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Abstract: In this report, 2×2 zero-sum games are studied under the following assumptions: (1) One of the players (the leader) commits to choose its actions by sampling a given probability measure (strategy); (2) The leader announces its action, which is observed by its opponent (the follower) through a binary channel; and (3) the follower chooses its strategy based on the knowledge of the leader's strategy and the noisy observation of the leader's action. Under these conditions, the equilibrium is shown to always exist. Interestingly, even subject to noise, observing the actions of the leader is shown to be either beneficial or immaterial for the follower. More specifically, the payoff at the equilibrium of this game is upper bounded by the payoff at the Stackelberg equilibrium (SE) in pure strategies; and lower bounded by the payoff at the Nash equilibrium, which is equivalent to the SE in mixed strategies. Finally, necessary and sufficient conditions for observing the payoff at equilibrium to be equal to its lower bound are presented. Sufficient conditions for the payoff at equilibrium to be equal to its upper bound are also presented.

Key-words: Zero-sum games, equilibria, commitments, Stackelberg, Nash, noisy observations

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Jeux 2×2 à Somme-Nulle avec Engagements et Observations Bruitées

Résumé : Dans ce rapport, les jeux à somme nulle 2×2 sont étudiés sous les hypothèses suivantes : (1) L'un des joueurs (le meneur) s'engage à choisir ses actions en échantillonnant une mesure de probabilité donnée (stratégie) ; (2) Le meneur annonce son action, qui est observée par son adversaire (le suiveur) à travers un canal binaire ; et (3) le suiveur choisit sa stratégie en fonction de la connaissance de la stratégie du meneur et de l'observation bruitée de l'action du meneur. Dans ces conditions, on montre que l'équilibre existe toujours. Fait intéressant, même sujette au bruit, l'observation des actions du leader s'avère soit bénéfique, soit immatérielle pour le suiveur. Plus précisément, la récompense à l'équilibre de ce jeu est majorée par la récompense à l'équilibre de Stackelberg (SE) en stratégies pures, et minorée par la récompense à l'équilibre de Nash, qui équivaut au SE en stratégies mixtes. Enfin, les conditions nécessaires et suffisantes pour observer que la récompense à l'équilibre est égale à sa borne inférieure sont présentées. Les conditions suffisantes pour observer que la récompense à l'équilibre est égale à sa borne supérieure sont aussi présentées.

Mots-clés : Jeux à somme nulle, équilibre, engagements, Stackelberg, Nash, observations bruitées.

Contents

1 Introduction

Zero-sum games (ZSGs) are mathematical models describing the interaction of mutually adversarial decision makers. Two solution concepts are often adopted for predicting the outcome of ZSGs: the Nash equilibrium (NE) [2] and the Stackelberg equilibrium (SE) [3]. The NE is a prediction observed under the assumption that both players simultaneously choose their strategies (probability measures over the set of possible actions). On the other hand, the SE describes the outcome in which one of the players (the leader) commits to use a particular strategy before its opponent (the follower). In such a case, the follower chooses its strategy as a best response to the commitment of the leader. Commitments are said to be in mixed strategies when the leader is allowed to commit to strategies whose support contains more than one action. In this case, the relevant solution concept is the SE in mixed strategies [4–7]. Interestingly, in ZSGs, the payoffs at the NE and the SE in mixed strategies are identical, as shown in [8]. The commitment is said to be in pure strategies when the leader is constrained to commit to play one action with probability one. This is assimilated to the case in which the follower perfectly observes the action played by the leader. The relevant solution concept under these assumptions is the SE in pure strategies [3, 9, 10]. The expected payoff at the SE in pure strategies is equal to the $\min\max$ or $\max\min$ solution, where the optimization is over the set of actions [11, 12]. In this case, the payoff at the SE in pure strategies might be significantly different from the payoff at the NE.

In a nutshell, the underlying assumption of the SE in mixed strategies is that the strategy to which the leader commits to is perfectly observed by the follower and the actions are unobservable. Alternatively, the assumption of the SE in pure strategies is that actions are perfectly observable, which makes the notion of commitment irrelevant. This is essentially because the follower can always respond with an optimal action to the action played by the leader, regardless of the commitment. Nonetheless, often, the actions of the leader are neither unobservable nor perfectly observed. Instead, observations might be obtained subject to noise.

1.1 Previous Works

The analysis of noisy observations of the actions played by a leader in ZSGs started in the realm of information theory [13]. Therein, an external entity referred to as *the informant* observes the action of the leader, encodes it and transmits it through a discrete memoryless channel (DMC) to the follower. The latter decodes the action of its opponent and thus, chooses its own action. In [13], commitments are not considered and the observation is noisy due to the impairments typical to data-transmission. In the realm of game theory, bi-matrix games with commitments and observability started with the work of Bagwell [14]. Therein, the leader is restricted to commit to a pure strategy, while the follower might observe a different pure strategy with positive probability. Note that this game is identical to a game without commitments in which the leader plays an action while the follower observes a different action with positive probability before choosing its own action, as described in numerous scenarios [15–21].

1.2 Contributions

For pedagogical purposes, the analysis is restricted to two-player two-action ZSGs, which capture all interesting challenges due to the noisy observations in the presence of commitments. One of the main contributions is a new game formulation in which the follower obtains a noisy observation of the action played by the leader, whereas the commitment is assumed to be perfectly observed. The game is proved to always possess an equilibrium. The optimal commitments are characterized and the set of best responses of the follower is thoroughly described. An explicit expression for the payoff at the equilibrium is derived. The payoff at equilibrium is greater than the payoff at the NE exclusively when the ZSG exhibits a unique NE in mixed strategies. In all other cases, e.g., ZSG exhibiting strategic dominance, unique NE in pure strategies, or infinitely many NEs, the payoffs with and without observations are identical. When the observation of the action of the leader is noiseless, the payoff at the equilibrium is the same as the payoff at the SE in pure strategies.

2 Game Formulation

Consider a two-player zero-sum game in normal form with a payoff matrix

$$\underline{\boldsymbol{u}}=\begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix}. \tag{1}$$

Let the elements of the set $\mathcal{K} \triangleq \{1,2\}$ represent the indices of the players; and let the elements of the set $\mathcal{A}_1 = \mathcal{A}_2 \triangleq \{a_1, a_2\}$ represent the actions of the players. Hence, for all $(i,j) \in \{1,2\}^2$, when Player 1 plays a_i and Player 2 plays a_j, the outcome of the game is $u_{i,j}$. Player 1 and Player 2 choose their actions to maximize and minimize their payoffs, respectively. When players simultaneously choose their actions in the absence of commitments, the game is represented by the tuple

$$\mathscr{G}\left(\underline{\boldsymbol{u}}\right)\triangleq\left(\mathcal{K}, \mathcal{A}_1, \mathcal{A}_2, \underline{\boldsymbol{u}}\right), \tag{2}$$

and the solution concept is the NE.

When the game is played with commitments and noisy observations, it unfolds in three stages. In the first stage, Player 2 announces its strategy to Player 1 and commits to choose its actions by using such a strategy. A strategy for Player 2 is a probability measure denoted by $P_{A_2} \in \Delta\left(\mathcal{A}_2\right)$. In stage two, Player 2 plays action $b \in \mathcal{A}_2$ with probability $P_{A_2}\left(b\right)$, while Player 1 observes action $\tilde{b} \in \mathcal{A}_2$ with probability $P_{\tilde{A}_2|A_2=b}\left(\tilde{b}\right)$. That is, Player 1 obtains a noisy observation of the action played by Player 2. The tuple of probability measures

$$P_{\tilde{A}_2|A_2} \triangleq \left(P_{\tilde{A}_2|A_2=a_1}, P_{\tilde{A}_2|A_2=a_2}\right) \in \Delta\left(\mathcal{A}_2\right)^2, \tag{3}$$

which is a parameter of the game, defines a discrete memoryless channel (DMC) as in [22, 23]. In the final stage, Player 1 plays the action $a \in \mathcal{A}_1$, with probability $P_{A_1|\tilde{A}_2=\tilde{b}}(a)$ and both players obtain their payoffs.

A strategy for Player 1 is a tuple of probability measures

$$P_{A_1|\tilde{A}_2} \triangleq \left(P_{A_1|\tilde{A}_2=a_1}, P_{A_1|\tilde{A}_2=a_2}\right) \in \Delta\left(\mathcal{A}_1\right)^2, \tag{4}$$

which is chosen based on the commitment (the probability measure P_{A_2}). Player 1 chooses its action by sampling the probability measure $P_{A_1|\tilde{A}_2=\tilde{b}}$, which is conditioned on the noisy observation $\tilde{b}$.

The expected payoff obtained by the players is determined by the function $v : \Delta\left(\mathcal{A}_1\right)^2 \times \Delta\left(\mathcal{A}_2\right) \to \mathbb{R}$, such that given the strategy $P_{A_1|\tilde{A}_2}$ in (4) of Player 1 and the strategy P_{A_2} of Player 2, the expected payoff is

$$v\left(P_{A_1|\tilde{A}_2}, P_{A_2}\right) = \sum_{(i,j)\in\{1,2\}^2} u_{i,j} \left(\sum_{\tilde{b}\in\mathcal{A}_2} P_{A_1|\tilde{A}_2=\tilde{b}}\left(a_i\right) P_{\tilde{A}_2|A_2=a_j}\left(\tilde{b}\right)\right) P_{A_2}\left(a_j\right). \tag{5}$$

Often, it is said that Player 2 acts as the leader and Player 1 acts as the follower to highlight the order in which players choose their actions.

The extension of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ in (2) to capture commitments and noisy observations through the DMC in (3) is represented by the tuple:

$$\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right) \triangleq \left(\mathcal{K}, \mathcal{A}_1, \mathcal{A}_2, \underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right). \tag{6}$$

The set of best responses of Player 1 to the commitment announced by Player 2 is determined by the correspondence $\mathrm{BR}_1 : \Delta\left(\mathcal{A}_2\right) \to \mathscr{F}\left(\Delta\left(\mathcal{A}_1\right)^2\right)$, where $\mathscr{F}\left(\Delta\left(\mathcal{A}_1\right)^2\right)$ denotes the power set of $\Delta\left(\mathcal{A}_1\right) \times \Delta\left(\mathcal{A}_1\right)$. In particular, the set of best responses to the commitment P_{A_2} is

$$\mathrm{BR}_1\left(P_{A_2}\right) = \arg \max_{Q_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1)^2} v(Q_{A_1|\tilde{A}_2}, P_{A_2}), \tag{7}$$

where the function v is defined in (5). Let the real-valued function $\hat{v} : \Delta\left(\mathcal{A}_2\right) \to \mathbb{R}$ be such that

$$\hat{v}\left(P_{A_2}\right) = \max_{Q_{A_1|\tilde{A}_2} \in \mathrm{BR}_1\left(P_{A_2}\right)} v\left(Q_{A_1|\tilde{A}_2}, P_{A_2}\right), \tag{8}$$

where the function v is defined in (5), and the correspondence BR_1 is defined in (7). Player 2 chooses its strategy (commitment) P_{A_2} assuming that Player 1 uses a best response to such strategy. Hence, the optimal commitments are the minimizers of $\hat{v}$ in (8).

Equipped with these objects, the solution concept for the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ in (6) is the following.

Definition 1 (Equilibrium) *The tuple* $\left(P_{A_1|\tilde{A}_2}, P_{A_2}\right) \in \Delta\left(\mathcal{A}_1\right)^2 \times \Delta\left(\mathcal{A}_2\right)$ *is said to form an equilibrium of the game* $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ *if*

$$P_{A_2} \in \arg\min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) \qquad \text{and} \tag{9}$$

$$P_{A_1|\tilde{A}_2} \in \mathrm{BR}_1\left(P_{A_2}\right), \tag{10}$$

where the function $\hat{v}$ *is in* (8), *and the correspondence* BR_1 *is in* (7).

