ contains at most n elements, so that $\textsc{supp}(\mu_v(m))$ is always non-empty. Our results would remain unaffected under an alternative specification of the strike mechanism in which the mechanism selects one among the unvetoed alternatives according to *any* probability distribution with full support over these alternatives.

In order to study the mechanism μ_v, we introduce the following notation. Let

$$g_v(\mathcal{M}_i, m_j) = \{X \in \mathcal{A} \mid \textsc{supp}(\mu_v(m_i, m_j)) = X \text{ for some } m_i \in \mathcal{M}_i\}$$

denote the attainable set for player i at m_j under μ_v. This set contains the different supports of the uniform lotteries that player i can induce when player j selects m_j

under the strike mechanism μ_v. Because of the number of vetoes at her disposal, player i can choose the support of the outcome by adequately casting her vetoes as described by the following result:

Lemma 1. *For each player i and each strategy $m_j \in \mathcal{M}_j$, the attainable set of the strike mechanism μ_v equals:*

$$g_v(\mathcal{M}_i, m_j) = \{X \subseteq A \setminus m_j \mid 1 \leq \#X \leq \min\{n+1-v_i, n+1-v_j\}\}.$$

Proof. Take some player i and some strategy $m_j \in \mathcal{M}_j$. Take first the case with $v_i < v_j$ so that $n+1-v_j < n+1-v_i$. We want to prove that for each non-empty $X \subseteq A \setminus m_j$ (hence with $\#X \leq n+1-v_j$), there is some $m_i \in \mathcal{M}_i$ with $\textsc{supp}(\mu_v(m_i, m_j)) = X$. Note that each non-empty subset of $A \setminus m_j$ is of the form $A \setminus (m_j \cup C)$ with $0 \leq \#C \leq v_i$. Thus, it suffices to pick m_i such that $m_i \setminus m_j = C$ which ensures that $\textsc{supp}(\mu_v(m_i, m_j)) = A \setminus (m_i \cup m_j) = A \setminus (m_j \cup C)$, as required. In the case $v_i \geq v_j$, take m_i with $m_i \setminus m_j = C$. Since $v_i \geq v_j$, it follows that $\#C \geq v_i - v_j$ and hence for each non-empty $X \subseteq A \setminus m_j$ with $\#X \leq n+1-v_j-(v_i-v_j) = n+1-v_i$, there is some $m_i \in \mathcal{M}_i$ with $\textsc{supp}(\mu_v(m_i, m_j)) = X$. $\square$

3.2 Best responses

Lemma 1's main implication is that player i can induce any singleton in $A \setminus m_j$ as the support of the outcome: formally, for any player i and any alternative $x \in A$:

$$x \in A \setminus m_j \Longrightarrow \{x\} \in g_v(\mathcal{M}_i, m_j).$$

Best responses can thus be easily described, as done in the following statement. For each strategy m_j of player j, let $x_i(m_j)$ be i's preferred unvetoed alternative so that $\{x_i(m_j)\} = \arg\max_{X \setminus m_j} \succ_i$.

Proposition 1. *Let the domain κ satisfies **BEB** in the range of μ_v. For each strategy m_j of her opponent, player i has a unique best response to m_j, denoted $m_i^*(m_j)$, with*

$$m_i^*(m_j) = X \setminus (m_j \cup \{x_i(m_j)\}) \quad \textit{and} \quad \mu_v(m_i^*(m_j), m_j) = \{x_i(m_j)\}.$$

Proof. The assumption that preferences over alternatives are strict implies that $x_i(m_j)$ is well-defined, and the assumption that the preferences over lotteries satisfy **BEB** implies that $x_i(m_j)$ is preferred to any other possible outcome. Lemma 1 indicates that $x_i(m_j)$ belongs to the attainable set $g_v(\mathcal{M}_i, m_j)$ for any $m_j \in \mathcal{M}_j$. It is thus the unique best possible outcome, and it is obtained by eliminating all other unvetoed alternatives, as indicated. □

3.3 Equilibrium

The first consequence of this property is that strike mechanisms are deterministic in equilibrium as long as the domain satisfies **BEB**.

Theorem 1. *For any strike mechanism μ_v, if the domain κ satisfies **BEB** in the range of μ_v, then μ_v is **DE**.*

Proof. Assume that there is some equilibrium $m = (m_1, m_2)$ with $\#\textsc{supp}(\mu_v(m)) > 1$. Consider some player i and some alternative $x \in \textsc{supp}(\mu_v(m))$ with $x \succeq_i y$ for all $y \in \textsc{supp}(\mu_v(m))$. Since $x \in A \setminus m_j$, Lemma 1 shows that $\{x\} \in g_v(\mathcal{M}_i, m_j)$. Thus, there is some $m_i' \in \mathcal{M}_i$ with $\mu_v(m_i', m_j) = \{x\}$. Furthermore, $\{x\} \succ_i^* \mu_v(m)$ due to **BEB**, which contradicts that m is an equilibrium. □

Since a strike mechanism is **DE**, no uncertainty remains in equilibrium: players veto disjoint sets of alternatives and a unique alternative is selected.

3.4 Best responses dynamics

As mentioned above, the equilibria of the considered game are pure and strict. This ensures that the usual game-theoretical refinement criteria are satisfied. However, what does this imply concerning the use of veto mechanisms in laboratory experiments or in real-life applications? Fudenberg and Levine [2016] argue that an equilibrium often fails to arise from introspection, but rather emerges from some non-equilibrium learning dynamics. Moreover, as they write, "in laboratory games do not usually resemble Nash equilibrium (except in some special cases); instead, there is abundant experimental evidence that play in many games moves toward equilibrium as subjects play the game repeatedly and receive feedback" (see Chan et al. [2017] for a recent treatment).

We consider the simplest learning dynamics: iterated best responses. Since there may be multiple equilibria in the game associated to a strike mechanism, there is no hope that the synchronous best response dynamics converge necessarily. If (m_1, m_2) and (m_1', m_2') are two different equilibria then the sequence

$$(m_1, m_2'), (m_1', m_2), (m_1, m_2'), (m_1', m_2), \ldots$$

is such that each player best-responses to her opponent's previous moves, but they never coordinate (This remark is very general: it holds for any two player game with multiple equilibria). We thus consider alternate best response dynamics and show that these processes lead to our equilibria. That point underlines the relevance of our mechanisms in applied settings.

Theorem 2. *Let the domain κ satisfies **BEB** in the range of μ_v. Let $m^0, m^1, m^2, m^3, \ldots$ be a sequence of messages from alternating players such that for all $t > 0$, m_t is a best response to m_{t-1}. Then (m_t, m_{t+1}) converges to an equilibrium in at most n steps: for all $t > n$, $m_{t+2} = m_t$.*

Proof. Say, for instance, and without loss of generality, that player 2 plays m^0, m^2, etc. For any $t \geq 1$, m^t is a best response (for player 1 if t is odd and for player 2 if t is even) to m^{t-1}. So $\#m^t$ is equal to v_1 for t odd and to v_2 for t even.

As stated in Proposition 1, thanks to our strict preferences assumption (**BEB**), best responses are unique. Precisely, when player i is facing a veto on the v_j alternatives m^{t-1}, her best response is to pick her unique preferred alternative among the remaining set $A \setminus m^{t-1}$ and to veto the other v_i alternatives. Thus the whole sequence is uniquely defined by its first element m^0, and we have, for any $t \geq 1$:

$$m^{t-1} \cap m^t = \emptyset. \tag{1}$$

Let r_t for $t \geq 1$ denote the outcome at date t; this is the unique alternative such that:

$$m^{t-1} \cup m^t \cup \{r_t\} = A.$$

By definition, both m^t and m^{t+2} contain v_j alternatives. However, as previously mentioned, m^t and m^{t+1} are disjoint, and so are m^{t+1} and m^{t+2}. Therefore, both m^t and m^{t+2} contain v_j alternatives from the set $A \setminus m^{t+1}$, which contains $n - v_i$ alternatives. Thus, since $v_i + v_j = n - 1$, m^t and m^{t+2} differ on at most one alternative. If $m^t = m^{t+2}$, an equilibrium is reached. If $m^t \neq m^{t+2}$ then m^t and m^{t+2} differ on one alternative exactly.

The following property of the best response correspondence is used in our proof of convergence. Suppose that one alternative, say a, is erased from the set A. In case $a \in m^{t-1}$, the best response to $\tilde{m}^{t-1} = m^{t-1} \setminus \{a\}$ is the same m^t. In case $a \notin m^{t-1}$ and $a \in m^{t+1}$, the best response to $\tilde{m}^{t-1} = m^{t-1}$ is $\tilde{m}^t = m^t \setminus \{a\}$, and is the best response of the same player in the modified game where a is not available and the player has one veto less.

We now prove that the sequence of best responses leads to an equilibrium in at most n iterations. Let a_n denote the worst alternative for player 1. If for some $k \geq 0$, $a_n \notin m^{2k}$, then the best response for player 1 implies to veto a_n, that is: $a_n \in m^{2k+1}$. This in turn implies (see (1)) that player 2 does not veto a_n at date $2k+2$. It follows that the following chain holds: for all $t \geq 0$

$$a_n \notin m^t \implies a_n \in m^{t+1} \implies a_n \notin m^{t+2} \implies \ldots$$

Consequently, a_n belongs either to all m^t for t odd and starting at 1 (call this case 1), or to all m^t for t even and starting at 2 (call this case 2).

Now consider the sequence of sets $\tilde{m}^t = m^t \setminus \{a_n\}$ for all $t \geq 1$. We claim that this new sequence is again a sequence of alternating best responses in the game where the set of alternatives is $A \setminus \{a_n\}$ and the numbers of vetoes are, in case 1, $v' = (v_1 - 1, v_2)$ and in case 2, $v' = (v_1, v_2 - 1)$. This is true in case 1 because, in the original sequence, player 1 always had to veto a_n that is her worst alternative and player 2 never had to block a_1 that is never available to her. This is also true in case 2 because, in the original sequence, player 1 never had to veto a_n that was never available to her, and player 2 always had to veto a_n.

