Birecognition of prime graphs, and minimal prime graphs

Houmem Belkhechine, Cherifa Ben Salha, Pierre Ille

Birecognition of prime graphs, and minimal prime graphs

Houmem Belkhechine [*] Cherifa Ben Salha [†] Pierre Ille[‡]

December 6, 2021

Abstract

Given a graph G, a subset M of $V(G)$ is a module of G if for each $v \in V(G) \setminus M$, v is adjacent to all the elements of M or to none of them. For instance, $V(G)$, $\varnothing$ and $\{v\}$ ($v \in V(G)$) are the trivial modules of G. A graph G is prime if $|V(G)| \geq 4$ and all its modules are trivial.

Given a prime graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Given a graph H such that $V(G) = V(H)$ and $G[X] = H[X]$, G and H are $G[X]$-similar if for each $W \subsetneq V(G) \setminus X$, $G[X \cup W]$ and $H[X \cup W]$ are both prime or not. The graph G is said to be $G[X]$-birecognizable if every graph, $G[X]$-similar to G, is prime. We study the graphs G that are not $G[X]$-birecognizable, where $X \subsetneq V(G)$ such that $G[X]$ is prime, by using the following notion of a minimal prime graph. Given a prime graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Given $v, w \in V(G) \setminus X$, G is $G[X \cup \{v, w\}]$-minimal if for each $W \subsetneq V(G)$ such that $X \cup \{v, w\} \subseteq W$, $G[W]$ is not prime.

Mathematics Subject Classifications (2010): 05C75

Key words: Module; Prime graph; Primality birecognition; Minimal prime graph

1 Introduction

We consider only finite graphs. A graph, with at least 4 vertices, is prime if it is indecomposable under modular decomposition. We study birecognition (or mutual recognition) of primality introduced as follows. We consider two graphs with the same vertex set. We suppose that they coincide on a prime induced subgraph. We fix this prime induced subgraph from which we introduce the

*University of Carthage, Bizerte Preparatory Engineering Institute, Bizerte, Tunisia; houmem.belkhechine@ipeib.rnu.tn.

†University of Carthage, Faculty of Sciences of Bizerte, Bizerte, Tunisia; cherifa.bensalha@fsb.u-carthage.tn.

‡Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France; pierre.ille@univ-amu.fr.

birecognition. We suppose also that the subgraphs of both graphs, induced by the same proper subset of the vertex set, are both prime or not, when they contain the prime induced subgraph. Under these assumptions, the graphs are birecognizable if both are prime or not.

We study the prime graphs, that are not birecognizable, by using the notion of a minimal prime graph introduced by Cournier and Ille [3]. In this manner, we answer [9, Problem 13] as well.

Consider two graphs with the same vertex set. Another type of primality birecognition consists in considering the subgraphs of both graphs induced by the same proper subset of the vertex set, with given sizes (for instance, see [1]).

Lastly, Ille and Villemaire [9] introduced an inner primality recognition. Precisely, they considered a graph that admits a prime induced subgraph. They introduced a digraph (called outside digraph, see Subsection 1.1) that yields a necessary and sufficient condition for G to be prime.

Next, we formalize our presentation. A *graph* $G = (V(G), E(G))$ consists of a finite *vertex* set $V(G)$ and of an *edge* set $E(G)$, where an edge is an unordered pair of distinct vertices. For a graph G, $v(G)$ denotes the cardinality of $V(G)$. Given a graph G, with each subset X of $V(G)$ associate the *subgraph* $G[X] = (X, E(G) \cap \binom{X}{2})$ of G induced by X. For convenience, given a subset X of $V(G)$, $G[V(G) \setminus X]$ is also denoted by $G - X$, and by $G - x$ if $X = \{x\}$.

Notation 1. Let G be a graph. Given $W \subsetneq V(G)$ and $v \in V(G) \setminus W$, $v \longleftrightarrow_G W$ means that

$$vw \in E(G) \text{ for every } w \in W \text{ or } vw \notin E(G) \text{ for every } w \in W.$$

The negation is denoted by $v \not\longleftrightarrow_G W$.

Given a graph G, a subset M of $V(G)$ is a *module* [11] (or a *closed set* [6, 10] or a *homogeneous* set [3]) of G if for every $v \in V(G) \setminus M$, we have $v \longleftrightarrow_G W$. For example, $\varnothing$, $V(G)$ and $\{v\}$, $v \in V(G)$, are modules of G called *trivial* modules. A graph is *indecomposable* if all its modules are trivial, otherwise it is *decomposable*. A graph G, with $v(G) \leq 2$, is indecomposable, whereas a graph G, with $v(G) = 3$, is decomposable. Hence, we introduce the following precision. A graph G is *prime* if it is indecomposable, with $v(G) \geq 4$. For instance, given $n \geq 4$, the path $P_n = (\{0, \ldots, n-1\}, \{i(i+1) : i \in \{0, \ldots, n-2\}\})$ is prime. Sumner [12] showed that every prime graph contains P_4 as an induced subgraph.

We introduce the main notions as follows. For convenience, we use the next notation.

Notation 2. Let G be a graph. For $W \subsetneq V(G)$, $\overline{W}$ denotes $V(G) \setminus W$.

Definition 3. Given graphs G and H such that $V(G) = V(H)$, G and H are *similar* (in terms of modular decomposability) if G and H are prime or G and H are decomposable. As previously noted, a graph of cardinality 3 is decomposable. Therefore, all the graphs of cardinality 3 are similar.

Now, consider graphs G and H such that $V(G) = V(H)$. Suppose that there exists $X \subsetneq V(G)$ such that $G[X]$ is prime and $G[X] = H[X]$. We say that G and H are $G[X]$-*similar* if for each $W \subsetneq \overline{X}$, the subgraphs $G[X \cup W]$ and $H[X \cup W]$ are similar.

Definition 4. Let G be a prime graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. The graph G is said to be $G[X]$-*birecognizable* if every graph, $G[X]$-similar to G, is prime as well.

Ille (1994) conjectured the following.

Conjecture 5. There exists an integer $k \geq 1$ satisfying the following. Let G be a prime graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. If $|\overline{X}| \geq k$, then G is $G[X]$-birecognizable.

Remark 6. In fact, Conjecture 5 is false. Indeed, let $n \geq 7$. Consider $G = P_n$ and $H = G - ((n-2)(n-1))$. Clearly, G is prime. Furthermore, $n-1$ is an isolated vertex of H, so H is decomposable. Set $X = \{0, 1, 2, 3\}$. We have $G[X] = P_4$ and $G[X] = H[X]$. Lastly, consider a proper and nonempty subset Y of $\overline{X}$. If $n-1 \notin Y$, then $G[X \cup Y] = H[X \cup Y]$. Hence, suppose that $n-1 \in Y$. Since $n-1$ is an isolated vertex of H, $n-1$ is an isolated vertex of $H[X \cup Y]$, so $H[X \cup Y]$ is decomposable. Moreover, since $Y \subsetneq \overline{X}$, we have $\{4, \dots, n-2\} \setminus Y \neq \varnothing$. Set $p = \min(\{4, \dots, n-2\} \setminus Y)$. We obtain that $\{0, \dots, p-1\}$ is a module of $G[X \cup Y]$, so $G[X \cup Y]$ is decomposable. Consequently, G and H are $G[X]$-similar.

Other types of primality recognition were studied. For instance, Boussaïri et al. [1] introduced the $\mathcal{N}$-recognition, where $\mathcal{N}$ is a set of negative integers, as follows. Consider graphs G and H such that $V(G) = V(H)$. We say that G and H are $\mathcal{N}$-*similar* if for each $X \subseteq V(G)$ such that $-|X| \in \mathcal{N}$ and $|X| \leq v(G) - 4$, $G - X$ and $H - X$ are similar. Let G be a prime graph. The graph G is said to be $\mathcal{N}$-*birecognizable* if every graph, $\mathcal{N}$-similar to G, is prime. It is easy to see that prime graphs are not $\{-1\}$-birecognizable. Boussaïri et al. [1] proved that prime graphs are not $\{-2\}$-birecognizable, but they are $\{-2, -1\}$-birecognizable. Lastly, note that the primality recognition under the assumptions of Ulam's reconstruction is demonstrated in [7].

Our purpose is to study the counter-examples to Conjecture 5. We use the notion of a minimal graph introduced by Cournier and Ille [4].

Definition 7. Let G be a prime graph. Given a nonempty subset W of $V(G)$, G is said to be $G[W]$-*minimal* if there does not exist $W' \subsetneq V(G)$ such that $W \subseteq W'$ and $G[W']$ is prime.

Cournier and Ille [4] characterized the graphs G that are $G[W]$-minimal when $|W| \leq 2$. Ille and Villemaire [9] characterized the graphs G that are $G[X \cup \{v\}]$-minimal when $X \subsetneq V(G)$, $G[X]$ is prime, and $v \in \overline{X}$ (see Theorem 16). To begin, we obtain the following result proved in Section 2.

Proposition 8. *Let G be a prime graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Suppose that $|\overline{X}| \geq 5$. If G is not $G[X]$-birecognizable, then one of the following statements holds*

- *there exists $v \in \overline{X}$ such that G is $G[X \cup \{v\}]$-minimal;*
- *for every $u \in \overline{X}$, G is not $G[X \cup \{u\}]$-minimal, but there exist distinct $v, w \in \overline{X}$ such that G is $G[X \cup \{v, w\}]$-minimal, $G - v$ is prime, and $G - w$ is prime.*

We have the opposite direction when the first statement of Proposition 8 holds. The proof of the next lemma is provided in Section 2.

Lemma 9. *Let G be a prime graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. If there exists $v \in \overline{X}$ such that G is $G[X \cup \{v\}]$-minimal, then G is not $G[X]$-birecognizable.*

Given Proposition 8 and Lemma 9, our aim is to characterize the graphs G that are $G[X \cup \{v, w\}]$-minimal, where $X \subsetneq V(G)$, $G[X]$ is prime, and v, w are distinct elements of $\overline{X}$ (see [9, Problem 13]). Furthermore, this characterization provides the following direct and natural recognition theorem of prime graphs.

Theorem 10 (Theorem 12 of [9])**.** *Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Suppose that $|\overline{X}| \geq 3$. The graph G is prime if and only if for any $v, w \in \overline{X}$, with $v \neq w$, there exists $Y \subseteq \overline{X}$ such that $v, w \in Y$ and $G[X \cup Y]$ is $G[X \cup \{v, w\}]$-minimal.*

1.1 Outside partition, graph, and digraph

Definition 11. Let G be a graph. With $X \subsetneq V(G)$ such that $G[X]$ is prime, associate the following subsets of $\overline{X}$

- $\mathrm{Ext}_G(X)$ is the set of $v \in \overline{X}$ such that $G[X \cup \{v\}]$ is prime;
- $\langle X \rangle_G$ is the set of $v \in \overline{X}$ such that X is a module of $G[X \cup \{v\}]$;
- Given $\alpha \in X$, $X_G(\alpha)$ is the set of $v \in \overline{X}$ such that $\{\alpha, v\}$ is a module of $G[X \cup \{v\}]$.

The family
$$\{\mathrm{Ext}_G(X), \langle X \rangle_G\} \cup \{X_G(\alpha) : \alpha \in X\}$$
is a partition of $\overline{X}$ (see [5, Lemma 6.3]). It is called the *outside partition* associated with the prime induced subgraph $G[X]$ of G. It is denoted by $p_{(G, \overline{X})}$.

We recall the classic parity theorem [5, Theorem 6.5].

Theorem 12. *Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime and $|\overline{X}| \geq 2$. If G is prime, then there exist $v, w \in \overline{X}$ such that $v \neq w$ and $G[X \cup \{v, w\}]$ is prime.*

The proof of Theorem 12 is based on the outside partition $p_{(G, \overline{X})}$. Theorem 12 led Ille [7] to consider the following graph.

Definition 13. Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. The *outside graph* $\Gamma_{(G,\overline{X})}$ is defined on $\overline{X}$ as follows. Given $v, w \in \overline{X}$, with $v \neq w$, $vw \in E(\Gamma_{(G,\overline{X})})$ if $G[X \cup \{v, w\}]$ is prime.

The outside graph is a usual tool to study prime graphs. Nevertheless, it is not precise enough to determine the primality of G even if it is sufficient under additional assumptions (see [9]). This led Ille and Villemaire [9] to introduce the following refinement. (Recall that a *digraph* $D = (V(D), A(D))$ consists of a finite *vertex* set $V(D)$ and of an *arc* set $A(D)$, where an arc is an ordered pair of distinct vertices.)

Definition 14. Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. The *outside digraph* $\Delta_{(G,\overline{X})}$ is defined on $\overline{X}$ as follows. Given $v, w \in \overline{X}$, with $v \neq w$, $vw \in A(\Delta_{(G,\overline{X})})$ if

$w \in \langle X \rangle_G$ and $X \cup \{v\}$ is not a module of $G[X \cup \{v, w\}]$

or

$w \in X_G(\alpha)$, where $\alpha \in X$, and $\{\alpha, w\}$ is not a module of $G[X \cup \{v, w\}]$.

The dipaths of the outside digraph play an important role. For instance, they allow a simple and concise characterization of graphs G that are $G[X \cup \{v\}]$-minimal, where $X \subsetneq V(G)$ such that $G[X]$ is prime, and $v \in \overline{X}$ (see Theorem 16).

Definition 15. Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Consider distinct $v_0, \dots, v_m \in \overline{X}$, where $m \geq 1$. We say that $v_0 \dots v_m$ is a *dipath* of $\Delta_{(G,\overline{X})}$ if $v_i v_{i+1} \in A(\Delta_{(G,\overline{X})})$ for $i \in \{0, \dots, m-1\}$. Moreover, a dipath $v_0 \dots v_m$ of $\Delta_{(G,\overline{X})}$ is *strict* if we have (when $m \geq 2$) $v_i v_j \notin A(\Delta_{(G,\overline{X})})$ for $i \in \{0, \dots, m-2\}$ and $j \in \{i+2, \dots, m\}$. Lastly, a strict dipath $v_0 \dots v_m$ of $\Delta_{(G,\overline{X})}$ is an *arrow* if there exists $B \in p_{(G,\overline{X})} \setminus \{\mathrm{Ext}_G(X)\}$ such that $v_0 \notin B$ and $\{v_1, \dots, v_m\} \subseteq B$.

Theorem 16 (Theorem 11 of [9]). *Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Given $v \in \overline{X} \setminus \mathrm{Ext}_G(X)$, G is $G[X \cup \{v\}]$-minimal if and only if the elements of $\overline{X}$ can be indexed as $v_0, \dots, v_m$ in such a way that $v_0 \dots v_m$ is an arrow of $\Delta_{(G,\overline{X})}$, with $v_m = v$.*

Notation 17. Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Consider $v \in \overline{X} \setminus \mathrm{Ext}_G(X)$. Suppose that there exists $Y \subseteq \overline{X}$ such that $v \in Y$ and $G[X \cup Y]$ is prime. By Theorem 16, there exists $m \geq 1$ such that $\Delta_{(G,\overline{X})}$ admits an arrow $v_0 \dots v_m$ with $v_m = v$. The smallest integer $m \geq 1$, for which such an arrow exists, is denoted by $\delta_{(G,\overline{X})}(v)$.

Definition 18. Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Consider a nonempty subset W of $\overline{X}$. An arrow $v_0 \dots v_m$ of $\Delta_{(G,\overline{X})}$ is a *W-dipath* if $W \subseteq \{v_0, \dots, v_m\}$, $v_m \in W$, and $\delta_{(G,\overline{X})}(v_m) = m$. When $W = \{v\}$, we say also that $v_0 \dots v_m$ is a *v-dipath*. Lastly, we say that W is *$G[X]$-reachable* (in $\Delta_{(G,\overline{X})}$) if $\Delta_{(G,\overline{X})}$ admits a W-dipath. When $W = \{v\}$, we say also that v is $G[X]$-reachable.