3 Preliminaries

The interest on the game $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ in (2) stems from the fact that its payoff at the NE is equivalent to the payoff at the equilibrium of the the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ in (6), under the assumption that Player 1 does not obtain any information about the action played by Player 2 from the output of the DMC. That is, $I\left(P_{\tilde{A}_2|A_2}; P\right) = 0$ for all $P \in \triangle\left(\mathcal{A}_2\right)$, where $I\left(\cdot;\cdot\right)$ is the mutual information. Let the expected payoff in the game $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ be represented by the function $u : \Delta\left(\mathcal{A}_1\right) \times \Delta\left(\mathcal{A}_2\right) \rightarrow \mathbb{R}$ such that, given the strategies P_{A_1} and P_{A_2},

$$u\left(P_{A_1}, P_{A_2}\right) = \sum_{(i,j)\in\{1,2\}^2} P_{A_1}\left(a_i\right) P_{A_2}\left(a_j\right) u_{i,j}. \tag{11}$$

The following lemma characterizes the payoff at the NE of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ and shows that 2×2 ZSGs exhibit either a unique NE or infinitely many NEs.

Lemma 1 (Theorem 1.5 **in [24])** *Let the probability measures* $P^{\star}_{A_1} \in \Delta\left(\mathcal{A}_1\right)$ *and* $P^{\star}_{A_2} \in \Delta\left(\mathcal{A}_2\right)$ *form a NE of the game* $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ *in* (2). *If the entries of the matrix* $\underline{\boldsymbol{u}}$ *in* (1) *satisfy*

$$\left(u_{1,1} - u_{1,2}\right)\left(u_{2,2} - u_{2,1}\right) > 0 \quad \text{and} \tag{12a}$$

$$\left(u_{1,1} - u_{2,1}\right)\left(u_{2,2} - u_{1,2}\right) > 0, \tag{12b}$$

then, the NE of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ *in* (2) *is unique, with*

$$P^{\star}_{A_1}(a_1) = \frac{u_{2,2} - u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} \in (0,1) \ \text{and} \tag{13a}$$

$$P^{\star}_{A_2}(a_1) = \frac{u_{2,2} - u_{1,2}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} \in (0,1). \tag{13b}$$

Moreover, the expected payoff at the NE is

$$u(P^{\star}_{A_1}, P^{\star}_{A_2}) = \frac{u_{1,1}u_{2,2} - u_{1,2}u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}}. \tag{14}$$

If the entries of the matrix $\underline{\boldsymbol{u}}$ *in* (1) *satisfy*

$$\left(u_{1,1} - u_{1,2}\right)\left(u_{2,2} - u_{2,1}\right) \leqslant 0 \quad \text{or} \tag{15a}$$

$$\left(u_{1,1} - u_{2,1}\right)\left(u_{2,2} - u_{1,2}\right) \leqslant 0, \tag{15b}$$

then, there exists either a unique NE or infinitely many NEs; and all NE strategies lead to the same payoff,

$$u(P^{\star}_{A_1}, P^{\star}_{A_2}) = \min_{j\in\{1,2\}} \max_{i\in\{1,2\}} u_{i,j} = \max_{i\in\{1,2\}} \min_{j\in\{1,2\}} u_{i,j}. \tag{16}$$

A payoff matrix $\underline{\boldsymbol{u}}$ that satisfies (12) represents a ZSG exhibiting a unique NE in strictly mixed strategies. Alternatively, a payoff matrix $\underline{\boldsymbol{u}}$ that satisfies (15) represents a ZSG exhibiting *strategic dominance*, a unique pure NE, or infinitely many NEs [24].

Let the function $\hat{u} : \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$ be such that for all $P \in \Delta(\mathcal{A}_2)$,

$$\hat{u}(P) = \max_{Q\in\Delta(\mathcal{A}_1)} u(Q, P), \tag{17}$$

where the function u is defined in (11). The function $\hat{u}$ in (17) determines the payoff $\hat{u}(P)$ in the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in (2) when Player 1 always plays an optimal strategy to the strategy P played by Player 2. Moreover, the minimum of the function $\hat{u}$ is the payoff at the NE.

4 Main Results

4.1 Characterization of the Equilibria

The following theorem ensures the existence of an equilibrium for the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ in (6).

Theorem 1 (Existence) *The game* $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ *in* (6) *always possesses an equilibrium.*

Proof: The proof is presented in Appendix A. ■

For characterizing the payoff at the equilibrium of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$, it is important to highlight that the set of optimal commitments for Player 2 are the strategies that minimize the function $\hat{v}$ in (8). Let $P^{(1)}$ and $P^{(2)}$ be two real numbers such that for all $i \in \{1,2\}$,

$$\begin{pmatrix}1\\0\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix}P^{(i)}\\1-P^{(i)}\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix}P^{(i)}\\1-P^{(i)}\end{pmatrix}, \tag{18}$$

where the 2×2 matrix $\underline{\boldsymbol{u}}^{(i)}$ satisfies,

$$\underline{\boldsymbol{u}}^{(i)} = \underline{\boldsymbol{u}} \begin{pmatrix}P_{\tilde{A}_2|A_2=a_1}(a_i) & 0\\ 0 & P_{\tilde{A}_2|A_2=a_2}(a_i)\end{pmatrix}, \tag{19}$$

with the matrix $\underline{\boldsymbol{u}}$ defined in (1); and the probability measures $P_{\tilde{A}_2|A_2}$ defined in (3). Using this notation, the following theorem characterizes the payoff at equilibrium.

Theorem 2 (Equilibrium Payoff) *Let the tuple* $\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right) \in \Delta\left(\mathcal{A}_1\right)^2 \times \Delta\left(\mathcal{A}_2\right)$ *form an equilibrium of the game* $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ *in* (6)*. If the matrix* $\underline{\boldsymbol{u}}$ *in* (1) *satisfies* (12)*, then*

$$v\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right)=\min\{\hat{v}\left(P_1\right), \hat{v}\left(P_2\right)\}, \tag{20}$$

where, the functions v *and* $\hat{v}$ *are defined in* (5) *and* (8)*, respectively, and for all* $i \in \{1,2\}$*, the probability measure* $P_i \in \Delta\left(\mathcal{A}_2\right)$ *is such that* $P_i\left(a_1\right) = P^{(i)}$*, with* $P^{(i)}$ *in* (18)*. Alternatively, if the entries of the matrix* $\underline{\boldsymbol{u}}$ *satisfy* (15)*, then*

$$v\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right)= \min_{j\in\{1,2\}} \max_{i\in\{1,2\}} u_{i,j}. \tag{21}$$

Proof: The proof is presented in Appendix B. ■

Theorem 2 characterizes the optimal commitment of Player 2. More specifically, when the payoff matrix $\underline{\boldsymbol{u}}$ in (1) is such that the game $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ in (2) possesses a unique NE in mixed strategies (conditions in (12)), the optimal commitment is one of the strategies P_1 or P_2 in (20). For all $i \in \{1,2\}$, the strategy P_i makes Player 1 indifferent to play any of its actions in the game $\mathscr{G}\left(\underline{\boldsymbol{u}}^{(i)}\right)$, with the matrix $\underline{\boldsymbol{u}}^{(i)}$ in (19). This follows from the construction in (18). Alternatively, when the payoff matrix $\underline{\boldsymbol{u}}$ in (1) is such that the game $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ in (2) does not possess a unique NE in mixed strategies (conditions in (15)), the optimal commitment for Player 2 is a pure strategy. This is equivalent to announcing to Player 1 that a given action would be played with probability one, which makes the noisy observation immaterial. Moreover, from Lemma 1, it follows that the payoffs at the NE and the SE in pure strategies of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ are identical to the payoff at the equilibrium of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$. That is, neither the fact that Player 2 commits before its opponent nor the fact that Player 1 obtains an observation of the action played by its opponent represent any benefit for either player.

4.2 The Set of Best Responses of Player 1

The following lemma shows that, given a commitment P_{A_2}, the set of best responses $\mathrm{BR}_1(P_{A_2})$ in (7) is the Cartesian product of two sets that can be independently described.

Lemma 2 *The correspondance* BR_1 *in* (7) *satisfies for all* $P \in \Delta\left(\mathcal{A}_2\right)$*,*

$$\mathrm{BR}_1\left(P\right)=\mathrm{BR}_{1,1}\left(P\right) \times \mathrm{BR}_{1,2}\left(P\right), \tag{22}$$

where for all $i \in \{1,2\}$*, the correspondence* $\mathrm{BR}_{1,i} : \Delta\left(\mathcal{A}_2\right) \to \mathscr{F}\left(\Delta\left(\mathcal{A}_1\right)\right)$ *is such that*

$$\mathrm{BR}_{1,i}\left(P\right)=\arg\max_{Q\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q\left(a_1\right) \\ Q\left(a_2\right) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix} P\left(a_1\right) \\ P\left(a_2\right) \end{pmatrix}, \tag{23}$$

where the matrix $\underline{\boldsymbol{u}}^{(i)}$ *is in* (19)*.*

Proof: The proof is presented in Appendix C. ■

The following lemma characterizes the sets $\mathrm{BR}_{1,1}(P)$ and $\mathrm{BR}_{1,2}(P)$ in (23).

Lemma 3 *Given a probability measure $P \in \Delta(\mathcal{A}_2)$, for all $i \in \{1,2\}$, the correspondence $\mathrm{BR}_{1,i}$ in (23) satisfies*

$$\mathrm{BR}_{1,i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } s_i > 0, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } s_i < 0, \\ \Delta(\mathcal{A}_1), & \text{if } s_i = 0, \end{cases} \quad (24)$$

where $s_i \in \mathbb{R}$ is

$$s_i \triangleq (u_{1,1} - u_{2,1}) P(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) + (u_{1,2} - u_{2,2}) P(a_2) P_{\tilde{A}_2|A_2=a_2}(a_i). \quad (25)$$

Proof: The proof is presented in Appendix D. ■

A first observation from Lemma 3 is that for all $i \in \{1,2\}$ and for all $P \in \Delta(\mathcal{A}_2)$, the cardinality of set $\mathrm{BR}_{1,i}(P)$ is either one or infinite. In the case in which $\mathrm{BR}_{1,i}(P)$ is a singleton, the only element is a pure strategy. Alternatively, when the cardinality is infinity, the set $\mathrm{BR}_{1,i}(P)$ is identical to the set of all possible probability measures on $\mathcal{A}_1$, i.e., $\mathrm{BR}_{1,i}(P) = \Delta(\mathcal{A}_1)$. That is, Player 1 chooses its actions either indifferently (all strategies are best responses) or deterministically (pure strategy). This contrasts with the case of bi-matrix Stakelberg games in which the existence of multiple best responses constraints the existence of equilibria [25].

Note also that for all $(i,j) \in \{1,2\}^2$, the expected payoff, when Player 1 plays a_j, Player 2 has committed to P_{A_2}, and the noisy observation is a_i, is $u_{j,1} P_{A_2}(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{j,2} P_{A_2}(a_2) P_{\tilde{A}_2|A_2=a_2}(a_i)$. Thus, the right-hand side of the equality in (25) is the difference between the expected payoff obtained when Player 1 plays a_1 and when it plays a_2, subject to the observation a_i and the commitment P_{A_2}.

The following lemma presents a different view of the correspondences $\mathrm{BR}_{1,1}$ and $\mathrm{BR}_{1,2}$ in (23). It suggests that Player 1 performs an estimation of the likelihood with which Player 2 might have played each of its actions based on the knowledge of the commitment and the noisy observation.

Lemma 4 *Given a probability measure $P \in \Delta(\mathcal{A}_2)$, for all $i \in \{1,2\}$, the correspondence $\mathrm{BR}_{1,i}$ in (23) satisfies*

$$\mathrm{BR}_{1,i}(P) = \arg\max_{Q \in \Delta(\mathcal{A}_1)} u\left(Q, P_{A_2|\tilde{A}_2=a_i}\right), \quad (26)$$

where the function u is defined in (11); the probability measure $P_{A_2|\tilde{A}_2=a_i}$ satisfies for all $j \in \{1,2\}$,

$$P_{A_2|\tilde{A}_2=a_i}(a_j) = \frac{P_{\tilde{A}_2|A_2=a_j}(a_i)\, P(a_j)}{\sum_{\ell \in \{1,2\}} P_{\tilde{A}_2|A_2=a_\ell}(a_i)\, P(a_\ell)}, \quad (27)$$

with the probability measures $P_{\tilde{A}_2|A_2=a_1}$ and $P_{\tilde{A}_2|A_2=a_2}$ defined in (3).

