GLS methods for Stokes equations under boundary condition of friction type: formulation-analysis-numerical schemes and simulations

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Abstract

In this paper, we present in two and three dimensional space Galerkin least squares (GLS) methods allowing the use of equal order approximation for both the velocity and pressure modeling the Stokes equations under Tresca's boundary condition. We propose and analyse two finite element formulations in bounded domains. Firstly, we construct the unique weak solution for each problem by using the method of regularization combined with the monotone operators theory and compactness properties. Secondly, we study the convergence of the finite element approximation by deriving a priori error estimate. Thirdly, we formulate three numerical algorithms namely; projection-like algorithm couple with Uzawa's iteration, the alternative direction method of multiplier and an active set strategy. Finally some numerical experiments are performed to confirm the theoretical findings and the efficiency of the schemes formulated.

Keywords: Tresca friction law; variational inequality; GLS; convergence; augmented Lagrangian; projection-like method; alternating direction method of multiplier, active set strategy.

AMS Subject Classification: 65N30. 35J85

1 Introduction

The objective in this work is to design a finite element procedure that allow the utilization of polynomial with equal degree for calculating the velocity and the pressure for the Stokes equations under nonlinear slip boundary condition of friction type. It is well documented (see [1, 2]) that the couple velocity-pressure for this problem comes with the so-called "inf-sup" condition which restricts the choice of the finite elements spaces needed. The numerical analysis of a flow model (for example Stokes or Navier-Stokes) with a non-classical boundary conditions has been the subject of intensive scrutiny over the years. One of the main reasons for this surge interest in our view is the fact that for applied mathematicians and numerical analysts, flows with such boundary conditions have always been a permanent source of challenging theoretical and computational questions. Given the boundary condition (2.6), it is well documented that the weak formulation associated to the problem is a variational inequality for which one of the early reference in the mathematical analysis is the book by Duvaut-Lions [3]. The two pillars of the solution methodology that we are going to describe (analyse) are; (i) mixed approach for the Galerkin least squares (GLS) formulation associated to the Stokes with Tresca's boundary condition with equal approximation order for both the velocity and pressure reminiscent of the one used in [4, 5], and (ii) iterative schemes based on Lagrange multiplier.

In [6, 7, 8, 9], just to cite a few, a priori error estimates of Stokes under Tresca's boundary condition are studied with the velocity and pressure being inf-sup stable. In this work, because we want to use equal order for polynomials approximating the velocity and the pressure, a sort of compensation is needed to bypass the inf-sup condition. For that purpose, we select the GLS approach, but we observe that many others techniques are possible and the readers interested in stabilization techniques can consult the excellent work of Brezzi-Fortin [10], where many stabilisations schemes are formulated and analysed.The GLS method has been introduced in the early 80's when T. J. R. Hughes and co-workers realized the lack of stability when analysing advection dominated diffusion problems. Then, they formulated and analysed new methods for advection-dominated diffusion problems and for incompressible flows in [11, 12, 13, 14, 15, 16, 17], and later extended their studies to compressible flows [18, 19]. In [20, 21] some stabilization procedures are formulated and analysed for Stokes problem with Dirichlet boundary condition. The main idea of this stabilization approach reside in the combination of the traditional formulation with the least squares terms of the differential equations. The main advantage of this approach is that the classes of finite element spaces that can be used are considerable big, and the mathematical foundations of the method is now well grounded.

The equations of Stokes or Navier-Stokes with Tresca's boundary condition has been considered with pressure stabilization in [22, 23, 24, 25, 26], but in our knowledge similar study with GLS stabilization has not yet been considered and it is the object of this work. Thus our challenge is to analyse how the added terms will affect the stability, convergence, and the actual computation. The GLS formulation correspond to this modified formulation in which the solution of the continuous problem is unchanged under some regularity assumptions, but the approximate solution is very different. The thinking behind the use of a "perturbed" formulation is that the discrete approximation has a better behavior with respect to stability issues and sometimes convergence. It should be made clear that the GLS method is an over-stabilization strategy following the terminology in [10], but it has the advantage that it does not change the symmetry/unsymmetry structure of the system. Two GLS methods are formulated in this work. The first one corresponds to the situation where all possible stabilization terms are added in a least squares manner following the presentation in [4], while the second GLS approach is a "reduced method" because only selected terms are added to the original variational formulation. It is worth mentioning that the second GLS method formulated here had been introduced and analysed in [17] in a particular context. It is clear that from the point of view of computation, the second strategy is more efficient because it has less terms. It should be mentioned that the structure of the "reduced formulation" may not necessarily be maintained with respect to symmetry/unsymmetry structure of the original system. In this work, the GLS are introduced within the context of the finite element discretization in a polygonal/polyhedron domain. After the formulation of the two methods, we show the existence and uniqueness of the finite element approximations without restriction on the data by using a technique based on the regularization-monotone-compactness. Also, on the theoretical front, we study convergence of the finite element solution by deriving a priori error estimates. It is then clear (see Theorem 5.1) that the error is dominated by the interpolation error on the friction zone. This is a classical result for variational inequalities of second kind (see [6, 7, 8, 9, 25, 27]). Thus one can say that the added terms do not increase the convergence rate, but instead we have ignored the inf-sup condition which plays a key role in mixed problems. Having in mind Theorem 5.1, it appears in particular that if piecewise linear approximations are used for both the velocity and the pressure, then optimal (sub-optimal) a priori error estimates for both GLS schemes are obtained depending on the regularity of the solution of the continuous problem on the friction zone. Now, describing our contribution on the computational side, we observe that the system of equations to be solved is nonlinear. Hence iterative or incremental method should be formulated for the actual computation of the solution. We formulate three iterative schemes namely; the projection-like algorithm, the alternating direction method of multiplier, and the primal dual active set algorithm. One notes at this level that the formulation of both the projection-like algorithm and the alternating direction method of multiplier borrow a lot from the presentation in [27, 28], while the active set strategy emanate from [29]. The projection-like method is based on the introduction of a new variable which permits to eliminate the inequality at the expense of adding a new equation. Thus Uzawa type iteration is used for the computation of the solution. The alternative direction method of multiplier and the active set strategy are based on the introduction of a functional for which the characterization of the saddle point is crucial. The iterative schemes discussed in this work make use of Lagrange multipliers with the common goal of "softening" the difficulties by introducing new unknowns. The convergence analysis of the projection-like algorithm and the alternating direction of multipliers formulated can be done by following the techniques presented in [27, 28], while the convergence analysis of the active set strategy is done by following [30, 31]. The rest of the paper is organized as follows:

- Section 2 is concerned with the governing equations and the continuous weak formulation.
- Section 3 is devoted to the formulation of GLS methods.
- Section 4 is devoted to the existence theory of the GLS methods formulated within the context of element approximations.
- Section 5 is about the error analysis together with convergence of GLS methods when the discretization parameter h tends to zero.
- Section 6 and Section 7 are devoted to the formulation of iterative schemes.
- Section 8 is concerned with the validation via numerical simulations of the theoretical findings and some conclusions are drawn in the last paragraph.