The same logic applies to the worst element for the other player as well. The argument can be repeated for player 1 or for player 2 until all vetoes are exhausted and about the sequences starting at m^1 then at $\tilde{m}^2$, then at $\tilde{\tilde{m}}^3$, etc. It follows that in the original sequence, for all $t \geq n$, $m^t = m^{t+2}$. We conclude that the iterative process of alternate best responses converges to an equilibrium in at most n steps. $\square$

4 Pareto efficient implementation

The previous section has studied the game-theoretical properties of the proposed mechanism. We now study its outcomes.

4.1 The Pareto-and-veto correspondence $\textsc{pv}_v$

Definition 2. *For any $v = (v_1, v_2) \in \{0, 1, \ldots, n\}^2$ with $v_1 + v_2 = n$, the **Pareto-and-veto** rule $\textsc{pv}_v : \mathcal{L}_A^2 \to \mathcal{A}$ is the SCR:*

$$\textsc{pv}_v(\succ) = \overbrace{\textsc{pe}(\succ)}^{\textit{Pareto}} \cap \overbrace{\underbrace{\{x \in A \mid \#L(x, \succ_1) \geq v_1\}}_{\textit{Best } n - v_1 \textit{ alternatives for 1}} \cap \underbrace{\{x \in A \mid \#L(x, \succ_2) \geq v_2\}}_{\textit{Best } n - v_2 \textit{ alternatives for 2}}}^{\textit{Veto}}.$$

The Pareto-and-veto rule $\textsc{pv}_v$ picks all Pareto efficient alternatives with a lower-contour set at least as large as v_i for every player i. The next Table fully describes the outcome of $\textsc{pv}_{(1,1)}$ in the case of three alternatives. In the Table, lines represent the preferences of player 1 and the columns the preferences of player 2, where for short abc stands for $a \succ_i b \succ_i c$ and so on.

	abc	acb	bac	bca	cab	cba
abc	a	a	$\{a,b\}$	b	a	b
acb	a	a	a	c	$\{a,c\}$	c
bac	$\{a,b\}$	a	b	b	a	b
bca	b	c	b	b	c	$\{b,c\}$
cab	a	$\{a,c\}$	a	c	c	c
cba	b	c	b	$\{b,c\}$	c	c

Table 1: The Pareto-and-veto rule $\text{PV}_{(1,1)}$.

Our first observation is that PV_v is non empty when $v_1 + v_2 \leq n$. To see this, just observe that eliminating n alternatives at most, out of $n+1$, leaves at least one, say a. If a is Pareto efficient, we are done. If not, a is Pareto-dominated by some $a' \in \text{PE}_v$, but since a' is at least as good as a for player i, a is still among her $n - v_i$ best alternatives. As soon as $v_1 + v_2$ is at least $n+1$, the example of completely opposed preferences shows that PV_v can be empty.

4.2 Implementation of PV_v

We now turn to the implementation of PV_v: the strike mechanism with veto vector v Nash-implements the Pareto-and-veto rule with the same veto vector v.

Theorem 3. *Let the domain κ satisfy **BEB** in Δ^{uni}. For any $v \in \{0, \ldots, n\}^2$ with $v_1 + v_2 = n$, the strike mechanism μ_v Nash-implements the Pareto-and-veto rule PV_v .*

Proof. (i) In order to check the inclusion $\text{PV}_v(\succ) \supseteq \bigcup_{m \in \mathcal{N}^{\mu_v}(\succeq^*)} \text{SUPP}(\mu_v(m))$, consider any equilibrium m. By Theorem 1 the support of $\mu_v(m)$ is a singleton $\{x\}$. Because player i can always veto her worst v_i alternatives in the mechanism μ_v any best response outcome, and thus any equilibrium outcome x is such that $\#L(x, \succ_i) \geq v_i \; \forall i \in N$. So x satisfies the veto conditions in the definition of PV. It remains to show that $x \in \text{PE}(\succ)$. Suppose not, i.e., there exists $y \in A$ with $y \succ_i x$ for all $i \in N$. Since $\mu_v(m) = \{x\}$, we have $m_1 \cap m_2 = \emptyset$. Thus, $y \in m_i$ for some $i \in N$, say $i = 1$, without loss of generality. It follows that $y \in A \setminus m_2$ and thus $\{y\} \in g_v(\mathcal{M}_1, m_2)$. Therefore, $\mu_v(m_1', m_2) = \{y\}$ for some m_1' and as $\{y\} \succ_1^* \mu_v(m) = \{x\}$, we contradict $m \in \mathcal{N}^{\mu_v}(\succeq^*)$.

(ii) For the reverse inclusion, take $x \in \mathrm{PV}_v(\succ)$. Because x is Pareto-optimal, any of the n other alternatives is either strictly better than x for one and only one player or strictly worse for both. So counting these $n = v_1 + v_2$ alternatives we obtain:

$$v_1 + v_2 = \#U(x, \succ_1) + \#U(x, \succ_2) + \#\left(L(x, \succ_1) \cap L(x, \succ_2)\right). \tag{2}$$

By definition of PV_v, $v_1 \leq \#L(x, \succ_1) = n - \#U(x, \succ_1)$. Therefore $v_2 \geq \#U(x, \succ_1)$, which means that player 2 has enough vetoes to block all the alternatives that player 1 strictly prefer to x. The same holds for player 1 with respect to player 2. Writing Equation 2 as:

$$[v_1 - \#U(x, \succ_2)] + [v_2 - \#U(x, \succ_1)] = \#\left(L(x, \succ_1) \cap L(x, \succ_2)\right),$$

one can see that it is possible to have players 1 and 2 respectively veto $v_1 - \#U(x, \succ_2)$ and $v_2 - \#U(x, \succ_1)$ different alternatives in $L(x, \succ_1) \cap L(x, \succ_2)$, so that all n alternatives are vetoed by one player or the other.

Let m_1 and m_2 be such strategies. We now prove that, under **BEB**, m_1 is a strict best response to m_2. To this end, recall that $U(x, \succ_1) \subseteq m_2$: any alternative strictly preferred by player 1 to x is vetoed by player 2. So when player 1 deviates to $m_1' \in \mathcal{M}_1$, the support $A \setminus (m_1' \cup m_2)$ of the outcome lottery excludes $U(x, \succ_1)$. Because of the constraints on the number of vetoes, $\mu(m_1', m_2) = \{x\}$ is impossible for $m_1' \neq m_1$. Therefore, for player 1, the support of $\mu(m_1', m_2)$ either contains only alternatives that are strictly worse than x, or contains x and some other alternatives that all are worse than x. By **BEB**, player 1 strictly prefers $\{x\}$ to such outcomes, so m_1 is the unique best response to m_2. The same holds for the other player, so that we proved that x is an equilibrium outcome. □

5 On the necessity of vetoes

This section shows that, under some richness conditions concerning the domain of preference extensions over lotteries, any Pareto efficient SCR that is implementable through a **DE** mechanism is a Pareto-and-veto rule.

We now define some conditions on the domain κ to be used throughout. The first one restricts admissible extensions in the same spirit as the **BEB** condition. A player with a worst-element bias (or simply **WEB**) prefers any lottery with support in X to the (sure) one that selects her worst element in X.

Worst-element bias: Let $\succ_i \in \mathcal{L}_A$ be a strict preference on A, and let $\bar{\Delta} \subseteq \Delta$ be a set of lotteries. An extension $\succeq_i^*$ of $\succ_i$ **exhibits the worst element bias in** $\bar{\Delta}$ when for any $X \in \mathcal{A}$ with $\#X > 1$ and any $x \in X$, if $y \succ_i x$ for any $y \in X \setminus \{x\}$, then $p \succ_i^* \{x\}$ for all $p \in \bar{\Delta}$ with $\textsc{supp}(p) \subseteq X$ and $p \neq \{x\}$.

As in the case of **BEB**, **WEB** is satisfied by virtually all preference extensions over lotteries.[7] We say that a domain κ satisfies **WEB** in $\bar{\Delta}$ iff **WEB** is satisfied in $\bar{\Delta}$ for all $\succeq^* \in \kappa(\succ)$, for all $\succ \in \mathcal{L}_A^2$.

The next condition, Priority Extension, deals with the richness of the domain of preference extensions. For any lottery $p \in \Delta$, we write $p[\cdot \succeq x] = \sum_{y:y \succeq x} p(y)$ to refer to the probability, according to p, of obtaining an alternative weakly preferred to x according to $\succ$.

Priority extension: Let $\succeq_i^*$ extend $\succ_i$ and let $x \in A$, the extension $\succeq_i^*$ is a priority extension **(PREX)** of $\succ_i$ for x in $\bar{\Delta}$ iff given any two lotteries $p, q \in \bar{\Delta}$, if $p[\cdot \succeq x] > 0$ and $q[\cdot \succeq x] = 0$, then $p \succ^* q$.

The interpretation of this property is clear: under a priority extension, each alternative is used as a grading benchmark: The individual prefers to reach the benchmark x, even with a tiny probability, than not reaching it. The argument "What is the best alternative I have some chance to obtain with that lottery?" has priority over the precise values of the probabilities. We say that a domain κ satisfies **PREX** in $\bar{\Delta}$ iff for all $\succ \in \mathcal{L}_A^2$, there is some $\succeq^* \in \kappa(\succ)$ that is a priority extension of $\succ$ in $\bar{\Delta}$ for all $x \in A$.[8]

[7] In fact, **BEB** and **WEB** are satisfied if one considers the well-known preference extension axioms of the literature (such as Gärdenfors [1976] or Kelly [1977]) and deduces preferences over lotteries through the preferences over their supports. If κ satisfies **BEB** and **WEB** (which are universally quantified), every sub-correspondence of κ satisfies them as well.

[8] Note that if x is bottom-ranked in $\succ$, there is no lottery q with $q[\cdot \succeq x] = 0$, so that any extension is (vacuously) a priority extension for x.