Proof: The proof is presented in Appendix E. ■

For all $(i, j) \in \{1, 2\}^2$, the likelihood with which Player 2 has chosen action a_j given the commitment P and the noisy observation a_i is $P_{A_2|\tilde{A}_2=a_i}(a_j)$ in (27). Hence, from Lemma 3 and Lemma 4, the optimal strategy of Player 1 to the observation a_i and the commitment P in the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ is identical to its optimal strategy in the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in (2) when its opponent plays the strategy $P_{A_2|\tilde{A}_2=a_i}$ in (27).

4.3 Relevance of Noisy Observations

The following lemma shows that the function $\hat{u}$ in (17) is upper bounded by the function $\hat{v}$ in (8). This implies that, granting observations to Player 1 of the actions played by Player 2 does not harm Player 1. On the contrary, in some cases it might significantly benefit it.

Lemma 5 *Let the probability measures $P^{\star}_{A_1} \in \Delta(\mathcal{A}_1)$ and $P^{\star}_{A_2} \in \Delta(\mathcal{A}_2)$ form one of the NEs of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in* (2)*. For all $P \in \Delta(\mathcal{A}_2)$, it holds that*

$$u(P^{\star}_{A_1}, P^{\star}_{A_2}) \leqslant \hat{u}(P) \leqslant \hat{v}(P) \leqslant \sum_{j\in\{1,2\}} P(a_j)\left(\max_{i\in\{1,2\}} u_{i,j}\right), \tag{28}$$

where the functions $\hat{v}$, u, and $\hat{u}$ are defined in (8)*,* (11)*, and* (17)*, respectively.*

Proof: The proof is presented in Appendix F. ■

The following lemma compares the payoffs at the equilibria of the games $\mathscr{G}(\underline{\boldsymbol{u}})$ in (2) and $\mathscr{G}(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2})$ in (6).

Lemma 6 *Let the probability measures $P^{\star}_{A_1} \in \Delta(\mathcal{A}_1)$ and $P^{\star}_{A_2} \in \Delta(\mathcal{A}_2)$ form one of the NEs of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in* (2)*. Let also the tuple $\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right) \in \Delta(\mathcal{A}_1)^2 \times \Delta(\mathcal{A}_2)$ form an equilibrium of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ in* (6)*. Then,*

$$u(P^{\star}_{A_1}, P^{\star}_{A_2}) \leqslant v\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right) \leqslant \min_{j\in\{1,2\}} \max_{i\in\{1,2\}} u_{i,j}. \tag{29}$$

Proof: The proof is presented in Appendix G. ■

Lemma 6 reveals that the payoff at the equilibria of the game $\mathscr{G}(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2})$ in (6) is lower bounded by the NE of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in (2), which coincides with the SE in mixed strategies; and is upper bounded by the SE in pure strategies of the game $\mathscr{G}(\underline{\boldsymbol{u}})$. The lower bound corresponds to the case in which the Player 1 does not observe the actions of its opponent, while the upper bound corresponds to the case in which Player 1 has perfect observations of the actions taken by Player 2.

The following lemma presents necessary and sufficient conditions under which the payoff at the equilibrium of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ is not greater than the NE of the game $\mathscr{G}(\underline{\boldsymbol{u}})$.

Lemma 7 *Let the tuple* $\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right) \in \Delta(\mathcal{A}_1)^2 \times \Delta(\mathcal{A}_2)$ *form an equilibrium of the game* $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ *in* (6). *Let also the tuple* $\left(P^{\star}_{A_1}, P^{\star}_{A_2}\right) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$ *form one of the NEs of the game* $\mathscr{G}(\underline{\boldsymbol{u}})$ *in* (2). *Then,*

$$v\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right) = u(P^{\star}_{A_1}, P^{\star}_{A_2}), \tag{30}$$

if and only if, (*a*) *the matrix* $\underline{\boldsymbol{u}}$ *in* (1) *satisfies* (15)*; or* (*b*) *the matrix* $\underline{\boldsymbol{u}}$ *in* (1) *satisfies* (12) *and the DMC in* (3) *satisfies for all* $P \in \triangle(\mathcal{A}_2)$*, that* $I\left(P_{\tilde{A}_2|A_2}; P\right) = 0$.

Proof: The proof is presented in Appendix H. ■

Lemma 7 establishes that granting Player 1 with noisy observations of the action played by Player 2 does not make any difference in two particular scenarios. First, in ZSGs with strategic dominance, NEs in pure strategies and infinitely many NEs (condition (*a*)). Second, in ZSGs when the DMC in (3) is such that Player 1 does not obtain any additional information about the action played by Player 2 by observing the output of the DMC.

Lemma 5 and Lemma 7 imply that granting Player 1 with relevant noisy observations of the action played by Player 2 makes a difference exclusively for ZSGs with a unique NE in mixed strategies. In this case, given the commitment of the leader P_{A_2}, relevant noisy observations refer to observations obtained through a DMC exhibiting positive mutual information between the channel input and the channel output. That is, $I\left(P_{\tilde{A}_2|A_2}; P_{A_2}\right) > 0$.

The following lemma describes a special class of channels

Lemma 8 *Let* $\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right) \in \Delta(\mathcal{A}_1)^2 \times \Delta(\mathcal{A}_2)$ *form an equilibrium of the game* $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ *in* (6). *If for all* $P \in \triangle(\mathcal{A}_2)$, $I\left(P_{\tilde{A}_2|A_2}; P\right) = H(P) = H\left(P_{\tilde{A}_2}\right)$*, with* $P_{\tilde{A}_2}(a_i) = \sum_{\ell \in \{1,2\}} P_{\tilde{A}_2|A_2 = a_\ell}(a_i) P(a_\ell)$ *and* $i \in \{1,2\}$*, then*

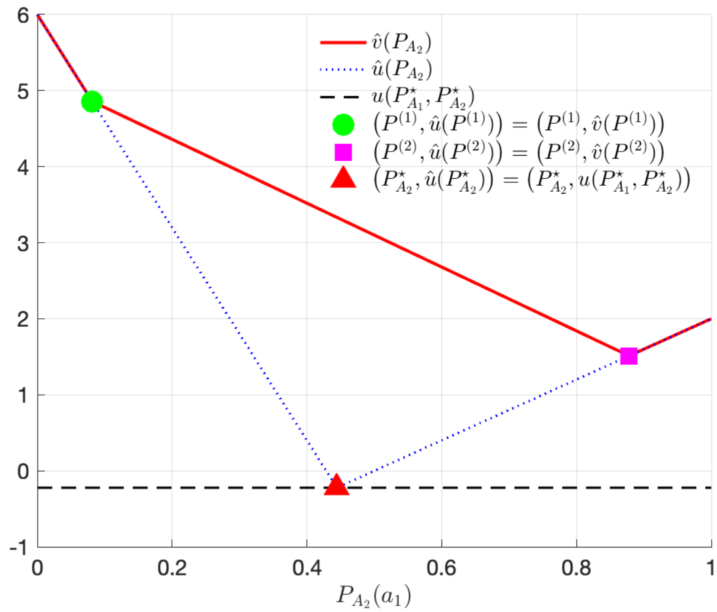
$$\hat{v}\left(P^{\dagger}_{A_2}\right) = \min_{j \in \{1,2\}} \max_{i \in \{1,2\}} u_{i,j}. \tag{31}$$

Proof: The proof is presented in Appendix I. ■

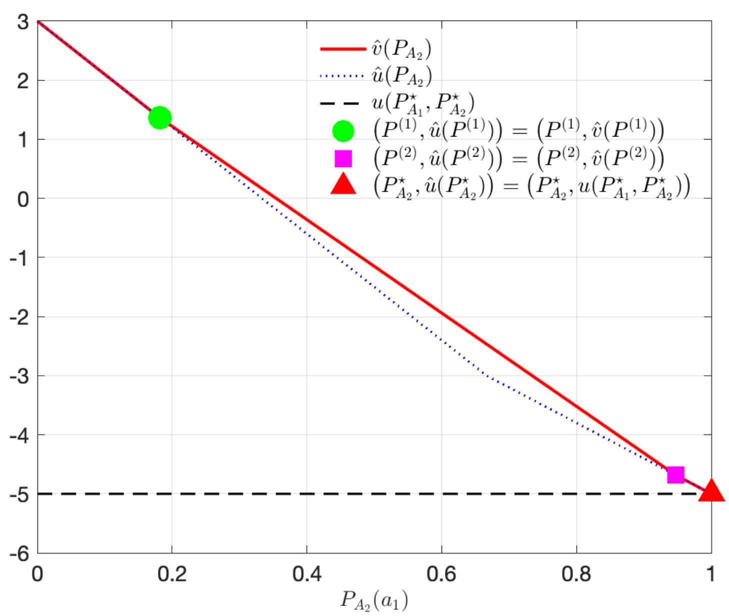
The condition that for all $P \in \triangle(\mathcal{A}_2)$, $I\left(P_{\tilde{A}_2|A_2}; P\right) = H(P) = H\left(P_{\tilde{A}_2}\right)$ implies that the DMC in (3) establishes a deterministic bijection between the channel input and the channel output. From this perspective, Lemma 8 strengthens the observation that under perfect observations of the action played by Player 2, the commitment becomes irrelevant and the payoff at the equilibrium of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ in (6) is identical to the SE in pure strategies of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in (2), i.e., the min max solution in pure strategies.

5 Examples

In Figure 1(a), the matrix $\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$ is such that the game $\mathscr{G}(\underline{\boldsymbol{u}})$ exhibits a unique NE in mixed strategies (Lemma 1). Hence, as announced by Lemma 5



(a) Payoff matrix $\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$ in (1).



(b) Payoff matrix $\underline{\boldsymbol{u}} = (-5, 1; -6, 3)$ in (1).

Figure 1: Plots of the function $\hat{v}$ in (8) and $\hat{u}$ in (17) as a function of the commitment P_{A_2} of Player 2 with a symmetric DMC $P_{\tilde{A}_2|A_2=a_1}(a_1) = P_{\tilde{A}_2|A_2=a_2}(a_2) = 0.9$ in (3).

and Lemma 7, there exists a strict inequality between the NE payoff $u(P^{\star}_{A_1}, P^{\star}_{A_2})$ of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ (red triangle) and the equilibrium payoff $v\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right)$ of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ (magenta square). Alternatively, in Figure 1(b), the matrix $\underline{\boldsymbol{u}} = (-5, 1; -6, 3)$ is such that the game $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ exhibits a unique NE in pure strategies (Lemma 1). Hence, as predicted by Lemma 7, the payoffs of the games $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$ and $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ are identical (red triangle). That is, $u(P^{\star}_{A_1}, P^{\star}_{A_2})$ $= v\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right)$.

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A Proof of Theorem 1

Let the function $f : [0, 1] \to \mathbb{R}$ be such that

$$\begin{aligned} f(\beta) = \max\big\{ & u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_1)\beta + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_1)\,(1-\beta)\,, \\ & u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_1)\beta + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_1)\,(1-\beta)\,\big\} \\ + \max\big\{ & u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_2)\beta + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_2)\,(1-\beta)\,, \\ & u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_2)\beta + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_2)\,(1-\beta)\,\big\}. \end{aligned} \tag{32}$$

Then from (8) and (32), it holds that given a probability measure $P \in \Delta(\mathcal{A}_2)$, with $P(a_1) = \beta \in [0, 1]$,

$$\hat{v}(P) = f(\beta). \tag{33}$$

Hence, it holds that

$$\arg \min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) = \left\{ P \in \Delta(\mathcal{A}_2) : P(a_1) \in \arg \min_{\beta \in [0,1]} f(\beta) \right\}. \tag{34}$$

Given the fact that function f is continuous piecewise linear, the following optimization problem always possesses a solution [26, Extreme Value Theorem]:

$$\min_{\beta \in [0,1]} f(\beta), \tag{35}$$

which, from (33), implies that the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, P_{\tilde{A}_2|A_2}\right)$ in (6) always possesses an equilibrium. This completes the proof.