2 Governing equations and variational formulation

Let $\Omega \subset \mathbb{R}^d$, d=2,3 be an open bounded set with boundary $\partial\Omega$ assume to be Lipschitz-continuous. We consider the steady incompressible Stokes equations modeled by the equations

$$-2\mu \operatorname{div} \varepsilon(\boldsymbol{u}) + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega\ , \tag{2.2}$$

where $\boldsymbol{u} = (u_i)_{i=1}^d$ is the velocity, pressure $p(\boldsymbol{x})$ and $\boldsymbol{f}(\boldsymbol{x})$ is the external body force applied to the fluid, while μ is the kinematic viscosity and $2\varepsilon(\boldsymbol{u}) = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\boldsymbol{T}}$ is the symmetric part of the velocity gradient. These equations are complemented by boundary conditions. For that purpose, we assume that $\partial\Omega$ is made of two components S and Γ, such that $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, with $S \cap \Gamma = \emptyset$. We assume the homogeneous Dirichlet condition on Γ, that is

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma\ . \tag{2.3}$$

Γ is the porous or artificial boundary where the fluid is prescribed. On the other part of the boundary S, we assume the impermeability condition

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \ \text{ on } \ S\,, \tag{2.4}$$

where $\boldsymbol{n} : S \longrightarrow \mathbb{R}^d$ is the normal outward unit vector to S, and S is an impermeable solid surface along which the fluid may slip. The force within the fluid is the Cauchy stress tensor $\boldsymbol{T}$ given by the relation

$$\boldsymbol{T} = 2\mu\varepsilon(\boldsymbol{u}) - p\boldsymbol{I} \text{ on } \Omega\,,$$

with $\boldsymbol{I}$ the d-dimensional identity matrix. In general for any vector $\boldsymbol{w}$ defined on $\partial\Omega$, we set $\boldsymbol{w_T} = \boldsymbol{w} - (\boldsymbol{w} \cdot \boldsymbol{n})\boldsymbol{n}$. Thus, the tangential stress $(\boldsymbol{Tn})_{\boldsymbol{T}}$ stands for the projection of the normal stress into the corresponding tangent plane. Taking the scalar product of $\boldsymbol{u}$ and the balance of linear momentum (2.1), we obtain

$$\int_\Omega \boldsymbol{T} : \varepsilon(\boldsymbol{u})dx + \int_S (-\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} d\sigma = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} dx\,, \tag{2.5}$$

with $d\sigma$ being the surface measure associated to S. The expression on the left hand side of (2.5) stand for the energy that is dissipated and transformed to other forms of energy. The expression $\int_\Omega \boldsymbol{T} : \varepsilon(\boldsymbol{u})dx$ is the dissipative energy mechanisms in bulk. We clearly observe that $\int_\Omega \boldsymbol{T} : \varepsilon(\boldsymbol{u})dx = 2\mu \int_\Omega |\varepsilon(\boldsymbol{u})|^2\, dx$ which is positive. The term $\int_S (-\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} d\sigma$ represent the dissipative energy on the boundary S. It is a non negative quantity. So we need a functional relation between $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$ and $\boldsymbol{u}_{\boldsymbol{\tau}}$ for the dissipative energy on S to be non negative. We note that if $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$ is zero, then one obtains the perfect slip boundary condition, while if the tangential velocity is zero, then one gets no-slip boundary condition. On the other hand, if the tangential component is expressed in the form $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}$, with $\alpha \geq 0$, then one gets Navier's slip boundary condition. In this text, we are interested in threshold-slip (or stick-slip) boundary condition. Thus we let $g : S \longrightarrow [0,\infty)$ be a non-negative function called threshold slip or barrier function. The stick-slip boundary condition we consider in this work was formulated by Fujita [32, 33] and reads as follows:

$$\left.\begin{array}{l} \text{If} \quad |(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}| < g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \\ \text{If} \quad |(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}| = g \Rightarrow \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0}\ , \ \text{and} \quad -(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} = g\dfrac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \end{array}\right\} \text{ on } \quad S\,, \tag{2.6}$$

where $|\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$ is the Euclidean norm. We note that $\int_S (-\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} d\sigma = \int_S g\, |\boldsymbol{u}_{\boldsymbol{\tau}}|\, d\sigma$ is positive. The stick-slip law (2.6) differ from the one formulated by Le Roux [34], which reads

$$\begin{array}{l} \text{If} \quad |(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}| > g,\ \text{then slip occurs and } (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} = -(g + \kappa |\boldsymbol{u}_{\boldsymbol{\tau}}|)\dfrac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \\ \text{If } |(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}| \leq g \text{ then no slip and } \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}\,. \end{array} \tag{2.7}$$

The most general relation between $\boldsymbol{u}_{\boldsymbol{\tau}}$ and $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$ is the implicit constitutive relation

$$\psi(\boldsymbol{u}_{\boldsymbol{\tau}}, (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}) = 0 \tag{2.8}$$

where ψ is function. We note that (2.6), or (2.7) are special cases of (2.8). For the mathematical setting of the problem, some notations need to be introduced and we refer to [35, 36]. We use standard notation on Lebesgue and Sobolev spaces, $(\cdot,\cdot)$ denotes the L^2 scalar product, and $\|\cdot\|$ the $L^2 - norm$. Having in mind the definition of the sub-differential, (2.6) reduces to

$$\text{for all vector } \boldsymbol{v} \quad , \quad g|\boldsymbol{v}_{\boldsymbol{\tau}}| - g|\boldsymbol{u}_{\boldsymbol{\tau}}| \geq -(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} \cdot (\boldsymbol{v}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}}) \ \text{ on } \ S\,. \tag{2.9}$$

In order to introduce the functions spaces for the analysis of the boundary value (2.1)–(2.6), we take once and for all that $g \in L^\infty(S)$ and $\boldsymbol{f} \in L^2(\Omega)^d$. (2.5) is reduced to

$$2\mu \int_\Omega |\varepsilon(\boldsymbol{u})|^2 dx + \int_S g|\boldsymbol{u}_{\boldsymbol{\tau}}| d\sigma - \int_\Omega p \operatorname{div} \boldsymbol{u} dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} dx\ . \tag{2.10}$$