Here is an example of a domain of extension that satisfies the condition in the set Δ^{uni} of uniform lotteries. Similar examples can be found for any finite set of lotteries. Consider the correspondence $\kappa^{\text{vNM}} : \mathcal{L}_A \to \mathcal{P}_{\Delta^{\text{uni}}}$ that allows any von Neumann and Morgenstern extension of $\succ$. In other words, for $\succ \in \mathcal{L}_A$, $\kappa^{\text{vNM}}(\succ)$ is the set of all $\succ^* \in \mathcal{P}_{\Delta^{\text{uni}}}$ such that there exists a vector $u \in \mathbb{R}^A$ with $a \succ b \iff u_a > u_b$ for all $a, b \in A$ and:

$$\forall p, q \in \Delta^{\text{uni}},\ p \succ^* q \iff \sum_{a \in A} p(a) u_a > \sum_{a \in A} q(a) u_a.$$

The domain $\kappa^{\text{vNM}}(\succ)$ contains priority extensions of $\succ$ to Δ^{uni}. To see this, label the alternatives in A according to the preference: $a_{n+1} \succ a_n \succ \ldots \succ a_1$ and let $u_{a_k} = (n+1)^k$ for any $a_k \in A$. Take any pair $p, q \in \Delta^{\text{uni}}$ with $p[\cdot \succeq a_k] > 0$ and $q[\cdot \succeq a_k] = 0$ for some a_k. The expected value of p, that is $\sum_{a \in A} p(a) u_a$, reaches its minimum when the lottery contains in its support a_k but no better alternative according to $\succeq$ (and hence has k alternatives in its support). The expected value $\sum_{a \in A} p(a) u_a$ is at least

$$\frac{u_{a_k}}{k} > \frac{u_{a_k}}{k+1} = \frac{(n+1)^k}{k+1} \geq (n+1)^{k-1}.$$

The expected value of q, $\sum_{a \in A} q(a) u_a$, reaches its maximum when $q = \{a_{k-1}\}$ and hence its value is at most $(n+1)^{k-1}$. Therefore, for any $a_k \in A$, $p[\cdot \succeq a_k] > 0$ and $q[\cdot \succeq a_k] = 0$ implies that $p \succ^* q$. Thus, uniform lotteries are ordered following the priority rule.

We are now ready to state the counterpart to Theorem 3, according to which, if one wants to implement a Pareto efficient SCR through a **DE** mechanism, the SCR must be a Pareto-and-veto rule. Precisely we prove the following:

Theorem 4. *Let f be a Pareto efficient SCR that is Nash-implementable by a* ***DE*** *mechanism μ on a domain κ. Let the domain κ satisfy* ***BEB***, ***WEB*** *and* ***PREX*** *in the range of μ. Then $f = \textsc{pv}_v$ for some $v \in \{0, \ldots, n\}^2$ with $v_1 + v_2 = n$.*

To prepare for the proof we provide two lemmas. For each player i, let

$$\text{veto}(\mu, i) = \{X \in \mathcal{A} \mid \exists m_i \in \mathcal{M}_i \text{ s.t. } \textsc{supp}(\mu(m_i, m_j)) \cap X = \emptyset \text{ for all } m_j \in \mathcal{M}_j\},$$

denote the veto set for player i. When $X \in \text{veto}(\mu, i)$, we say that player i has veto power over the set X of alternatives, i.e., she has a strategy that ensures that no alternative in this set belongs to the support of the outcome independently of the strategy of her opponent. We first state a result on the structure of the veto power that **DE** mechanisms generate.

Lemma 2. *Under the hypothesis of Theorem 4, for any partition* $\{X, Y\}$ *of* A *with* $X \in \overline{\mathcal{A}}$*, either* $Y \in \text{veto}(\mu, 1)$ *or* $X \in \text{veto}(\mu, 2)$ *but not both.*

Proof. Let $\mu : \mathcal{M} \to \Delta$ be admissible and **DE** and let $X \in \overline{\mathcal{A}}$. Write $Y = A \backslash X$. Pick some $\succ \in \mathcal{L}_A^2$ such that $\forall x \in X$, $\forall y \in Y$, $x \succ_1 y$ and $y \succ_2 x$. The existence of such preference $\succ$ is ensured by our assumption that the domain contains all strict preferences on alternatives. Take also $\succeq^* \in \kappa(\succ)$ such that $p \succ_1^* q$ for all $p, q \in \mu(\mathcal{M})$ with $p[X] > 0$ and $q[X] = 0$, and such that $p \succ_2^* q$ for all $p, q \in \mu(\mathcal{M})$ with $p[Y] > 0$ and $q[Y] = 0$. The existence of such extended preference $\succeq^*$ is ensured by **PREX**. Now suppose, for a contradiction, that $Y \notin \text{veto}(\mu, 1)$ and $X \notin \text{veto}(\mu, 2)$. Because μ is admissible and **DE**, there exists an equilibrium $m = (m_1, m_2) \in \mathcal{N}^\mu(\succeq^*)$ with $\mu(m) = \{a\}$ for some $a \in A$. Two cases are possible:

- If $a \in X$. As $Y \notin \text{veto}(\mu, 1)$, $\exists m_2' \in \mathcal{M}_2$ such that $\textsc{supp}\big(\mu(m_1, m_2')\big) \cap Y \neq \varnothing$, hence $\mu(m_1, m_2') \succ_2^* \{a\}$ due to **WEB**, contradicting $m \in N^\mu(\succeq^*)$.
- If $a \in Y$. As $X \notin \text{veto}(\mu, 2)$, $\exists m_1' \in \mathcal{M}_1$ such that $\textsc{supp}\big(\mu(m_1', m_2)\big) \cap X \neq \varnothing$, hence $\mu(m_1', m_2) \succ_1^* \{a\}$, again contradicting $m \in \mathcal{N}^\mu(\succeq^*)$.

Thus, $Y \in \text{veto}(\mu, 1)$ or $X \in \text{veto}(\mu, 2)$. Because the mechanism is well-defined, it is impossible that a set belongs to $\text{veto}(\mu, 1)$ and its complement belongs to $\text{veto}(\mu, 2)$. Therefore either $Y \in \text{veto}(\mu, 1)$ or $X \in \text{veto}(\mu, 2)$ but not both.

□

The next lemma shows that we can restrict attention to mechanisms that are "neutral on their vetoes". A mechanism μ is **neutral on its vetoes** for player i iff for any $X \in \mathcal{A}$ and any permutation $\rho : A \to A$, $X \in \text{veto}(\mu, i) \iff \rho(X) \in \text{veto}(\mu, i)$. This does not mean that any player has any veto power ($\text{veto}(\mu, i)$ can be empty) nor does it mean that the μ is neutral (μ does not have to treat alternative in a symmetric way), it just means that if a set with a given cardinality belongs to veto(μ, i) then any other set with the same cardinality belongs to veto(μ, i) as well. Note that a player with veto power over X has also veto power over any $X' \subset X$. Hence, the veto set for player i can be written as:

$$\text{veto}(\mu, i) = \{X \in \mathcal{A} \mid \#X \leq v_i\},$$

where the integer v_i stands for the cardinality of the highest cardinality set over which i has veto power.

Lemma 3. *Under the hypothesis of Theorem 4, μ is neutral on its vetoes for both players.*

Proof. Let $X \in \text{veto}(\mu, 1)$, $x \in X$ and $x' \in A \setminus X$.[9] Thus, there exists $m_1 \in \mathcal{M}_1$ that vetoes X. The set $X' = X \setminus \{x\} \cup \{x'\}$ has the same cardinality as X. Write $Y = A \setminus (X \cup \{x'\})$, so that we have a partition

$$A = (X \setminus \{x\}) \cup \{x\} \cup \{x'\} \cup Y.$$

Suppose, for a contradiction, that $X' \notin \text{veto}(\mu, 1)$. Lemma 2 then implies that $Y \cup \{x\} \in \text{veto}(\mu, 2)$. Therefore there exists $m_2 \in \mathcal{M}_2$ that vetoes $Y \cup \{x\}$. Since x' is neither vetoed by m_1 nor by m_2, $\mu(m_1, m_2) = \{x'\}$. Now consider a unanimous preference profile $\succ = (\succ_1, \succ_2)$ such that $x \succ_i x' \succ_i y$ for all $y \neq x, x'$and for $i = 1, 2$. For this preference profile, the second-best alternative x' is Pareto-dominated by x but, at (m_1, m_2), both players veto x. Thus, no unilateral deviation can obtain, with any probability, a better outcome than x'. Thanks to **BEB**, that implies that (m_1, m_2) is a Nash equilibrium, in contradiction with the Pareto efficiency assumption.

The proof of the proposition is established by noting that given any $X, X' \in \overline{\mathcal{A}}$ with $\#X = \#X'$, there is a finite sequence of sets $X = X_1, \ldots, X_s = X'$ with $\#(X_i \cap X_{i+1}) = \#X - 1$ for each $i \in \{1, \ldots, s-1\}$ and applying repeatedly the argument above. □

We can now proceed to a complete proof of the Theorem.

[9] The two extreme cases $\text{veto}(\mu, 1) = \{\{\emptyset\}\}$ and $A \in \text{veto}(\mu, 1)$ are trivial.