Warning. We use an environment called *Layout*. We use it nine times: Layout 1 to Layout 9. In Layouts 1, 2, and 3, we consider the two cases where $\{v,w\} \cap \mathrm{Ext}_G(X) \neq \varnothing$. In Layout 4, we consider the case where there exist distinct $B, C \in p_{(G,\overline{X})} \setminus \{\mathrm{Ext}_G(X)\}$ such that $v \in B$ and $w \in C$. The last five layouts are devoted to the three cases where there exists $B \in p_{(G,\overline{X})} \setminus \{\mathrm{Ext}_G(X)\}$ such that $v, w \in B$. The discussion is based on the notion of separated vertices (see Definition 40).

1.2 The first two results

Layout 1. In this subsection, we consider a graph G, a proper subset X of $V(G)$ such that $G[X]$ is prime, and distinct elements v, w of $\overline{X}$. We suppose that $\{v,w\} \cap \mathrm{Ext}_G(X) \neq \varnothing$.

Remark 19. We are looking for a characterization of G when G is $G[X \cup \{v,w\}]$-minimal. One of our tools for this characterization is the outside digraph $\Delta_{(G,\overline{X})}$ (see Definition 14). If there exists $u \in \overline{X}$ such that G is $G[X \cup \{u\}]$-minimal, then Theorem 16 yields such a characterization. Consequently, we can assume that G is neither $G[X \cup \{v\}]$-minimal nor $G[X \cup \{w\}]$-minimal.

Lastly, observe that if G is $G[X \cup \{v,w\}]$-minimal and $\{v,w\}$ is $G[X]$-reachable, then G is $G[X \cup \{v\}]$-minimal or $G[X \cup \{w\}]$-minimal. Therefore, since G is supposed to be $G[X \cup \{v,w\}]$-minimal, we assume also that $\{v,w\}$ is not $G[X]$-reachable.

Layout 2. We suppose that $v, w \in \mathrm{Ext}_G(X)$.

Theorem 20. *Suppose that $|\overline{X}| \geq 3$. Set $Y = X \cup \{v\}$ and $Z = X \cup \{w\}$. The graph G is $G[X \cup \{v,w\}]$-minimal if and only if the following statement holds*

(S1) *the elements of $\overline{X} \setminus \{v,w\}$ can be indexed as $u_0, \dots, u_p$ in such a way that $u_0 \dots u_p w$ is a w-dipath of $\Delta_{(G,\overline{Y})}$, and $u_0 \dots u_p v$ is a v-dipath of $\Delta_{(G,\overline{Z})}$.*

Layout 3. Now, we suppose that $v \notin \mathrm{Ext}_G(X)$ and $w \in \mathrm{Ext}_G(X)$. Hence, there exists $B \in p_{(G,\overline{X})} \setminus \{\mathrm{Ext}_G(X)\}$ such that $v \in B$. Following Remark 19, we suppose also that G is not $G[X \cup \{v\}]$-minimal.

Theorem 21. *The graph G is $G[X \cup \{v,w\}]$-minimal if and only if the following statement holds*

(S2) *there exist distinct elements $v_1, \dots, v_m$ of B and $v_0 \in \overline{X} \setminus B$ such that*

- *$w \in \overline{X} \setminus \{v_0, \dots, v_m\}$ and $\overline{X} = \{w\} \cup \{v_0, \dots, v_m\}$;*
- *$v_0 \dots v_m$ is a v-dipath of $\Delta_{(G,\overline{X})}$;*
- *for $i \in \{1, \dots, m\}$, $wv_i \notin E(\Gamma_{(G,\overline{X})})$.*

Let B and C be distinct elements of $p_{(G,\overline{X})} \setminus \{\mathrm{Ext}_G(X)\}$. In Theorem 39, we consider the case where $v \in B$ and $w \in C$. Difficulties appear when v and w belong to the same block of $p_{(G,\overline{X})} \setminus \{\mathrm{Ext}_G(X)\}$. We have to introduce the notion of separated vertices (see Definition 40). The situation becomes arduous when $\{v,w\}$ is a module of $G[X \cup \{v,w\}]$ (see Defintion 24 below). In this case, v and w are indistinguishable from $G[X]$. The statements are too long to be provided in this section.

2 Partially critical graphs

Breiner et al. [2] introduced the following definition.

Definition 22. Let G be a prime graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. The graph G is said to be $G[X]$-*critical* if $G - x$ is decomposable for each $x \in \overline{X}$. A prime graph G is *partially critical* if it is $G[X]$-critical for some $X \subsetneq V(G)$ such that $G[X]$ is prime.

The next result follows from [2, Lemma 4.1].

Fact 23. *Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Suppose that $|\overline{X}| \geq 2$. If G is $G[X]$-critical, then we have*

$$\text{for every } Y \subseteq \overline{X}, \text{ if } |Y| = 1 \text{ or } 3, \text{ then } G[X \cup Y] \text{ is decomposable.} \tag{1}$$

Now, we refine the partition $p_{(G,\overline{X})}$ in the following way.

Definition 24. Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Let $\varepsilon_{(G,\overline{X})}$ be the graph defined on $\overline{X}$ as follows. Given $v, w \in \overline{X}$, with $v \neq w$, we have $vw \in E(\varepsilon_{(G,\overline{X})})$ if $\{v,w\}$ is a module of $G[X \cup \{v,w\}]$. Note that the components of $\varepsilon_{(G,\overline{X})}$ are complete. Let $q_{(G,\overline{X})}$ be the partition of $\overline{X}$ given by the vertex sets of the components of $\varepsilon_{(G,\overline{X})}$. Observe that the partition $q_{(G,\overline{X})}$ of $\overline{X}$ is finer than $p_{(G,\overline{X})}$. A partition similar to $q_{(G,\overline{X})}$ is directly defined from $p_{(G,\overline{X})}$ in [2].

The next theorem follows from [9, Theorem 17, Lemma 36, and Corollary 38]. The partition $q_{(G,\overline{X})}$ is used to state [9, Theorem 17 and Corollary 38]. It is also used in Sections 4 and 5.

Theorem 25. *Given a graph G, consider $X \subsetneq V(G)$ such that $|\overline{X}| \geq 2$ and $G[X]$ is prime. Suppose that (1) holds. The graph G is prime if and only if for each connected component C of $\Gamma_{(G,\overline{X})}$, we have $v(C) = 2$ or C is prime.*

As shown by the next result, partially critical graphs are birecognizable. This fact is useful to prove Proposition 8.

Proposition 26. *Let G be a prime graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Suppose that $|\overline{X}| \geq 4$. If G is $G[X]$-critical, then G is $G[X]$-birecognizable.*

Proof. Consider a graph H such that G and H are $G[X]$-similar. We have to show that H is prime. Since G is $G[X]$-critical, with $|\overline{X}| \geq 4$, it follows from

Fact 23 that G satisfies (1). Since G and H are $G[X]$-similar, with $|\overline{X}| \geq 4$, H satisfies (1) as well. Furthermore, since G and H are $G[X]$-similar, with $|\overline{X}| \geq 3$, we have

$$\Gamma_{(G,\overline{X})} = \Gamma_{(H,\overline{X})}. \tag{2}$$

It follows from Theorem 25 applied to G that for each connected component C of $\Gamma_{(G,\overline{X})}$, we have $v(C) = 2$ or C is prime. By (2), the same holds for H. By Theorem 25 applied to H, H is prime. □

The next result is the first step in the proof of Proposition 8.

Lemma 27. *Let G be a prime graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Suppose that $|\overline{X}| \geq 4$. If G is not $G[X]$-birecognizable, then there exists $v \in \overline{X}$ such that G is $G[X \cup \{v\}]$-minimal or there exist distinct $u, v \in \overline{X}$ such that G is $G[X \cup \{u, v\}]$-minimal.*

Proof. Since G is not $G[X]$-birecognizable, there exists a graph H such that G and H are $G[X]$-similar, and H is decomposable. Moreover, it follows from Proposition 26 that there exists $v \in \overline{X}$ such that $G - v$ is prime. Since G and H are $G[X]$-similar, $H - v$ is prime as well. Set $Y = \overline{\{v\}}$. Since H is decomposable, we have $v \in \langle Y \rangle_H$ or $v \in Y_H(\alpha)$, where $\alpha \in Y$ (see Definition 11). We distinguish the following three cases.

1. Suppose that $v \in \langle Y \rangle_H$. In this case, G is $G[X \cup \{v\}]$-minimal. Indeed, consider $W \subsetneq V(G)$ such that $X \cup \{v\} \subseteq W$. Since Y is a module of H, $W \cap Y = W \setminus \{v\}$ is a module of $H[W]$. Hence, $H[W]$ is decomposable. Since G and H are $G[X]$-similar, $G[W]$ is decomposable.

2. Suppose that $v \in Y_H(\alpha)$, where $\alpha \in X$. In this case, G is $G[X \cup \{v\}]$-minimal. Indeed, consider $W \subsetneq V(G)$ such that $X \cup \{v\} \subseteq W$. Since $\{\alpha, v\}$ is a module of H, $\{\alpha, v\} \cap W = \{\alpha, v\}$ is a module of $H[W]$. Hence, $H[W]$ is decomposable. Since G and H are $G[X]$-similar, $G[W]$ is decomposable.

3. Suppose that $v \in Y_H(\alpha)$, where $\alpha \in Y \setminus X$. In this case, G is $G[X \cup \{\alpha, v\}]$-minimal. Indeed, consider $W \subsetneq V(G)$ such that $X \cup \{\alpha, v\} \subseteq W$. Since $\{\alpha, v\}$ is a module of H, $\{\alpha, v\} \cap W = \{\alpha, v\}$ is a module of $H[W]$. Hence, $H[W]$ is decomposable. Since G and H are $G[X]$-similar, $G[W]$ is decomposable. □

In the proof of the next proposition, we use [2, Corollary 4.5], stated as follows.

Fact 28. *Let G be a prime graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. If G is $G[X]$-critical, then $\Gamma_{(G,\overline{X})}$ does not have isolated vertices.*

Proposition 29. *Let G be a prime graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Suppose that $|\overline{X}| \geq 5$. Given distinct $v, w \in \overline{X}$, if G is $G[X \cup \{v, w\}]$-minimal, then $G - v$ or $G - w$ is prime.*

Proof. Consider distinct $v, w \in \overline{X}$, and suppose that G is $G[X \cup \{v, w\}]$-minimal. For a contradiction, suppose that G is $G[X]$-critical. By Fact 28, there exists $v' \in \overline{X} \smallsetminus \{v\}$ such that $vv' \in E(\Gamma_{(G,\overline{X})})$. Set $Y = X \cup \{v, v'\}$, so $G[Y]$ is prime. Since G is $G[X \cup \{v, w\}]$-minimal and $|\overline{X}| \geq 3$, we obtain $w \notin Y$. Since G is $G[X]$-critical, G is $G[Y]$-critical too. By Fact 28, there exists $w' \in \overline{Y} \smallsetminus \{w\}$ such that $ww' \in E(\Gamma_{(G,\overline{Y})})$. Therefore, $G[Y \cup \{w, w'\}]$ is prime. Since G is $G[X \cup \{v, w\}]$-minimal, we obtain $V(G) = Y \cup \{w, w'\}$, that is, $V(G) = X \cup \{v, v', w, w'\}$, which contradicts $|\overline{X}| \geq 5$. It follows that G is not $G[X]$-critical. Hence, there exists $u \in \overline{X}$ such that $G - u$ is prime. Since G is $G[X \cup \{v, w\}]$-minimal, we have $u = v$ or $u = w$. □

Lemma 30. *Let G be a prime graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Let v, w be distinct elements of $\overline{X}$. Suppose that G is $G[X \cup \{v, w\}]$-minimal. Suppose also that $G - v$ is prime, and G is not $G[X \cup \{v\}]$-minimal. If $G - w$ is decomposable, then G is $G[X]$-birecognizable.*

Proof. Consider a graph H which is $G[X]$-similar to G. We have to show that H is prime. Since $G - w$ is decomposable, $H - w$ is decomposable. Since $G - v$ is prime, $H - v$ is prime. Thus

$$H - v \not\simeq H - w. \tag{3}$$

Set $Y = \overline{\{v\}}$. Since G is not $G[X \cup \{v\}]$-minimal, there exists $Z \subsetneq \overline{X}$ such that $v \in Z$ and $G[X \cup Z]$ is prime. Thus, $H[X \cup Z]$ is too. It follows that $v \notin \langle Y \rangle_H$. For a contradiction, suppose that there exists $\alpha \in Y$ such that $v \in Y_H(\alpha)$. We obtain that $H[(Y \smallsetminus \{\alpha\}) \cup \{v\}]$ is prime, so $G[(Y \smallsetminus \{\alpha\}) \cup \{v\}]$ is as well. Since G is $G[X \cup \{v, w\}]$-minimal, we obtain $w = \alpha$. It follows that $H - v \simeq H - w$ contradicting (3). Consequently, $v \notin Y_H(\alpha)$ for every $\alpha \in Y$. It follows that $v \in \mathrm{Ext}_H(Y)$, so H is prime. □

Proposition 8 is a simple consequence of Lemma 27, Proposition 29, and Lemma 30. We complete this section with the proof of Lemma 9.

Proof of Lemma 9. Consider the graph H defined on $V(G) = V(H)$ by $H - v = G - v$, and v is an isolated vertex of H. Since $\overline{\{v\}}$ is a module of H, H is decomposable. We verify that G and H are $G[X]$-similar. Let $Y \subsetneq \overline{X}$. If $v \notin Y$, then $G[X \cup Y] = H[X \cup Y]$ because $H - v = G - v$. Now, suppose that $v \in Y$. Since $\overline{\{v\}}$ is a module of H, $\overline{\{v\}} \cap (X \cup Y) = (X \cup Y) \smallsetminus \{v\}$ is a module of $H[X \cup Y]$. Hence, $H[X \cup Y]$ is decomposable. Since G is $G[X \cup \{v\}]$-minimal, $G[X \cup Y]$ is decomposable too. Therefore, G and H are $G[X]$-similar. Consequently, G is not $G[X]$-birecognizable. □

3 Proofs of Theorems 20 and 21

Using Notation 17, we obtain the following simple consequences of Theorem 16 that are often used.

Corollary 31. *Given a prime graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime.*

1. *For each $v \in \overline{X} \smallsetminus \mathrm{Ext}_G(X)$, $\Delta_{(G,\overline{X})}$ admits a v-dipath.*
2. *For each $v \in \overline{X}$, if $v_0 \dots v_m$ is a v-dipath of $\Delta_{(G,\overline{X})}$ (or an arrow of $\Delta_{(G,\overline{X})}$), then $G[X \cup \{v_0, \dots, v_m\}]$ is prime.*

A simple and important tool follows.

Lemma 32. *Let G be a graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Consider a subset W of $V(G)$ such that $X \subsetneq W$. Let W' be a nonempty subset of $W \smallsetminus X$. Suppose that there exists $B \in p_{(G,\overline{X})} \smallsetminus \{\mathrm{Ext}_G(X)\}$ such that $W' \subseteq B$. If*

$$ww' \notin A(\Delta_{(G,\overline{X})}) \text{ for } w \in (W \smallsetminus X) \smallsetminus W' \text{ and } w' \in W', \tag{4}$$

then one of the following assertions holds

- *$B = \langle X \rangle_G$, and $W \smallsetminus W'$ is a nontrivial module of $G[W]$;*
- *$B = X_G(\alpha)$, where $\alpha \in X$, and $\{\alpha\} \cup W'$ is a nontrivial module of $G[W]$.*

Consequently, if (4) holds, then $G[W]$ is decomposable.