B Proof of Theorem 2

The proof is divided into two parts. Subsection B.1 introduces preliminary results in the form of lemmas. Subsection B.2 presents the proof using the preliminary results.

B.1 Preliminaries

Lemma 9 *Let $\underline{\boldsymbol{w}} \in \mathbb{R}^{2\times 2}$ be such that*

$$\underline{\boldsymbol{w}} \triangleq \begin{pmatrix} P_{\tilde{A}_2|A_2=a_1}(a_1) & P_{\tilde{A}_2|A_2=a_2}(a_1) \\ P_{\tilde{A}_2|A_2=a_1}(a_2) & P_{\tilde{A}_2|A_2=a_2}(a_2) \end{pmatrix}. \tag{36}$$

Then, the determinant of matrix $\underline{\boldsymbol{w}}$ satisfies

$$\det \underline{\boldsymbol{w}} = P_{\tilde{A}_2|A_2=a_1}(a_1) - P_{\tilde{A}_2|A_2=a_2}(a_1) \tag{37}$$

$$= P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_1}(a_2). \tag{38}$$

Proof: From (36), it follows that

$$\det \underline{\boldsymbol{w}} = P_{\tilde{A}_2|A_2=a_1}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_2}(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) \quad (39)$$
$$= P_{\tilde{A}_2|A_2=a_1}(a_1) - P_{\tilde{A}_2|A_2=a_2}(a_1) \quad (40)$$

and

$$P_{\tilde{A}_2|A_2=a_1}(a_1) - P_{\tilde{A}_2|A_2=a_2}(a_1) = P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_1}(a_2), \quad (41)$$

which completes the proof. ■

Lemma 10 *For all tuples $(Q_1, Q_2) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)$ and for all probability measures $P \in \Delta(\mathcal{A}_2)$, the function v in (5) satisfies*

$$v(Q_1, Q_2, P) = P_{\tilde{A}_2}(a_1)\begin{pmatrix} Q_1(a_1) \\ Q_1(a_2)\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_1}(a_1) \\ P_{A_2|\tilde{A}_2=a_1}(a_2)\end{pmatrix} + P_{\tilde{A}_2}(a_2)\begin{pmatrix} Q_2(a_1) \\ Q_2(a_2)\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_2}(a_1) \\ P_{A_2|\tilde{A}_2=a_2}(a_2)\end{pmatrix}, \quad (42)$$

where the probability measure $P_{\tilde{A}_2}$ satisfies for all $i \in \{1, 2\}$,

$$P_{\tilde{A}_2}(a_i) = \sum_{\ell \in \{1,2\}} P_{\tilde{A}_2|A_2=a_\ell}(a_i)\, P(a_\ell); \quad (43)$$

and the probability measure $P_{A_2|\tilde{A}_2=a_i}$ is in (27).

Proof: From (5), it holds that for all $(Q_1, Q_2) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)$ and for all $P \in \Delta(\mathcal{A}_2)$,

$$\begin{aligned} v(Q_1, Q_2, P) &= \begin{pmatrix} Q_1(a_1) \\ Q_2(a_2)\end{pmatrix}^{\mathsf{T}} \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2}\end{pmatrix} \begin{pmatrix} P_{A_2,\tilde{A}_2}(a_1, a_1) \\ P_{A_2,\tilde{A}_2}(a_2, a_1)\end{pmatrix} \\ &\quad + \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2)\end{pmatrix}^{\mathsf{T}} \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2}\end{pmatrix} \begin{pmatrix} P_{A_2,\tilde{A}_2}(a_1, a_2) \\ P_{A_2,\tilde{A}_2}(a_2, a_2)\end{pmatrix} \quad (44) \\ &= \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2)\end{pmatrix}^{\mathsf{T}} \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2}\end{pmatrix} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_1}(a_1) \\ P_{A_2|\tilde{A}_2=a_1}(a_2)\end{pmatrix} P_{\tilde{A}_2}(a_1) \\ &\quad + \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2)\end{pmatrix}^{\mathsf{T}} \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2}\end{pmatrix} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_2}(a_1) \\ P_{A_2|\tilde{A}_2=a_2}(a_2)\end{pmatrix} P_{\tilde{A}_2}(a_2), \quad (45) \end{aligned}$$

which completes the proof. ■

Lemma 11 *Let the probability measures $P^\star_{A_1} \in \Delta(\mathcal{A}_1)$ and $P^\star_{A_2} \in \Delta(\mathcal{A}_2)$ form one of the NEs of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in (2). Given a probability measure $P \in \Delta(\mathcal{A}_2)$, the equality*

$$u(P^\star_{A_1}, P^\star_{A_2}) = \hat{v}(P), \quad (46)$$

with the functions u in (11) *and $\hat{v}$ in* (8)*, holds if and only if for all $i \in \{1,2\}$, the probability measure $P_{A_2|\tilde{A}_2=a_i}$ in* (27) *satisfy*

$$P_{A_2|\tilde{A}_2=a_i} \in \arg \min_{P\in\Delta(\mathcal{A}_2)} \max_{P_0\in\Delta(\mathcal{A}_1)} u(P_0, P). \tag{47}$$

In particular, if the game $\mathscr{G}(\underline{\boldsymbol{u}})$ possesses a unique NE, the equality in (46) *holds if and only if*

$$P(a_1) = P^{\star}_{A_2}(a_1) \in \{0,1\}, \qquad \text{or} \tag{48a}$$
$$P(a_1) = P^{\star}_{A_2}(a_1) \in (0,1) \text{ and } \det \underline{\boldsymbol{w}} = 0. \tag{48b}$$

Proof: From Lemma 10, for all $P \in \Delta(\mathcal{A}_2)$, it holds that

$$\begin{aligned}
\hat{v}(P) = {}& P_{\tilde{A}_2}(a_1) \max_{Q_1\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q_1(a_1) \\ Q_1(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_1}(a_1) \\ P_{A_2|\tilde{A}_2=a_1}(a_2) \end{pmatrix} \\
& + P_{\tilde{A}_2}(a_2) \max_{Q_2\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q_2(a_1) \\ Q_2(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_2}(a_1) \\ P_{A_2|\tilde{A}_2=a_2}(a_2) \end{pmatrix} &(49)\\
= {}& P_{\tilde{A}_2}(a_1) \max_{Q_1\in\Delta(\mathcal{A}_1)} u\left(Q_1, P_{A_2|\tilde{A}_2=a_1}\right) \\
& + P_{\tilde{A}_2}(a_2) \max_{Q_2\in\Delta(\mathcal{A}_1)} u\left(Q_2, P_{A_2|\tilde{A}_2=a_2}\right) &(50)\\
\geqslant {}& \min\left\{ \max_{Q_1\in\Delta(\mathcal{A}_1)} u\left(Q_1, P_{A_2|\tilde{A}_2=a_1}\right), \max_{Q_2\in\Delta(\mathcal{A}_1)} u\left(Q_2, P_{A_2|\tilde{A}_2=a_2}\right)\right\} &(51)\\
= {}& \min_{i\in\{1,2\}} \max_{Q_0\in\Delta(\mathcal{A}_1)} u\left(Q_0, P_{A_2|\tilde{A}_2=a_i}\right) &(52)\\
\geqslant {}& \min_{P_0\in\Delta(\mathcal{A}_1)} \max_{Q_0\in\Delta(\mathcal{A}_1)} u(Q_0, P_0) &(53)\\
= {}& u\left(P^{\star}_{A_1}, P^{\star}_{A_2}\right), &(54)
\end{aligned}$$

where the equality in (54) follows from the minmax theorem [8].

Note that if for all $i \in \{1,2\}$, the probability measure P in (49) is such that

$$\max_{Q_1\in\Delta(\mathcal{A}_1)} u\left(Q_1, P_{A_2|\tilde{A}_2=a_1}\right) = \max_{Q_2\in\Delta(\mathcal{A}_1)} u\left(Q_2, P_{A_2|\tilde{A}_2=a_2}\right), \tag{55}$$

then (51) holds with equality. Moreover, if there exists an $i \in \{1,2\}$ such that (47) holds, then the inequality in (53) holds with equality. From these observations, it holds that if for all $i \in \{1,2\}$, the probability measure P in (49) satisfies (47), then the inequalities in (51) and (53) hold with equality, which implies the equality in (46).

Alternatively, if the equality in (46) holds, then, both inequalities in (51) and (53) hold with equality, which implies that for all $i \in \{1,2\}$, the probability measure P in (49) satisfies (47).

In particular, under the assumption that there exists a unique NE in game $\mathscr{G}(\underline{\boldsymbol{u}})$, the inequalities (51) and (53) hold with equality if and only if for all $i \in \{1,2\}$,

$$P_{A_2|\tilde{A}_2=a_i} = P^{\star}_{A_2}. \tag{56}$$

The measure $P_{A_2|\tilde{A}_2=a_1}$ in (56) satisfies

$$P_{A_2|\tilde{A}_2=a_1}(a_1)=\frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)}{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)+P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1)} \tag{57}$$

$$=\frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)}{P_{\tilde{A}_2|A_2=a_1}(a_1)-\left(P_{\tilde{A}_2|A_2=a_1}(a_1)-P_{\tilde{A}_2|A_2=a_2}(a_1)\right)P(a_2)} \tag{58}$$

$$=\frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)}{P_{\tilde{A}_2|A_2=a_1}(a_1)-\det\underline{\boldsymbol{w}}P(a_2)}. \tag{59}$$

Similarly, the measure $P_{A_2|\tilde{A}_2=a_2}$ in (56) satisfies

$$P_{A_2|\tilde{A}_2=a_2}(a_1)=\frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2)}{P_{\tilde{A}_2|A_2=a_1}(a_2)+\det\underline{\boldsymbol{w}}P(a_2)}. \tag{60}$$

Hence, from (59) and (60), the equality $P_{A_2|\tilde{A}_2=a_1}(a_1)=P_{A_2|\tilde{A}_2=a_2}(a_1)$ implies

$$\frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)}{P_{\tilde{A}_2|A_2=a_1}(a_1)-\det\underline{\boldsymbol{w}}P(a_2)}=\frac{P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2)}{P_{\tilde{A}_2|A_2=a_1}(a_2)+\det\underline{\boldsymbol{w}}P(a_2)}, \tag{61}$$

which can be rewritten as

$$\begin{aligned}0=&\, P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_1)\left(P_{\tilde{A}_2|A_2=a_1}(a_2)+\det\underline{\boldsymbol{w}}P(a_2)\right)\\ &-P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2)\left(P_{\tilde{A}_2|A_2=a_1}(a_1)-\det\underline{\boldsymbol{w}}P(a_2)\right) &(62)\\ =&\det\underline{\boldsymbol{w}}P(a_1)P(a_2)\left(P_{\tilde{A}_2|A_2=a_1}(a_1)+P_{\tilde{A}_2|A_2=a_1}(a_2)\right) &(63)\\ =&\det\underline{\boldsymbol{w}}P(a_1)P(a_2). &(64)\end{aligned}$$

Hence, the equality $P_{A_2|\tilde{A}_2=a_1}(a_1)=P_{A_2|\tilde{A}_2=a_2}(a_1)=P^{\star}_{A_2}(a_1)$ in (56) holds if and only if one of the conditions in (48) holds. This completes the proof. ■

Lemma 12 *Let the probability measures $P^{\star}_{A_1}\in\Delta(\mathcal{A}_1)$ and $P^{\star}_{A_2}\in\Delta(\mathcal{A}_2)$ form one of the NEs of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in* (2)*. Under the assumption that the entries of the matrix $\underline{\boldsymbol{u}}$ in* (1) *satisfy* (12)*, for all $i\in\{1,2\}$, the real $P^{(i)}$ in* (18) *satisfies*

$$P^{(i)}=\frac{P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P^{\star}_{A_2}(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i)+P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}\in[0,1]. \tag{65}$$