Proof of Theorem 4. **(1)** We first establish the existence of v such that $f \subseteq \text{PV}_v$. Let $\mathcal{M}$ be the joint message space of μ that **DE**-implements f. Take any preference profile $(\succ_1, \succ_2)$ and let $\succ_i^*$ be a priority extension of $\succ_i$ at all $x \in A$. Thus, for all $p, q \in \mu(\mathcal{M})$ and for $i = 1, 2$, if $p[\cdot \succ_i x] > 0$ and $q[\cdot \succ_i x] = 0$ then $p \succ_i^* q$. Take any $x \in f(\succ)$. By assumption, μ admits a Nash equilibrium (m_1, m_2) with $\mu(m_1, m_2) = \{x\}$. By the definition of an equilibrium, player 2 has no better response to m_1 than m_2. However, under $\succ_2^*$, a deviation m_2' is profitable for player 2 iff $\text{SUPP}(\mu(m_1, m_2')) \cap U(\succ_2, x) \neq \emptyset$. Therefore:

$$\forall m_2 \in \mathcal{M}_2, \ \text{SUPP}(\mu(m_1, m_2)) \cap U(\succ_2, x) = \emptyset$$

and likewise for player 1. In other words, m_1 makes the set $U(\succ_2, x)$ unattainable for player 2 under μ. We say that m_1 **gives player** 1 **veto power on** the set $U(\succ_2, x)$, and likewise for player 2.

$$U(\succ_2, x) \in \text{veto}(\mu, 1), \ U(\succ_1, x) \in \text{veto}(\mu, 2).$$

From Lemma 3, if a player has veto power on some set, she has also veto power on any set of the same cardinality. Let v_i be the largest number of outcomes that i can veto. For the mechanism to be well-defined, one needs $v_1 + v_2 \leq n$, so that not all the $n+1$ alternatives can be vetoed simultaneously. Now consider the case of opposed preferences: for any $x \in A$, $U(\succ_1, x) \cap U(\succ_2, x) = \emptyset$. The existence of a deterministic equilibrium (an equilibrium with a singleton outcome) in that case shows that $v_1 + v_2 \geq n$. Hence $v_1 + v_2 = n$.

Clearly, an outcome that would be among the v_i worse alternatives for a player i cannot be an equilibrium outcome under μ because i could then veto her v_i worse alternatives. Due to **WEB**, a player prefers any lottery with support not included in the v_i worst alternatives to any lottery that selects (surely) one of the worst v_i alternatives. Hence f being implementable imposes the required veto conditions on the ranks of the implemented alternatives in the individual preferences. Since we assumed that f is also efficient, we obtain $f \subseteq \text{PV}_v$.

(2) For this v, we now prove the reverse inclusion. Given $\succ = (\succ_1, \succ_2)$, let $x \in \textsc{pv}_v(\succ)$. Consider the profile $\succ'$ defined as follows.

Label the $n+1$ alternatives in two ways: $a_{n+1} \succ_1 a_n \succ_1 \ldots \succ_1 a_1$ and $b_{n+1} \succ_2 b_n \succ_2 \ldots \succ_2 b_1$. Write $a_{w_1} = b_{w_2} = x$. The veto conditions in the definition of $\textsc{pv}_v$ are that $w_1 > v_1$ and $w_2 > v_2$, which implies that:

$$a_{n+1} \succ_1 \ldots \succ_1 a_{w_1} = x \succ_1 \ldots \succ_1 a_{v_1} \succ_1 \ldots \succ_1 a_1,$$
$$b_{n+1} \succ_2 \ldots \succ_2 b_{w_2} = x \succ_2 \ldots \succ_2 b_{v_2} \succ_2 \ldots \succ_2 b_1.$$

The preference $\succ_1'$ is obtained by lowering the ranks of all those, among the alternatives a_{v_1+1}, ..., a_{w_1-1}, which are preferred to x by the other player, player 2. If $w_1 = v_1 + 1$ we simply let $\succ_1' = \succ_1$. If $w_1 \geq v_1 + 2$, consider the set

$$H_1 = \left\{a_{v_1+1}, \ldots, a_{w_1-1}\right\} \cap \{b_{w_2+1}, \ldots, b_{n+1}\}$$

and observe that

$$\# H_1 \leq n - w_2 \leq n - v_2 = v_1.$$

Starting from $\succ_1$, we define $\succ_1'$ by switching in the ranking the first elements a_1,... $a_{\# H_1}$ with the elements of H_1, where a_1 is switched with the most preferred element of H_1 of player 1, a_2 is switched with the second most preferred element of H_1 of player 1 and so on...

We now claim that if $x \in f(\succ_1', \succ_2)$ then $x \in f(\succ)$. Let μ **DE**-implement f. If $x \in f(\succ_1', \succ_2)$, there exists a pure strategy equilibrium (m_1', m_2') for the game with preferences $(\succ_1', \succ_2)$ with $\{x\} = \mu(m_1', m_2')$. With the initial preferences $(\succ_1, \succ_2)$, m_2 is also a best response since player 2 does not change her preference, and m_1' is also a best response for player 1 because her preferences differ only below x. As previously argued, m_2 gives player 2 veto power on the set $U(\succ_1, x)$. Since $U(\succ_1', x) = U(\succ_1, x)$ by construction, it follows that the support of any lottery that player 1 can attain given m_2 is included in $A \setminus U(\succ_1, x)$. Hence, due to **BEB**, m_1 is a best response for player 1 since $\mu(m_1, m_2) = \{x\}$. Therefore this equilibrium for $(\succ_1', \succ_2)$ is also an equilibrium for $(\succ_1, \succ_2)$, that is: $x \in f(\succ') \implies x \in f(\succ)$.

The same construction for player 2 yields the preference profile $\succ'' = (\succ'_1, \succ'_2)$ with the property:

$$x \in f(\succ'') \implies x \in f(\succ). \tag{3}$$

But notice that, by construction of $\succ'_1$, all the alternatives y such that $y \succ_2 x$ are now among the v_1 worse alternatives according to $\succ'_1$. Therefore x is the preferred alternative, according to $\succ'_2$, among the alternatives in the intersection of the top $n - v_1$ alternatives for player 1 and $n - v_2$ alternatives for player 2 in $\succ'$. Since the same is true for the other player, we find that x is the unique Pareto optimum in the alternatives among the top $n - v_1$ alternatives for player 1 and the top $n - v_2$ alternatives for player 2 in $\succ''$. Since f itself is assumed to be efficient and is selecting in PV_v, we obtain that $f(\succ'') = \{x\}$. From (2) it follows that $x \in f(v)$ as requested. □

Theorem 4 shows the existence of a strong link between implementation through **DE** mechanisms and veto power. Indeed, it shows that under the conditions **BEB**, **WEB**, and **PREX**, a SCR has to admit some veto structure in order to be both Pareto efficient and implementable. This theorem is related to the impossibility result by Hurwicz and Schmeidler [1978] in the following sense. Hurwicz and Schmeidler [1978] show that the only SCRs which are both Pareto efficient and implementable (through a deterministic mechanism) are the dictatorial ones. Note that a dictatorial SCR corresponds to PV_v with $v = (n, 0)$ (if player 1 is the dictator) or $v = (0, n)$ (if player 2 is the dictator). Our theorem shows that by allowing lotteries as off-equilibrium punishments, the Pareto-and-veto rules appear as a class of intermediate and, interestingly, non dictatorial SCRs.

Note that PV_v is neutral for any $v \in \{0..., n\}^2$ and that it is anonymous if and only if $v_1 = v_2$. Thus, under the assumptions of Theorem 4, the following observations trivially follow. With an odd number of alternatives, an anonymous, neutral and Pareto efficient SCR f is Nash-implementable by a **DE** mechanism iff f is a Pareto-and-veto rule with $v_1 = v_2$. On the contrary, with an even number of alternatives, there exist no anonymous, neutral and Pareto efficient SCR that is Nash-implementable by a **DE** mechanism.

6 Ex-ante Pareto efficiency

This section shows that ensuring ex-ante Pareto efficient equilibria through **DE** mechanisms is in general not possible. It presents two separate impossibility results for two notions: ex-ante efficiency for mechanisms (Section 6.1) and for SCRs (Section 6.2). The first one shows that no ex-ante Pareto efficient admissible mechanism ensures a positive veto power to each player. The second one proves that any ex-ante Pareto efficient and implementable SCR is a dictatorship.

6.1 Ex-ante efficient mechanisms

Ex-ante efficiency means that efficiency is observed at the level of lotteries, before their realization. Received knowledge on this issue (see, for instance, Börgers and Postl [2009]) highlights that ex-ante efficiency is difficult to obtain. An example is published (Núñez and Laslier [2015]) of an ex-ante Pareto efficient two-player mechanism for three alternatives; this mechanism, called Approval mechanism, is not **DE** and fails to be efficient for four alternatives or more. The existence of a non-**DE** efficient mechanism for many alternatives remains an open problem.

The difficulty can be described by the following argument. Let $A = \{a, b, c\}$ with $a \succ_1 b \succ_1 c$ and $c \succ_2 b \succ_2 a$. Consider the strike mechanism that gives one veto to each player. If the domain κ satisfies **BEB**, the unique equilibrium outcome is b. Now, assume that both players prefer a non degenerate lottery with support $\{a, c\}$ to the pure outcome b. This is the case when both players extend their preference over alternatives to uniform lotteries through expected utility and their intensity of preference for b is low. In this case, the unique equilibrium outcome is Pareto dominated by a lottery, that is a possible outcome of the mechanism, therefore non dictatorial ex-ante Pareto efficiency cannot be reached with deterministic outcomes. Our first result is a negative result, that generalizes this observation to veto rules, as studied in this paper.

Instead of social choice rules, defined on profiles of preferences over pure alternatives, we are here dealing with **social lottery rules** (SLR), defined on profiles of preferences over lotteries. For such a preference profile, $\succeq^*$, the SLR F defines a set of lotteries $F(\succeq^*) \subseteq \Delta$. We will consider SLRs that are defined on the same domains that were used in the previous sections: preferences over pure alternatives are strict, and all strict preferences are admitted, and the preferences on lotteries are described by a product correspondence κ.

For a mechanism μ and a profile of preferences over alternatives $\succeq^*$, let $F_\mu(\succeq^*)$ denote the set of **Nash outcomes**: $F_\mu(\succeq^*) = \{\mu(m) : m \in \mathcal{N}_\mu(\succeq^*)\}$. This is a subset of $\mu(\mathcal{M})$, the range of μ. **A mechanism μ is ex-ante Pareto efficient** on the domain κ if for any $\succ \in \mathcal{L}_A^2$ and any $\succeq^* \in \kappa(\succ)$ there is no $p \in \mu(\mathcal{M})$ and $q \in F_\mu(\succeq^*)$ such that $p \succeq_i^* q$ for all i with at least one strict preference. Say that μ is **DE** at $\succeq^*$ if all its Nash outcomes are deterministic, that is, with our loose notation: $F_\mu(\succeq^*) \subseteq A$. A mechanism μ is a **dictatorship** iff there is some $i \in N$ such that for each $x \in A$, there exists $m_i \in \mathcal{M}_i$ such that $\mu(m_i, m_j) = \{x\}$ for all $m_j \in \mathcal{M}_j$.