Proof. Suppose that (4) holds. To begin, suppose that $B = \langle X \rangle_G$. Let $w \in (W \smallsetminus X) \smallsetminus W'$ and $w' \in W'$. Since $ww' \notin A(\Delta_{(G,\overline{X})})$, $X \cup \{w\}$ is a module of $G[X \cup \{w, w'\}]$. It follows that $W \smallsetminus W'$ is a module of $G[W]$.

Now, suppose that $B = X_G(\alpha)$, where $\alpha \in X$. Let $w \in (W \smallsetminus X) \smallsetminus W'$ and $w' \in W'$. Since $ww' \notin A(\Delta_{(G,\overline{X})})$, $\{\alpha, w'\}$ is a module of $G[X \cup \{w, w'\}]$. It follows that $\{\alpha\} \cup W'$ is a module of $G[W]$. □

Proof of Theorem 20. To begin, suppose that G is $G[X \cup \{v, w\}]$-minimal. Since $|\overline{X}| \geq 3$, $w \notin \mathrm{Ext}_G(Y)$. Since $w \in \mathrm{Ext}_G(X)$, we have $w \in Y_G(v)$. Since G is prime, it follows from Corollary 31 that $\Delta_{(G,\overline{Y})}$ admits a w-dipath $w_0 \dots w_n$, and $G[Y \cup \{w_0, \dots, w_n\}]$ is prime. Since G is $G[X \cup \{v, w\}]$-minimal, we obtain $V(G) = Y \cup \{w_0, \dots, w_n\}$. Similarly, $v \in Z_G(w)$, and $\Delta_{(G,\overline{Z})}$ admits a v-dipath $v_0 \dots v_m$. Moreover, we have $V(G) = Z \cup \{v_0, \dots, v_m\}$. Therefore, $\{v\} \cup \{w_0, \dots, w_n\} = \{w\} \cup \{v_0, \dots, v_m\}$, so $m = n$. Set $p = m - 1$. Since $w_0 \dots w_n$ is a w-dipath of $\Delta_{(G,\overline{Y})}$, and $w \in Y_G(v)$, w_p is the unique element of $\overline{X} \smallsetminus \{v, w\}$ such that $w_p \not\leftrightarrow_G \{v, w\}$ (see Notation 1). Similarly, v_p is the unique element of $\overline{X} \smallsetminus \{v, w\}$ such that $v_p \not\leftrightarrow_G \{v, w\}$. It follows that $v_p = w_p$. Set $u_p = v_p$. To continue, suppose that $p \geq 1$. We obtain that w_{p-1} is the unique element of $\overline{X} \smallsetminus \{v, w, u_p\}$ such that $w_{p-1} \not\leftrightarrow_G \{u_p, v, w\}$. Similarly, v_{p-1} is the unique element of $\overline{X} \smallsetminus \{v, w, u_p\}$ such that $v_{p-1} \not\leftrightarrow_G \{u_p, v, w\}$. It follows that $v_{p-1} = w_{p-1}$. Set $u_{p-1} = v_{p-1}$. By proceeding by induction, we obtain the sequence $u_0, \dots, u_p$, where $u_i = v_i$ and $u_i = w_i$ for every $i \in \{0, \dots, p\}$.

Conversely, suppose that Statement (S1) holds. By Corollary 31, $G[Y \cup \{u_0, \dots, u_p\} \cup \{w\}] = G[X \cup \{u_0, \dots, u_p\} \cup \{v, w\}]$ is prime. Since $\overline{X} = \{u_0, \dots, u_p\} \cup$

$\{v,w\}$, G is prime. To show that G is $G[X \cup \{v,w\}]$-minimal, consider $W \subsetneq V(G)$ such that $X \cup \{v,w\} \subseteq W$. We have to verify that $G[W]$ is decomposable. Since $X \cup \{v,w\} \subseteq W \subsetneq V(G)$, there exists $i \in \{0,\dots,p\}$ such that $u_i \notin W$. Set $W' = \{u_{i+1},\dots,u_{p+1}\} \cap W$, where u_{p+1} denotes w. By Lemma 32 applied to $G[Y]$, $G[W]$ is decomposable. □

Proof of Theorem 21. To begin, suppose that G is $G[X \cup \{v,w\}]$-minimal. Since G is prime, it follows from Corollary 31 that $\Delta_{(G,\overline{X})}$ admits a v-dipath $v_0 \dots v_m$. Set $Y = X \cup \{v_0,\dots,v_m\}$. By Theorem 16, $G[Y]$ is $G[X \cup \{v\}]$-minimal. Since G is $G[X \cup \{v,w\}]$-minimal without being $G[X \cup \{v\}]$-minimal, we have $w \notin Y$. Since $w \in \mathrm{Ext}_G(X)$, we have $w \notin \langle Y \rangle_G$, and $w \notin Y_G(\alpha)$ for $\alpha \in X$. Furthermore, since $v_1,\dots,v_m \notin \mathrm{Ext}_G(X)$ and $w \in \mathrm{Ext}_G(X)$, we obtain $w \notin Y_G(v_i)$ for $i \in \{1,\dots,m\}$. Now, suppose for a contradiction that there exists $i \in \{1,\dots,m\}$ such that $wv_i \in E(\Gamma_{(G,\overline{X})})$. Let I be the largest element of $\{1,\dots,m\}$ such that $wv_I \in E(\Gamma_{(G,\overline{X})})$. We obtain that $wv_I \dots v_m$ is an arrow of $\Delta_{(G,\overline{X})}$. By Corollary 31, $G[X \cup \{w\} \cup \{v_I,\dots,v_m\}]$ is prime, which contradicts the fact that G is $G[X \cup \{v,w\}]$-minimal. Consequently, $wv_i \notin E(\Gamma_{(G,\overline{X})})$ for every $i \in \{1,\dots,m\}$. In particular, we have $wv_1 \notin E(\Gamma_{(G,\overline{X})})$. It follows that $w \notin Y_G(v_0)$. Therefore, $w \in \mathrm{Ext}_G(Y)$. It follows that $G[X \cup \{w\} \cup \{v_0,\dots,v_m\}]$ is prime. Since G is $G[X \cup \{v,w\}]$-minimal, we obtain $\overline{X} = \{w\} \cup \{v_0,\dots,v_m\}$. Thus, Statement (S2) holds.

Conversely, suppose that Statement (S2) holds. Set $Y = X \cup \{v_0,\dots,v_m\}$. By Corollary 31, $G[Y]$ is prime. Since $w \in \mathrm{Ext}_G(X)$, we have $w \notin \langle Y \rangle_G$, and $w \notin Y_G(\alpha)$ for $\alpha \in X$. Furthermore, since $v_1,\dots,v_m \notin \mathrm{Ext}_G(X)$ and $w \in \mathrm{Ext}_G(X)$, we obtain $w \notin Y_G(v_i)$ for $i \in \{1,\dots,m\}$. Since $wv_1 \notin E(\Gamma_{(G,\overline{X})})$ and $v_0v_1 \in E(\Gamma_{(G,\overline{X})})$, we obtain $w \notin Y_G(v_0)$. It follows that $w \in \mathrm{Ext}_G(Y)$, so G is prime. Lastly, we verify that G is $G[X \cup \{v,w\}]$-minimal. Consider $W \subsetneq V(G)$ such that $X \cup \{v,w\} \subseteq W$. There exists $i \in \{0,\dots,m\}$ such that $v_i \notin W$. Note that $i < m$ because $v_m = v$ and $v \in W$. Set $W' = \{v_{i+1},\dots,v_m\} \cap W$. By Lemma 32, $G[W]$ is decomposable. □

4 Technical preliminaries

To begin, we recall the definition of a module of a digraph. Given a digraph D, a subset of M of $V(D)$ is a *module* of D if for $x,y \in M$ and $v \in V(D) \setminus M$, we have: $xv \in A(D)$ (resp. $vx \in A(D)$) if and only if $yv \in A(D)$ (resp. $vy \in A(D)$). More weakly, we say that M is an *absorbing subset* of D if for $x,y \in M$ and $v \in V(D) \setminus M$, we have: $vx \in A(D)$ if and only if $vy \in A(D)$. The proof of the next lemma can be deduced from that of [9, Lemma 28].

Lemma 33. *Given a graph G, consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Consider also $M \subseteq \overline{X}$ such that $|M| \geq 2$. The following two statements hold*

1. *if M is a module of G, then M is contained in a block of $q_{(G,\overline{X})}$, and M is a module of $\Delta_{(G,\overline{X})}$;*

2. *if M is an absorbing subset of $\Delta_{(G,\overline{X})}$ such that $M \cap \mathrm{Ext}_G(X) = \varnothing$, and M is contained in a block of $q_{(G,\overline{X})}$, then M is a module of G.*

We complete the section with five technical results.

Claim 34. *Let G be a graph. Consider $W_1, W_2 \subseteq V(G)$ such that $W_1 \cup W_2 = V(G)$ and $|W_1 \cap W_2| \geq 2$. Suppose that $G[W_1]$ and $G[W_2]$ are prime. Suppose also that G is decomposable. For each nontrivial module M of G, there exist $w_1 \in W_1 \smallsetminus W_2$ and $w_2 \in W_2 \smallsetminus W_1$ such that $M = \{w_1, w_2\}$.*

Proof. Let M be a nontrivial module of G. For a contradiction, suppose that $W_1 \subseteq M$. Since $|W_1 \cap W_2| \geq 2$, we obtain $|M \cap W_2| \geq 2$. Since $M \cap W_2$ is a module of $G[W_2]$, we obtain $M \cap W_2 = W_2$. Hence $M = V(G)$, which contradicts the fact that M is a nontrivial module of G. It follows that $|M \cap W_1| \leq 1$. If $M \cap W_1 = \varnothing$, then M is a nontrivial module of $G[W_2]$, which contradicts the fact that $G[W_2]$ is prime. Consequently, there exists $w_1 \in W_1$ such that $M \cap W_1 = \{w_1\}$. Similarly, there exists $w_2 \in W_2$ such that $M \cap W_2 = \{w_2\}$. Since $|M| \geq 2$, we obtain $w_1 \in W_1 \smallsetminus W_2$, $w_2 \in W_2 \smallsetminus W_1$, and $M = \{w_1, w_2\}$. □

Fact 35. *Let G be a graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Suppose that there exist distinct elements B and C of $p_{(G,\overline{X})} \smallsetminus \{\mathrm{Ext}_G(X)\}$. Consider distinct elements $v_1, \ldots, v_m$ of B, and distinct elements $w_1, \ldots, w_n$ of C. Suppose that $V(G) = X \cup \{v_1, \ldots, v_m\} \cup \{w_1, \ldots, w_n\}$. If $w_1 v_1 \ldots v_m$ and $v_1 w_1 \ldots w_n$ are strict dipaths of $\Delta_{(G,\overline{X})}$, then G is prime.*

Proof. Set $W_1 = X \cup \{w_1\} \cup \{v_1, \ldots, v_m\}$. Since $v_1, \ldots, v_m \in B$ and $w_1 \notin B$, $w_1 v_1 \ldots v_m$ is an arrow of $\Delta_{(G,\overline{X})}$. By Corollary 31, $G[W_1]$ is prime. Similarly, by setting $W_2 = X \cup \{v_1\} \cup \{w_1, \ldots, w_n\}$, we obtain that $G[W_2]$ is prime. If $m = 1$, then $V(G) = W_1$, so G is prime. Hence, suppose that $m \geq 2$. Similarly, suppose that $n \geq 2$. Since $V(G) = X \cup \{v_1, \ldots, v_m\} \cup \{w_1, \ldots, w_n\}$, $V(G) = W_1 \cup W_2$. Since $X \subseteq W_1 \cap W_2$, we have $|W_1 \cap W_2| \geq 2$. Moreover, we have $W_1 \smallsetminus W_2 = \{v_2, \ldots, v_m\}$ and $W_2 \smallsetminus W_1 = \{w_2, \ldots, w_n\}$. Let $i \in \{2, \ldots, m\}$ and $j \in \{2, \ldots, n\}$. Since v_i and w_j do not belong to the same block of $p_{(G,\overline{X})}$, $\{v_i, w_j\}$ is not a module of G. It follows from Claim 34 that G is prime. □

Fact 36. *Let G be a graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Suppose that there exist distinct elements B and C of $p_{(G,\overline{X})} \smallsetminus \{\mathrm{Ext}_G(X)\}$. Consider distinct elements $v_1, \ldots, v_m$ of B, and distinct elements $w_1, \ldots, w_n$ of C. Suppose that there exists $u \in \overline{X} \smallsetminus (B \cup C)$ such that $V(G) = X \cup \{u\} \cup \{v_1, \ldots, v_m\} \cup \{w_1, \ldots, w_n\}$. If $u v_1 \ldots v_m$ and $u w_1 \ldots w_n$ are strict dipaths of $\Delta_{(G,\overline{X})}$, then G is prime.*

Since the proof of Fact 36 is close to that of Fact 35, we omit it.

Fact 37. *Let G be a graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Suppose that there exist distinct elements B and C of $p_{(G,\overline{X})} \smallsetminus \{\mathrm{Ext}_G(X)\}$. Consider distinct elements $v_1, \ldots, v_m$ of B, and distinct elements $w_1, \ldots, w_n$ of C. Suppose that there exist $t \in \overline{X} \smallsetminus (B \cup \{w_1, \ldots, w_n\})$ and $u \in \overline{X} \smallsetminus (C \cup \{v_1, \ldots, v_m\})$ such that $t \neq u$ and $V(G) = X \cup \{t, u\} \cup \{v_1, \ldots, v_m\} \cup \{w_1, \ldots, w_n\}$. Under these assumptions, G is prime if the following statements hold*

- *$tv_1 \dots v_m$ and $uw_1 \dots w_n$ are strict dipaths of $\Delta_{(G,\overline{X})}$;*
- *for $i \in \{1, \dots, m\}$, $v_i w_1 \notin E(\Gamma_{(G,\overline{X})})$;*
- *for $j \in \{1, \dots, n\}$, $w_j v_1 \notin E(\Gamma_{(G,\overline{X})})$;*
- *$uv_1 \notin A(\Delta_{(G,\overline{X})})$ or $tw_1 \notin A(\Delta_{(G,\overline{X})})$.*

Proof. Set $W_1 = X \cup \{t\} \cup \{v_1, \dots, v_m\}$ and $W_2 = X \cup \{u\} \cup \{w_1, \dots, w_n\}$. Since $v_1, \dots, v_n \in B$ and $t \in \overline{X} \setminus B$, $tv_1 \dots v_m$ is an arrow of $\Delta_{(G,\overline{X})}$. By Corollary 31, $G[W_1]$ is prime. Similarly, $G[W_2]$ is prime. Since $V(G) = X \cup \{t, u\} \cup \{v_1, \dots, v_m\} \cup \{w_1, \dots, w_n\}$, $V(G) = W_1 \cup W_2$. We have $|W_1 \cap W_2| \geq 2$ because $W_1 \cap W_2 = X$. Furthermore, $W_1 \setminus W_2 = \{t\} \cup \{v_1, \dots, v_m\}$ and $W_2 \setminus W_1 = \{u\} \cup \{w_1, \dots, w_n\}$. Let $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. Since v_i and w_j do not belong to the same block of $p_{(G,\overline{X})}$, $\{v_i, w_j\}$ is not a module of G. We verify that $\{u, v_i\}$ is not a module of G. Since $uw_1 \dots w_n$ is a strict dipath of $\Delta_{(G,\overline{X})}$, $uw_1 \in A(\Delta_{(G,\overline{X})})$. Furthermore, since v_i and w_1 do not belong to the same block of $p_{(G,\overline{X})}$, and $v_i w_1 \notin E(\Gamma_{(G,\overline{X})})$, we obtain $v_i w_1 \notin A(\Delta_{(G,\overline{X})})$. Therefore, $\{u, v_i\}$ is not a module of $\Delta_{(G,\overline{X})}$. By Lemma 33, $\{u, v_i\}$ is not a module of G. Similarly, $\{t, w_j\}$ is not a module of G. Lastly, we verify that $\{u, t\}$ is not a module of G. Since $tv_1 \dots v_m$ and $uw_1 \dots w_n$ are strict dipaths of $\Delta_{(G,\overline{X})}$, $tv_1, uw_1 \in A(\Delta_{(G,\overline{X})})$. Since $uv_1 \notin A(\Delta_{(G,\overline{X})})$ or $tw_1 \notin A(\Delta_{(G,\overline{X})})$, $\{t, u\}$ is not a module of $\Delta_{(G,\overline{X})}$. By Lemma 33, $\{t, u\}$ is not a module of G. It follows from Claim 34 that G is prime. □

We end this section with the following claim, which is useful to examine non separated vertices (see Definition 40 and Lemma 43).