Proof: From Lemma 1, it follows that if the entries of the matrix $\underline{\boldsymbol{u}}$ in (1) satisfy (12), the probability measures $P^{\star}_{A_1}\in\Delta(\mathcal{A}_1)$ and $P^{\star}_{A_2}\in\Delta(\mathcal{A}_2)$ form the unique NE of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in (2) and moreover, for all $(i,j)\in\{1,2\}^2$, $P^{\star}_{A_i}(a_j)\in(0,1)$. Hence, for all $i\in\{1,2\}$, it follows from (18) that

$$P^{(i)}=\frac{(u_{2,2}-u_{1,2})P_{\tilde{A}_2|A_2=a_2}(a_i)}{(u_{1,1}-u_{2,1})P_{\tilde{A}_2|A_2=a_1}(a_i)+(u_{2,2}-u_{1,2})P_{\tilde{A}_2|A_2=a_2}(a_i)} \tag{66}$$

$$=\frac{P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P^{\star}_{A_2}(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i)+P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}, \tag{67}$$

where the equality in (67) follows from the fact that $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \neq 0$ when the entries of the matrix $\underline{\boldsymbol{u}}$ satisfy (12), and from (13). Furthermore, note that

$$
\begin{aligned}
0 \leq & P^{\star}_{A_2}(a_1) P_{\tilde{A}_2|A_2=a_2}(a_i) && (68)\\
\leq & P^{\star}_{A_2}(a_1) P_{\tilde{A}_2|A_2=a_2}(a_i) + P^{\star}_{A_2}(a_2) P_{\tilde{A}_2|A_2=a_1}(a_i), && (69)
\end{aligned}
$$

which follows from the fact that for all $(i,j) \in \{1,2\}^2$, it holds that $P_{\tilde{A}_2|A_2=a_i}(a_j) \geq 0$ and $P^{\star}_{A_2}(a_i) \in (0,1)$. Hence, from (67) and (68), for all $i \in \{1,2\}$, it holds that

$$
P^{(i)} \in [0,1]. \tag{70}
$$

This completes the proof. ∎

Lemma 13 *Let the probability measures $P^{\star}_{A_1} \in \Delta(\mathcal{A}_1)$ and $P^{\star}_{A_2} \in \Delta(\mathcal{A}_2)$ form one of the NEs of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in* (2)*. Assume that the entries of the matrix $\underline{\boldsymbol{u}}$ in* (1) *satisfy* (12)*. Hence, the reals $P^{(1)}$ and $P^{(2)}$ in* (18) *satisfy*

$$
0 \leqslant \min\left\{P^{(1)}, P^{(2)}\right\} \leqslant P^{\star}_{A_2}(a_1) \leqslant \max\left\{P^{(1)}, P^{(2)}\right\} \leqslant 1. \tag{71}
$$

Proof: From Lemma 1, it follows that if the entries of the matrix $\underline{\boldsymbol{u}}$ in (1) satisfy (12), the probability measures $P^{\star}_{A_1} \in \Delta(\mathcal{A}_1)$ and $P^{\star}_{A_2} \in \Delta(\mathcal{A}_2)$ form the unique NE of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ in (2). Hence, from Lemma 12, it follows that for all $i \in \{1,2\}$,

$$
P^{(i)} - P^{\star}_{A_2}(a_1) = \frac{P^{\star}_{A_2}(a_1) P^{\star}_{A_2}(a_2)\left(P_{\tilde{A}_2|A_2=a_2}(a_i) - P_{\tilde{A}_2|A_2=a_1}(a_i)\right)}{P^{\star}_{A_2}(a_2) P_{\tilde{A}_2|A_2=a_1}(a_i) + P^{\star}_{A_2}(a_1) P_{\tilde{A}_2|A_2=a_2}(a_i)}. \tag{72}
$$

From (72), it follows that if $P^{(1)} \geq P^{\star}_{A_2}(a_1)$, then, it holds that

$$
P^{(2)} \leq P^{\star}_{A_2}(a_1). \tag{73}
$$

Similarly, if $P^{(1)} \leq P^{\star}_{A_2}(a_1)$, then, it holds that

$$
P^{(2)} \geq P^{\star}_{A_2}(a_1). \tag{74}
$$

This implies that

$$
\min\left\{P^{(1)}, P^{(2)}\right\} \leqslant P^{\star}_{A_2}(a_1) \leqslant \max\left\{P^{(1)}, P^{(2)}\right\}. \tag{75}
$$

The proof is completed by noticing that if the entries of the matrix $\underline{\boldsymbol{u}}$ in (1) satisfy (12), from Lemma 12, it holds that for all $i \in \{1,2\}$, $P^{(i)} \in [0,1]$. ∎

Lemma 14 *Assume that the entries of the matrix $\underline{\boldsymbol{u}}$ in* (1) *satisfy* (12)*. The reals $P^{(1)}$ and $P^{(2)}$ in* (18) *satisfy*

$$
\min\left\{P^{(1)}, P^{(2)}\right\} = \begin{cases} P^{(1)}, & \text{if} \quad \det \underline{\boldsymbol{w}} > 0,\\ P^{(2)}, & \text{if} \quad \det \underline{\boldsymbol{w}} < 0,\\ P^{(1)} = P^{(2)} = P^{\star}_{A_2}(a_1), & \text{if} \quad \det \underline{\boldsymbol{w}} = 0, \end{cases} \tag{76}
$$

with $\underline{\boldsymbol{w}}$ being the matrix in (36)*.*

Proof: First, consider the case in which $\det \underline{\boldsymbol{w}} < 0$. From Lemma 9, if $\det \underline{\boldsymbol{w}} < 0$, then it holds that

$$P_{\tilde{A}_2|A_2=a_2}(a_1) - P_{\tilde{A}_2|A_2=a_1}(a_1) > 0. \tag{77}$$

Under the assumption that the entries of the matrix $\underline{\boldsymbol{u}}$ in (1) satisfy (12), from (72) and (77), it holds that $P^{(1)} > P^{\star}_{A_2}(a_1)$ and $P^{\star}_{A_2}(a_1) > P^{(2)}$. Hence,

$$\min\left\{P^{(1)}, P^{(2)}\right\} = P^{(2)}. \tag{78}$$

Then consider the case in which $\det \underline{\boldsymbol{w}} > 0$. From Lemma 9, if $\det \underline{\boldsymbol{w}} > 0$, then it holds that

$$P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_1}(a_2) > 0. \tag{79}$$

Under the assumption that the entries of the matrix $\underline{\boldsymbol{u}}$ in (1) satisfy (12), from (72) and (79), it holds that $P^{(2)} > P^{\star}_{A_2}(a_1)$ and $P^{\star}_{A_2}(a_1) > P^{(1)}$. Hence,

$$\min\left\{P^{(1)}, P^{(2)}\right\} = P^{(1)}. \tag{80}$$

Finally, consider the case in which $\det \underline{\boldsymbol{w}} = 0$. From Lemma 9, if $\det \underline{\boldsymbol{w}} = 0$, then it holds that

$$P_{\tilde{A}_2|A_2=a_2}(a_2) - P_{\tilde{A}_2|A_2=a_1}(a_2) = 0. \tag{81}$$

Under the assumption that the entries of the matrix $\underline{\boldsymbol{u}}$ in (1) satisfy (12), from (72) and (81), it holds that $P^{(2)} = P^{\star}_{A_2}(a_1)$ and $P^{\star}_{A_2}(a_1) = P^{(1)}$. Hence,

$$\min\left\{P^{(1)}, P^{(2)}\right\} = P^{(1)} = P^{(2)} = P^{\star}_{A_2}(a_1), \tag{82}$$

which completes the proof. ∎

Lemma 15 *Assume that the entries of the matrix $\underline{\boldsymbol{u}}$ in* (1) *satisfy* (12)*. If $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$, then it holds that for all $i \in \{1,2\}$ and for all $P \in \Delta(\mathcal{A}_2)$,*

$$\mathrm{BR}_{1,i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } P(a_1) > P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } P(a_1) < P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0,1]\}, & \text{if } P(a_1) = P^{(i)}, \end{cases} \tag{83}$$

where $P^{(i)}$ is in (18)*. Otherwise, if $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0$, then it holds that for all $i \in \{1,2\}$ and for all $P \in \Delta(\mathcal{A}_2)$,*

$$\mathrm{BR}_{1,i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } P(a_1) < P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } P(a_1) > P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0,1]\}, & \text{if } P(a_1) = P^{(i)}. \end{cases} \tag{84}$$

Proof: Note that if the entries of the matrix $\underline{\boldsymbol{u}}$ in (1) satisfy (12), then $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \neq 0$. Furthermore, for all $i \in \{1, 2\}$, the real s_i in (25) satisfies

$$\begin{aligned} s_i =&(u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}) \\ &\Big(P^{\star}_{A_2}(a_2)P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) - P^{\star}_{A_2}(a_1)P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i)\Big), \end{aligned} \tag{85}$$

where (85) follows from Lemma 1. For all $i \in \{1, 2\}$, let B_i be the following constant

$$B_i = P^{\star}_{A_2}(a_2)P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) - P^{\star}_{A_2}(a_1)P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i) \tag{86}$$

$$= P^{\star}_{A_2}(a_2)P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) - P^{\star}_{A_2}(a_1)\Big(1 - P(a_1)\Big)P_{\tilde{A}_2|A_2=a_2}(a_i) \tag{87}$$

$$\begin{aligned} &= P(a_1)\Big(P^{\star}_{A_2}(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)\Big) \\ &-P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i). \end{aligned} \tag{88}$$

For all $i \in \{1, 2\}$, plugging B_i into (85) yields

$$s_i = (u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2})\, B_i. \tag{89}$$

First, consider the case in which $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$. If $P(a_1)$ in (88) satisfies

$$P(a_1) > \frac{P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P^{\star}_{A_2}(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \tag{90}$$

$$= P^{(i)}, \tag{91}$$

then, it holds that $B_i > 0$. The equality in (91) follows from Lemma 12. Under the assumptions that $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ and (91), it holds that $s_i > 0$, which from Lemma 3, further implies that

$$\mathrm{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \tag{92}$$

If $P(a_1)$ in (88) satisfies

$$P(a_1) < \frac{P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P^{\star}_{A_2}(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \tag{93}$$

$$= P^{(i)}, \tag{94}$$

then, it holds that $B_i < 0$. The equality in (94) follows from Lemma 12. Under the assumptions that $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ and (94), it holds that $s_i < 0$, which from Lemma 3, further implies that

$$\mathrm{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \tag{95}$$

If $P(a_1)$ in (88) satisfies

$$P(a_1) = \frac{P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)}{P^{\star}_{A_2}(a_2)P_{\tilde{A}_2|A_2=a_1}(a_i) + P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_i)} \tag{96}$$

$$= P^{(i)}, \tag{97}$$

then, it holds that $B_i = 0$. The equality in (97) follows from Lemma 12. Under the assumptions that $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$ and (97), it holds that $s_i = 0$, which from Lemma 3, further implies that

$$\mathrm{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0,1]\}. \tag{98}$$

This proves the equality in (83).

The proof is completed by noticing that the equality in (84) is proved by following the same steps as before.. ■

Lemma 16 *Assume that the entries of the matrix* $\underline{\boldsymbol{u}}$ *in* (1) *satisfy* (12) *and* $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$. *For all* $P \in \Delta(\mathcal{A}_2)$ *such that* $P(a_1) \geqslant \max\{P^{(1)}, P^{(2)}\}$, *with* $P^{(1)}$ *and* $P^{(2)}$ *in* (18), *it holds that*

$$\hat{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2). \tag{99}$$

If $\det \underline{\boldsymbol{w}} > 0$, *with* $\underline{\boldsymbol{w}}$ *in* (36), *then for all* $P \in \Delta(\mathcal{A}_2)$ *such that* $P^{(1)} < P(a_1) < P^{(2)}$, *it holds that*

$$\begin{aligned}\hat{v}(P) = &\left(u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2)\right)P(a_1)\\ &+\left(u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2)\right)P(a_2).\end{aligned} \tag{100}$$

If $\det \underline{\boldsymbol{w}} \leqslant 0$, *then for all* $P \in \Delta(\mathcal{A}_2)$ *such that* $P^{(2)} < P(a_1) < P^{(1)}$, *it holds that*

$$\begin{aligned}\hat{v}(P) = &\left(u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1)\right)P(a_1)\\ &+\left(u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1)\right)P(a_2).\end{aligned} \tag{101}$$

Finally, for all $P \in \Delta(\mathcal{A}_2)$ *such that* $P(a_1) \leqslant \min\{P^{(1)}, P^{(2)}\}$, *it holds that*

$$\hat{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \tag{102}$$

Proof: For all probability measures $P \in \Delta(\mathcal{A}_2)$ such that $P(a_1) \geq \max\{P^{(1)}, P^{(2)}\}$, from Lemma 15, it holds that, for all $i \in \{1, 2\}$,

$$\mathrm{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \tag{103}$$

Furthermore, from (8), it holds that

$$\hat{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2), \tag{104}$$

which proves the equality in (99).