We introduce from (2.10), (2.3), and (2.4), we the following functions spaces

$$\mathbb{V} = \{\boldsymbol{u} \in H^1(\Omega)^2, \ \ \boldsymbol{u}|_\Gamma = \boldsymbol{0}\ ,\ \boldsymbol{u} \cdot \boldsymbol{n}|_S = 0\},$$

$$M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \text{ with } \int_\Omega q\ dx = 0 \right\}\ .$$

With the spaces $\mathbb{V}$ and M, one can introduce the weak formulation of the boundary value (2.1)–(2.6). We thus multiply (2.2) by $q \in L^2(\Omega)$ and integrate over Ω. Next, we take the dot product between (2.1) and $\boldsymbol{v} - \boldsymbol{u}$ with $\boldsymbol{v} \in \mathbb{V}$, integrate the resulting equation over Ω, apply Green's formula and the boundary conditions (2.3), (2.4) and (2.9). We obtain the following variational problem:

$$\begin{cases} \text{Find } (\boldsymbol{u}, p) \in \mathbb{V} \times M \text{ such that for all } (\boldsymbol{v}, q) \in \mathbb{V} \times M \ , \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq \ell(\boldsymbol{v} - \boldsymbol{u}) \ , \\ b(\boldsymbol{u}, q) = 0 \end{cases} \tag{2.11}$$

where we used the identity $\sum_{1 \leq i,j \leq d} \varepsilon_{ij}(\boldsymbol{u}) \dfrac{\partial v_i}{\partial x_j} = \sum_{1 \leq i,j \leq d} \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{ij}(\boldsymbol{v})$ which is the consequence of the symmetry of $\varepsilon(\boldsymbol{u})$. We recall that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= 2\mu \int_\Omega \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) dx \ , \quad b(\boldsymbol{v}, q) = \int_\Omega q \operatorname{div} \boldsymbol{v} dx, \\ j(\boldsymbol{v}) &= \int_S g |\boldsymbol{v}_{\boldsymbol{T}}| \, d\sigma \quad , \quad \ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx \ , \end{aligned} \tag{2.12}$$

with $\boldsymbol{A} : \boldsymbol{B} = \sum_{1 \leq i,j \leq d} A_{ij} B_{ij}$.

Remark 2.1 *In some contributions, the conservation of the momentum is expressed in terms of the Laplacian of the velocity which gives rise to the bilinear form* $\widetilde{a}(\boldsymbol{u}, \boldsymbol{v}) = 2\mu \, (\nabla \boldsymbol{v}, \nabla \boldsymbol{u})$, *instead of* $a(\cdot, \cdot)$ *defined before. Although at a continuous level*

$$\operatorname{div} \boldsymbol{u} = 0 \quad \textit{implies that} \ \operatorname{div} \varepsilon(\boldsymbol{u}) = \Delta \boldsymbol{u},$$

from a modeling viewpoint it may be important to work with symmetric tensor. For instance, the problem (2.1)—(2.6) gives directly the natural boundary condition in term of the force (traction force) exerted by the fluid on its boundary.

It is worth recalling that the existence of solutions of (2.11) is well established in the literature (see [37]). One needs in particular the following inf-sup condition to hold: there exists β such that

$$\beta \|q\| \leq \sup_{0 \neq \boldsymbol{v} \in \mathbb{V}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \quad \text{for all } q \in M \, . \tag{2.13}$$

In fact (2.13) is obtained by observing that $H_0^1(\Omega)^d \subset \mathbb{V}$ and the pair $(H_0^1(\Omega)^d, M)$ is inf-sup stable (see [1, 2]), hence there exists γ such that

$$\text{for all } q \in M \quad , \quad \sup_{0 \neq \boldsymbol{v} \in \mathbb{V}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \geq \sup_{0 \neq \boldsymbol{v} \in H_0^1(\Omega)^d} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \geq \gamma \|q\| \, .$$

From the numerical point of view, (2.13) should also be satisfied in the finite element subspaces and choosing equal order approximations for $\boldsymbol{u}$ and p, does not lead to a stable scheme. The GLS is exactly designed to avoid the condition (2.13) by adding extra terms to the variational formulation. We discuss next the stabilization procedures for the utilization of equal order approximations for the velocity and pressure within the finite element context.

3 Galerkin Least squares methods

3.1 First stabilized approach

We assume that Ω is a polygon when $d = 2$ and a polyhedron when $d = 3$, so that it can be completely meshed. Now, we describe the discretization space. A regular (see Ciarlet [38]) family of triangulations $(\mathcal{T}_h)_h$ of Ω, is a set of closed non degenerate triangles or tetrahedra, called elements, satisfying

(i)
$$\overline{\Omega} = \bigcup_{1\le i\le M} K_i\,;$$

(ii) The intersection of two distinct elements K in $\mathcal{T}_h$ is either empty, a common vertex, or an entire common edge or face;

(iii) the ratio of the diameter of an element K in $\mathcal{T}_h$ to the diameter of its inscribed circle or ball is bounded by a constant independent of h, this is to say that there exists a constant σ, independent of h and K, such that
$$\text{for all } K\in\mathcal{T}_h,\ \frac{h_K}{\rho_K} = \sigma_K \le \sigma, \tag{3.1}$$
where h_K is the diameter of K and ρ_K is the diameter of the circle (ball) inscribed in K.