Claim 38. *Let G be a graph. Consider $X \subsetneq V(G)$ such that $G[X]$ is prime. Suppose that there exists $B \in p_{(G,\overline{X})} \setminus \{\mathrm{Ext}_G(X)\}$ such that $|B| \geq 2$. Consider distinct elements v and w of B. Suppose that $\{v, w\}$ is not $G[X]$-reachable (see Definition 18). Consider a v-dipath $v_0 \dots v_m$ of $\Delta_{(G,\overline{X})}$, and a w-dipath $w_0 \dots w_n$ of $\Delta_{(G,\overline{X})}$. Under these assumptions, the following assertions hold*

1. *if there exist $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$ such that $v_i = w_j$, then $i = j$, $i < m$, and $i < n$;*
2. *if there exist $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$ such that $v_i \neq w_j$ and $\{v_i, w_j\}$ is a module of $G[X \cup \{v_0, \dots, v_m\} \cup \{w_0, \dots, w_n\}]$, then $i = j$, and*

$$\begin{cases} i = m = n \\ \text{or} \\ i < m \text{ and } i < n. \end{cases}$$

Proof. For the first assertion, suppose that there exist $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$ such that $v_i = w_j$. Suppose that $i = 0$. Since $v_0 \notin B$ and $w_l \in B$ for

$l \in \{1, \ldots, n\}$, we obtain $j = 0$. Similarly, if $j = 0$, then $i = 0$. Thus, suppose that $i \geq 1$ and $j \geq 1$. For a contradiction, suppose that $i = m$. Hence $v = w_j$. Since $v \neq w$, we obtain $j < n$, which contradicts the fact that $\{v, w\}$ is not $G[X]$-reachable. It follows that $i \in \{1, \ldots, m-1\}$. Analogously, $j \in \{1, \ldots, n-1\}$. The sequence $v_0 \ldots v_{i-1} w_j w_{j+1} \ldots w_n$ is a dipath of $\Delta_{(G,\overline{X})}$. We can extract from $v_0 \ldots v_{i-1} w_j w_{j+1} \ldots w_n$ an arrow $u_0 \ldots u_p$ of $\Delta_{(G,\overline{X})}$ such that $u_0 = v_0$, $u_p = w_n$, and $p \leq n - j + i$. Since $\delta_{(G,\overline{X})}(w) = n$, we obtain $i \geq j$. Analogously, we have $j \geq i$. Therefore $i = j$.

For the second assertion, set $H = G[X \cup \{v_0, \ldots, v_m\} \cup \{w_0, \ldots, w_n\}]$. Note that

$$\Delta_{(H,\overline{X})} = \Delta_{(G,\overline{X})}[\{v_0, \ldots, v_m\} \cup \{w_0, \ldots, w_n\}]. \tag{5}$$

It follows that

$$v_0 \ldots v_m \text{ is a } v\text{-dipath of } \Delta_{(H,\overline{X})} \text{ and } w_0 \ldots w_n \text{ is a } w\text{-dipath of } \Delta_{(H,\overline{X})}. \tag{6}$$

We obtain that v and w are $H[X]$-reachable. It follows that

$$\delta_{(H,\overline{X})}(v) = m \text{ and } \delta_{(H,\overline{X})}(w) = n. \tag{7}$$

Now, we suppose that there exist $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$ such that $v_i \neq w_j$ and $\{v_i, w_j\}$ is a module of H. By Lemma 33 applied to H, $\{v_i, w_j\}$ is a module of $\Delta_{(H,\overline{X})}$. Observe that if $i = 0$, then $j = 0$ because v_0 and w_0 are the only elements of $\{v_0, \ldots, v_m\} \cup \{w_0, \ldots, w_n\}$ that do not belong to B. Similarly, if $j = 0$, then $i = 0$. Now, suppose that $i > 0$ and $j > 0$. We distinguish the following two cases.

First, suppose that $i = m$. Suppose for a contradiction that $1 \leq j \leq n - 1$. By (6), $w_0 \ldots w_n$ is a w-dipath of $\Delta_{(H,\overline{X})}$. Since $\{v, w_j\}$ is a module of $\Delta_{(H,\overline{X})}$, $w_0 \ldots w_{j-1} v w_{j+1} \ldots w_n$ is an arrow of $\Delta_{(H,\overline{X})}$. Therefore, $\{v, w\}$ is $H[X]$-reachable, which contradicts the fact that $\{v, w\}$ is not $G[X]$-reachable. It follows that $j = n$. Thus, $\{v, w\}$ is a module of $\Delta_{(H,\overline{X})}$. It follows from (6) that $v_0 \ldots v_{m-1} w$ is an arrow of $\Delta_{(H,\overline{X})}$. Hence, $m \geq \delta_{(H,\overline{X})}(w)$. It follows from (7) that $m \geq n$. Similarly, $n \geq m$. Therefore, we obtain $m = n$. Consequently, if $i = m$, then $j = n$, and $m = n$. Analogously, if $j = n$, then $i = m$, and $m = n$.

Second, suppose that $m \geq 2$, $n \geq 2$, $i \in \{1, \ldots, m-1\}$, and $j \in \{1, \ldots, n-1\}$. We have to show that $i = j$. Since $\{v_i, w_j\}$ is a module of $\Delta_{(H,\overline{X})}$, $v_0 \ldots v_{i-1} w_j w_{j+1} \ldots w_n$ is a dipath of $\Delta_{(H,\overline{X})}$. From $v_0 \ldots v_{i-1} w_j w_{j+1} \ldots w_n$, we can extract an arrow $u_0 \ldots u_p$ of $\Delta_{(H,\overline{X})}$ such that $u_0 = v_0$, $u_p = w_n$, and $p \leq n - j + i$. Since $\delta_{(H,\overline{X})}(w) = n$ by (7), we obtain $n - j + i \geq p \geq n$, so $i \geq j$. Similarly, $j \geq i$. Therefore $i = j$. $\square$

5 The other results

Layout 4. In this section, we consider a graph G, a proper subset X of $V(G)$ such that $G[X]$ is prime, and distinct elements v, w of $\overline{X}$. In Theorems 20 and

21, we considered the case where $\{v,w\} \cap \mathrm{Ext}_G(X) \neq \varnothing$. Now, we suppose that $\{v,w\} \cap \mathrm{Ext}_G(X) = \varnothing$. Hence, there exist $B, C \in p_{(G,\overline{X})} \setminus \{\mathrm{Ext}_G(X)\}$ such that $v \in B$ and $w \in C$. To begin, we suppose that $B \neq C$.

Theorem 39. *The graph G is $G[X \cup \{v,w\}]$-minimal if and only if there exist distinct elements $v_1, \dots, v_m$ of B, and distinct elements $w_1, \dots, w_n$ of C such that one of the following statements holds*

(S3) $\overline{X} = \{v_1, \dots, v_m\} \cup \{w_1, \dots, w_n\}$ *and*

- $w_1 v_1 \dots v_m$ *is a v-dipath of* $\Delta_{(G,\overline{X})}$,
- $v_1 w_1 \dots w_n$ *is a w-dipath of* $\Delta_{(G,\overline{X})}$,
- *for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, if $i \geq 2$ or $j \geq 2$, then $v_i w_j \notin E(\Gamma_{(G,\overline{X})})$;*

(S4) *there exists $u \in \overline{X} \setminus (B \cup C)$ such that $\overline{X} = \{u\} \cup \{v_1, \dots, v_m\} \cup \{w_1, \dots, w_n\}$, and*

- $uv_1 \dots v_m$ *is a v-dipath of* $\Delta_{(G,\overline{X})}$,
- $uw_1 \dots w_n$ *is a w-dipath of* $\Delta_{(G,\overline{X})}$,
- *for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, $v_i w_j \notin E(\Gamma_{(G,\overline{X})})$;*

(S5) *there exist $t \in \overline{X} \setminus (B \cup \{w_1, \dots, w_n\})$ and $u \in \overline{X} \setminus (C \cup \{v_1, \dots, v_m\})$ such that $t \neq u$, $\overline{X} = \{t,u\} \cup \{v_1, \dots, v_m\} \cup \{w_1, \dots, w_n\}$, and*

- $tv_1 \dots v_m$ *is a v-dipath of* $\Delta_{(G,\overline{X})}$,
- $uw_1 \dots w_n$ *is a w-dipath of* $\Delta_{(G,\overline{X})}$,
- *for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, $v_i w_j \notin E(\Gamma_{(G,\overline{X})})$,*
- *for $i \in \{1, \dots, m\}$, $uv_i \notin A(\Delta_{(G,\overline{X})})$,*
- *for $j \in \{1, \dots, n\}$, $tw_j \notin A(\Delta_{(G,\overline{X})})$.*

Proof. To begin, suppose that G is $G[X \cup \{v,w\}]$-minimal. Since G is prime, it follows from Corollary 31 that $\Delta_{(G,\overline{X})}$ admits a v-dipath $v_0 \dots v_m$, and a w-dipath $w_0 \dots w_n$. We distinguish the two following cases.

1. Suppose that there exist $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ such that $v_i w_j \in E(\Gamma_{(G,\overline{X})})$. Denote by I the largest $i \in \{1, \dots, m\}$ such that there exists $j \in \{1, \dots, n\}$ with $v_i w_j \in E(\Gamma_{(G,\overline{X})})$. Now, denote by J the largest $j \in \{1, \dots, n\}$ such that $v_I w_j \in E(\Gamma_{(G,\overline{X})})$. We obtain that $w_J v_I \dots v_m$ and $v_I w_J \dots w_n$ are strict dipaths of $\Delta_{(G,\overline{X})}$. It follows from Fact 35 that $G[X \cup \{v_I, \dots, v_m\} \cup \{w_J, \dots, w_n\}]$ is prime. Since G is $G[X \cup \{v,w\}]$-minimal, we obtain $V(G) = X \cup \{v_I, \dots, v_m\} \cup \{w_J, \dots, w_n\}$. It follows that $I = 1$, $J = 1$, $v_0 = w_1$, and $w_0 = v_1$. Consequently, Statement (S3) holds.

2. Suppose that $v_i w_j \notin E(\Gamma_{(G,\overline{X})})$ for $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. In particular, we obtain $v_0 \notin \{w_1,\dots,w_n\}$, and $w_0 \notin \{v_1,\dots,v_m\}$. Thus

$$v_0, w_0 \in \overline{X} \smallsetminus (\{v_1,\dots,v_m\} \cup \{w_1,\dots,w_n\}). \tag{8}$$

For a contradiction, suppose that there exists $j \in \{2,\dots,n\}$ with $v_0 w_j \in A(\Delta_{(G,\overline{X})})$. Denote by J the largest $j \in \{2,\dots,n\}$ such that $v_0 w_j \in A(\Delta_{(G,\overline{X})})$. We obtain that $v_0 v_1 \dots v_m$ and $v_0 w_J \dots w_n$ are strict dipaths of $\Delta_{(G,\overline{X})}$. If $v_0 \in C$, then $G[X \cup \{v_0,\dots,v_m\} \cup \{w_J,\dots,w_n\}]$ is prime by Fact 35, which contradicts the fact that G is $G[X \cup \{v,w\}]$-minimal because $w_0 \notin X \cup \{v_0,\dots,v_m\} \cup \{w_J,\dots,w_n\}$ by (8). If $v_0 \notin C$, then $G[X \cup \{v_0,\dots,v_m\} \cup \{w_J,\dots,w_n\}]$ is prime by Fact 36, which contradicts the fact that G is $G[X \cup \{v,w\}]$-minimal because $w_1 \notin X \cup \{v_0,\dots,v_m\} \cup \{w_J,\dots,w_n\}$. It follows that $v_0 w_j \notin A(\Delta_{(G,\overline{X})})$ for $j \in \{2,\dots,n\}$. Similarly, $w_0 v_i \notin A(\Delta_{(G,\overline{X})})$ for $i \in \{2,\dots,m\}$. We distinguish the following three subcases.

2.1. Suppose that $v_0 w_1 \in A(\Delta_{(G,\overline{X})})$. If $v_0 \in C$, then $G[X \cup \{v_0\} \cup \{v_1,\dots,v_m\} \cup \{w_1,\dots,w_n\}]$ is prime by Fact 35. If $v_0 \notin C$, then $G[X \cup \{v_0\} \cup \{v_1,\dots,v_m\} \cup \{w_1,\dots,w_n\}]$ is prime by Fact 36. It follows that $G[X \cup \{v_0\} \cup \{v_1,\dots,v_m\} \cup \{w_1,\dots,w_n\}]$ is prime. Since G is $G[X \cup \{v,w\}]$-minimal, we obtain $V(G) = X \cup \{v_0\} \cup \{v_1,\dots,v_m\} \cup \{w_1,\dots,w_n\}$. Since $w_0 \notin \{v_1,\dots,v_m\} \cup \{w_1,\dots,w_n\}$ by (8), we obtain $w_0 = v_0$. Consequently, $v_0 \notin B \cup C$, and hence Statement (S4) holds.

2.2. Suppose that $w_0 v_1 \in A(\Delta_{(G,\overline{X})})$. Similarly, we obtain $w_0 = v_0$, and Statement (S4) holds.

2.3. Suppose that $v_0 w_1 \notin A(\Delta_{(G,\overline{X})})$ and $w_0 v_1 \notin A(\Delta_{(G,\overline{X})})$. Clearly, we have $v_0 \neq w_0$. By Fact 37, $G[X \cup \{v_0,\dots,v_m\} \cup \{w_0,\dots,w_n\}]$ is prime. Since G is $G[X \cup \{v,w\}]$-minimal, we obtain $V(G) = X \cup \{v_0,\dots,v_m\} \cup \{w_0,\dots,w_n\}$. Hence Statement (S5) holds.

Conversely, suppose that there exist distinct elements $v_1,\dots,v_m$ of B, and distinct elements $w_1,\dots,w_n$ of C such that one of Statements (S3), (S4) or (S5) holds. We distinguish the following three cases.

(i) Suppose that Statement (S3) holds. By Fact 35, G is prime. Consider $W \subsetneq V(G)$ such that $X \cup \{v,w\} \subseteq W$. We have to verify that $G[W]$ is decomposable. For instance, assume that $v_i \notin W$, where $i \in \{1,\dots,m\}$. Since $v_m = v$, $i < m$. Set $W' = \{v_{i+1},\dots,v_m\} \cap W$. Clearly, $v_m \in W'$ and $W' \subseteq B$. By Lemma 32, $G[W]$ is decomposable.

(ii) Suppose that Statement (S4) holds. By Fact 36, G is prime. Consider $W \subsetneq V(G)$ such that $X \cup \{v,w\} \subseteq W$. We have to verify that $G[W]$ is decomposable. To begin, suppose that $u \notin W$. Set $W' = \{v_1,\dots,v_m\} \cap W$. Clearly, $v_m \in W'$ and $W' \subseteq B$. By Lemma 32, $G[W]$ is decomposable.

Now, suppose that $u \in W$. Since $X \cup \{v, w\} \subseteq W \subsetneq V(G)$, we can assume that $v_i \notin W$, where $i \in \{1, \dots, m\}$. Since $v_m = v$, $i < m$. Set $W' = \{v_{i+1}, \dots, v_m\} \cap W$. Clearly, $v_m \in W'$ and $W' \subseteq B$. By Lemma 32, $G[W]$ is decomposable.