If $\det \underline{\boldsymbol{w}} > 0$, from Lemma 14, it holds that $P^{(1)} < P^{(2)}$. For all probability measures $P \in \Delta(\mathcal{A}_2)$ such that $P^{(1)} < P(a_1) < P^{(2)}$, from Lemma 15, it holds that

$$\mathrm{BR}_{1,1}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\} \quad \text{and} \tag{105}$$

$$\mathrm{BR}_{1,2}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \tag{106}$$

From (8), it holds that

$$\hat{v}(P)= \left(u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1)+u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2)\right)P(a_1) + \left(u_{1,2}(a_2)P_{\tilde{A}_2|A_2=a_2}(a_1)+u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2)\right)P(a_2), \tag{107}$$

which proves the equality in (100).

If $\det \underline{\boldsymbol{w}} \leq 0$, from Lemma 14, it holds that $P^{(1)} \geq P^{(2)}$. For all probability measures $P \in \Delta(\mathcal{A}_2)$ such that $P^{(1)} > P(a_1) > P^{(2)}$, from Lemma 15, it holds that

$$\mathrm{BR}_{1,1}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\} \quad \text{and} \tag{108}$$
$$\mathrm{BR}_{1,2}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}. \tag{109}$$

From (8), it holds that

$$\hat{v}(P) = \left(u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2)+u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1)\right)P(a_1) + \left(u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2)+u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1)\right)Pa_2), \tag{110}$$

which proves the equality in (101).

For all probability measures $P \in \Delta(\mathcal{A}_2)$ such that $P(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$, from Lemma 15, it holds that, for all $i \in \{1, 2\}$,

$$\mathrm{BR}_{1,i}(P) = \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}. \tag{111}$$

From (8), it holds that

$$\hat{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2), \tag{112}$$

which proves the equality in (102), and completes the proof. ∎

Lemma 17 *Assume that the entries of the matrix* $\underline{\boldsymbol{u}}$ *in* (1) *satisfy* (12) *and* $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leqslant 0$. *For all* $P \in \Delta(\mathcal{A}_2)$ *such that* $P(a_1) \leqslant \min\{P^{(1)}, P^{(2)}\}$, *with* $P^{(1)}$ *and* $P^{(2)}$ *in* (18), *it holds that*

$$\hat{v}\,(P)=u_{1,1}P(a_1) + u_{1,2}P(a_2). \tag{113}$$

If $\det \underline{\boldsymbol{w}} > 0$, *with* $\underline{\boldsymbol{w}}$ *in* (36), *then for all* $P \in \Delta(\mathcal{A}_2)$ *such that* $P^{(1)} < P(a_1) < P^{(2)}$ *, it holds that*

$$\hat{v}\,(P)= \left(u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2)+u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1)\right)P(a_1) + \left(u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2)+u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1)\right)P(a_2). \tag{114}$$

If $\det \underline{\boldsymbol{w}} \leqslant 0$, *then for all* $P \in \Delta(\mathcal{A}_2)$ *such that* $P^{(2)} < P(a_1) < P^{(1)}$ *, it holds that*

$$\hat{v}\,(P)= \left(u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1)+u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2)\right)P(a_1) + \left(u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1)+u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2)\right)P(a_2). \tag{115}$$

Finally, for all $P \in \Delta(\mathcal{A}_2)$ *such that* $P(a_1) \geqslant \max\{P^{(1)}, P^{(2)}\}$*, it holds that*

$$\hat{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2). \tag{116}$$

Proof: The proof follows the same steps of the proof of Lemma 16. ■

Lemma 18 *Assume that the entries of the matrix* $\underline{\boldsymbol{u}}$ *in* (1) *satisfy* (12)*. For all tuples* $(P, Q) \in \Delta(\mathcal{A}_2) \times \Delta(\mathcal{A}_2)$*, if* $0 \leq P(a_1) < Q(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$*, then it holds that*

$$\hat{v}(P) > \hat{v}(Q). \tag{117}$$

Alternatively, if $\max\{P^{(1)}, P^{(2)}\} \leq P(a_1) < Q(a_1) \leq 1$*, then it holds that*

$$\hat{v}(P) < \hat{v}(Q). \tag{118}$$

Proof: Two cases are considered. First, the case in which $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$; Second, the case in which $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0$.

Consider the case in which $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$. From Lemma 1, if the entries of the matrix $\underline{\boldsymbol{u}}$ in (1) satisfy (12), then one of the following conditions holds:

$$u_{1,1} - u_{1,2} > 0 \quad \text{and} \quad u_{2,1} - u_{2,2} < 0, \qquad \text{or} \tag{119}$$

$$u_{1,1} - u_{1,2} < 0 \quad \text{and} \quad u_{2,1} - u_{2,2} > 0. \tag{120}$$

Nonetheless, only the first condition yields $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$. Hence, from Lemma 16, if $0 \leq P(a_1) < Q(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$, then it holds that

$$\hat{v}(P) > \hat{v}(Q); \tag{121}$$

and if $\max\{P^{(1)}, P^{(2)}\} \leq P(a_1) < Q(a_1) \leq 1$, then it holds that

$$\hat{v}(P) < \hat{v}(Q). \tag{122}$$

Alternatively, consider the case in which $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0$. From Lemma 1, if the entries of the matrix $\underline{\boldsymbol{u}}$ in (1) satisfy (12), then only the case in which

$$u_{1,1} - u_{1,2} < 0 \quad \text{and} \quad u_{2,1} - u_{2,2} > 0. \tag{123}$$

yields $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0$. Hence, from Lemma 16, if $0 \leq P(a_1) < Q(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$, then it holds that

$$\hat{v}(P) > \hat{v}(Q); \tag{124}$$

and if $\max\{P^{(1)}, P^{(2)}\} \leq P(a_1) < Q(a_1) \leq 1$, then it holds that

$$\hat{v}(P) < \hat{v}(Q). \tag{125}$$

This completes the proof. ■

B.2 The Proof

Two cases are considered: First, the case in which the payoff matrix $\underline{\boldsymbol{u}}$ satisfies (12); Second, the case in which the payoff matrix $\underline{\boldsymbol{u}}$ satisfies (15).

Consider the case in which the payoff matrix $\underline{\boldsymbol{u}}$ satisfies (12). From Lemma 1, there is a unique NE in strictly mixed strategies in the game $\mathscr{G}(\underline{\boldsymbol{u}})$. From Lemma 18, it holds that for all $P \in \Delta(\mathcal{A}_2)$, with $0 \leq P(a_1) \leq \min\{P^{(1)}, P^{(2)}\}$, it holds that

$$\hat{v}(P) \geq f\left(\min\{P^{(1)}, P^{(2)}\}\right), \tag{126}$$

where the function f is in (32); and equality holds if $P(a_1) = \min\{P^{(1)}, P^{(2)}\}$. Alternatively, for all $P \in \Delta(\mathcal{A}_2)$ such that $\max\{P^{(1)}, P^{(2)}\} \leq P(a_1) \leq 1$,

$$\hat{v}(P) \geq f\left(\max\{P^{(1)}, P^{(2)}\}\right), \tag{127}$$

where equality holds if $P(a_1) = \max\{P^{(1)}, P^{(2)}\}$. Given the fact that the function f in (32) is continuous piecewise linear, from (34) , it holds that

$$\hat{v}\left(P_{A_2}^{\dagger}\right) = \min_{P_{A_2} \in \Delta(\mathcal{A}_2)} \hat{v}(P_{A_2}) = \min\left\{\hat{v}\left(P_1\right), \hat{v}\left(P_2\right)\right\}, \tag{128}$$

in which for all $i \in \{1, 2\}$, the probability measures $P_i \in \Delta(\mathcal{A}_2)$ are such that $P_i(a_1) = P^{(i)}$, with $P^{(i)}$ in (18).

Alternatively, consider the case in which the payoff matrix $\underline{\boldsymbol{u}}$ satisfies (15). From Lemma 1, there is a unique NE in pure strategies or infinitely many NE in the game $\mathscr{G}(\underline{\boldsymbol{u}})$. If there is a unique NE in pure strategies, from Lemma 11, it holds that

$$\begin{aligned} v\left(P_{A_1|\tilde{A}_2}^{\dagger}, P_{A_2}^{\dagger}\right) &= u\left(P_{A_1}^{\star}, P_{A_2}^{\star}\right) && (129)\\ &= \min\left\{\max\left\{u_{1,1}, u_{2,1}\right\}, \max\left\{u_{1,2}, u_{2,2}\right\}\right\}, && (130) \end{aligned}$$

where the equality in (130) follows from (16).

If there are infinitely many NE, then there exists a probability measure $P_{A_2}^{\star} \in \Delta(\mathcal{A}_2)$ such that

$$P_{A_2}^{\star} \in \left\{P \in \Delta(\mathcal{A}_2) : P(a_1) \in \{0, 1\}\right\}. \tag{131}$$

Hence, it holds that for all $(i, j) \in \{1, 2\}^2$,

$$P_{A_2|\tilde{A}_2=a_i}(a_j) = \frac{P_{\tilde{A}_2|A_2=a_j}(a_i)\, P_{A_2}^{\star}(a_j)}{\sum_{\ell \in \{1,2\}} P_{\tilde{A}_2|A_2=a_\ell}(a_i)\, P_{A_2}^{\star}(a_\ell)} = P_{A_2}^{\star}(a_j). \tag{132}$$

Note that the equalities in (132) imply that

$$P_{A_2|\tilde{A}_2=a_1} = P_{A_2|\tilde{A}_2=a_2} = P_{A_2}^{\star}. \tag{133}$$

Hence, from Lemma 11, it holds that

$$v\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right) = u\left(P^{\star}_{A_1}, P^{\star}_{A_2}\right) \tag{134}$$
$$= \min\left\{\max\left\{u_{1,1}, u_{2,1}\right\}, \max\left\{u_{1,2}, u_{2,2}\right\}\right\}, \tag{135}$$

where the equality in (135) follows from (16).

This completes the proof.

C Proof of Lemma 2

From (7) and Lemma 10, it holds that for all probability measures $P \in \Delta(\mathcal{A}_2)$,

$$\begin{aligned}
&\mathrm{BR}_1(P)\\
&= \arg\max_{(Q_1,Q_2)\in\Delta(\mathcal{A}_1)\times\Delta(\mathcal{A}_1)} v(Q_1,Q_2,P)\\
&= \arg\max_{Q\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1)\\ Q(a_2)\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(1)} \begin{pmatrix} P(a_1)\\ P(a_2)\end{pmatrix} \times \arg\max_{Q\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1)\\ Q(a_2)\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(2)} \begin{pmatrix} P(a_1)\\ P(a_2)\end{pmatrix}\\
&= \mathrm{BR}_{1,1}(P) \times \mathrm{BR}_{1,2}(P),
\end{aligned}$$

which completes the proof.