Theorem 5. *Let the domain κ satisfy **PREX** and **WEB**. On κ, any admissible **DE** mechanism that is ex-ante Pareto-efficient is a dictatorship.*

Proof. Suppose first that the mechanism μ is not purely deterministic, that is there exists a strategy combination $m^* \in \mathcal{M}$ and two distinct alternatives $a_1, a_{n+1} \in A$ such that $\{a_1, a_{n+1}\} \subseteq \textsc{supp}(\mu(m^*))$. Since μ is **DE**, it follows that $F^\mu(\succeq^*) \subseteq A$. Write $A = \{a_1, a_2, ..., a_{n+1}\}$ (recall that $n+1 \geq 3$) and consider the opposed preferences $\succ = (\succ_1, \succ_2)$ with $a_1 \succ_1 a_2 \succ_1 ... \succ_1 a_{n+1}$ and $a_{n+1} \succ_2 a_n \succ_2 ... \succ_2 a_1$. Let the players' preferences over lotteries, $\succ_1^*$ and $\succ_2^*$, be such that, for any $p, q \in \mu(\mathcal{M})$, if $p[\cdot \succeq_1 a_1] > 0$ and $q[\cdot \succeq_1 a_1] = 0$ then $p \succ_1^* q$, and if $p[\cdot \succeq_2 a_{n+1}] > 0$ and $q[\cdot \succeq_2 a_{n+1}] = 0$ then $p \succ_2^* q$. Such a profile exists because the domain κ satisfies **PREX**. Therefore, since $\{a_1, a_{n+1}\} \subseteq \textsc{supp}(\mu(m^*))$, $\mu(m^*) \succ_i^* \{x\}$ for $i = 1, 2$ and any $x \neq a_1, a_{n+1}$. Since μ is ex-ante Pareto efficient, it follows that $F^\mu(\succeq^*) \subseteq \{a_1, a_{n+1}\}$.

Therefore, at this profile, the mechanism μ admits either $\{a_1\}$ or $\{a_{n+1}\}$ or both as equilibrium outcome. Assume w.l.og. that μ admits some equilibrium $\tilde{m}$ with $\mu(\tilde{m}) = \{a_1\}$. By definition of equilibrium, $\{a_1\} = \mu(\tilde{m}) \succeq_2^* \mu(\tilde{m}_1, m_2')$ for any $m_2' \in \mathcal{M}_2$. Yet, since κ satisfies **WEB**, then $p \succ_2^* \{a_1\}$ for all $p \in \mu(\mathcal{M})$ with $p \neq \{x\}$. Therefore, $\mu(\tilde{m}_1, m_2') = \{a_1\}$ for any $m_2' \in \mathcal{M}_2$. It follows that for every $a \in A$, there is either some $m_1 \in \mathcal{M}_1$ such that $\mu(m_1, m_2') = \{a\}$ for any $m_2' \in \mathcal{M}_2$ or some $m_2 \in \mathcal{M}_2$ such that

$\mu(m'_1, m_2) = \{a\}$ for any $m'_1 \in \mathcal{M}_1$. It follows that either for every $a \in A$ there is some $m_1 \in \mathcal{M}_1$ such that $\mu(m_1, m'_2) = \{a\}$ for any $m'_2 \in \mathcal{M}_2$ or for every $a \in A$ there is some $m_2 \in \mathcal{M}_2$ such that $\mu(m'_1, m_2) = \{a\}$ for any $m'_1 \in \mathcal{M}_1$. In the first case, player 1 is the dictator and in the second case player 2 is the dictator.

Suppose now that there is no $m^* \in \mathcal{M}$ such that $\textsc{supp}(\mu(m^*)) \supset \{x, y\}$ for some pair of alternatives $\{x, y\} \in A$. Then, for any $m \in M$, $\mu(m) \in A$ so that μ is a deterministic mechanism and $\mu(\mathcal{M}) \subseteq A$. Thus, for any $\mathcal{N}^{\mu}(\succeq^*) = \mathcal{N}^{\mu}(\succeq)$ for any $\succeq^* \in \kappa(\succ)$ and any $\succ \in \mathcal{L}^2_A$. Hence, ex-ante Pareto efficient is equivalent to Pareto efficiency w.r.t. $\succ$. Thus, the two-person implementation problem (as stated by Hurwicz and Schmeidler [1978] and Maskin [1999]) applies: the only mechanisms that are admissible and Pareto efficient are dictatorships. $\square$

6.2 Ex-ante efficiency of implementable social choice rules

The literature on implementation has concentrated on social choice rules (SCRs) which, by definition use only ordinal information: a preference profile $\succ$ on alternatives, and not a preference profile $\succeq^*$ over lotteries. Since we consider mechanisms that can outcome lotteries, some definitions are useful in order to make the link with this literature.

So consider a SCR f: for all $\succ \in \mathcal{L}(A)$, $f(\succ) \subseteq A$. A mechanism μ that is **DE** on a domain κ is said to implement the SCR f on κ iff:

$$\forall \succ \in \mathcal{L}(A),\ \forall \succeq^* \in \kappa(\succ),\ F_{\mu}(\succeq^*) = f(\succ).$$

Note that, for a mechanism to implement a social choice rule, it is required that the outcomes of the mechanism not only are deterministic, but also are independent of the precise preferences over lotteries. The following definition presents a concept of ex-ante Pareto efficient SCR that is suitable for the study of the implementation of SCRs by mechanisms that can output lotteries. It should not be confused with the concept of an ex-ante Pareto efficient mechanism defined above.

Given a set of lotteries $\overline{\Delta} \subseteq \Delta$ **a SCR f is ex-ante Pareto efficient in the range $\overline{\Delta}$** iff given any $\succ \in \mathcal{L}^2_A$ and any $\succeq^*$ in $\kappa(\succ)$, any $X \in f(\succ)$ and any $x \in X$, there is no $p \in \overline{\Delta}$

such that $p \succeq_i^* x$ for all $i \in N$ with at least one strict preference.

We show that the notions of ex-ante Pareto efficiency and admissibility clash, hence extending the two-player implementation problem to the setting with lotteries and **DE** mechanisms. This shows that ex-ante Pareto efficiency is too restrictive in our setting.

Theorem 6. *Let f be a SCR that is Nash-implementable by a **DE** mechanism μ on a domain κ. Suppose that κ satisfies **PREX** and **WEB** in the range of μ. If f is ex-ante Pareto efficient in the range of μ, then μ is a dictatorship.*

Proof. Let f be an ex-ante Pareto efficient SCR that is Nash-implementable by a **DE** mechanism μ on a domain κ. Borrowing the vocabulary of Hurwicz and Schmeidler [1978], we think of the mechanism μ as a matrix where player 1 controls rows and player 2 controls columns. We hence write, for every $x \in X$, an $\{x\}$-row is a row that contains only $\{x\}$ as an outcome and similarly for an $\{x\}$-column.

Take any profile $\succ \in \mathcal{L}_A^2$. Let a and b respectively denote the best outcomes for player 1 and 2 at $\succ$. Take $\succ^* \in \kappa(\succ)$ such that $p \succ_1^* q$ for all $p, q \in \mu(\mathcal{M})$ with $p(a) > 0$ and $q(a) = 0$ and such that $p \succ_2^* q$ for all $p, q \in \mu(\mathcal{M})$ with $p(b) > 0$ and $q(b) = 0$. The existence of $\succ^*$ is ensured by **PREX**. Take any alternative $x \neq a, b$. According to $\succ^*$ both players strictly prefer a lottery with support $\{a, b\}$ to the pure alternative x. Ex-ante Pareto efficiency thus implies that $x \notin f(\succ)$. Indeed, if $x \in f(\succ)$, then μ admits an equilibrium m^* with $\mu(m^*) = \{x\}$ (since μ is **DE**). However, both players prefer the lottery $\{a, b\}$ to x, contradicting ex-ante Pareto efficiency. So $f(\succ) \subseteq \{a, b\}$. Thus, an ex-ante Pareto optimal and admissible **DE** mechanism gives equilibrium outcomes from the union of tops.

Now consider a preference profile $\succ$ where the players' preferences are completely opposed. Relabel the alternatives as a_1, a_2,..., a_{n+1}. Take a preference profile where a_1 and a_2 are respectively the best and last alternatives for player 1 while a_2 and a_1 are, respectively, the best and last alternatives for player 2. So the equilibrium outcomes of μ belong to $\{a_1, a_2\}$. Note that μ is **DE**, so no lottery with support $\{a_1, a_2\}$ is an equilibrium outcome. Let, without loss of generality, a_1 be an equilibrium outcome. This is the worst element for player 2 and also the worst lottery (due to **WEB**), hence player 1 must have an $\{a_1\}$-row.

Now, take a preference profile where a_2 and a_3 are, respectively, the best and last alternatives for player 1 while a_3 and a_2 are the respective top and bottom outcomes for player 2. So the equilibrium outcomes of μ belong to $\{a_2, a_3\}$. We first show that a_3 cannot be an equilibrium outcome. Suppose it is. As a_3 is the worst element and lottery for player 1, player 2 must have an $\{a_3\}$-column, due to **WEB**, which contradicts player 1 has an $\{a_1\}$-row. As a result, a_2 is an equilibrium outcome and we argue, mutatis mutandis, player 1 has an a_2-row.

Iterate by making the arguments for $a_3, a_4, \ldots, a_n, a_{n+1}$, proves that for each $a \in A$, player 1 has an $\{a\}$-row, showing that player 1 is the dictator. Repeating the argument assuming that a_2 is an equilibrium outcome shows that player 2 is the dictator. □

7 Connections to two-player Nash implementation theory

A complete characterization of Nash implementable SCRs with two players was independently achieved by both Moore and Repullo [1990] and Dutta and Sen [1991]. In order to clarify the connection between our results and these characterizations, we quote condition β of Dutta and Sen [1991] (whose equivalent version is called condition μ_2 in Moore and Repullo [1990]) which is necessary and sufficient for a SCR to be Nash implementable with two players.