(iii) Suppose that Statement (S5) holds. By Fact 37, G is prime. Consider $W \subsetneq V(G)$ such that $X \cup \{v, w\} \subseteq W$. We have to verify that $G[W]$ is decomposable. To begin, suppose that $\{t, u\} \setminus W \neq \varnothing$. For instance, assume that $t \notin W$. Set $W' = \{v_1, \dots, v_m\} \cap W$. Clearly, $v_m \in W'$ and $W' \subseteq B$. By Lemma 32, $G[W]$ is decomposable. Now, suppose that $\{t, u\} \subseteq W$. Since $X \cup \{v, w\} \subseteq W \subsetneq V(G)$, we can assume that $v_i \notin W$, where $i \in \{1, \dots, m\}$. Since $v_m = v$, $i < m$. Set $W' = \{v_{i+1}, \dots, v_m\} \cap W$. Clearly, $v_m \in W'$ and $W' \subseteq B$. By Lemma 32, $G[W]$ is decomposable. □

Layout 5. In the sequel, we suppose that $B = C$.

We use the following notion of separated vertices.

Definition 40. We say that v and w are *separated* if for every v-dipath $v_0 \dots v_m$ of $\Delta_{(G,\overline{X})}$, and for every w-dipath $w_0 \dots w_n$ of $\Delta_{(G,\overline{X})}$, both assertions below hold

$$\begin{cases} \{v_0, \dots, v_m\} \cap \{w_0, \dots, w_n\} = \varnothing \\ \text{and} \\ G[X \cup \{v_0, \dots, v_m\} \cup \{w_0, \dots, w_n\}] \text{ is prime.} \end{cases} \tag{9}$$

Layout 6. To begin, we suppose that v and w are separated.

We use the following notation.

Notation 41. Let G be a graph. Consider subsets X and Y of $V(G)$ such that $X \subsetneq Y$, $G[X]$ is prime, and $G[Y]$ is prime. Let $B \in p_{(G,\overline{X})} \setminus \{\text{Ext}_G(X)\}$. Set

$$\widetilde{B}_Y = \begin{cases} \langle Y \rangle_G \text{ if } B = \langle X \rangle_G, \\ Y_G(\alpha) \text{ if } B = X_G(\alpha), \text{ where } \alpha \in X. \end{cases}$$

Clearly, we have $\widetilde{B}_Y \subseteq B$.

Theorem 42. *The graph G is $G[X \cup \{v, w\}]$-minimal if and only if there exist distinct elements $v_1, \dots, v_m$ of B and $v_0 \in \overline{X} \setminus B$, and there exist distinct elements $w_1, \dots, w_n$ of B and $w_0 \in \overline{X} \setminus B$, satisfying the following statements*

(A1) *$v_0 \dots v_m$ is a v-dipath of $\Delta_{(G,\overline{X})}$,*

(A2) *$w_0 \dots w_n$ is a w-dipath of $\Delta_{(G,\overline{X})}$,*

(A3) *$\overline{X} = \{v_0, \dots, v_m\} \cup \{w_0, \dots, w_n\}$,*

and satisfying one of the following statements

(S6) • *$\{v\} \cup \{w_1, \dots, w_n\}$ is a module of $G - w_0$,*

- *for* $i \in \{1,\dots,m\}$ *and* $j \in \{0,\dots,n\}$, $w_j v_i \notin A(\Delta_{(G,\overline{X})})$;

(S7) *Statement (S7) is obtained from Statement (S6) by interchanging the roles of* v *and* w;

(S8)
- *for* $i \in \{1,\dots,m\}$ *and* $j \in \{0,\dots,n\}$, $w_j v_i \notin A(\Delta_{(G,\overline{X})})$,
- *for* $i \in \{0,\dots,m\}$ *and* $j \in \{1,\dots,n\}$, $v_i w_j \notin A(\Delta_{(G,\overline{X})})$.

Proof. To begin, suppose that G is $G[X \cup \{v,w\}]$-minimal. By Corollary 31, $\Delta_{(G,\overline{X})}$ admits a v-dipath $v_0 \dots v_m$, and a w-dipath $w_0 \dots w_n$. Since v and w are separated, (9) holds. Since G is $G[X \cup \{v,w\}]$-minimal, we obtain $\overline{X} = \{v_0,\dots,v_m\} \cup \{w_0,\dots,w_n\}$. Set $Y = X \cup \{v_0,\dots,v_m\}$, and $Z = X \cup \{w_0,\dots,w_n\}$. By Corollary 31, $G[Y]$ and $G[Z]$ are prime. We verify that if $w \notin Y_G(v)$, then

$$\begin{cases} w \in \widetilde{B}_Y \text{ (see Notation 41)}, \\ \text{and} \\ v_i w_j \notin A(\Delta_{(G,\overline{X})}) \text{ for } i \in \{0,\dots,m\} \text{ and } j \in \{1,\dots,n\}. \end{cases} \tag{10}$$

Indeed, suppose that $w \notin Y_G(v)$. Since G is $G[X \cup \{v,w\}]$-minimal and $n \geq 1$, we obtain $w \notin \mathrm{Ext}_G(Y)$. Since $v_0 \notin B$, we have $w \notin Y_G(v_0)$. Lastly, suppose that $m \geq 2$, and consider $i \in \{1,\dots,m-1\}$. For a contradiction, suppose that $w \in Y_G(v_i)$. We obtain that $G[(Y \setminus \{v_i\}) \cup \{w\}]$ is prime, which contradicts the fact that G is $G[X \cup \{v,w\}]$-minimal. Therefore, $w \notin Y_G(v_i)$. It follows that $w \in \langle Y \rangle_G$ or $w \in Y_G(\alpha)$, where $\alpha \in X$. If $w \in \langle Y \rangle_G$, then $w \in \langle X \rangle_G$. Moreover, if $w \in Y_G(\alpha)$, where $\alpha \in X$, then $w \in X_G(\alpha)$. Consequently,

$$w \in \widetilde{B}_Y. \tag{11}$$

Furthermore, since G is prime, it follows from Corollary 31 applied to $G[Y]$ that $\Delta_{(G,\overline{Y})}$ admits a w-dipath $z_0 \dots z_q$, and $G[Y \cup \{z_0,\dots,z_q\}]$ is prime. It follows from (11) that

$$\{z_1 \dots z_q\} \subseteq \widetilde{B}_Y. \tag{12}$$

Since G is $G[X \cup \{v,w\}]$-minimal, we obtain $\overline{X} = \{v_0,\dots,v_m\} \cup \{z_0,\dots,z_q\}$. Therefore,

$$\{w_0,\dots,w_n\} = \{z_0,\dots,z_q\}.$$

In particular, we have $n = q$. If $w_0 \in \{z_1,\dots,z_q\}$, then $w_0 \in \widetilde{B}_Y$, and hence $w_0 \in B$. Thus, $w_0 = z_0$, and hence $\{w_1,\dots,w_n\} = \{z_1,\dots,z_q\}$. It follows from (12) that

$$\{w_1,\dots,w_n\} \subseteq \widetilde{B}_Y.$$

Let $i \in \{0,\dots,m\}$ and $j \in \{1,\dots,n\}$. Since $w_j \in \widetilde{B}_Y$, we obtain $v_i w_j \notin A(\Delta_{(G,\overline{X})})$. Consequently, (10) holds. Similarly, if $v \notin Z_G(w)$, then

$$\begin{cases} v \in \widetilde{B}_Z, \\ \text{and} \\ w_j v_i \notin A(\Delta_{(G,\overline{X})}) \text{ for } i \in \{1,\dots,m\} \text{ and } j \in \{0,\dots,n\}. \end{cases} \tag{13}$$

We distinguish the following three cases.

1. Suppose that $w \in Y_G(v)$. Since G is prime, it follows from Corollary 31 that $\Delta_{(G,\overline{Y})}$ admits a w-dipath $z_0 \dots z_q$, and $G[Y \cup \{z_0, \dots, z_q\}]$ is prime. Since G is $G[X \cup \{v, w\}]$-minimal, we obtain $\overline{X} = \{v_0, \dots, v_m\} \cup \{z_0, \dots, z_q\}$. Therefore, $\{w_0, \dots, w_n\} = \{z_0, \dots, z_q\}$. In particular, we have $n = q$ and $Z = X \cup \{z_0, \dots, z_q\}$. If $w_0 \in \{z_1, \dots, z_q\}$, then $w_0 \in Y_G(v)$, and hence $w_0 \in B$ because $v \in B$. It follows that $w_0 = z_0$, so $\{w_1, \dots, w_n\} = \{z_1, \dots, z_q\}$, and hence $\{w_1, \dots, w_n\} \subseteq Y_G(v)$. It follows that $\{v\} \cup \{w_1, \dots, w_n\}$ is a module of $G - w_0$.

   Now, since $z_0 \dots z_q$ is a strict dipath of $\Delta_{(G,\overline{Y})}$ such that $z_q = w$, we have $z_{q-1}w \in A(\Delta_{(G,\overline{Y})})$. Since $w \in Y_G(v)$, we obtain that $\{v, w\}$ is not a module of $G[Y \cup \{z_{q-1}, w\}]$. Therefore, we have $v \notin Z_G(w)$. It follows from (13) that for $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, n\}$, $w_j v_i \notin A(\Delta_{(G,\overline{X})})$. Hence, Statement (S6) holds.

2. Suppose that $v \in Z_G(w)$. We obtain that Statement (S7) holds.

3. Suppose that $v \notin Z_G(w)$, and $w \notin Y_G(v)$. It follows from (10) and (13) that Statement (S8) holds.

Conversely, suppose that there exist distinct elements $v_1, \dots, v_m$ of B and $v_0 \in \overline{X} \setminus B$, and there exist distinct elements $w_1, \dots, w_n$ of B and $w_0 \in \overline{X} \setminus B$ satisfying Statements (A1), (A2), (A3), and one of Statements (S6) or (S8). Since Statement (A1) holds, $v_0 \dots v_m$ is a v-dipath of $\Delta_{(G,\overline{X})}$. By Corollary 31, $G[X \cup \{v_0, \dots, v_m\}]$ is prime. Similarly, since Statement (A2) holds, $G[X \cup \{w_0, \dots, w_n\}]$ is prime. Since Statement (S6) or (S8) holds, we have $w_j v_i \notin A(\Delta_{(G,\overline{X})})$ for $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, n\}$. For a contradiction, suppose that G is decomposable. Recall that $\overline{X} = \{v_0, \dots, v_m\} \cup \{w_0, \dots, w_n\}$ because Statement (A3) holds. By Claim 34 applied with $W_1 = X \cup \{v_0, \dots, v_m\}$ and $W_2 = X \cup \{w_0, \dots, w_n\}$, there exist $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$ such that $\{v_i, w_j\}$ is a module of G. By Lemma 33, $\{v_i, w_j\}$ is a module of $\Delta_{(G,\overline{X})}$. We cannot have $i \leq m - 1$ because $v_i v_{i+1} \in A(\Delta_{(G,\overline{X})})$ and $w_j v_{i+1} \notin A(\Delta_{(G,\overline{X})})$. Furthermore, we cannot have $i = m$ and $j \in \{1, \dots, n\}$ because $w_{j-1} w_j \in A(\Delta_{(G,\overline{X})})$ and $w_{j-1} v_m \notin A(\Delta_{(G,\overline{X})})$. Lastly, we cannot have $i = m$ and $j = 0$ because $v_m \in B$ and $w_0 \notin B$. It follows that G is prime. Lastly, we have to verify that G is $G[X \cup \{v, w\}]$-minimal. We distinguish the following two cases.

First, suppose that Statement (S6) holds. Set $Y = X \cup \{v_0, \dots, v_m\}$. We verify that G is $G[Y \cup \{w\}]$-minimal. Recall that $G[Y]$ is prime because Assertion (A1) holds. We show that

$$\begin{cases} w_0 \notin Y_G(v_m), \\ \{w_1, \dots, w_n\} \subseteq Y_G(v_m), \\ \text{and} \\ w_0 \dots w_n \text{ is an arrow of } \Delta_{(G,\overline{Y})}. \end{cases} \tag{14}$$

Since $v_m \in B$ and $w_0 \in \overline{X} \setminus B$, we have $w_0 \notin Y_G(v_m)$. Furthermore, since Statement (S6) holds, $\{v\} \cup \{w_1, \dots, w_n\}$ is a module of $G - w_0$. Therefore,

$\{w_1,\dots,w_n\} \subseteq Y_G(v_m)$, and $\{v_m\} \cup \{w_1,\dots,w_n\}$ is contained in a block of $q_{(G,\overline{X})}$. Consider $k \in \{0,\dots,n-1\}$. Since $w_0 \dots w_n$ is a strict dipath of $\Delta_{(G,\overline{X})}$, we have $w_k w_{k+1} \in A(\Delta_{(G,\overline{X})})$. Since Statement (S6) holds, we have $w_k v_m \notin A(\Delta_{(G,\overline{X})})$. It follows from Lemma 33 applied to $G[X \cup \{v_m, w_k, w_{k+1}\}]$ that $w_k \not\longleftrightarrow_G \{v_m, w_{k+1}\}$. Since $w_{k+1} \in Y_G(v_m)$, we obtain $w_k w_{k+1} \in A(\Delta_{(G,\overline{Y})})$. Suppose that $k+2 \le n$, and consider $l \in \{k+2,\dots,n\}$. Since $w_0 \dots w_n$ is a strict dipath of $\Delta_{(G,\overline{X})}$, $w_k w_l \notin A(\Delta_{(G,\overline{X})})$. Since Statement (S6) holds, $w_k v_m \notin A(\Delta_{(G,\overline{X})})$. Since v_m and w_l belong to the same block of $q_{(G,\overline{X})}$, it follows from Lemma 33 that $w_k \longleftrightarrow_G \{v_m, w_l\}$. Since $w_l \in Y_G(v_m)$, we obtain $w_k w_l \notin A(\Delta_{(G,\overline{Y})})$. Therefore, $w_0 \dots w_n$ is an arrow of $\Delta_{(G,\overline{Y})}$. Consequently, (14) holds. By Theorem 16, G is $G[Y \cup \{w\}]$-minimal.

To continue, we verify that G is $G[X \cup \{v,w\}]$-minimal. Consider a subset W of $V(G)$ such that $X \cup \{v,w\} \subseteq W \subsetneq V(G)$.

1. Suppose that there exists $i \in \{0,\dots,m-1\}$ such that $v_i \notin W$. Set $W' = \{v_{i+1},\dots,v_m\} \cap W$. By Lemma 32, $G[W]$ is decomposable.

2. Suppose that $\{v_0,\dots,v_m\} \subseteq W$. Hence, there exists $k \in \{0,\dots,n-1\}$ such that $w_k \notin W$. Since G is $G[Y \cup \{w\}]$-minimal, $G[W]$ is decomposable.

Second, suppose that Statement (S8) holds. We verify that G is $G[X \cup \{v,w\}]$-minimal. Consider a subset W of $V(G)$ such that $X \cup \{v,w\} \subseteq W \subsetneq V(G)$. By exchanging v and w if necessary, we can assume that there exists $i \in \{0,\dots,m-1\}$ such that $v_i \notin W$. Set $W' = \{v_{i+1},\dots,v_m\} \cap W$. It follows from Lemma 32 that $G[W]$ is decomposable. □

Layout 7. Now, we suppose that v and w are not separated. It follows that v and w are $G[X]$-reachable (see Definition 18). Therefore, $\delta_{(G,\overline{X})}(v)$ and $\delta_{(G,\overline{X})}(w)$ are well-defined (see Theorem 16 and Notation 17). For convenience, set $m = \delta_{(G,\overline{X})}(v)$ and $n = \delta_{(G,\overline{X})}(w)$. Moreover, following Remark 19, we assume that $\{v,w\}$ is not $G[X]$-reachable.