D Proof of Lemma 3

From Lemma 2, it holds that for all $i \in \{1,2\}$ and for all probability measures $P \in \Delta(\mathcal{A}_2)$,

$$\mathrm{BR}_{1,i}(P) = \arg\max_{Q\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1)\\ Q(a_2)\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix} P(a_1)\\ P(a_2)\end{pmatrix}. \tag{136}$$

Moreover,

$$\begin{pmatrix} Q(a_1)\\ Q(a_2)\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix} P(a_1)\\ P(a_2)\end{pmatrix}$$
$$= \begin{pmatrix} Q(a_1)\\ Q(a_2)\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}} \begin{pmatrix} P_{\tilde{A}_2|A_2=a_1}(a_i) & 0\\ 0 & P_{\tilde{A}_2|A_2=a_2}(a_i)\end{pmatrix} \begin{pmatrix} P(a_1)\\ P(a_2)\end{pmatrix} \tag{137}$$
$$= \begin{pmatrix} Q(a_1)\\ 1-Q(a_1)\end{pmatrix}^{\mathsf{T}} \begin{pmatrix} u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i)\\ u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i)\end{pmatrix} \tag{138}$$
$$= \Big(u_{1,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{1,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i) - u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) - u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i)\Big) Q(a_1) \tag{139}$$
$$+ u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i) \tag{140}$$
$$= s_i Q(a_1) + u_{2,1}P(a_1)P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2}P(a_2)P_{\tilde{A}_2|A_2=a_2}(a_i), \tag{141}$$

where s_i is in (25).

Note that if $s_i > 0$ in (141), it holds that

$$\begin{aligned}&\begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}\\ \leqslant & s_i + u_{2,1} P(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2} P(a_2) P_{\tilde{A}_2|A_2=a_2}(a_i),\end{aligned} \tag{142}$$

where the equality holds only if $Q(a_1) = 1$.

Alternatively, if $s_i = 0$ in (141), it holds that for all $Q \in \Delta(\mathcal{A}_1)$,

$$\begin{aligned}&\begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}\\ = & u_{2,1} P(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2} P(a_2) P_{\tilde{A}_2|A_2=a_2}(a_i).\end{aligned} \tag{143}$$

Finally, if $s_i < 0$ in (141), it holds that

$$\begin{aligned}&\begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}\\ \leqslant & u_{2,1} P(a_1) P_{\tilde{A}_2|A_2=a_1}(a_i) + u_{2,2} P(a_2) P_{\tilde{A}_2|A_2=a_2}(a_i),\end{aligned} \tag{144}$$

where the equality holds only if $Q(a_1) = 0$. This completes the proof.

E Proof of Lemma 4

From Lemma 2, it holds that for all $i \in \{1,2\}$ and for all probability measures $P \in \Delta(\mathcal{A}_2)$,

$$\begin{aligned}
&\mathrm{BR}_{1,i}(P)\\
=&\arg\max_{Q\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}\\
=&\arg\max_{Q\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}} \begin{pmatrix} P_{\tilde{A}_2|A_2=a_1}(a_i) & 0 \\ 0 & P_{\tilde{A}_2|A_2=a_2}(a_i) \end{pmatrix} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}\\
=&\arg\max_{Q\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}} \begin{pmatrix} P_{A_2,\tilde{A}_2}(a_1,a_i) \\ P_{A_2,\tilde{A}_2}(a_2,a_i) \end{pmatrix}\\
=&\arg\max_{Q\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_i}(a_1) \\ P_{A_2|\tilde{A}_2=a_i}(a_2) \end{pmatrix} \left(\sum_{\ell\in\{1,2\}} P_{\tilde{A}_2|A_2=a_\ell}(a_i) P_{A_2}(a_\ell) \right)\\
=&\arg\max_{Q\in\Delta(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}} \begin{pmatrix} P_{A_2|\tilde{A}_2=a_i}(a_1) \\ P_{A_2|\tilde{A}_2=a_i}(a_2) \end{pmatrix},
\end{aligned}$$

where the measure $P_{A_2,\tilde{A}_2}$ satisfies for all $(\ell,k) \in \{1,2\}^2$,

$$P_{A_2,\tilde{A}_2}(a_\ell,a_k) = P_{\tilde{A}_2|A_2=a_\ell}(a_k) P(a_\ell). \tag{145}$$

This completes the proof.

F Proof of Lemma 5

The proof of the first inequality in (28) is as follows. For all probability measures $P \in \Delta(\mathcal{A}_2)$, it holds that

$$\begin{aligned}
\hat{u}(P) &= \max_{Q \in \Delta(\mathcal{A}_1)} u(Q, P) && (146)\\
&\geq \min_{P_0 \in \Delta(\mathcal{A}_2)} \max_{Q \in \Delta(\mathcal{A}_1)} u(Q, P_0) && (147)\\
&= u\left(P^{\star}_{A_1}, P^{\star}_{A_2}\right), && (148)
\end{aligned}$$

which follows from the minmax theorem.

The proof of the second inequality in (28) is as follows. Note that from (11) and (5), it holds that for all $P \in \Delta(\mathcal{A}_2)$,

$$\begin{aligned}
\hat{v}(P) &= \max_{(Q_1,Q_2) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_1)} v\left(Q_1, Q_2, P\right) && (149)\\
&\geq \max_{Q_3 \in \Delta(\mathcal{A}_1)} v\left(Q_3, Q_3, P\right) && (150)\\
&= \max_{Q_3 \in \Delta(\mathcal{A}_1)} u\left(Q_3, P\right) && (151)\\
&= \hat{u}(P). && (152)
\end{aligned}$$

The proof of the third inequality in (28) is as follows. From Lemma 10, for all probability measures $P \in \Delta(\mathcal{A}_2)$, the function $\hat{v}$ satisfies

$$\begin{aligned}
&\hat{v}\left(P\right)\\
&= \max_{Q_1 \in \Delta(\mathcal{A}_2)} \begin{pmatrix} Q_1\left(a_1\right) \\ Q_1\left(a_2\right) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(1)} \begin{pmatrix} P\left(a_1\right) \\ P\left(a_2\right) \end{pmatrix} + \max_{Q_2 \in \Delta(\mathcal{A}_2)} \begin{pmatrix} Q_2\left(a_1\right) \\ Q_2\left(a_2\right) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(2)} \begin{pmatrix} P\left(a_1\right) \\ P\left(a_2\right) \end{pmatrix}\\
&= \max_{Q_1 \in \Delta(\mathcal{A}_2)} \begin{pmatrix} Q_1\left(a_1\right) \\ Q_1\left(a_2\right) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_1) P(a_1) + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_1) P(a_2) \\ u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_1) P(a_1) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_1) P(a_2) \end{pmatrix}\\
&\quad + \max_{Q_2 \in \Delta(\mathcal{A}_2)} \begin{pmatrix} Q_2\left(a_1\right) \\ Q_2\left(a_2\right) \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_2) P(a_1) + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_2) P(a_2) \\ u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_2) P(a_1) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_2) P(a_2) \end{pmatrix}\\
&= \max \left\{ u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_1) P(a_1) + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_1) P(a_2), \right.\\
&\qquad \left. u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_1) P(a_1) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_1) P(a_2) \right\}\\
&\quad + \max \left\{ u_{1,1} P_{\tilde{A}_2|A_2=a_1}(a_2) P(a_1) + u_{1,2} P_{\tilde{A}_2|A_2=a_2}(a_2) P(a_2), \right.\\
&\qquad \left. u_{2,1} P_{\tilde{A}_2|A_2=a_1}(a_2) P(a_1) + u_{2,2} P_{\tilde{A}_2|A_2=a_2}(a_2) P(a_2) \right\}. && (153)
\end{aligned}$$

Note that the equality in (153) can be written for all $P \in \Delta(\mathcal{A}_2)$ as follows:

$$\begin{aligned}\hat{v}\,(P)=&\max\Big\{u_{1,1}P(a_1)+u_{1,2}P(a_2),\\ &\Big(u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1)+u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2)\Big)P(a_1)\\ &+\Big(u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1)+u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2)\Big)P(a_2),\\ &\Big(u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2)+u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1)\Big)P(a_1)\\ &+\Big(u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2)+u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1)\Big)P(a_2),\\ &u_{2,1}P(a_1)+u_{2,2}P(a_2)\Big\} \qquad (154)\\ \leqslant&\max\Big\{u_{1,1}P(a_1)+u_{1,2}P(a_2),\\ &\max\{u_{1,1},u_{2,1}\}P(a_1)+\max\{u_{1,2},u_{2,2}\}P(a_2),\\ &u_{2,1}P(a_1)+u_{2,2}P(a_2)\Big\}\\ =&\max\{u_{1,1},u_{2,1}\}P(a_1)+\max\{u_{1,2},u_{2,2}\}P(a_2), \qquad (155)\end{aligned}$$

which completes the proof.

G Proof of Lemma 6

The proof of the first inequality is as follows. From Lemma 5, it follows that for all $P \in \Delta(\mathcal{A}_2)$, $u(P^\star_{A_1}, P^\star_{A_2}) \leqslant \hat{v}(P)$. Hence, given that $\hat{v}(P^\dagger_{A_2}) = v\left(P^\dagger_{A_1|\tilde{A}_2}, P^\dagger_{A_2}\right)$, it follows that $u(P^\star_{A_1}, P^\star_{A_2}) \leqslant v\left(P^\dagger_{A_1|\tilde{A}_2}, P^\dagger_{A_2}\right)$. The proof of the second inequality is as follows. From Lemma 1, it follows that if the matrix $\underline{\boldsymbol{u}}$ in (1) satisfies (15), then from Theorem 2, it holds that $v\left(P^\dagger_{A_1|\tilde{A}_2}, P^\dagger_{A_2}\right) = \min_{j\in\{1,2\}} \max_{i\in\{1,2\}} u_{i,j}$.

Alternatively, if the matrix $\underline{\boldsymbol{u}}$ in (1) satisfies (12), it holds that

$$\begin{aligned} v\left(P^\dagger_{A_1|\tilde{A}_2}, P^\dagger_{A_2}\right)=&\hat{v}\left(P^\dagger_{A_2}\right) && (156)\\ =&\min\{\hat{v}\,(P_1)\,,\hat{v}\,(P_2)\} && (157)\\ \leq&\min\{\hat{v}\,(P_0)\,,\hat{v}\,(P_3)\} && (158)\\ =&\min\{\max\{u_{2,1},u_{2,2}\},\max\{u_{1,1},u_{1,2}\}\} && (159)\\ =&\min_{j\in\{1,2\}}\max_{i\in\{1,2\}} u_{i,j}, && (160)\end{aligned}$$

where the inequality in (157) follows from Theorem 2; the inequality in (158) follows from Lemma 18, in which $P_0 \in \Delta(\mathcal{A}_2)$ such that $P_0(a_1) = 0$ and $P_3 \in \Delta(\mathcal{A}_2)$ such that $P_3(a_1) = 1$; and the inequality in (159) follows from the definition of $\hat{v}$ in (8).

This completes the proof.

H Proof of Lemma 7

The proof is divided into two parts. The first part proves a preliminary result; the second part proves the lemma using the preliminary result.

H.1 Preliminary Result

Lemma 19 *For all $P \in \triangle(\mathcal{A}_2)$ it holds that*

$$I\left(P_{\tilde{A}_2|A_2}; P\right)=0, \tag{161}$$

if and only if $\det \underline{\boldsymbol{w}} = 0$*, with the matrix* $\underline{\boldsymbol{w}}$ *in* (36)*.*

Proof: The proof is divided into two parts. The first part proves that if the equality in (161) holds for all $P \in \triangle(\mathcal{A}_2)$, then $\det \underline{\boldsymbol{w}} = 0$. The second part proves the converse.

