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Catalan’s constant is irrational

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Abstract

In mathematics, Catalan’s constant $G$ is defined by

$$G = \beta(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots,$$

where $\beta$ is the Dirichlet beta function.

Catalan’s constant has been called arguably the most basic constant whose irrationality and transcendence (though strongly suspected) remain unproven. In this paper we show that $G$ is indeed irrational.

Proof

Keeping in mind the Riemann series theorem (also called the Riemann rearrangement theorem), we have

$$\begin{array}{c|c}
\frac{1}{1^2} & - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots \\
- \frac{2}{3^2} & + \frac{2}{5^2} - \frac{2}{7^2} + \frac{2}{9^2} - \cdots \\
+ \frac{2}{7^2} & - \frac{2}{9^2} + \frac{2}{11^2} - \cdots \\
- \frac{2}{9^2} & + \frac{2}{11^2} - \cdots \\
+ \frac{2}{11^2} & - \cdots \\
\vdots & \vdots \\
\frac{1}{t} & - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \\
\end{array}$$

$$G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots,$$

$$2G = \frac{2}{1^2} - \frac{2}{3^2} + \frac{2}{5^2} - \frac{2}{7^2} + \frac{2}{9^2} - \cdots.$$

Notice that the Leibniz formula for $\pi$ states that

$$\frac{\pi}{4} = \beta(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

Moreover, it is easy to see that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ is conditionally convergent. On the another hand, $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is absolutely convergent and we are able to rearrange the terms as we want.

Let’s assume the contrary: $G$ is a rational number $\frac{a}{b}$, where $t$ is odd. Hence, we have

$$\begin{align*}
stG &= st \sum_{n=0, nt}^{\infty} \frac{(-1)^n}{(2n+1)^2} + st \sum_{m=0}^{\infty} \frac{(-1)^m + (t/2)}{t^2(2m+1)^2} \\
&= st \sum_{n=0, nt}^{\infty} \frac{(-1)^n}{(2n+1)^2} + ((-1)^{t/2})2^k G \sum_{m=0}^{\infty} \frac{((-1)^m)^t}{(2m+1)^2} \\
&= st \sum_{n=0, nt}^{\infty} \frac{(-1)^n}{(2n+1)^2} + ((-1)^{t/2})2^k G^2.
\end{align*}$$
In other words, we obtain the following quadratic equation for $G$:

$$G^2 - (-1)^{t/2} \frac{s t}{2^k} G + (-1)^{t/2} \frac{s t}{2^k} \sum_{n=0, n \not| t}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$ 

The last is equal to

$$G^2 - (-1)^{t/2} \frac{s t}{2^k} G + (-1)^{t/2} t^2 G \sum_{n=0, n \not| t}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$ 

Since $G \neq 0$, we have the next equation

$$G = (-1)^{t/2} \frac{s t}{2^k} - (-1)^{t/2} t^2 \sum_{n=0, n \not| t}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$ 

Indeed, we have

$$G = (-1)^{t/2} \frac{s t}{2^k} - (-1)^{t/2} t^2(G + \epsilon),$$

$$G = (-1)^{t/2} t^2 G - (-1)^{t/2} t^2(G + \epsilon),$$

where

$$\epsilon = - \sum_{m=0}^{\infty} \frac{(-1)^{mt+\lfloor t/2 \rfloor}}{t^2(2m+1)^2} = -(-1)^{\lfloor t/2 \rfloor} \frac{G}{t^2}.$$ 

According to the above, we consider the following quadratic equation for $t$:

$$G = (-1)^{t/2} \frac{s t}{2^k} - (-1)^{t/2} t^2(G + \epsilon),$$

$$t^2 - \frac{s}{2^k(G + \epsilon)} t + (-1)^{t/2} \frac{G}{(G + \epsilon)} = 0.$$ 

Since $\frac{s}{2^k(G + \epsilon)} > 0$ due to $t > 1$ ($G$ can not be $\frac{s}{2^k}$ for natural $s, k$: it goes around with the representation\n
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2},$$

we get

$$t = \frac{s}{2^k(G + \epsilon)}(1 \pm \sqrt{1 - \frac{4(-1)^{\lfloor t/2 \rfloor} G(G + \epsilon)^2 2^{2k}}{(G + \epsilon)s^2}}) =$$

$$= \frac{s}{2^k(G + \epsilon)}(1 \pm \sqrt{1 - \frac{(-1)^{\lfloor t/2 \rfloor} G(G + \epsilon)^2 2^{2k+2}}{s^2}}).$$ 

Using the Taylor series of $\sqrt{1 + x}$, we come to

$$t_+ \cong \frac{s}{2^k(G + \epsilon)} - \frac{(-1)^{\lfloor t/2 \rfloor} G 2^k}{s}, \quad t_- \cong \frac{(-1)^{\lfloor t/2 \rfloor} G 2^k}{s},$$

where $t_-$ is impossible as $G = \frac{s}{2^k}$ and $t > 1$.