As usual, h stands for the maximum of the diameter of all elements of $\mathcal{T}_h$.
For each non-negative integer l and any K in $\mathcal{T}_h$, $\mathcal{P}_l(K)$ is the space of restrictions to K of polynomials with d variables and total degree less than or equal to l. In what follows, c is a generic constant which may vary from line to line but is always independent of h. For a given triangulation $\mathcal{T}_h$, the velocity and pressure are approximated with continuous polynomials of order $l\ge 1$, that is
$$\begin{aligned}\mathbb{V}_h^l &= \{\boldsymbol{v}^h\in\mathcal{C}(\overline{\Omega})^d\cap\mathbb{V}:\ \text{ for all } K\in\mathcal{T}_h,\ \boldsymbol{v}^h|_K\in\mathcal{P}_l(K)^d\}\ ,\\ M_h^l &= \{q^h\in M\cap\mathcal{C}(\overline{\Omega}),\ \text{ for all } K\in\mathcal{T}_h,\ \ q^h|_K\in\mathcal{P}_l(K)\}\,.\end{aligned}\tag{3.2}$$
We recall that for the choice given in (3.2), the discrete version of (2.13) does not hold as pointed out in [1, 2]. Following [11], we introduce the augmented functional
$$\begin{aligned}J_\alpha(\boldsymbol{v},q) =& \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) + j(\boldsymbol{v}) - b(\boldsymbol{v},q) - \ell(\boldsymbol{v})\\ &-\alpha\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{2}\int_K |2\mu\,\mathrm{div}\,\varepsilon(\boldsymbol{v}) - \nabla q + \boldsymbol{f}|^2\,dx\ .\end{aligned}\tag{3.3}$$
The saddle point problem associated with $J_\alpha(\cdot,\cdot)$ reads:
$$\begin{aligned}&\text{Find } \boldsymbol{u},p\ \in\mathbb{V}\times M \text{ such that}\\ &J_\alpha(\boldsymbol{u},q)\le J_\alpha(\boldsymbol{u},p)\le J_\alpha(\boldsymbol{v},q)\ \text{ for all } \boldsymbol{v},q\in\mathbb{V}\times M\ .\end{aligned}\tag{3.4}$$
With the problem (3.4) in mind, the corresponding finite element problem reads as follows: Find $(\boldsymbol{u}_h,p_h)\in\mathbb{V}_h^l\times M_h^l$ such that for all $(\boldsymbol{v},q)\in\mathbb{V}_h^l\times M_h^l$,
$$\begin{aligned}&A(\boldsymbol{u}_h,\boldsymbol{v}-\boldsymbol{u}_h) + B(\boldsymbol{v}-\boldsymbol{u}_h,p_h) + j(\boldsymbol{v}) - j(\boldsymbol{u}_h) \ge \ell_1(\boldsymbol{v}-\boldsymbol{u}_h)\ ,\\ &B(\boldsymbol{u}_h,q) - C(q,p_h) = \ell_2(q)\,,\end{aligned}\tag{3.5}$$
with (see (2.12))
$$A(\boldsymbol{u},\boldsymbol{v}) = a(\boldsymbol{u},\boldsymbol{v}) - \alpha\sum_K h_K^2\int_K 2\mu\,\mathrm{div}\,\varepsilon(\boldsymbol{u})\cdot 2\mu\,\mathrm{div}\,\varepsilon(\boldsymbol{v})dx,$$
$$B(\boldsymbol{v},q) = -b(\boldsymbol{v},q) + \alpha\sum_K h_K^2\int_K 2\mu\,\mathrm{div}\,\varepsilon(\boldsymbol{v})\cdot\nabla q dx$$
$$C(p,q) = \alpha\sum_K h_K^2\int_K \nabla p\cdot\nabla q dx\quad ,$$
$$\ell_1(\boldsymbol{v}) = \ell(\boldsymbol{v}) + \alpha\sum_K h_K^2\int_K \boldsymbol{f}\cdot 2\mu\,\mathrm{div}\,\varepsilon(\boldsymbol{v})dx\ ,$$
$$\ell_2(q) = -\alpha\sum_K h_K^2\int_K \boldsymbol{f}\cdot\nabla q dx\,.$$

Remark 3.1 *One notes that if $(\boldsymbol{u}, p)$ is the solution of (2.11) with $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and $p \in H^1(\Omega)$ then (3.5) is reduced to*

$$\begin{aligned} &a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq \ell(\boldsymbol{v} - \boldsymbol{u}) \ , \\ &b(\boldsymbol{u}, q) = 0 \,. \end{aligned}$$

3.2 Second stabilized approach

One notes from (3.5) that the crucial term added that permits to avoid the compatibility condition between the velocity and the pressure is the expression $C(p, q)$. The following formulation can be regarded as a reduced GLS (see [17]) because it has less stabilizing expressions. It reads as follows: find $(\boldsymbol{u}_h, p_h) \in \mathbb{V}_h^l \times M_h^l$ such that for all $(\boldsymbol{v}, q) \in \mathbb{V}_h^l \times M_h^l$

$$\begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v} - \boldsymbol{u}_h) - b(\boldsymbol{v} - \boldsymbol{u}_h, p_h) + j(\boldsymbol{v}) - j(\boldsymbol{u}_h) \geq \ell(\boldsymbol{v} - \boldsymbol{u}_h) \ , \\ B(\boldsymbol{u}_h, q) - C(q, p_h) = \ell_2(q) \,. \end{cases} \tag{3.6}$$

Remark 3.2 *The second equation in fact is*

$$-b(\boldsymbol{u}_h, q) + \alpha \sum_{K \in \mathcal{T}_h} \int_K h_K^2 \left(2\mu \operatorname{div} \varepsilon(\boldsymbol{u}_h) - \nabla p_h + \boldsymbol{f}\right) \cdot \nabla q dx = 0 \,.$$

Thus if the solution $(\boldsymbol{u}, p)$ of (2.11) belong to $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)$, then $(\boldsymbol{u}, p)$ solves (3.6) regardless of $q \in M_h^l$.

Remark 3.3 *(3.6) has less terms than (3.5), hence computationally, it is more attractive. (3.5) has the symmetry structure of the original system for any degree of approximation and can be re-written as optimization of a lower semi-continuous, and non-differentiable convex functional. Thus its belong to "convex analysis" for which one has a numerous algorithms for its resolution. In conclusion, these two choices present some interesting aspects.*

We study next the existence theory of problems (3.6) and (3.5).

4 Existence of solutions

In this section, we will address the solvability of both (3.5) and (3.6). In what follows, c is a positive constant that may vary from one line to the next but always independent of h. The following inverse inequality will be used throughout

$$\sum_{K \in \mathcal{T}_h} h_K^2 \| \operatorname{div} \varepsilon(\boldsymbol{v})\|_K^2 \leq c_I^2 \|\varepsilon(\boldsymbol{v})\|^2 \quad \text{for all} \;\; \boldsymbol{v} \in \mathbb{V}_h^l \ . \tag{4.1}$$

We introduce the discrete-norm

$$||q||_h^2 = \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla q\|_K^2 \quad , \quad \text{for } q \in M_h^l \ .$$

The continuity requirement in M_h^l, together with the zero mean-value condition easily imply that this is a norm on M_h^l for which the following inverse inequality is valid: there exists c independent of h such that

$$||q||_h \leq c\|q\| \text{ for all } \;\; q \in M_h^l \ . \tag{4.2}$$

We claim that

Proposition 4.1 *There are positive constant c_1, c_2, c_3 independent of h such that for all $(\boldsymbol{v}, \boldsymbol{u}, q, p) \in \mathbb{V}_h^l \times \mathbb{V}_h^l \times M_h^l \times M_h^l$*

$$\begin{aligned} &A(\boldsymbol{u}, \boldsymbol{v}) \leq c_1 \|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_1 \ , \\ &B(\boldsymbol{v}, q) \leq c_2 \|\boldsymbol{v}\|_1 \|q\| \ , \\ &C(p, q) \leq \alpha \|p\|_h \|q\|_h \leq c_3 \|p\| \|q\| \ , \end{aligned}$$