The first part of the proof is as follows. If the equality in (161) holds for all $P \in \triangle(\mathcal{A}_2)$, it follows from [27, pp. 28] that the channel input and channel output are independent. If such is the case, from (36), it holds that

$$\begin{aligned}
&\det \underline{\boldsymbol{w}}\\
&= (1-P_{\tilde{A}_2|A_2=a_1}(a_2))(1-P_{\tilde{A}_2|A_2=a_2}(a_1))-P_{\tilde{A}_2|A_2=a_2}(a_1)P_{\tilde{A}_2|A_2=a_1}(a_2) &(162)\\
&= 1 - P_{\tilde{A}_2|A_2=a_1}(a_2) - P_{\tilde{A}_2|A_2=a_2}(a_1) &(163)\\
&= P_{\tilde{A}_2|A_2=a_1}(a_1) - P_{\tilde{A}_2|A_2=a_2}(a_1) &(164)\\
&= P_{\tilde{A}_2}(a_1) - P_{\tilde{A}_2}(a_1) &(165)\\
&= 0, &(166)
\end{aligned}$$

where (164) follows from the fact that $P_{\tilde{A}_2|A_2=a_1}(a_1)+P_{\tilde{A}_2|A_2=a_1}(a_2) = 1$; (165) follows from the fact that $P_{\tilde{A}_2|A_2=a_1}(a_1) = P_{\tilde{A}_2|A_2=a_2}(a_1) = P_{\tilde{A}_2}(a_1)$ due to independence. Then the determinant of $\underline{\boldsymbol{w}}$ is equal to 0 when the channel input and the channel output are independent. This completes the first part of the proof.

The second part of the proof is as follows. If the determinant $|\underline{\boldsymbol{w}}|$ equals to 0, from (164), it holds that

$$P_{\tilde{A}_2|A_2=a_1}(a_1) = P_{\tilde{A}_2|A_2=a_2}(a_1). \tag{167}$$

As a result, it holds that for all $j \in \{0, 1\}$ and for all $P \in \triangle(\mathcal{A}_2)$,

$$\begin{aligned}
P_{\tilde{A}_2}(a_1)&=\sum_{i=1}^{2} P_{\tilde{A}_2|A_2=a_i}(a_1)P(a_i) &(168)\\
&= P_{\tilde{A}_2|A_2=a_j}(a_1)\sum_{i=1}^{2} P(a_i) &(169)\\
&= P_{\tilde{A}_2|A_2=a_j}(a_1), &(170)
\end{aligned}$$

where (169) follows from (167). Similarly, it holds that for all $j \in \{0, 1\}$

$$P_{\tilde{A}_2}(a_2) = P_{\tilde{A}_2|A_2=a_j}(a_2). \tag{171}$$

From (170) and (171), it follows that the channel input and channel output are independent. This completes the second part of the proof.

■

H.2 The Proof

The proof considers two cases. First, the case in which the entries in the payoff matrix $\underline{\boldsymbol{u}}$ satisfy (12). Second, the case in which the entries in the payoff matrix $\underline{\boldsymbol{u}}$ satisfy (15).

The proof of the first case is as follows. If the entries in the payoff matrix $\underline{\boldsymbol{u}}$ satisfy (12), from Lemma 1, it holds that

$$P^{\star}_{A_2}(a_1) \in (0, 1). \tag{172}$$

Hence, from Lemma 11, the equality in (30) holds if and only if $\det \underline{\boldsymbol{w}} = 0$. Note that from Lemma 19, $\det \underline{\boldsymbol{w}} = 0$ holds if and only if for all $P \in \triangle(\mathcal{A}_2)$, $I\left(P_{\tilde{A}_2|A_2}; P\right) = 0$. Then, the equality in (30) holds if and only if for all $P \in \triangle(\mathcal{A}_2)$, $I\left(P_{\tilde{A}_2|A_2}; P\right) = 0$.

The proof of the second case is as follows. From Theorem 2, it follows that

$$\begin{aligned} \hat{v}\left(P^{\dagger}_{A_2}\right) &= \min\{\max\{u_{1,1}, u_{2,1}\}, \max\{u_{1,2}, u_{2,2}\}\}, && (173)\\ &= u(P^{\star}_{A_1}, P^{\star}_{A_2}), && (174) \end{aligned}$$

where the equality in (174) follows from (16). Thus, there is nothing to prove in this case. This completes the proof.

I Proof of Lemma 8

The proof is divided into two parts. The first part proves a preliminary result; the second part proves the lemma using the preliminary result

I.1 Preliminary Result

Lemma 20 *The matrix $\underline{\boldsymbol{w}}$ in* (36) *satisfies $|\det \underline{\boldsymbol{w}}| = 1$ if and only if for all $P \in \triangle(\mathcal{A}_2)$ it holds that*

$$I\left(P_{\tilde{A}_2|A_2}; P\right) = H(P) = H\left(P_{\tilde{A}_2}\right), \tag{175}$$

where $P_{\tilde{A}_2}(a_i) = \sum_{\ell \in \{1,2\}} P_{\tilde{A}_2|A_2=a_\ell}(a_i) P(a_\ell)$, with $i \in \{1, 2\}$.

Proof: From [27, Theorem 2.4.1], the equalities $I\left(P_{\tilde{A}_2|A_2}; P\right) = H(P)$ and $I\left(P_{\tilde{A}_2|A_2}; P\right) = H\left(P_{\tilde{A}_2}\right)$ simultaneously hold for all $P \in \triangle(\mathcal{A}_2)$ if and only if for all $P \in \triangle(\mathcal{A}_2)$

$$\sum_{a \in \tilde{A}_2} P_{\tilde{A}_2}(a) H\left(P_{A_2|\tilde{A}_2=a}\right) = \sum_{a \in \mathcal{A}_2} P(a) H\left(P_{\tilde{A}_2|A_2=a}\right) = 0, \tag{176}$$

where for all $i \in \{1,2\}$, the measure $P_{A_2|\tilde{A}_2=a_i}$ is in (27). From [27, (2.12)], the equalities in (176) can be rewritten as follows:

$$-\sum_{i \in \{1,2\}} P_{A_2}(a_i) \sum_{j \in \{1,2\}} P_{\tilde{A}_2|A_2=a_i}(a_j) \log P_{\tilde{A}_2|A_2=a_i}(a_j) = 0 \tag{177}$$

and

$$-\sum_{i \in \{1,2\}} P_{\tilde{A}_2}(a_i) \sum_{j \in \{1,2\}} P_{A_2|\tilde{A}_2=a_i}(a_j) \log P_{A_2|\tilde{A}_2=a_i}(a_j) = 0. \tag{178}$$

The left-hand side of the equalities in (177) and (178) are sums of nonnegative terms. Hence, the equalities in (177) and (178) hold for all $P \in \triangle(\mathcal{A}_2)$ if and only if for all $(i,j) \in \{1,2\}^2$,

$$P_{\tilde{A}_2|A_2=a_i}(a_j) \log P_{\tilde{A}_2|A_2=a_i}(a_j) = 0, \text{ and} \tag{179}$$

$$P_{A_2|\tilde{A}_2=a_i}(a_j) \log P_{A_2|\tilde{A}_2=a_i}(a_j) = 0, \tag{180}$$

which holds if and only if for all $(i,j) \in \{1,2\}^2$,

$$P_{\tilde{A}_2|A_2=a_i}(a_j) \in \{0,1\} \text{ and} \tag{181a}$$

$$P_{A_2|\tilde{A}_2=a_i}(a_j) \in \{0,1\}. \tag{181b}$$

Note that here the convention $0 \log 0 = 0$ is used. Note also that for all $(i,j) \in \{1,2\}^2$, the probability $P_{A_2|\tilde{A}_2=a_i}(a_j)$ satisfies (27). Thus, $P_{\tilde{A}_2|A_2=a_i}(a_j) \in \{0,1\}$, if and only if $P_{A_2|\tilde{A}_2=a_j}(a_i) = P_{\tilde{A}_2|A_2=a_i}(a_j)$. This implies that the equalities in (177) and (178) imply each other.

Finally, the equality $H(P) = H\left(P_{\tilde{A}_2}\right)$ holds for all $P \in \triangle(\mathcal{A}_2)$, subject to (181), if and only if one of the following statements is true:

$$P_{\tilde{A}_2|A_2=a_1}(a_1) = 1 \text{ and } P_{\tilde{A}_2|A_2=a_2}(a_2) = 1, \tag{182}$$

or

$$P_{\tilde{A}_2|A_2=a_1}(a_1) = 0 \text{ and } P_{\tilde{A}_2|A_2=a_2}(a_2) = 0. \tag{183}$$

Note that the equalities in (182) and (183) imply that $\det \underline{\boldsymbol{w}} = 1$ and $\det \underline{\boldsymbol{w}} = -1$, respectively. As a result, if the equality in (175) holds for all $P \in \triangle(\mathcal{A}_2)$, then $|\det \underline{\boldsymbol{w}}| = 1$.

The converse is as follows. If $|\det \underline{\boldsymbol{w}}| = 1$, then either (182) or (183) holds, which implies both (177) and (178). Hence, if $|\det \underline{\boldsymbol{w}}| = 1$, the equality in (175) holds. This completes the proof.

■

I.2 The Proof

The proof considers two cases. First, the case in which the entries in the payoff matrix $\underline{\boldsymbol{u}}$ satisfy (15). Second, the case in which the entries in the payoff matrix $\underline{\boldsymbol{u}}$ satisfy (12).

Consider the case in which the entries in the payoff matrix $\underline{\boldsymbol{u}}$ satisfy (15). From Theorem 2, the equality in (31) holds regardless of whether for all $P \in \triangle(\mathcal{A}_2)$, $I\left(P_{\tilde{A}_2|A_2}; P\right) = H(P) = H\left(P_{\tilde{A}_2}\right)$, which ends the proof of this case.

Alternatively consider the case in which the entries in the payoff matrix $\underline{\boldsymbol{u}}$ satisfy (12). From Lemma 20, if for all $P \in \triangle(\mathcal{A}_2)$, $I\left(P_{\tilde{A}_2|A_2}; P\right) = H(P) = H\left(P_{\tilde{A}_2}\right)$, it holds that either $\det \underline{\boldsymbol{w}} = 1$ or $\det \underline{\boldsymbol{w}} = -1$. Consider the case $\det \underline{\boldsymbol{w}} = 1$ first. From Lemma 9, it holds that

$$P_{\tilde{A}_2|A_2=a_1}(a_1) = 1 \text{ and } P_{\tilde{A}_2|A_2=a_2}(a_1) = 0, \tag{184}$$

which implies that $P^{(1)}$ and $P^{(2)}$ in (65) satisfy

$$P^{(1)} = \frac{P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_1)}{P^{\star}_{A_2}(a_2)P_{\tilde{A}_2|A_2=a_1}(a_1) + P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_1)} = 0 \quad \text{and} \tag{185}$$

$$P^{(2)} = \frac{P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_2)}{P^{\star}_{A_2}(a_2)P_{\tilde{A}_2|A_2=a_1}(a_2) + P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_2)} = 1. \tag{186}$$

Hence, from Theorem 2, it holds that

$$\hat{v}\left(P^{\dagger}_{A_2}\right) = \min\left\{\hat{v}(P_1), \hat{v}(P_2)\right\}, \tag{187}$$

$$= \min\{\max\{u_{1,1}, u_{2,1}\}, \max\{u_{1,2}, u_{2,2}\}\}, \tag{188}$$

where the equality in (188) follows from (154).

On the other hand, if $\det \underline{\boldsymbol{w}} = -1$, then it holds that

$$P_{\tilde{A}_2|A_2=a_1}(a_1) = 0 \text{ and } P_{\tilde{A}_2|A_2=a_2}(a_1) = 1, \tag{189}$$

which implies that $P^{(1)}$ and $P^{(2)}$ in (65) satisfy

$$P^{(1)} = \frac{P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_1)}{P^{\star}_{A_2}(a_2)P_{\tilde{A}_2|A_2=a_1}(a_1) + P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_1)} = 1 \quad \text{and} \tag{190}$$

$$P^{(2)} = \frac{P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_2)}{P^{\star}_{A_2}(a_2)P_{\tilde{A}_2|A_2=a_1}(a_2) + P^{\star}_{A_2}(a_1)P_{\tilde{A}_2|A_2=a_2}(a_2)} = 0. \tag{191}$$

Hence, from Theorem 2, it holds that

$$\hat{v}\left(P^{\dagger}_{A_2}\right) = \min\left\{\hat{v}(P_1), \hat{v}(P_2)\right\}, \tag{192}$$

$$= \min\{\max\{u_{1,1}, u_{2,1}\}, \max\{u_{1,2}, u_{2,2}\}\}, \tag{193}$$

where the equality in (193) follows from (154).

This completes the proof.

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