Substituting $G = \frac{s}{2^k t_+}$, we derive

$$t_+ \cong \frac{s}{2^k(G + \epsilon)} - \frac{(-1)^{\lfloor t/2 \rfloor} G 2^k}{s} = \frac{s}{2^k(G + \epsilon)} - \frac{(-1)^{\lfloor t/2 \rfloor}}{t_+} = \frac{t_+ G}{(G + \epsilon)} - \frac{(-1)^{\lfloor t/2 \rfloor}}{t_+}.$$ 

According to the above, we consider the following quadratic equation for $t_+$:

$$t_+^2 - \frac{\epsilon}{(G + \epsilon)} + (-1)^{\lfloor t/2 \rfloor} \cong 0.$$ 

Substituting $\epsilon = -(-1)^{\lfloor t/2 \rfloor} \frac{G}{t_+^2}$, we derive

$$-\frac{G}{(G + \epsilon)} + 1 \cong 0.$$ 

So, on the one hand, $\epsilon$ can not be close to 0 with any accuracy (it is $1/t^2$), but, on the other hand, accuracy of $\cong$ in the Taylor expansion is $O(1/t^4)$. Note that $1/(1 \pm x)$ and $\sqrt{1 \pm x}$ are different as series. Hence, the last equation can not be fulfilled. Q.E.D.
Remark 1. There exists the following integration

\[ \int_0^\infty \frac{1}{1+x^2} \cos(kx)dx = \frac{\pi}{2} e^{-k}. \]

One way to see it is via the Fourier inversion theorem: we know that the Fourier transform of a function has a unique inverse. This carries over to the cosine transform as well. Moreover, the unique continuous function on the positive real axis with Fourier transform \( \frac{1}{1+x^2} \) is \( e^{-k} \).

Notice that if

\[ I_n = \int \frac{x^n}{1+x^2} dx, \]

then

\[ I_{n+2} + I_n = \frac{x^n}{n+1} + C. \]

Remark 2. Are all \( \{1, \pi \mid n \in \mathbb{N} \} \) linearly independent over \( \mathbb{Q} \), where “\( x \) is tetration?” Meaning none of exponents is an integer (we have not known that \( \pi^{1/2} \) (56 digits) is not an integer).

Moreover, at least one of \( e^x \) and \( e^{2x} \) must be transcendental due to W. D. Brownawell.

Remark 3. Is \( e + \pi \) irrational?

Note that \((x - e)(x - \pi) = x^2 - (e + \pi)x + e\pi\). So, at least one of the coefficients \( e + \pi, e\pi \) must be irrational.

Remark 4. Is \( \ln(\pi) \) irrational?

There exists such representation

\[ \frac{\sin(x)}{x} = \prod_{n=1}^\infty \left(1 - \frac{x^2}{n^2\pi^2}\right). \]

Let \( x = \frac{\pi}{2} \) and then we have the Wallis product formulae for \( \frac{\pi}{2} \):

\[ \frac{\pi}{2} = \prod_{n=1}^\infty \frac{2n}{2n-1} = \frac{2n}{2n+1}. \]

Taking logarithms of this, we come to

\[ \ln(\pi) = \ln(2) + \sum_{n=1}^\infty \left(2\ln(2n) - \ln(2n-1) - \ln(2n+1)\right). \]

Remark 5. Is the Euler–Mascheroni constant \( \gamma \) irrational?

\[ \gamma = \lim_{n \to \infty} \left( \sum_{m=1}^n \frac{1}{m} - \log(n) \right). \]

Remark 6. Is the Khinchin’s constant \( K_0 \) irrational?

\[ K_0 = \prod_{n=1}^\infty \left(1 + \frac{1}{n(n+2)}\right)^{\log_2 n}. \]

References