Let the stabilization parameter α such that $\alpha < (2\mu c_I^2)^{-1}$ with c_I given by (4.1). Then one can find a positive constant c_4 independent of h such that for all $\boldsymbol{v} \in \mathbb{V}_h^l$

$$A(\boldsymbol{v}, \boldsymbol{v}) \geq c_4 \|\boldsymbol{v}\|_1^2 \ .$$

Proof. Using Cauchy-Schwarz's inequality, Hölder's inequality together with (4.1), we obtain

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v}) &\leq 2\mu \|\varepsilon(\boldsymbol{u})\| \|\varepsilon(\boldsymbol{v})\| + 4\mu^2 \alpha \left(\sum_K h_K^2 \| \operatorname{div} \varepsilon(\boldsymbol{v}) \|_K^2 \right)^{1/2} \left(\sum_K h_K^2 \| \operatorname{div} \varepsilon(\boldsymbol{u}) \|_K^2 \right)^{1/2} \\ &\leq 2\mu \|\varepsilon(\boldsymbol{u})\| \|\varepsilon(\boldsymbol{v})\| + 4\mu^2 \alpha c \|\varepsilon(\boldsymbol{v})\| \|\varepsilon(\boldsymbol{u})\| \\ &\leq (2\mu + 4\mu^2 \alpha c) \|\boldsymbol{v}\|_1 \|\boldsymbol{u}\|_1 \ . \end{aligned}$$

Next from Cauchy-Schwarz's inequality, Hölder's inequality together with (4.2), we obtain

$$\begin{aligned} B(\boldsymbol{v}, q) &\leq \|\boldsymbol{v}\|_1 \|q\| + 2\mu\alpha \left(\sum_K h_K^2 \| \operatorname{div} \varepsilon(\boldsymbol{v}) \|^2 \right)^{1/2} \left(\sum_K h_K^2 \|\nabla q\|_K^2 \right)^{1/2} \\ &\leq (1 + 2c\mu\alpha) \|\boldsymbol{v}\|_1 \|q\| \ . \end{aligned}$$

Thirdly Cauchy-Schwarz's inequality, Hölder's inequality together with (4.2), gives

$$C(p, q) \leq \alpha \|p\|_h \|q\|_h \leq c_3 \|p\| \|q\| \ .$$

Finally

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{u}) &\geq 2\mu \|\varepsilon(\boldsymbol{u})\|^2 - 4c_I^2 \mu^2 \alpha \|\varepsilon(\boldsymbol{u})\|^2 \\ &\geq 2\mu(1 - 2c_I^2 \mu\alpha) \|\varepsilon(\boldsymbol{u})\|^2 . \end{aligned}$$

So, it suffice to take α such that $\alpha < (2\mu c_I^2)^{-1}$ with c_I given by (4.1) and apply Korn's inequality. □

Remark 4.1 *It is manifest that for piecewise linear approximation, we have coercivity of $A(\cdot, \cdot)$ without restriction on the stability parameter α because $\operatorname{div} \varepsilon(\boldsymbol{u}_h) = 0$.*

The next result is the a priori bounds of the solutions of (3.5). In fact we claim that

Proposition 4.2 *Let $\mathcal{T}_h$ satisfies (3.1), and assume that for $d = 3$, the boundary $\partial\Omega$ is Lipschitz-continuous. Let the stabilization parameter α satisfies the inequality $\alpha < (2\mu c_I^2)^{-1}$ with c_I given by (4.1). Let $(\boldsymbol{u}_h, p_h)$ be the solution of (3.5). Then, there is c independent of h such that*

$$\|\boldsymbol{u}_h\|_1^2 + \|p_h\|_h^2 + \|p_h\|^2 + j(\boldsymbol{u}_h) \leq c \|\boldsymbol{f}\|^2 .$$

Proof. Recall that $(\boldsymbol{u}_h, p_h)$ is solution of

$$\begin{cases} \text{for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h^l \times M_h^l \ , \\ K(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h - \boldsymbol{u}_h, q_h) + j(\boldsymbol{v}_h) - j(\boldsymbol{u}_h) \geq \ell_1(\boldsymbol{v}_h - \boldsymbol{u}_h) - \ell_2(q_h) . \end{cases} \tag{4.3}$$

with $K(\boldsymbol{u},p;\boldsymbol{v},q) = A(\boldsymbol{u},\boldsymbol{v}) + B(\boldsymbol{v},p) - B(\boldsymbol{u},q) + C(p,q)$. We take $\boldsymbol{v}_h$ in (4.3) such that $\boldsymbol{v}_h - \boldsymbol{u}_h = \pm\boldsymbol{w}_h \in \mathbb{V}_h^l \cap \{\boldsymbol{v}_h|_S = 0\} = \mathbb{W}_h$. Thus one obtains

$$\begin{cases}\text{for all } (\boldsymbol{w}_h, q_h) \in \mathbb{W}_h \times M_h^l \ , \\ K(\boldsymbol{u}_h, p_h; \boldsymbol{w}_h, q_h) = \ell_1(\boldsymbol{w}_h) - \ell_2(q_h) \, . \end{cases}$$

Next, from [4] (see Lemma 3.2), there exists c independent of h such that

$$\begin{aligned} c\left(\|\boldsymbol{u}_h\|_1^2 + \|p_h\|^2\right)^{1/2} &\leq \sup_{0\neq(\boldsymbol{w}_h,q_h)\in\mathbb{W}_h\times M_h^l} \frac{K(\boldsymbol{u}_h, p_h; \boldsymbol{w}_h, q_h)}{\left(\|\boldsymbol{w}_h\|_1^2 + \|q_h\|^2\right)^{1/2}} \\ &\leq \sup_{0\neq(\boldsymbol{w}_h,q_h)\in\mathbb{W}_h\times M_h^l} \frac{\ell_1(\boldsymbol{w}_h) - \ell_2(q_h)}{\left(\|\boldsymbol{w}_h\|_1^2 + \|q_h\|^2\right)^{1/2}} \, . \end{aligned}$$

Using Cauchy-Schwarz's inequality,(4.1), (4.2) and the fact that $\mathcal{T}_h$ satisfies (3.1), one gets

$$\begin{aligned} \ell_1(\boldsymbol{w}_h) - \ell_2(q_h) \leq & \|\boldsymbol{f}\|\|\boldsymbol{w}_h\|_1 + 2\mu\alpha c_I \left(\sum_K h_K^2 \|\boldsymbol{f}\|_K^2\right)^{1/2} \|\boldsymbol{w}_h\|_1 + \alpha c \left(\sum_K h_K^2 \|\boldsymbol{f}\|_K^2\right)^{1/2} \|q_h\| \\ \leq & c\|\boldsymbol{f}\| \left(\|\boldsymbol{w}_h\|_1^2 + \|q_h\|^2\right)^{1/2} \, . \end{aligned}$$