For any $i \in N$, let $\widetilde{L}(x, \succ_i) = L(x, \succ_i) \cup \{x\}$ be the weak lower contour set of $x \in A$ at $\succ_i \in \mathcal{L}_A$ and $M(C, \succ_i) = \{a \in C \mid a \succ_i c \ \forall c \in C \setminus \{a\}\}$ be the singleton set containing the maximal elements of $C \subseteq A$ with respect to $\succ_i \in \mathcal{L}_A$.

Definition 3. *A SCR f satisfies condition β iff there exists a set A^* which contains the range of f, and for each $i \in N$, $\succ \in \mathcal{L}_A^2$ and $a \in f(\succ)$, there exists a set $C_i(a, \succ) \subseteq A^*$, with $x \in C_i(a, \succ) \subseteq \widetilde{L}(a, \succ_i)$ such that for all $\succ' \in \mathcal{L}_A^2$, we have:*

(i) (a) for all $b \in f(\succ')$, $C_1(a, \succ) \cap C_2(b, \succ') \neq \emptyset$.
(b) Moreover, there exists $x \in C_1(a, \succ) \cap C_2(b, \succ')$ such that if for some $\succ'' \in \mathcal{L}_A^2$, $x \in M(C_1(a, \succ), \succ''_1) \cap M(C_2(b, \succ'), \succ''_2)$, then $x \in f(\succ'')$.

(ii) if $a \notin f(\succ')$*, then there exist* $j \in N$ *and* $b \in C_j(a,\succ)$ *such that* $b \notin \widetilde{L}(a,\succ')$*.*

(iii) $M_i\left(C_i(a,\succ) \setminus \{a\},\succ'\right) \cap M_j(A^*,\succ') \subseteq f(\succ')$ $\forall i \in N$ *and* $j \neq i$*.*

(iv) $M(A^*,\succ_1') \cap M(A^*,\succ_2') \subseteq f(\succ')$*.*

Without restrictions on the domain of preferences, only dictatorial SCRs satisfy condition β (in line with the impossibility results of Hurwicz and Schmeidler [1978] and Maskin [1999]). As Moore and Repullo [1990] notes, parts (ii), (iii) and (iv) of condition β are necessary and sufficient for Nash implementation with three or more players. Among these, part (ii) corresponds to Maskin monotonicity; part (iv) is a unanimity condition while part (iii) is a relaxation of the no-veto power condition. On the other hand, condition (i) is central for the situation with two players. However, $(i)(a)$, which has been referred to as a self-selection constraint or simply *intersection property* (see Abreu and Sen [1991] for a discussion) turns out to be a critical condition for different implementation concepts such as virtual implementation (Abreu and Sen [1991]) or implementation with partially honest players (Dutta and Sen [2012]). Busetto and Colognato [2009] has shown that the different parts of condition β exhibit problems of logical dependence.

For the sake of precision, we introduce the definitions of the intersection property and Maskin monotonicity formally, respectively implied by conditions $\beta(i)(a)$ and $\beta(ii)$.

Definition 4. *A SCR* f *satisfies the intersection property (IP) iff for all* $\succ,\succ' \in \mathcal{L}_A^2$ *and* $x,y \in A$ *with* $x \in f(\succ)$ *and* $y \in f(\succ')$*, we have* $\widetilde{L}(x,\succ_i) \cap \widetilde{L}(y,\succ_j') \neq \emptyset$ *for any* $i \in N \setminus \{j\}$*.*

Definition 5. *A SCR* f *satisfies Maskin monotonicity (MM) iff for all* $\succ,\succ' \in \mathcal{L}_A^2$ *and* $x \in A$ *with* $\widetilde{L}(x,\succ_i) \subseteq \widetilde{L}(x,\succ_i')$ $\forall i \in N$*, we have* $x \in f(\succ) \Longrightarrow f(\succ')$*.*

We first observe that the necessity of MM and IP prevails when **DE** mechanisms are used. Theorem 4 has shown that when the domain satisfies **BEB**, **WEB** and **PREX**, a Pareto efficient SCR that is Nash-implementable by a **DE** mechanism is a Pareto-and-veto rule. Thus, the necessity of MM and IP for **DE** mechanisms can be seen by establishing that Pareto-and-veto rules satisfy both conditions:

Proposition 2. *For any veto vector v, the Pareto-and-veto rule* $\textsc{pv}_v$ *satisfies IP and MM.*

Proof. In order to check IP, for any veto vector v, take any pair $\succ, \succ' \in \mathcal{L}_A^2$ and any $x \in \textsc{pv}_v(\succ)$ and $y \in \textsc{pv}_v(\succ')$. By definition of $\textsc{pv}_v$, $\#\widetilde{L}(x, \succ_i) \geq v_i + 1$ and $\#\widetilde{L}(y, \succ_j) \geq v_j + 1$ for $j \neq i$. However, since $v_1 + v_2 = n$, it follows that $\#\widetilde{L}(x, \succ_i) + \#\widetilde{L}(y, \succ_j) \geq n + 2$ and hence $\widetilde{L}(x, \succ_i) \cap \widetilde{L}(y, \succ_j') \neq \emptyset$, which shows that IP holds.

In order to check MM, for any veto vector v, take any $\succ \in \mathcal{L}_A^2$ and any $x \in \textsc{pv}_v(\succ)$. Let $\succ' \in \mathcal{L}_A^2$ be some profile with $L(x, \succ_i) \subseteq L(x, \succ_i')$ $\forall i \in N$. Note that $x \in \textsc{pe}(\succ)$ implies that $x \in \textsc{pe}(\succ')$. Moreover, $\#L(x, \succ_i') \geq \#L(x, \succ_i)$ for each $i \in N$ (by construction of $\succ'$) and $\#L(x, \succ_i) \geq v_i$ $\forall i \in N$ (by the definition of $\textsc{pv}_v$). Thus $x \in \textsc{pv}_v(\succ')$, as desired. □

Interestingly, MM and IP pave the way towards a full characterization of the class of Pareto-and-veto rules.

Definition 6. *Under a SCR f, player i has veto power over the set $X \subset A$ at the profile $\succ \in \mathcal{L}_A^2$ iff for any $Y \subseteq X$, if $z \succ_i y$ $\forall z \in A \setminus Y, \forall y \in Y$ then $f(\succ) \cap Y = \emptyset$. When i has veto power over X for any $\succ \in \mathcal{L}_A^2$, we say that i has veto power over X under f.*

Remark that if player i can veto X under f, she can also veto any subset Y of X whenever Y consists of her least preferred alternatives. When player i does not have veto power over X under f, this means that there exists a profile $\succ$ in which X are the least preferred alternatives of player i in $\succ_i$ and such that $f(\succ) \cap X \neq \emptyset$.

Definition 7. *We say that a SCR f is neutral-on-its-vetoes iff whenever f gives veto power to player i over a set X, f gives veto power to i over every set Y with $\#Y = \#X$.*

Note that when f is neutral-on-its-vetoes, the veto power of player i can be expressed by an integer $v_i \in \{0, ..., n\}$ which is the cardinality of the largest set that i can veto.

Proposition 3. *If f satisfies IP and is Pareto efficient and neutral-on-its-vetoes, then $f \subseteq \textsc{pv}_v$.*

Proof. Take some f which is neutral-on-its-vetoes, satisfies IP and is Pareto efficient. Assume that at some profile $\succ$, $x \in f(\succ)$ with $\#L(x, \succ_i) = k$ for $k = 0, \ldots, n$. IP implies that any profile $\succ'$, $f(\succ') \subseteq A \backslash L(x, \succ_2')$. Therefore, if player 1 cannot prevent x at $f(\succ)$, then player 2 can ensure at any profile that $L(x, \succ_2') \geq n-k$. Again, this observation can be generalized since f neutral-on-its-vetoes: $v_1 = k$ implies that $v_2 = n-k$.

Since f is Pareto-efficient and only selects alternatives such that $v_1 + v_2 = n$, it follows that $f \subseteq \textsc{pv}_v$ as required. □

Proposition 4. *For any veto vector v, $f \subseteq \textsc{pv}_v$ satisfies MM if and only if $f = \textsc{pv}_v$*

Proof. Note that the proof is immediate if either $v_1 = n$ or $v_2 = n$ since, in both cases, $\textsc{pv}_v$ is singleton valued for each preference profile $\succ$. Thus, $f \subseteq \textsc{pv}_v$ directly implies that $f = \textsc{pv}_v$ and hence is MM. We assume that $0 < v_1, v_2 < n$ in the sequel of the proof.

We show first that for any $f \subseteq \textsc{pv}_v$, any $\succ \in \mathcal{L}_A^2$ and any $x \in \textsc{pe}(\succ)$, if $\#L(x, \succ_i) = v_i$ $\forall i \in N$, then $f(\succ) = \{x\}$. Take any $f \subseteq \textsc{pv}_v$, any $\succ \in \mathcal{L}_A^2$ and any $x \in \textsc{pe}(\succ)$ with $\#L(x, \succ_i) = v_i$ $\forall i \in N$. Assume by contradiction that there is some $y \in f(\succ)$ with $y \in A \setminus \{x\}$. Since $f \subseteq \textsc{pv}_v$, $\#L(y, \succ_i) \geq v_i$ $\forall i \in N$, and as $\#L(x, \succ_i) = v_i$ $\forall i \in N$, it follows that $\#L(y, \succ_i) > v_i$ $\forall i \in N$. But this implies that $y \succ_i x$ $\forall i \in N$, contradicting $x \in \textsc{pe}(\succ)$. Hence, since $f(\succ)$ is non-empty, $f(\succ) = \{x\}$ as wanted.