We use the following lemma.

Lemma 43. *There exists a v-dipath $v_0 \dots v_m$ of $\Delta_{(G,\overline{X})}$, and there exist distinct elements $w_{i+1},\dots,w_n$ of $B \setminus \{v_0,\dots,v_m\}$, where $i \in \{0,\dots,m-1\} \cap \{0,\dots,n-1\}$, such that the following assertions hold*

- *$v_0 \dots v_i w_{i+1} \dots w_n$ is a w-dipath of $\Delta_{(G,\overline{X})}$;*
- *if $G[X \cup \{v_0,\dots,v_m\} \cup \{w_{i+1},\dots,w_n\}]$ is decomposable, then $m = n$, $i = m-1$, and $\{v_m, w_n\}$ is the only nontrivial module of $G[X \cup \{v_0,\dots,v_m\} \cup \{w_n\}]$.*

Proof. Since v and w are not separated, there exist a v-dipath $v_0 \dots v_m$ of $\Delta_{(G,\overline{X})}$, and a w-dipath $w_0 \dots w_n$ of $\Delta_{(G,\overline{X})}$ that do not satisfy (9). Since $v_0 \dots v_m$ is a v-dipath of $\Delta_{(G,\overline{X})}$, $G[X \cup \{v_0,\dots,v_m\}]$ is prime by Corollary 31. Similarly, $G[X \cup \{w_0,\dots,w_n\}]$ is prime.

First, suppose that $\{v_0,\dots,v_m\} \cap \{w_0,\dots,w_n\} \neq \varnothing$. We show that there exists $k \in \{0,\dots,m-1\} \cap \{0,\dots,n-1\}$ such that

$$\begin{cases} \{v_0,\dots,v_m\} \cap \{w_{k+1},\dots,w_n\} = \varnothing \\ \text{and} \\ v_0 \dots v_k w_{k+1} \dots w_n \text{ is a } w\text{-dipath of } \Delta_{(G,\overline{X})}. \end{cases} \tag{15}$$

There exist $p \in \{0,\dots,m\}$ and $q \in \{0,\dots,n\}$ such that $v_p = w_q$. Set

$$J = \max(\{q \in \{0,\dots,n\} : w_q \in \{v_0,\dots,v_m\}\}).$$

It follows from the maximality of J that $\{v_0,\dots,v_m\} \cap \{w_{J+1},\dots,w_n\} = \varnothing$. Furthermore, since $w_J \in \{v_0,\dots,v_m\}$, there exists $I \in \{0,\dots,m\}$ such that $v_I = w_J$. By Claim 38, we have $I = J$, $I < m$, and $I < n$. Thus, $v_0 \dots v_I w_{I+1} \dots w_n$ is a dipath of $\Delta_{(G,\overline{X})}$. Since $\delta_{(G,\overline{X})}(w) = n$, we obtain that $v_0 \dots v_I w_{I+1} \dots w_n$ is a w-dipath of $\Delta_{(G,\overline{X})}$. Therefore (15) holds for $k = I$. Let K be the largest $k \in \{0,\dots,m-1\} \cap \{0,\dots,n-1\}$ such that (15) holds.

Lastly, suppose that $G[X \cup \{v_0,\dots,v_m\} \cup \{w_{K+1},\dots,w_n\}]$ is decomposable. Consider a nontrivial module M of $G[X \cup \{v_0,\dots,v_m\} \cup \{w_{K+1},\dots,w_n\}]$. Recall that $G[X \cup \{v_0,\dots,v_K, w_{K+1},\dots,w_n\}]$ is prime by Corollary 31. By Claim 34 applied with $W_1 = X \cup \{v_0,\dots,v_m\}$ and $W_2 = X \cup \{v_0,\dots,v_K, w_{K+1},\dots,w_n\}$, there exist $k \in \{K+1,\dots,m\}$ and $l \in \{K+1,\dots,n\}$ such that $M = \{v_k, w_l\}$. By Claim 38, we have $k = l$. Moreover, we have $k = m = n$ or $k < m$ and $k < n$. For a contradiction, suppose that $k < m$ and $k < n$. By Lemma 33, $\{v_k, w_k\}$ is a module of $\Delta_{(H,\overline{X})}$, where $H = G[X \cup \{v_0,\dots,v_m\} \cup \{w_{K+1},\dots,w_n\}]$. It follows that $v_0 \dots v_k w_{k+1} \dots w_n$ is a dipath of $\Delta_{(G,\overline{X})}$. In fact, $v_0 \dots v_k w_{k+1} \dots w_n$ is a w-dipath of $\Delta_{(G,\overline{X})}$ because $\delta_{(G,\overline{X})}(w) = n$. Furthermore, since $\{v_0,\dots,v_m\} \cap \{w_{K+1},\dots,w_n\} = \varnothing$ and $k \in \{K+1,\dots,m\}$, we obtain $\{v_0,\dots,v_m\} \cap \{w_k,\dots,w_n\} = \varnothing$. Hence (15) holds for k, which contradicts the maximality of K. It follows that $k = m = n$. Consequently, $\{v_m, w_m\}$ is the unique nontrivial module of $G[X \cup \{v_0,\dots,v_m\} \cup \{w_{K+1},\dots,w_n\}]$. We obtain that $v_0 \dots v_{m-1} w_m$ is a strict dipath of $\Delta_{(G,\overline{X})}$. In fact, $v_0 \dots v_{m-1} w_m$ is a w-dipath of $\Delta_{(G,\overline{X})}$ because $\delta_{(G,\overline{X})}(w) = n$ and $m = n$. Moreover, $w_m \notin \{v_0,\dots,v_m\}$ because $\{v_0,\dots,v_m\} \cap \{w_{K+1},\dots,w_n\} = \varnothing$. It follows that (15) holds for $k = m-1$. We obtain $K = m-1$ by maximality of K.

Second, suppose that $\{v_0,\dots,v_m\} \cap \{w_0,\dots,w_n\} = \varnothing$. Since (9) does not hold, $G[X \cup \{v_0,\dots,v_m\} \cup \{w_0,\dots,w_n\}]$ is decomposable. Consider a nontrivial module M of $G[X \cup \{v_0,\dots,v_m\} \cup \{w_0,\dots,w_n\}]$. By Claim 34, there exist $r \in \{0,\dots,m\}$ and $s \in \{0,\dots,n\}$ such that $M = \{v_r, w_s\}$. By Claim 38, we have $r = s$. Moreover, we have $r = m = n$ or $r < m$ and $r < n$. We distinguish the following two cases.

1. Suppose that $r = s$ and $r = m = n$. Thus, $\{v_m, w_m\}$ is a nontrivial module of $G[X \cup \{v_0,\dots,v_m\} \cup \{w_m\}]$. Since $G[X \cup \{v_0,\dots,v_m\}]$ is prime, $\{v_m, w_m\}$ is the only one. Therefore, $v_0 \dots v_{m-1} w_m$ is a strict dipath of $\Delta_{(G,\overline{X})}$. Since $\delta_{(G,\overline{X})}(w) = n$ and $m = n$, $v_0 \dots v_{m-1} w_m$ is

a w-dipath of $\Delta_{(G,\overline{X})}$. Moreover, $w_m \notin \{v_0, \ldots, v_m\}$ because $\{v_0, \ldots, v_m\} \cap \{w_0, \ldots, w_n\} = \varnothing$. Consequently, $i = m-1$ is suitable to conclude.

2. Suppose that $r = s$, $r < m$, and $r < n$. We obtain that $v_0 \ldots v_r w_{r+1} \ldots w_n$ is a dipath of $\Delta_{(G,\overline{X})}$. Since $\delta_{(G,\overline{X})}(w) = n$, $v_0 \ldots v_r w_{r+1} \ldots w_n$ is a w-dipath of $\Delta_{(G,\overline{X})}$. Consider the largest element I of $\{0, \ldots, m-1\} \cap \{0, \ldots, n-1\}$ such that $v_0 \ldots v_I w_{I+1} \ldots w_n$ is a w-dipath of $\Delta_{(G,\overline{X})}$. By Corollary 31, $G[X \cup \{v_0, \ldots, v_I, w_{I+1}, \ldots, w_n\}]$ is prime.

   Finally, suppose that $G[X \cup \{v_0, \ldots, v_m\} \cup \{w_{I+1}, \ldots, w_n\}]$ is decomposable. Consider a nontrivial module M of $G[X \cup \{v_0, \ldots, v_m\} \cup \{w_{I+1}, \ldots, w_n\}]$. By Claim 34 applied with $W_1 = X \cup \{v_0, \ldots, v_I, v_{I+1}, \ldots, v_m\}$ and $W_2 = X \cup \{v_0, \ldots, v_I, w_{I+1}, \ldots, w_n\}$, there exist $k \in \{I+1, \ldots, m\}$ and $l \in \{I+1, \ldots, n\}$ such that $M = \{v_k, w_l\}$. By Claim 38, we have $k = l$. Moreover, we have $k = m = n$ or $k < m$ and $k < n$. For a contradiction, suppose that $k < m$ and $k < n$. Since $\{v_k, w_k\}$ is a module of $G[X \cup \{v_0, \ldots, v_m\} \cup \{w_{I+1}, \ldots, w_n\}]$, it follows from Lemma 33 that $v_0 \ldots v_k w_{k+1} \ldots w_n$ is a dipath of $\Delta_{(G,\overline{X})}$. Since $\delta_{(G,\overline{X})}(w) = n$, $v_0 \ldots v_k w_{k+1} \ldots w_n$ is a w-dipath of $\Delta_{(G,\overline{X})}$, which contradicts the maximality of I. Consequently, we have $k = l$ and $k = m = n$. Hence, $\{v_m, w_m\}$ is the unique nontrivial module of $G[X \cup \{v_0, \ldots, v_m\} \cup \{w_{I+1}, \ldots, w_m\}]$. We obtain that $v_0 \ldots v_{m-1} w_m$ is a dipath of $\Delta_{(G,\overline{X})}$. In fact, $v_0 \ldots v_{m-1} w_m$ is a w-dipath of $\Delta_{(G,\overline{X})}$ because $\delta_{(G,\overline{X})}(w) = n$. By maximality of I, we have $I = m-1$. Therefore, $i = m-1$ is suitable to conclude. □

Notation 44. The largest element i of $\{0, \ldots, m-1\} \cap \{0, \ldots, n-1\}$ such that Lemma 43 holds is denoted by I.

Layout 8. We suppose that

$$I \leq m-2 \text{ or } I \leq n-2. \tag{16}$$

Theorem 45. *The graph G is $G[X \cup \{v, w\}]$-minimal if and only if there exist distinct elements $v_1, \ldots, v_m$ of B, $v_0 \in \overline{X} \smallsetminus B$, and distinct elements $w_{I+1}, \ldots, w_n$ of $B \smallsetminus \{v_0, \ldots, v_m\}$ satisfying the following statements*

(A4) $\overline{X} = \{v_0, \ldots, v_m\} \cup \{w_{I+1}, \ldots, w_n\}$,

(A5) $v_0 \ldots v_m$ *is a v-dipath of* $\Delta_{(G,\overline{X})}$,

(A6) $v_0 \ldots v_I w_{I+1} \ldots w_n$ *is a w-dipath of* $\Delta_{(G,\overline{X})}$,

and one of the following statements

(S9) $I \leq n-2$, $n < m$, *and*

- *for* $i \in \{I+2, \ldots, m\}$ *and* $j \in \{I+1, \ldots, n\}$, *we have* $w_j v_i \notin A(\Delta_{(G,\overline{X})})$,
- $\{v_m\} \cup \{w_{I+2}, \ldots, w_n\}$ *is a module of* $G - w_{I+1}$;

(S10) *Statement (S10) is obtained from Statement (S9) by interchanging the roles of v and w;*

(S11)
- *if $I \leq m-2$, then for $i \in \{I+2,\dots,m\}$ and $j \in \{I+1,\dots,n\}$, we have $w_j v_i \notin A(\Delta_{(G,\overline{X})})$,*
- *if $I \leq n-2$, then for $i \in \{I+1,\dots,m\}$ and $j \in \{I+2,\dots,n\}$, we have $v_i w_j \notin A(\Delta_{(G,\overline{X})})$.*

Proof. To begin, suppose that G is $G[X \cup \{v,w\}]$-minimal. By Lemma 43, $\Delta_{(G,\overline{X})}$ admits a v-dipath $v_0 \dots v_m$ and a w-dipath $v_0 \dots v_I w_{I+1} \dots w_n$, where $w_{I+1},\dots,w_n$ are distinct elements of $B \smallsetminus \{v_0,\dots,v_m\}$. Hence, Assertions (A5) and (A6) hold. Furthermore, if $G[X \cup \{v_0,\dots,v_m\} \cup \{w_{I+1},\dots,w_n\}]$ is decomposable, then it follows from Lemma 43 that $I = m-1$ and $m = n$, which contradicts (16). Thus, $G[X \cup \{v_0,\dots,v_m\} \cup \{w_{I+1},\dots,w_n\}]$ is prime. Since G is $G[X \cup \{v,w\}]$-minimal, we have $\overline{X} = \{v_0,\dots,v_m\} \cup \{w_{I+1},\dots,w_n\}$. Hence Assertion (A4) holds. Set $Y = X \cup \{v_0,\dots,v_m\}$ and $Z = X \cup \{v_0,\dots,v_I\} \cup \{w_{I+1},\dots,w_n\}$. By Corollary 31, $G[Y]$ and $G[Z]$ are prime. We show that

$$\text{if } w_n \in Y_G(v_m), \text{ then } m > n. \tag{17}$$

Indeed, suppose that $w_n \in Y_G(v_m)$. It follows from Lemma 33 that $v_0 \dots v_{m-1} w_n$ is a strict dipath of $\Delta_{(G,\overline{X})}$. Thus $m \geq n$. For a contradiction, suppose that $m = n$. We obtain that $m-1$ satisfies Lemma 43. By maximality of I, we obtain $I = m-1$, which contradicts (16). Since $m \geq n$, it follows that $m > n$. Similarly,

$$\text{if } v_m \in Z_G(w_n), \text{ then } n > m. \tag{18}$$

Now, we show that if $I \leq m-2$ and $v_m \notin Z_G(w_n)$, then

$$\text{for } i \in \{I+2,\dots,m\} \text{ and } j \in \{I+1,\dots,n\},\ w_j v_i \notin A(\Delta_{(G,\overline{X})}). \tag{19}$$

Indeed, suppose that $I \leq m-2$ and $v_m \notin Z_G(w_n)$. We have $v_{I+1} \notin Z \cup \{v_m\}$ because $I \leq m-2$. Since G is $G[X \cup \{v,w\}]$-minimal, we obtain $v_m \notin Z_G(u)$ for $u \in Z \smallsetminus X$. Furthermore, since G is $G[X \cup \{v,w\}]$-minimal and $m \geq I+2$, we obtain $v_m \notin \mathrm{Ext}_G(Z)$. It follows that $v_m \in \widetilde{B}_Z$. Since G is prime, it follows from Corollary 31 that $\Delta_{(G,\overline{Z})}$ admits a v-dipath $y_0 \dots y_p$, and $G[Z \cup \{y_0,\dots,y_p\}]$ is prime. Since G is $G[X \cup \{v,w\}]$-minimal, we obtain $\{y_0,\dots,y_p\} = \{v_{I+1},\dots,v_m\}$. Since $v_I v_{I+1} \in A(\Delta_{(G,\overline{X})})$ and $v_{I+1} \in B$, we obtain $v_{I+1} \notin \widetilde{B}_Z$. It follows that $y_0 = v_{I+1}$ and $\{y_1,\dots,y_p\} = \{v_{I+2},\dots,v_m\}$. Therefore, $\{v_{I+2},\dots,v_m\} \subseteq \widetilde{B}_Z$. Since $\{v_{I+2},\dots,v_m\} \subseteq B$, we obtain $w_j v_i \notin A(\Delta_{(G,\overline{X})})$ for $i \in \{I+2,\dots,m\}$ and $j \in \{I+1,\dots,n\}$. Similarly, if $I \leq n-2$ and $w_n \notin Y_G(v_m)$, then

$$\text{for } i \in \{I+1,\dots,m\} \text{ and } j \in \{I+2,\dots,n\},\ v_i w_j \notin A(\Delta_{(G,\overline{X})}). \tag{20}$$

To conclude, we distinguish the following three cases.