Hence

$$\|\boldsymbol{u}_h\|_1^2 + \|p_h\|^2 \leq c\|\boldsymbol{f}\|^2 \, . \tag{4.4}$$

Next, we take $q_h = p_h$, $\boldsymbol{v}_h = \boldsymbol{0}$ and $\boldsymbol{v}_h = 2\boldsymbol{u}_h$ in (4.3) which yields

$$A(\boldsymbol{u}_h, \boldsymbol{u}_h) + C(p_h, p_h) + j(\boldsymbol{u}_h) = \ell_1(\boldsymbol{u}_h) - \ell_2(p_h) \, .$$

Now applying proposition 4.1, Cauchy-Schwarz's inequality,(4.1), (4.2), Young's inequality and the fact that $\mathcal{T}_h$ satisfies (3.1), one gets

$$\|\boldsymbol{u}_h\|_1^2 + \|p_h\|_h^2 + j(\boldsymbol{u}_h) \leq c\|\boldsymbol{f}\|^2 \, ,$$

which together with (4.4) leads to the asserted result. □

The variational problem (3.5) is a mixed variational inequality of second kind. Its existence theory will be analysed by making use of; regularization, properties of monotone operator, a priori estimates and passage to the limit. We claim that

Proposition 4.3 *Assume that the mesh $\mathcal{T}_h$ satisfies (3.1), and let the stabilization parameter α such that $\alpha < (2\mu c_I^2)^{-1}$ with c_I given by (4.1). Then the variational problem (3.5) admits exactly one solution $(\boldsymbol{u}_h, p_h)$ in $\mathbb{V}_h^l \times M_h^l$.*

Proof. It is done in three steps.
Step 1: Regularization. The functional j is non differentiable at zero. Let $\varepsilon > 0$, approaching zero and define the functional $j_\varepsilon : \mathbb{V}_h^l \longrightarrow \mathbb{R}$ as follows

$$j_\varepsilon(\boldsymbol{v}) = \int_S g\sqrt{|\boldsymbol{v}_{\boldsymbol{\tau}}|^2 + \varepsilon^2} \; d\sigma.$$

One observes that

$$\lim_{\varepsilon\to 0}(j_\varepsilon(\boldsymbol{v}) - j(\boldsymbol{v})) = \lim_{\varepsilon\to 0} \varepsilon^2 \int_S \frac{g}{\sqrt{|\boldsymbol{v}_{\boldsymbol{\tau}}|^2 + \varepsilon^2} + |\boldsymbol{v}_{\boldsymbol{\tau}}|} d\sigma = 0 \, .$$

The functional j_ε is lower semi-continuous and twice Gateaux-differentiable with

$$\begin{aligned} Dj_\varepsilon(\boldsymbol{u}) \cdot \boldsymbol{v} &= \int_S g \frac{\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}}}{\sqrt{|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 + \varepsilon^2}} \; d\sigma \ , \\ D^2 j_\varepsilon(\boldsymbol{u})(\boldsymbol{v}, \boldsymbol{w}) &= \int_S g \frac{(\boldsymbol{v}_{\boldsymbol{\tau}} \cdot \boldsymbol{w}_{\boldsymbol{\tau}})(|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 + \varepsilon^2) - (\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{w}_{\boldsymbol{\tau}})(\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}})}{(|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 + \varepsilon^2)^{3/2}} d\sigma \, . \end{aligned} \tag{4.5}$$

With (4.5) in mind, one easily show that $D^2 j_\varepsilon(\boldsymbol{u})$ is symmetric that is

$$D^2 j_\varepsilon(\boldsymbol{u})(\boldsymbol{v},\boldsymbol{w}) = D^2 j_\varepsilon(\boldsymbol{u})(\boldsymbol{w},\boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{v},\boldsymbol{w}\,,$$

and positive definite that is

$$D^2 j_\varepsilon(\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v}) \geq 0 \quad \text{for all } \boldsymbol{v}\,.$$

Also because j_ε is convex and differentiable, then Dj_ε is monotone that is

$$\langle Dj_\varepsilon(\boldsymbol{u}) - Dj_\varepsilon(\boldsymbol{v}), \boldsymbol{u}-\boldsymbol{v}\rangle \geq 0 \quad \text{for all} \quad \boldsymbol{v},\boldsymbol{u} \in \mathbb{V}\,. \tag{4.6}$$

The regularized problem reads (with obvious notation)

$$\begin{cases} \text{Find } (\boldsymbol{u}^\varepsilon, p^\varepsilon) \in \mathbb{V}_h^l \times M_h^l \text{ such that for all } (\boldsymbol{v},q) \in \mathbb{V}_h^l \times M_h^l, \\ A(\boldsymbol{u}^\varepsilon, \boldsymbol{v}-\boldsymbol{u}^\varepsilon) + B(\boldsymbol{v}-\boldsymbol{u}^\varepsilon, p^\varepsilon) + j_\varepsilon(\boldsymbol{v}) - j_\varepsilon(\boldsymbol{u}^\varepsilon) \geq \ell_1(\boldsymbol{v}-\boldsymbol{u}^\varepsilon)\,, \\ B(\boldsymbol{u}^\varepsilon, q) - C(q,p^\varepsilon) = \ell_2(q)\,. \end{cases} \tag{4.7}$$

Since j_ε is differentiable, then (4.7) is equivalent to the variational problem (see [3] where similar examples are treated)

$$\begin{cases} \text{Find } (\boldsymbol{u}^\varepsilon, p^\varepsilon) \in \mathbb{V}_h^l \times M_h^l \text{ such that for all } (\boldsymbol{v},q) \in \mathbb{V}_h^l \times M_h^l, \\ A(\boldsymbol{u}^\varepsilon, \boldsymbol{v}) + B(\boldsymbol{v}, p^\varepsilon) + Dj_\varepsilon(\boldsymbol{u}^\varepsilon)\cdot\boldsymbol{v} = \ell_1(\boldsymbol{v})\,, \\ B(\boldsymbol{u}^\varepsilon, q) - C(q,p^\varepsilon) = \ell_2(q)\,, \end{cases} \tag{4.8}$$

which is re-written as follows:

$$\begin{cases} \text{Find } (\boldsymbol{u}^\varepsilon, p^\varepsilon) \in \mathbb{V}_h^l \times M_h^l \text{ such that for all } (\boldsymbol{v},q) \in \mathbb{V}_h^l \times M_h^l \\ A(\boldsymbol{u}^\varepsilon, \boldsymbol{v}) + B(\boldsymbol{v}, p^\varepsilon) + Dj_\varepsilon(\boldsymbol{u}^\varepsilon)\cdot\boldsymbol{v} - B(\boldsymbol{u}^\varepsilon, q) + C(q,p^\varepsilon) = \ell_1(\boldsymbol{v}) - \ell_2(q)\,. \end{cases} \tag{4.9}$$