We now show that for any $\succ \in \mathcal{L}_A^2$ and any $x \in \textsc{pv}_v(\succ)$, if f is MM and $f \subseteq \textsc{pv}_v$, then $x \in f(\succ)$. For each $x \in A$ and each veto vector v, let $B_v^x = \{\succ \in \mathcal{L}_A^2 \mid x \in \textsc{pe}(\succ)$ with $\#L(x, \succ_i) = v_i \forall i \in N\}$. Since the preferences are unrestricted, for any partition (X, Y) of $A \setminus \{x\}$ with $\#X = v_i$ and $\#Y = v_j$, there is some $\succ \in B_v^x$ such that $L(x, \succ_i) = X$ and $L(x, \succ_j) = Y$.

As shown before, we know that for any $f \subseteq \textsc{pv}_v$ and any $\succ \in B_v^x$, $f(\succ) = \{x\}$ with $\widetilde{L}(x, \succ_1) \cup \widetilde{L}(x, \succ_2) = A$ since $x \in \textsc{pe}(\succ)$ with $\#L(x, \succ_1) + \#L(x, \succ_2) = n$ (since $v_1 + v_2 = n$) .

Consider now any profile $\succ'$ with $x \in \textsc{pv}_v(\succ')$. Assume by contradiction that $x \notin f(\succ')$. Since $x \in \textsc{pv}_v(\succ')$, it follows that $x \in \textsc{pe}(\succ')$ and $\#L(x, \succ_i') \geq v_i$ for all $i \in N$. Note that there is at least some strict inequality since otherwise $\succ' \in B_v^x$ and hence $x \in f(\succ')$, a contradiction. Since $x \in \textsc{pe}(\succ')$, $L(x, \succ_1') \cup L(x, \succ_2') \cup \{x\} = A$ whereas $\#L(x, \succ_1') + \#L(x, \succ_2') > n$. Since $L(x, \succ_1') \cup L(x, \succ_2') \cup \{x\} = A$, it follows that:

$$\#\Big(L(x, \succ_1') \setminus L(x, \succ_2')\Big) + \#\Big(L(x, \succ_2') \setminus L(x, \succ_1')\Big) + \#\Big(L(x, \succ_1') \cap L(x, \succ_2')\Big) = n = v_1 + v_2, \quad (4)$$

where $L(x,\succ'_1) \cap L(x,\succ'_2)$ denote the set of alternatives that x Pareto dominates. Since $\#L(x,\succ'_1) + \#L(x,\succ'_2) > n$, note that $L(x,\succ'_1) \cap L(x,\succ'_2) \neq \emptyset$.

Since $\#\Big(L(x,\succ'_1) \setminus L(x,\succ'_2)\Big) \leq v_1$[10] and $\#\Big(L(x,\succ'_2) \setminus L(x,\succ'_1)\Big) \leq v_2$, we can find a partition (X,Y) of $A \setminus \{x\}$ with

$$X \subseteq L(x,\succ'_1) \text{ and } Y \subseteq L(x,\succ'_2) \text{ with } X \cap Y = \emptyset, \#X = v_1 \text{ and } \#Y = v_2.$$

It follows that there is some $\succ^* \in B_v^x$ with $\widetilde{L}(x,\succ_i^*) \subseteq \widetilde{L}(x,\succ'_i)$ since $\widetilde{L}(x,\succ_i^*) = v_i + 1$ and $\widetilde{L}(x,\succ'_i) \geq v_i + 1 \ \forall i \in N$. Moreover, $x \in f(\succ^*)$ since $\succ^* \in B_v^x$. Hence MM implies that $x \in f(\succ')$, as desired.

We have shown that any alternative that could be selected by a Pareto-and-veto rule is selected by any MM sub-correspondence which shows the desired result. □

We are now in the position to characterize the class of Pareto-and-veto rules by the conditions of IP, MM, Pareto efficiency and neutrality-on-its vetoes. These conditions are independent as shown in the appendix.

Theorem 7. *A SCR f satisfies IP and is neutral-on-its vetoes, MM and Pareto efficient if and only if f is a Pareto-and-veto rule.*

Proof. Take some f that satisfies IP and is neutral-on-its vetoes, MM and Pareto efficient. Proposition 3 implies that f is a sub-correspondence of a Pareto-a-veto rule. Moreover, Proposition 4 shows that the only MM subcorrespondence of a Pareto-a-veto rule is the Pareto-and-veto rule itself, proving the if claim. The converse implication follows directly from Proposition 2. □

To conclude our comments on the classical Moore-Repullo-Dutta-Sen characterization we point precisely which condition, in this result, is not satisfied by the Pareto-and-veto rules.

Proposition 5. *For any veto vector v, the Pareto-and-veto rule PV_v fails condition $\beta(i)(b)$*

Proof. We provide a proof for three alternatives and for the Pareto-and-veto rule with veto vector $v = (1,1)$. It can be easily generalized to any Pareto-and-veto rule

[10] Note that $L(x,\succ'_1) \leq n$ and $L(x,\succ'_2) \geq v_2$. Hence, $L(x,\succ'_1) - L(x,\succ'_2) \leq n - v_2 = v_1 + v_2 - v_2 = v_1$.

and any number of alternatives. Let $\succ = (\succ_1, \succ_2)$ and $\succ' = (\succ'_1, \succ'_2)$ be two preference profiles such that: (1) $c \succ_1 a \succ_1 b$ and $a \succ_1 b \succ_1 c$ and (2) $b \succ'_1 a \succ'_1 c$ and $c \succ'_1 b \succ'_1 a$. For these profiles, $\mathrm{PV}_v(\succ) = \{a\}$ and $\mathrm{PV}_v(\succ') = \{b\}$. Since $\widetilde{L}(a, \succ_1) = \{a, b\}$ and $\widetilde{L}(b, \succ'_2)$ $= \{a, b\}$ as well, we are going to find a violation of condition $\beta(i)(b)$ for profiles $\succ''$ that are unanimous ($\succ''_1 = \succ''_2$) and in favor of c ($c \succ''_i a$ and $c \succ''_i b$ for $i = 1, 2$), so that $\mathrm{PV}_v(\succ'') = \{c\}$

Since $C_1(a, \succ)$ and $C_2(b, \succ')$ are subsets of $\{a, b\}$ with a non-empty intersection, as stated by condition $\beta(i)(a)$, the following cases have to be considered:

Case 1: $C_1(a, \succ) = C_2(b, \succ)$.
In this case, since $\succ''$ is unanimous, $M(C_1(a, \succ), \succ''_1) = M(C_2(b, \succ'), \succ''_2) \subseteq \{a, b\}$. Therefore c does not belong to the intersection $M(C_1(a, \succ), \succ''_1) \cap M(C_2(b, \succ'), \succ''_2)$, in contradiction with $\beta(i)(b)$.

Case 2: $C_1(a, \succ) = \{a\}$ and $C_2(b, \succ') = \{a, b\}$, or $C_1(a, \succ) = \{a, b\}$ and $C_2(b, \succ') = \{a\}$. Take then $c \succ''_i a \succ''_i b$ for $i = 1, 2$; for this $\succ''$: $M(C_1(a, \succ), \succ''_1) \cap M(C_2(b, \succ'), \succ''_2) = \{a\}$, again a contradiction.

Case 3: $C_1(a, \succ) = \{b\}$ and $C_2(b, \succ') = \{a, b\}$, or $C_1(a, \succ) = \{a, b\}$ and $C_2(b, \succ') = \{b\}$. The same contradiction appears for the unanimous profile such that $c \succ''_i b \succ''_i a$. $\square$

8 Concluding comments

This section provides a short review of the two-player implementation problem (see Dutta [2019] for a recent and complete survey) and some concluding comments on the strike mechanisms.

As argued in the introduction, the pioneering works (Hurwicz and Schmeidler [1978] and Maskin [1999]) provide a provocative result: dictatorships are the only Pareto efficient rules that can be Nash implemented. Their proof builds on three key assumptions: (i) the preference domain is universal (any preference profile is allowed) while implementing mechanisms are (ii) simultaneous and (iii) deterministic.

The literature has explored the consequences of weakening each of these assumption.[11] The first strand relaxes condition (i), Dutta and Sen [1991] and Moore and Repullo [1990] are the key papers in this direction. They identify the domain restrictions under which one can design Pareto efficient and non-dictatorial Nash-implementable rules. While the full characterization is rather complex, the sufficient domain conditions for implementation often rely in the Euclidean space (see Section 5 in Dutta and Sen [1991] for instance); in the current work, we do not impose any structure on the alternatives or the preferences, beyond the assumption that preferences over alternatives are strict.

A second strand is concerned with (ii), that is, replacing simultaneous with dynamic mechanisms. This literature, in which Moore and Repullo [1988], Abreu and Sen [1991] and Herrero and Srivastava [1992] play a key role, shows that introducing an order of play expands the set of implementable rules with more than two players. No characterization of implementable rules via subgame-perfect or via backward induction is available. By altering the notion of implementation (role-robust implementation), De Clippel et al. [2014][12] show that a possibility arises with dynamic vetoes and randomized order of play (see also Barberà and Coelho [2019] who consider the implementation of the fallback-bargaining solution). However, while ex-ante fairness is achieved by randomizing the order of play, ex-post fairness fails. The order of play matters for determining the outcome, creating first, or second, mover advantages. As Moulin [1981] puts it, "voting by veto procedures introduce a strong asymmetry among agents: ... the ordering of the agents has a strong influence on the outcome".

The third and final strand of the literature deals with assumption (iii), as does the current work: it explores the consequences of modifying the type of mechanisms.[13]

[11] Other approaches have modified the rationality notion, using "partial honesty"; see Dutta and Sen [2012] among others.

[12] A classic literature considers sequential voting by veto with many players (see Mueller [1978], Moulin [1981], Bloom and Cavanagh [1986], Felsenthal and Machover [1992] and Anbarci [2006]) where each player is assigned a certain number of vetoes to be distributed freely among the alternatives. See also the rules of k-names in Barberà and Coelho [2010].