1. Suppose that $w_n \in Y_G(v_m)$. We prove that Statement (S9) holds. Since $\{v_m, w_n\}$ is not a module of G, we obtain $v_m \notin Z_G(w_n)$. By (17), $m > n$. It follows from (16) that $I \leq m-2$. By (19), we obtain $w_j v_i \notin A(\Delta_{(G,\overline{X})})$ for $i \in \{I+2, \dots, m\}$ and $j \in \{I+1, \dots, n\}$. Since G is prime, it follows from Corollary 31 that $\Delta_{(G,\overline{Y})}$ admits a w-dipath $z_0 \dots z_q$, and $G[Y \cup \{z_0, \dots, z_q\}]$ is prime. Since G is $G[X \cup \{v, w\}]$-minimal, we obtain $\{z_0, \dots, z_q\} = \{w_{I+1}, \dots, w_n\}$. Hence, we have $n \geq I+2$ because $q \geq 1$. Furthermore, since $m \geq I+2$, $v_I v_m \notin A(\Delta_{(G,\overline{X})})$. Since $v_I w_{I+1} \in A(\Delta_{(G,\overline{X})})$, it follows from Lemma 33 that $w_{I+1} \notin Y_G(v_m)$. Thus, $z_0 = w_{I+1}$, and hence $\{z_1, \dots, z_q\} = \{w_{I+2}, \dots, w_n\}$. Therefore $\{w_{I+2}, \dots, w_n\} \subseteq Y_G(v_m)$. It follows that $\{v_m\} \cup \{w_{I+2}, \dots, w_n\}$ is a module of $G - w_{I+1}$. Consequently, Statement (S9) holds.

2. Suppose that $v_m \in Z_G(w_n)$. We obtain that Statement (S10) holds.

3. Suppose that $v_m \notin Z_G(w_m)$ and $w_n \notin Y_G(v_m)$. It follows from (19) and (20) that Statement (S11) holds.

Conversely, suppose that there exist distinct elements $v_1, \dots, v_m$ of B, $v_0 \in \overline{X} \smallsetminus B$, and distinct elements $w_{I+1}, \dots, w_n$ of $B \smallsetminus \{v_0, \dots, v_m\}$ satisfying Assertions (A4), (A5), and (A6). Moreover, suppose that Statement (S9) or (S11) holds.

First, we prove that G is prime. For a contradiction, suppose that $G[X \cup \{v_0, \dots, v_m\} \cup \{w_{I+1}, \dots, w_n\}]$ is decomposable. Since I satisfies Lemma 43, we obtain $m = n$ and $I = m-1$, which contradicts (16). It follows that $G[X \cup \{v_0, \dots, v_m\} \cup \{w_{I+1}, \dots, w_n\}]$ is prime.

Second, we prove that $G[X \cup \{v_0, \dots, v_m\} \cup \{w_{I+1}, \dots, w_n\}]$ is $G[X \cup \{v, w\}]$-minimal. Consider a subset W of $V(G)$ such that $X \cup \{v, w\} \subseteq W \subsetneq V(G)$. We have to show that $G[W]$ is decomposable. Suppose that there exists $i \in \{0, \dots, I\}$ such that $v_i \notin W$. Since Assertions (A5) and (A6) hold, it suffices to apply Lemma 32 with $W' = (\{v_{i+1}, \dots, v_m\} \cup \{w_{I+1}, \dots, w_n\}) \cap W$. Lastly, suppose that $\{v_0, \dots, v_I\} \subseteq W$. We distinguish the following cases.

1. Suppose that Statement (S9) holds. We have $I+2 \leq m$ and $w_j v_i \notin A(\Delta_{(G,\overline{X})})$ for $i \in \{I+2, \dots, m\}$ and $j \in \{I+1, \dots, n\}$. Therefore, if there exists $i \in \{I+1, \dots, m\}$ such that $v_i \notin W$, then it suffices to apply Lemma 32 with $W' = \{v_{i+1}, \dots, v_m\} \cap W$. Hence, suppose that $\{v_0, \dots, v_m\} \subseteq W$. There exists $j \in \{I+1, \dots, n\}$ such that $w_j \notin W$. Since $w \in W$, we have $j \leq n-1$. Set $M = \{v_m\} \cup \{w_{j+1}, \dots, w_n\}$. We prove that M is a module of $G - w_j$. Since Statement (S9) holds, $\{v_m\} \cup \{w_{I+2}, \dots, w_n\}$ is a module of $G - w_{I+1}$. Hence, M is a module of $G - w_j$ if $j = I+1$. Thus, suppose that $j \geq I+2$. Let $k \in \{I+1, \dots, j-1\}$. Since Assertion (A6) holds, we have $w_k w_l \notin A(\Delta_{(G,\overline{X})})$ for $l \in \{j+1, \dots, n\}$. Furthermore, since Statement (S9) holds, $w_k v_m \notin A(\Delta_{(G,\overline{X})})$. Since $\{v_m\} \cup \{w_{I+2}, \dots, w_n\}$ is a module of $G - w_{I+1}$, v_m and $w_{j+1}, \dots, w_n$ belong to the same block of $q_{(G,\overline{X})}$. It follows from Lemma 33 that $w_k \longleftrightarrow_G M$. Consequently, M is a module of $G - w_j$. It follows that $M \cap W$ is a module of $G[W]$. We have $|M \cap W| \geq 2$

because $v, w \in M \cap W$. Furthermore, $M \cap W \subsetneq W$ because $X \subseteq W \setminus M$. Therefore, $M \cap W$ is a nontrivial module of $G[W]$.

2. Suppose that Statement (S11) holds. By exchanging v and w if necessary, we can assume that there exists $i \in \{I+1, \dots, m\}$ such that $v_i \notin W$. In particular, we obtain $I + 2 \leq m$ because $v_m \in W$. Since Statement (S11) holds, we have $w_j v_i \notin A(\Delta_{(G,\overline{X})})$ for $i \in \{I+2, \dots, m\}$ and $j \in \{I+1, \dots, n\}$. As previously, we conclude by applying Lemma 32 with $W' = \{v_{i+1}, \dots, v_m\} \cap W$. □

Layout 9. In what follows, we suppose that $m = n$ and $I = m - 1$.

Theorem 46. *Suppose that v and w do not belong to the same block of $q_{(G,\overline{X})}$. The graph G is $G[X \cup \{v, w\}]$-minimal if and only if there exist distinct elements $v_1, \dots, v_m$ of $B \setminus \{w\}$ and $v_0 \in \overline{X} \setminus B$ satisfying the following statement*

(S12) *$v_0 \dots v_m$ is a v-dipath of $\Delta_{(G,\overline{X})}$, $v_0 \dots v_{m-1} w$ is a w-dipath of $\Delta_{(G,\overline{X})}$, and $\overline{X} = \{v_0, \dots, v_m\} \cup \{w\}$.*

Proof. To begin, suppose that G is $G[X \cup \{v, w\}]$-minimal. Since $m = n$ and $I = m - 1$, $\Delta_{(G,\overline{X})}$ admits a v-dipath $v_0 \dots v_m$ such that $v_0 \dots v_{m-1} w$ is a w-dipath of $\Delta_{(G,\overline{X})}$. By Corollary 31, $G[X \cup \{v_0, \dots, v_m\}]$ is prime. Set $Y = X \cup \{v_0, \dots, v_m\}$. Since $\{v, w\}$ is not $G[X]$-reachable, we have $w \notin Y_G(v_i)$ for $0 \leq i \leq m-1$. Moreover, since v and w do not belong to the same block of $q_{(G,\overline{X})}$, we have $w \notin Y_G(v_m)$. Lastly, since $v_{m-1} w \in A(\Delta_{(G,\overline{X})})$, we have $w \notin \widetilde{B}_Y$. It follows that $w \in \mathrm{Ext}_G(Y)$. Consequently, $G[Y \cup \{w\}]$ is prime. Since G is $G[X \cup \{v, w\}]$-minimal, we obtain $\overline{X} = \{v_0, \dots, v_m\} \cup \{w\}$.

Conversely, suppose that there exist distinct elements $v_1, \dots, v_m$ of $B \setminus \{w\}$ and $v_0 \in \overline{X} \setminus B$ such that Statement (S12) holds. For a contradiction, suppose that G is decomposable. Since $v_0 \dots v_m$ is a v-dipath of $\Delta_{(G,\overline{X})}$, $G[X \cup \{v_0, \dots, v_m\}]$ is prime by Corollary 31. Similarly, $G[X \cup \{v_0, \dots, v_{m-1}\} \cup \{w\}]$ is prime. It follows from Claim 34 that $\{v_m, w\}$ is a module of G, which contradicts the fact that v and w do not belong to the same block of $q_{(G,\overline{X})}$. It follows that G is prime. Finally, we show that G is $G[X \cup \{v, w\}]$-minimal. Consider a subset W of $V(G)$ such that $X \cup \{v, w\} \subseteq W \subsetneq V(G)$. There exists $i \in \{0, \dots, m-1\}$ such that $v_i \notin W$. Set $W' = (\{v_{i+1}, \dots, v_m\} \cup \{w\}) \cap W$. By Lemma 32, $G[W]$ is decomposable. □

Theorem 47. *Suppose that v and w belong to the same block D of $q_{(G,\overline{X})}$. The graph G is $G[X \cup \{v, w\}]$-minimal if and only if there exist distinct elements $v_1, \dots, v_m$ of $B \setminus \{w\}$, $v_0 \in \overline{X} \setminus B$, and distinct elements $z_0, \dots, z_q$ of $\overline{X} \setminus \{v_0, \dots, v_m\}$ satisfying the following assertions*

(A7) *$v_0 \dots v_m$ is a v-dipath of $\Delta_{(G,\overline{X})}$,*

(A8) *$q \geq 1$, $z_q = w$, and $\overline{X} = \{v_0, \dots, v_m\} \cup \{z_0, \dots, z_q\}$,* (A9) *for* $r \in \{0, \ldots, q-1\}$, $\{v\} \cup \{z_{r+1}, \ldots, z_q\}$ *is a module of* $G - z_r$, *but not of* G,

(A10) *by exchanging* v *and* w *if necessary, we can assume that* $z_{q-1}v \notin A(\Delta_{(G,\overline{X})})$ *and* $z_{q-1}z_q \in A(\Delta_{(G,\overline{X})})$,

(A11) $\{z_0, v_m\}$ *is not a module of* $G[X \cup \{v_0, \ldots, v_m\} \cup \{z_0\}]$,

and satisfying one of the following statements

(S13) $z_0 \notin B$, *and*

- $z_0 \ldots z_q$ *is an arrow of* $\Delta_{(G,\overline{X})}$,
- $z_j v_i \notin A(\Delta_{(G,\overline{X})})$ *for* $i \in \{1, \ldots, m\}$ *and* $j \in \{0, \ldots, q\}$;

(S14) $z_0 \in B$, $m \geq 2$, *and there exists* $k \in \{0, \ldots, m-2\}$ *such that*

- $v_0 \ldots v_k z_0 \ldots z_q$ *is an arrow of* $\Delta_{(G,\overline{X})}$,
- $z_j v_i \notin A(\Delta_{(G,\overline{X})})$ *for* $i \in \{k+2, \ldots, m\}$ *and* $j \in \{0, \ldots, q\}$;

(S15) $z_0 \in B$, $m \geq 2$, *and* $v_i z_0 \notin A(\Delta_{(G,\overline{X})})$ *for* $i \in \{0, \ldots, m-2\}$;

(S16) $z_0 \in B$, *and* $m = 1$.

Proof. To begin, suppose that G is $G[X \cup \{v, w\}]$-minimal. Since $m = n$ and $I = m - 1$, $\Delta_{(G,\overline{X})}$ admits a v-dipath $v_0 \ldots v_m$ such that $v_0 \ldots v_{m-1}w$ is a w-dipath of $\Delta_{(G,\overline{X})}$. Hence, Assertion (A7) holds. By Corollary 31, $G[X \cup \{v_0, \ldots, v_m\}]$ is prime. Set $Y = X \cup \{v_0, \ldots, v_m\}$. Since $v_0 \ldots v_m$ and $v_0 \ldots v_{m-1}w$ are strict dipaths of $\Delta_{(G,\overline{X})}$, $\{v_m, w\}$ is a module of $\Delta_{(G,\overline{X})}[\{v_0, \ldots, v_m\} \cup \{w\}]$. Since $v_m, w \in D$, it follows from Lemma 33 that $w \in Y_G(v_m)$. Since G is prime, it follows from Corollary 31 that $\Delta_{(G,\overline{Y})}$ admits a w-dipath $z_0 \ldots z_q$ and $G[Y \cup \{z_0, \ldots, z_q\}]$ is prime. Since G is $G[X \cup \{v, w\}]$-minimal, we obtain $\overline{Y} = \{z_0, \ldots, z_q\}$. Hence, Assertion (A8) holds. Since $z_0 \ldots z_q$ is a w-dipath of $\Delta_{(G,\overline{Y})}$, Assertions (A9) holds. Since Assertion (A9) holds, we have $z_{q-1} \not\longleftrightarrow_G \{v, w\}$. Since $v, w \in D$, it follows from Lemma 33 that $|\{z_{q-1}v, z_{q-1}w\} \cap A(\Delta_{(G,\overline{X})})| = 1$. Therefore, Assertion (A10) holds. Since $z_0 \ldots z_q$ is a w-dipath of $\Delta_{(G,\overline{Y})}$ and $w \in Y_G(v_m)$, we have $z_0 \notin Y_G(v_m)$, that is, Assertion (A11) holds.

Since $z_{q-1}w \in A(\Delta_{(G,\overline{X})})$ by Assertion (A10), we consider the smallest element L of $\{0, \ldots, q-1\}$ such that

$$z_L v \in A(\Delta_{(G,\overline{X})}) \text{ or } z_L z_q \in A(\Delta_{(G,\overline{X})}). \tag{21}$$

We verify that

$$\text{if } L < q-1, \text{ then } z_L v, z_L z_q \in A(\Delta_{(G,\overline{X})}). \tag{22}$$

Indeed, suppose that $L < q - 1$. Since Assertion (A9) holds, we have $z_L \longleftrightarrow_G \{v, z_q\}$. It follows from Lemma 33 that $z_L v, z_L z_q \in A(\Delta_{(G,\overline{X})})$ or $z_L v, z_L z_q \notin A(\Delta_{(G,\overline{X})})$. By (21), we have $z_L v, z_L z_q \in A(\Delta_{(G,\overline{X})})$. Hence (22) holds.