(4.9) is a nonlinear monotone problem and to study it, it is convenient to introduce the mapping $(\boldsymbol{u},p) \longrightarrow \mathcal{H}(\boldsymbol{u},p)$ such that

$$\langle \mathcal{H}(\boldsymbol{u},p);(\boldsymbol{v},q)\rangle = A(\boldsymbol{u},\boldsymbol{v}) + B(\boldsymbol{v},p) + Dj_\varepsilon(\boldsymbol{u})\cdot\boldsymbol{v} - B(\boldsymbol{u},q) + C(q,p)\,.$$

Hence for the existence of solutions of (4.9), we need to show the following conditions (see [39], Chap 2)

(a) $\mathcal{H}$ is monotone, i.e for all $\boldsymbol{u},\boldsymbol{v} \in \mathbb{V}_h^l \times \mathbb{V}_h^l$, and $p,q \in M_h^l \times M_h^l$

$$\langle \mathcal{H}(\boldsymbol{u},p) - \mathcal{H}(\boldsymbol{v},q);(\boldsymbol{u}-\boldsymbol{v},p-q)\rangle \geq 0\,.$$

Indeed one has

$$\begin{aligned}\langle \mathcal{H}(\boldsymbol{u},p) - \mathcal{H}(\boldsymbol{v},q);(\boldsymbol{u}-\boldsymbol{v},p-q)\rangle =& A(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}) + \langle Dj_\varepsilon(\boldsymbol{u}) - Dj_\varepsilon(\boldsymbol{v}),\boldsymbol{u}-\boldsymbol{v}\rangle \\ &+ C(p-q,p-q)\end{aligned}$$

which is non negative because of proposition 4.1, (4.6) and $C(p,p)$ is non-negative.

(b) $\mathcal{H}$ is coercive meaning that for all $(\boldsymbol{v},q) \in \mathbb{V}_h^l \times M_h^l$

$$\left(\frac{1}{(\|\boldsymbol{v}\|_1^2 + \|q\|_h^2)^{1/2}}\langle \mathcal{H}(\boldsymbol{v},q);(\boldsymbol{v},q)\rangle\right) \to \infty \quad \text{if} \quad (\|\boldsymbol{v}\|_1^2 + \|q\|_h^2)^{1/2} \to \infty\,.$$

Indeed from (4.6), $\alpha < (2\mu c_I^2)^{-1}$ and using Korn's inequality

$$\begin{aligned}\langle \mathcal{H}(\boldsymbol{v},q);(\boldsymbol{v},q)\rangle &= A(\boldsymbol{v},\boldsymbol{v}) + C(q,q) + \langle Dj_\varepsilon(\boldsymbol{v}),\boldsymbol{v}\rangle \\ &\geq 2\mu(1-2\mu\alpha c_I^2)\|\varepsilon(\boldsymbol{v})\|^2 + \alpha\|q\|_h^2 \\ &\geq \min\left(2\mu(1-2\mu\alpha c_I^2),\alpha\right)\left(\|\boldsymbol{v}\|_1^2 + \|q\|_h^2\right)\end{aligned}$$

from which we deduce the coercivity of $\mathcal{H}$.

(c) $\mathcal{H}$ is hemi-continuous in $\mathbb{V}_h^l \times M_h^l$, i.e for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}_h^l \times \mathbb{V}_h^l$, and $p, q \in M_h^l \times M_h^l$ the mapping

$$t \longrightarrow \langle \mathcal{H}(\boldsymbol{u} + t\boldsymbol{v}, p + tq); (\boldsymbol{v}, q)\rangle \text{ is continuous from } \mathbb{R} \text{ into } \mathbb{R} .$$

Indeed

$$\begin{aligned}
&\langle \mathcal{H}(\boldsymbol{u} + t_1\boldsymbol{v}, p + t_1 q) - \mathcal{H}(\boldsymbol{u} + t_2\boldsymbol{v}, p + t_2 q); (\boldsymbol{v}, q)\rangle \\
=&(t_1 - t_2)\left[A(\boldsymbol{v}, \boldsymbol{v}) + B(\boldsymbol{v}, p) - B(\boldsymbol{v}, q) + C(q, q)\right] + \left(Dj_\varepsilon(\boldsymbol{u} + t_1\boldsymbol{v}) - Dj_\varepsilon(\boldsymbol{u} + t_2\boldsymbol{v})\right)\boldsymbol{v} \\
=&(t_1 - t_2)\left[A(\boldsymbol{v}, \boldsymbol{v}) + C(q, q) + B(\boldsymbol{v}, p) - B(\boldsymbol{v}, q) + \int_0^1 D^2 j_\varepsilon(\boldsymbol{u} + t_2\boldsymbol{v} - \theta(t_2 - t_1)\boldsymbol{v})(\boldsymbol{v}, \boldsymbol{v})d\theta\right] ,
\end{aligned}$$

which tends to zero with $t_1 - t_2$ because $\boldsymbol{u}, \boldsymbol{v}, p, q$ are fixed.

We then conclude partially that (4.9) has a solution $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) \in \mathbb{V}_h^l \times M_h^l$. But because $\mathcal{H}$ is strictly monotone,then the solution $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ is unique.
In the next lines, we study the limit when ε approaches zero of $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ solution of (4.9).
Step 2: a priori estimates and passage to the limit. The a priori estimates obtained in proposition 4.2 are valid due to the equivalence between (4.8) and (4.7). Hence

$$\|\boldsymbol{u}_h^\varepsilon\|_1^2 + \|p_h^\varepsilon\|^2 + \sum_{K\in\mathcal{T}_h} h_K^2 \int_K |\nabla p_h^\varepsilon|^2 dx + j(\boldsymbol{u}_h^\varepsilon) \le c\|\boldsymbol{f}\|^2 .$$

We deduce that the sequences $(\boldsymbol{u}_h^\varepsilon)_\varepsilon$ and $(p_h^\varepsilon)_\varepsilon$ are respectively $\boldsymbol{H}^1$ and L^2 bounded. Moreover one has

$$h_K^2\|\nabla p_h^\varepsilon\|_K^2 < \sum_{K\in\mathcal{T}_h} h_K^2 \int_K |\nabla p_h^\varepsilon|^2 dx \le c\|\boldsymbol{f}\|^2 .$$