[13] See also the papers on approval voting with two players as Núñez and Laslier [2015] and Laslier et al. [2017]. See also Jackson and Sonnenschein [2007] who show that linking decisions (that is, a common decision on several independent problems) can help overcoming incentive constraints in

As mentioned in the introduction, Sanver [2006], Bochet [2007] and Benoît and Ok [2008] exploit the idea of allowing lotteries/awards off-equilibrium. The main idea of these works is that, with at least three players, monotonicity fully characterizes the class of Nash implementable rules under a domain restriction so that the no-veto power condition is dispensable. This is in line with the results present in this paper in which **DE** mechanisms expand the set of implementable rules. Yet, this paper is the first one to consider this idea with two players. This strand of literature is related to the one on virtual implementation, a reformulation of the original implementation problem. A social choice rule is virtually implementable if there exists a game form G, such that for all preference profiles G admits a unique equilibrium outcome (a lottery) which is ε-close to the outcome prescribed by the rule at this preference profile and this holds for every $\varepsilon > 0$. Following this approach, Matsushima [1988] and Abreu and Sen [1991] provide a strong possibility result: with at least three players, any rule is implementable. With two players, the result is more nuanced but some SCRs are virtually implementable, among which the Pareto-and-veto rule described in the current work. However, under the virtual implementation approach, "any alternative can be the outcome of the game as it receives positive probability in the equilibrium lottery" (Bochet and Maniquet [2010]). In other words, in order to virtually implement a social choice rule, one constructs game forms whose equilibrium outcome at every preference profile is a full-support lottery, arbitrarily close to the outcome prescribed by the social choice rule. This represents a threat to the relevance of these solutions since it involves that socially undesirable alternative, even with a small probability, can be selected.

Strike mechanisms arise as a solution to the two-person implementation problem. This solution is obtained by altering two key elements of the classic framework: (i) considering mechanisms that allow in equilibrium pure alternatives and off equilibrium lotteries and (ii) restricting efficiency to the ex-post Pareto notion.

Our class of **DE** mechanisms is a simultaneous version of the dynamic veto mechanisms (see Moulin [1981]) which, by allowing off-equilibrium set-valued outcomes, resolve the unfairness generated by dynamic mechanisms. To see the difference be-

Bayesian collective decision problems.

tween our solution and the one based on dynamic veto mechanisms, consider a dynamic game that allows player 1 to veto $n+1-k$ alternatives and player 2 to veto $k-1$ of the remaining k alternatives, where $k \in \{1, \ldots, n+1\}$. At each preference profile $\succ$, the subgame perfect equilibrium outcome of this game is the most preferred alternative of player 1 among $\text{PV}_v(\succ)$ where $v_1 = n+1-k$ and $v_2 = k-1$. In other words, this dynamic veto mechanism subgame perfect implements a sub-correspondence of PV_v by refining it with respect to the true preference of the first mover. One could argue that fairness here could be achieved by selecting randomly the first-mover. Yet, this needs qualification since this randomization prevents some alternatives to arise as the following example shows. When $A = \{a, b, c, d, e\}$, at the preference profile $a \succ_1 b \succ_1 c \succ_1 d \succ_1 e$ and $c \succ_2 b \succ_2 a \succ_2 d \succ_2 e$, the dynamic veto mechanism which gives 2 vetoes to each voter implements, by alternating first movers, either a or c but excludes b. However, PV_v picks all three of a, b and c. Thus, our simultaneous direct veto mechanisms allow for the implementation of the compromise alternative b whereas their dynamic counterparts fail to do so. This constitutes a strong argument in favor of using simultaneous mechanisms.

We close by noting three limitations of our analysis. First, it is restricted to Nash implementation in pure strategies. Allowing for mixed strategies and exploring the existence of interesting **DE** mechanisms for settings with two or more players seems to be a promising research avenue (see Mezzetti and Renou [2012]). Second, the set of implementable SCRs expands if one considers implementation through non-**DE** mechanisms. Indeed, as long as **BEB** holds, the game-form associated to plurality rule Nash implements the union of tops[14] which selects at each preference profile all alternatives that are top-ranked by at least one player.[15] Third, we have considered implementation through ex-post Pareto efficient **DE** mechanisms. Other notions of efficiency are present in the literature such as stochastic dominance. Whether other SCRs can be Nash implemented through **DE** mechanisms by considering different notions of efficiency remains to be explored.

[14]See Yeh [2008] for an axiomatization of this rule.

[15]In this game form, each player announces a single alternative and one of them is selected randomly. Since it is a dominant strategy to announce one's best alternative, this mechanism is not **DE** as we may have several alternatives selected with positive probability in equilibrium.

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A Independence of the conditions

We discuss in this section the independence of the four conditions, namely MM, IP, neutral-on-its-vetoes (N) and Pareto (P) used to characterize Pareto-and-veto rules.

Lemma A.1. *N, P and MM do not imply IP.*

Proof. Take $f = \mathrm{PV}_v$ with 0 vetoes and consider the profiles $\succ$ with $a \succ_1 b \succ_1 c$ and $b \succ_2 c \succ_2 a$ and $\succ'$ with $a \succ'_1 b \succ'_1 c$ and $b \succ'_2 a \succ'_2 c$. It follows that $f(\succ) = \{a,b,c\}$ and $f(\succ') = \{a,b\}$. Yet, $L(a,\succ_2) = a$ and $L(b,\succ'_1) = \{b,c\}$, contradicting IP. □

Lemma A.2. *N, MM, IP do not imply P.*

Proof. Take f that selects all alternatives not ranked last by some player. . In the profile $\succ$ with $a \succ_1 b \succ_1 c$ and $a \succ_2 b \succ_2 c$, $f(\succ) = \{a,b\}$ while only a is Pareto efficient. □

Lemma A.3. *N, P, IP do not imply MM.*

Proof. This is a direct consequence of Proposition 4. □

Lemma A.4. *MM, P and IP do not imply N.*

Proof. Let $A = \{a,b,c\}$. Let f be the SCR depicted in the following Table. In the Table, the lines represent the preferences of player 1 and the columns the preferences of player 2. For short, abc stands for $a \succ_i b \succ_i c$ and so on. The rule f is constructed as follows. For each i if $\{a\}$, $\{c\}$ or $\{a,c\}$ are ranked last for i, a, c or both are eliminated. The unvetoed alternatives are shown in the Table in parenthesis next to the preferences. Then $f(\succ)$ contains all remaining Pareto efficient alternatives. It is indicated in the corresponding cell of the Table.

For instance $f(bac,acb) = \{b\}$ since player 1 has veto power over $\{a,c\}$ whereas 2 can't veto $\{b\}$. Similarly, $f(cab,acb) = \{a,c\}$ since b is Pareto dominated by both a and c and no player has veto power over $\{b\}$, the common least preferred alternative.

	abc (ab)	acb (acb)	bac (b)	bca (b)	cab (cab)	cba (cb)
abc (ab)	a	a	b	b	a	b
acb (acb)	a	a	b	b	$\{a,c\}$	c
bac (b)	b	b	b	b	b	b
bca (b)	b	b	b	b	b	b
cab (cab)	a	$\{a,c\}$	b	b	c	c
cba (cb)	b	c	b	b	c	c

Table 2: f that satisfies MM, P and IP but fails N.

This SCR is well-defined since a non-empty set is associated to each preference profile. By construction each player has veto power over $\{a\}$ and $\{c\}$ under f. For each player it maybe the case that b is chosen when b is her worst alternative; thus no player has veto power over $\{b\}$. Consequently f does not satisfy N.

The rule f satisfies P by definition.

The condition MM also holds since a candidate not going down in the voter's rankings is not harmed with the rule f. Indeed, if $x \in f(\succ)$, then x is Pareto efficient and neither of the players can veto $\{x\}$ in $\succ$. For any $\succ'$ with $\widetilde{L}(x,\succ_i) \subseteq \widetilde{L}(x,\succ'_i)$ $\forall i \in N$, x remains Pareto efficient and neither of the players can veto $\{x\}$ in $\succ'_i$, which implies that $x \in f(\succ')$ which implies that MM holds.

We now prove that f satisfies IP. Take $x \in f(\succ)$ and $y \in f(\succ')$. We wish to prove that $\widetilde{L}(x,\succ_i) \cap \widetilde{L}(y,\succ'_j) \neq \emptyset$ and this is obvious if $x = y$; so let $x \neq y$.

Consider first the case where neither x nor y equals b. Let, without loss of generality, $x = a$ and $y = c$. Take some player i. By definition of f, $a = f(\succ)$ is ranked last by no player at $\succ$, so $\widetilde{L}(a,\succ_i)$ contains some z other than a. In case $a \succ_i c$, the condition for IP is satisfied as $c = f(\succ')$ and $\widetilde{L}(c,\succ'_j)$ contains c. In case $c \succ_i a$, z must be b, so $\widetilde{L}(a,\succ_i) = \{a,b\}$. Again by definition of f, $\widetilde{L}(c,\succ'_j)$ contains some z other than c and the condition for IP holds whether z is a or b.

Consider second the remaining case where, without loss of generality, $x = b$ and $y \in \{a,c\}$. Say $y = a$ without loss of generality. Take player i. There are two subcases. In the first one $\widetilde{L}(b,\succ_i) = \{b\}$. Note that at $\succ'_j$, b cannot be ranked at top, as j has veto power over $\{a,c\}$ which would contradict that $a \in f(\succ')$. If $\succ'_j$ ranks b the second best, then a must be ranked top at $\succ'_j$, as otherwise a would be ranked bottom and wouldn't be picked at $\succ'$ by the veto power of j on a. Thus, $\widetilde{L}(a,\succ'_j)$ contains b which was in $\widetilde{L}(b,\succ_i)$, ensuring the IP condition. Now consider the other subcase where $\widetilde{L}(b,\succ_i)$ contains some z other than b. In case $z = a$, the condition for IP is satisfied. Now let z be c. So $a \succ_i b \succ_i c$. For the condition to fail, a must be ranked last by $\succ'_j$ which contradicts that $a \in f(\succ')$. We therefore conclude that f satisfies IP.

$\square$