To continue, we verify that $z_0 \dots z_L z_q$ is a strict dipath of $\Delta_{(G,\overline{X})}$. If $L = q-1$, then $z_{q-1}z_q \in A(\Delta_{(G,\overline{X})})$ by Assertion (A10). If $L < q-1$, then $z_L z_q \in A(\Delta_{(G,\overline{X})})$ by (22). Now, suppose that $L \geq 1$, and consider $p \in \{0, \dots, L-1\}$. It follows from Assertion (A9) that $\{v\} \cup \{z_{p+2}, \dots, z_q\}$ is a module of $G - z_{p+1}$. Thus, $z_p \longleftrightarrow_G \{v\} \cup \{z_{p+2}, \dots, z_q\}$. By minimality of L, we have $z_p v \notin A(\Delta_{(G,\overline{X})})$ and $z_p z_q \notin A(\Delta_{(G,\overline{X})})$. It follows from Lemma 33 that $z_p u \notin A(\Delta_{(G,\overline{X})})$ for every $u \in \{v\} \cup \{z_{p+2}, \dots, z_q\}$. Furthermore, it follows from Assertion (A9) that $\{v\} \cup \{z_{p+1}, \dots, z_q\}$ is a module of $G - z_p$, but not of G. We obtain $z_p \not\longleftrightarrow_G \{z_{p+1}, v\}$ and $z_{p+1}, v \in D$. Since $z_p v \notin A(\Delta_{(G,\overline{X})})$, it follows from Lemma 33 that $z_p z_{p+1} \in A(\Delta_{(G,\overline{X})})$. Consequently, $z_0 \dots z_L z_q$ is a strict dipath of $\Delta_{(G,\overline{X})}$.

We distinguish the following cases.

1. Suppose that $z_0 \notin B$. We prove that Statement (S13) holds. We have $z_1, \dots, z_q \in Y_G(v_m)$, so $z_1, \dots, z_q \in B$. Since $z_0 \notin B$, $z_0 \dots z_L z_q$ is an arrow of $\Delta_{(G,\overline{X})}$. By Corollary 31, $G[X \cup \{z_0, \dots, z_L\} \cup \{z_q\}]$ is prime. Set $Z = X \cup \{z_0, \dots, z_L\} \cup \{z_q\}$. Since G is $G[X \cup \{v, w\}]$-minimal and $v_0 \notin Z \cup \{v_m\}$, we have $v_m \notin \mathrm{Ext}_G(Z)$. Since G is $G[X \cup \{v, w\}]$-minimal, $v_m \notin Z_G(z_r)$ for $0 \leq r \leq L$. For a contradiction, suppose that $v_m \in Z_G(z_q)$. Recall that $z_q \in Y_G(v_m)$. Since $\{v, z_q\}$ is not a module of G, we have $L < q-1$. Since G is prime, it follows from Corollary 31 that $\Delta_{(G,\overline{Z})}$ admits a v-dipath $y_0 \dots y_k$, and $G[Z \cup \{y_0, \dots, y_k\}]$ is prime. Since G is $G[X \cup \{v, w\}]$-minimal, we have $V(G) = Z \cup \{y_0, \dots, y_k\}$. Therefore, we have $\overline{Z} \setminus Z_G(z_q) = \{y_0\}$. Since $Z_G(z_q) \subseteq B$ and $v_0 \notin B$, we obtain $v_0 \in \overline{Z} \setminus Z_G(z_q)$. Thus, $y_0 = v_0$, so $\overline{Z} \setminus Z_G(z_q) = \{v_0\}$. Moreover, since $L < q-1$, it follows from Assertion (A9) that $z_L \not\longleftrightarrow_G \{z_{L+1}, z_q\}$. Therefore, $z_{L+1} \notin Z_G(z_q)$. Clearly, $z_{L+1} \neq v_0$, which contradicts $\overline{Z} \setminus Z_G(z_q) = \{v_0\}$. It follows that $v_m \notin Z_G(z_q)$. Consequently, we have $v_m \in \widetilde{B}_Z$. It follows that $z_L v_m \notin A(\Delta_{(G,\overline{X})})$. By (22), $L = q-1$. Hence $Z = X \cup \{z_0, \dots, z_q\}$, and $\overline{Z} = \{v_0, \dots, v_m\}$ by Assertion (A8). Since G is $G[X \cup \{v, w\}]$-minimal, we obtain $\overline{Z} = \{y_0, \dots, y_k\}$. It follows that $\{v_0, \dots, v_m\} = \{y_0, \dots, y_k\}$. Since $y_0 \dots y_k$ is a v-dipath of $\Delta_{(G,\overline{Z})}$, we have $\{y_1, \dots, y_k\} \subseteq \widetilde{B}_Z \subseteq B$. Since $v_0 \notin B$, we obtain $v_0 = y_0$. Thus $\{v_1, \dots, v_m\} \subseteq \widetilde{B}_Z$. It follows that $z_j v_i \notin A(\Delta_{(G,\overline{X})})$ for $i \in \{1, \dots, m\}$ and $j \in \{0, \dots, q\}$. Hence Statement (S13) holds.

2. Suppose that $z_0 \in B$. If $m = 1$, then Statement (S16) holds. Therefore, suppose that $m \geq 2$. If $v_i z_0 \notin A(\Delta_{(G,\overline{X})})$ for $i \in \{0, \dots, m-2\}$, then Statement (S15) holds. Lastly, suppose that there exists $i \in \{0, \dots, m-2\}$ such that $v_i z_0 \in A(\Delta_{(G,\overline{X})})$. We consider the smallest element k of $\{0, \dots, m-2\}$ such that $v_k z_0 \in A(\Delta_{(G,\overline{X})})$. We prove that Statement (S14) holds. Let $j \in \{1, \dots, q\}$. As previously seen, $z_j \in Y_G(v_m)$. Let $l \in \{0, \dots, k\}$. Since $v_l v_m \notin A(\Delta_{(G,\overline{X})})$, it follows from Lemma 33 that $v_l z_j \notin A(\Delta_{(G,\overline{X})})$. Moreover, when $k \geq 1$, it follows from the minimality of k that $v_l z_0 \notin A(\Delta_{(G,\overline{X})})$ for $l \in \{0, \dots, k-1\}$. Therefore, $v_0 \dots v_k z_0 \dots z_L z_q$

is a strict dipath of $\Delta_{(G,\overline{X})}$. Since $v_0 \notin B$ and $\{v_1,\dots,v_k\} \cup \{z_0,\dots,z_L\} \cup \{z_q\} \subseteq B$, $v_0 \dots v_k z_0 \dots z_L z_q$ is an arrow of $\Delta_{(G,\overline{X})}$. By Corollary 31, $G[X \cup \{v_0,\dots,v_k\} \cup \{z_0,\dots,z_L\} \cup \{z_q\}]$ is prime. Set $Z = X \cup \{v_0,\dots,v_k\} \cup \{z_0,\dots,z_L\} \cup \{z_q\}$. Since G is $G[X \cup \{v,w\}]$-minimal, $v_m \notin Z_G(u)$ for $u \in \{v_0,\dots,v_k\} \cup \{z_0,\dots,z_L\}$. Since $m \geq k+2$, $v_{k+1} \in \overline{Z} \setminus \{v_m\}$. It follows that $v_m \notin \mathrm{Ext}_G(Z)$ because G is $G[X \cup \{v,w\}]$-minimal. For a contradiction, suppose that $v_m \in Z_G(z_q)$. Recall that $z_q \in Y_G(v_m)$. Since $\{v, z_q\}$ is not a module of G, we obtain $L < q-1$. Since G is prime, it follows from Corollary 31 applied with $G[Z]$ that $\Delta_{(G,\overline{Z})}$ admits a v_m-dipath $y_0 \dots y_l$, and $G[Z \cup \{y_0,\dots,y_l\}]$ is prime. Since G is $G[X \cup \{v,w\}]$-minimal, we obtain $\overline{Z} = \{y_0,\dots,y_l\}$. It follows that $\overline{Z} \setminus Z_G(z_q) = \{y_0\}$. Since $L < q-1$, we have $z_L v, z_L z_q \in A(\Delta_{(G,\overline{X})})$ by (22). It follows from Lemma 33 and Assertion (A9) that $z_L z_{L+1} \notin A(\Delta_{(G,\overline{X})})$, so $z_{L+1} \notin Z_G(z_q)$. Furthermore, since $m \geq k+2$, we have $v_k z_q \notin A(\Delta_{(G,\overline{X})})$. Since $v_k v_{k+1} \in A(\Delta_{(G,\overline{X})})$, it follows from Lemma 33 that $v_{k+1} \notin Z_G(z_q)$. Therefore, $v_{k+1} \neq z_{L+1}$ and $v_{k+1}, z_{L+1} \notin Z_G(z_q)$, which contradicts $\overline{Z} \setminus Z_G(z_q) = \{y_0\}$. Consequently, $v_m \notin Z_G(z_q)$, and hence $v_m \in \widetilde{B}_Z$. If $L < q-1$, then $z_L v \in A(\Delta_{(G,\overline{X})})$ by (22), which contradicts $v_m \in \widetilde{B}_Z$. Hence $L = q-1$. Since G is prime, it follows from Corollary 31 applied with $G[Z]$ that $\Delta_{(G,\overline{Z})}$ admits a v_m-dipath $y_0 \dots y_l$, and $G[Z \cup \{y_0,\dots,y_l\}]$ is prime. Since G is $G[X \cup \{v,w\}]$-minimal, we obtain $\{v_{k+1},\dots,v_m\} = \{y_0,\dots,y_l\}$. Therefore, we have $\overline{Z} \setminus \widetilde{B}_Z = \{y_0\}$. Since $v_k v_{k+1} \in A(\Delta_{(G,\overline{X})})$, we have $v_{k+1} \notin \widetilde{B}_Z$. It follows that $y_0 = v_{k+1}$, so $\{v_{k+2},\dots,v_m\} = \{y_1,\dots,y_l\}$, and hence $\{v_{k+2},\dots,v_m\} \subseteq \widetilde{B}_Z$. We obtain that $z_j v_i \notin A(\Delta_{(G,\overline{X})})$ for $i \in \{k+2,\dots,m\}$ and $j \in \{0,\dots,q\}$. Consequently, Statement (S14) holds.

Conversely, suppose that there exist distinct elements $v_1,\dots,v_m$ of $B \setminus \{w\}$, $v_0 \in \overline{X} \setminus B$, and distinct elements $z_0,\dots,z_q$ of $\overline{X} \setminus \{v_0,\dots,v_m\}$ satisfying Assertions (A7),...,(A11), and one of Statements (S13), (S14), (S15) or (S16).

First, we show that G is prime. By Assertion (A7), $v_0 \dots v_m$ is a v-dipath of $\Delta_{(G,\overline{X})}$. By Corollary 31, $G[X \cup \{v_0,\dots,v_m\}]$ is prime. Set $Y = X \cup \{v_0,\dots,v_m\}$. It follows from Assertion (A9) that $z_1,\dots,z_q \in Y_G(v_m)$. By Assertion (A11), $z_0 \notin Y_G(v_m)$. Lastly, it follows from Assertion (A9) that $z_0 \dots z_q$ is a strict dipath of $\Delta_{(G,\overline{Y})}$. Since $z_0 \notin Y_G(v_m)$ and $\{z_1,\dots,z_q\} \subseteq Y_G(v_m)$, $z_0 \dots z_q$ is an arrow of $\Delta_{(G,\overline{Y})}$. By Corollary 31 applied with $G[Y]$, $G[Y \cup \{z_0,\dots,z_q\}]$ is prime. It follows from Assertion (A8) that G is prime.

Second, we prove that G is $G[X \cup \{v,w\}]$-minimal. Consider a subset W of $V(G)$ such that $X \cup \{v,w\} \subseteq W \subsetneq V(G)$. Suppose that $\{v_0,\dots,v_m\} \subseteq W$. There exists $r \in \{0,\dots,q-1\}$ such that $z_r \notin W$. As previously seen, $z_q \in Y_G(v_m)$ and $z_0 \dots z_q$ is a w-dipath of $\Delta_{(G,\overline{Y})}$. Set $W' = W \cap \{z_{r+1},\dots,z_q\}$. It follows from Lemma 32 applied to $G[Y]$ that $G[W]$ is decomposable. Finally, suppose that there exists $i \in \{0,\dots,m\}$ such that $v_i \notin W$. Since $v \in W$, $i \leq m-1$. We distinguish the following cases.

1. Suppose that Statement (S13) holds. Set $W' = W \cap \{v_{i+1},\dots,v_m\}$. By

Lemma 32, $G[W]$ is decomposable.

2. Suppose that Statement (S14) holds.

   - Suppose that $0 \le i \le k$. Set $W' = W \cap (\{v_{i+1}, \dots, v_m\} \cup \{z_0, \dots, z_q\})$. By Lemma 32, $G[W]$ is decomposable.
   - Suppose that $k + 1 \le i \le m - 1$. Set $W' = W \cap \{v_{i+1}, \dots, v_m\}$. By Lemma 32, $G[W]$ is decomposable.

3. Suppose that Statement (S15) holds. Set $W' = W \cap (\{v_{i+1}, \dots, v_m\} \cup \{z_0, \dots, z_q\})$. By Lemma 32, $G[W]$ is decomposable.

4. Suppose that Statement (S16) holds. Since $m = 1$, we have $i = 0$. We obtain $W \smallsetminus X \subseteq B$. It follows that $G[W]$ is decomposable. $\square$

6 Conclusion

Initially, our aim is to study the counter-examples to Conjecture 5. Proposition 8 leads us to characterize the prime graphs G that are $G[X \cup \{v, w\}]$-minimal, where X is a proper subset of $V(G)$ such that $G[X]$ is prime, and $v, w \in \overline{X}$. Theorem 16 provides a nice characterization when $v = w$ in terms of the outside digraph $\Delta_{(G,\overline{X})}$ (see Definiton 14). In Sections 1.2 and 5, we treat the case $v \neq w$. We consider seven situations (see Theorems 20, 21, 39, 42, 45, 46, and 47) that are induced from the locations of v and w in $\overline{X}$, described by using the outside partition $p_{(G,\overline{X})}$ (see Definition 11), and its refinement $q_{(G,\overline{X})}$ (see Definition 24). It is not difficult to provide graphs that correspond with each of these seven situations. Nevertheless, we did not complete our initial characterization. Indeed, for each of these seven situations, we have still to identify the counter-examples to Conjecture 5.

Acknowledgements

The authors thank the referee for his constructive suggestions that allow for notable improvements to the manuscript.

References

[1] A. Boussaïri, A. Chaïchaâ, P. Ille, Indecomposability graph and indecomposability recognition, Proceedings of ROGICS'08, European J. Combin. 37 (2014) 32–42.

[2] A. Breiner, J. Deogun, P. Ille, Partially critical indecomposable graphs, Contrib. Discrete Math. 3 (2008) 40–59.

[3] A. Cournier, M. Habib, An efficient algorithm to recognize prime undirected graphs, Graph-theoretic Concepts in Computer Science, Lecture Notes in Computer Science 657, E. W. Mayr, (Editor), Springer, Berlin (1993), pp. 212–224.

[4] A. Cournier, P. Ille, Minimal indecomposable graph, Discrete Math. 183 (1998) 61–80.

[5] A. Ehrenfeucht, T. Harju, G. Rozenberg, The Theory of 2-Structures, A Framework for Decomposition and Transformation of Graphs (World Scientific, Singapore, 1999).

[6] T. Gallai, Transitiv orientierbare Graphen, Acta Math. Acad. Sci. Hungar. 18 (1967) 25–66.

[7] P. Ille, Recognition problem in reconstruction for decomposable relations, Finite and Infinite Combinatorics in Sets and Logic B. Sands, N. Sauer, R. Woodrow, (Editors), Kluwer Academic Publishers (1993), pp. 189–198.

[8] P. Ille, Indecomposable graphs, Discrete Math 173 (1997) 71–78.

[9] P. Ille, R. Villemaire, Recognition of prime graphs from a prime subgraph, Discrete Math. 327 (2014) 76–90.

[10] F. Maffray, M. Preissmann, A translation of Tibor Gallai's paper: Transitiv orientierbare Graphen, Perfect Graphs J.L. Ramirez-Alfonsin and B.A. Reed, (Editors), Wiley, New York (2001), pp. 25–66.

[11] J. Spinrad, P_4-trees and substitution decomposition, Discrete Appl. Math. 39 (1992) 263–291.

[12] D. P. Sumner, Graphs indecomposable with respect to the X-join, Discrete Math. 6 (1973) 281–298.