Thus, for h_K fixed, one has

$$\|\nabla p_h^\varepsilon\|^2 = \sum_{K\in\mathcal{T}_h} \|\nabla p_h^\varepsilon\|_K^2 \le \sum_{K\in\mathcal{T}_h} \frac{c}{h_K^2}\|\boldsymbol{f}\|^2 < \infty .$$

Hence we can find a subsequence, denoted also $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) \in \mathbb{V}_h^l \times M_h^l$, such that

$$\begin{aligned}
\boldsymbol{u}_h^\varepsilon &\to \boldsymbol{u}_h \quad \text{weakly in } \boldsymbol{H}^1(\Omega) \\
p_h^\varepsilon &\to p_h \quad \text{weakly in } \ H^1(\Omega) .
\end{aligned} \tag{4.10}$$

One notes that the regularized problem (4.7) is re-written as follows

$$\begin{aligned}
&\text{for all } (\boldsymbol{v}, q) \in \mathbb{V}_h^l \times M_h^l , \\
&A(\boldsymbol{u}_h^\varepsilon, \boldsymbol{u}_h^\varepsilon) + B(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) + j_\varepsilon(\boldsymbol{u}_h^\varepsilon) \\
\le\ \ &A(\boldsymbol{u}_h^\varepsilon, \boldsymbol{v}) + B(\boldsymbol{v}, p_h^\varepsilon) + j_\varepsilon(\boldsymbol{v}) - \ell_1(\boldsymbol{v} - \boldsymbol{u}_h^\varepsilon)
\end{aligned} \tag{4.11}$$

$$B(\boldsymbol{u}_h^\varepsilon, q) - C(q, p_h^\varepsilon) = \ell_2(q) . \tag{4.12}$$

The weak convergence properties in (4.10) allows one to pass to the limit in (4.12) and one obtains

$$\text{for all } q \in M_h^l, \quad B(\boldsymbol{u}_h, q) - C(q, p_h) = \ell_2(q) . \tag{4.13}$$

Owing to the compactness of the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, there exits a subsequence, still denoted by $(\boldsymbol{u}_h^\varepsilon)$, such that

$$\boldsymbol{u}_h^\varepsilon \to \boldsymbol{u}_h \quad \text{weakly in } \boldsymbol{H}^1(\Omega) \ \text{ and } \ \boldsymbol{u}_h^\varepsilon \to \boldsymbol{u}_h \quad \text{strongly in } \boldsymbol{L}^4(\Omega) . \tag{4.14}$$

For the right hand side of (4.11), one notes that $j_\varepsilon \to j$ when $\varepsilon \to 0$ together with (4.10) leads to

$$\begin{aligned}
&\liminf_\varepsilon\,[A(\boldsymbol{u}_h^\varepsilon, \boldsymbol{v}) + B(\boldsymbol{v}, p_h^\varepsilon) + j_\varepsilon(\boldsymbol{v}) - \ell_1(\boldsymbol{v} - \boldsymbol{u}_h^\varepsilon)] \\
\le&A(\boldsymbol{u}_h, \boldsymbol{v}) + B(\boldsymbol{v}, p_h) + j(\boldsymbol{v}) - \ell_1(\boldsymbol{v} - \boldsymbol{u}_h) .
\end{aligned} \tag{4.15}$$

For the left hand side of (4.11), the convergence properties (4.14), (4.10) and the fact that $j_\varepsilon \to j$ when $\varepsilon \to 0$ yield

$$A(\boldsymbol{u}_h, \boldsymbol{u}_h) + B(\boldsymbol{u}_h, p_h) + j(\boldsymbol{u}_h) \leq \liminf_{\varepsilon} \left[A(\boldsymbol{u}_h^\varepsilon, \boldsymbol{u}_h^\varepsilon) + B(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) + j_\varepsilon(\boldsymbol{u}_h^\varepsilon)\right] . \tag{4.16}$$

Putting together (4.15) and (4.16) implies that

$$\begin{cases} \text{for all } \boldsymbol{v} \in \mathbb{V}_h^l \\ A(\boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}) + B(\boldsymbol{u}_h - \boldsymbol{v}, p_h) + j(\boldsymbol{u}_h) - j(\boldsymbol{v}) \leq \ell_1(\boldsymbol{u}_h - \boldsymbol{v}) . \end{cases} \tag{4.17}$$

Whence the existence of solutions of (3.5) which is (4.13) and (4.17).

Having constructed the weak solution $(\boldsymbol{u}_h, p_h)$ of (3.5), we now address its unique solvability. **Step 3: uniqueness**. Let $(\boldsymbol{u}_1, p_1)$ and $(\boldsymbol{u}_2, p_2)$ be the solutions of (3.5). A classical algebraic manipulation reveals that

$$\begin{cases} A(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) - B(\boldsymbol{u}_2 - \boldsymbol{u}_1, p_1 - p_2) \leq 0 \\ C(p_1 - p_2, p_1 - p_2) + B(\boldsymbol{u}_2 - \boldsymbol{u}_1, p_1 - p_2) = 0 . \end{cases}$$

We then deduce that

$$A(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) + C(p_1 - p_2, p_1 - p_2) \leq 0 .$$

So by coercivity of $A(\cdot, \cdot)$ one obtains

$$c_3 \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1^2 + \|p_1 - p_2\|_h^2 \leq 0$$

which implies that $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $p_1 - p_2 = c_K$ in each element K of $\mathcal{T}_h$. Having in mind that $p_1 - p_2$ is an element of M_h^l, hence continuous, it appears that $c_K = c$, the same constant throughout. Next, knowing that $\int_\Omega (p_1 - p_2) = 0$, we deduce that $c = 0$ and $p_1 = p_2$. Hence the solution is unique. □

We now turn to the existence theory of (3.6) and claim that

Proposition 4.4 *Assume that the mesh $\mathcal{T}_h$ satisfies (3.1). Let α the stabilization parameter satisfying the relation $\alpha < (2\mu c_I^2)^{-1}$ with c_I given by (4.1). Then the variational problem (3.6) admits exactly one solution $(\boldsymbol{u}_h, p_h) \in \mathbb{V}_h^l \times M_h^l$ and there is a positive constant c independent of h such that*

$$\|\boldsymbol{u}_h\|_1^2 + \|p_h\|_h^2 + \|p_h\|^2 + j(\boldsymbol{u}_h) \leq c\|\boldsymbol{f}\|^2 . \tag{4.18}$$

Proof. We start with the a priori estimate.
We take respectively $\boldsymbol{v}_h = \boldsymbol{0}$ and $2\boldsymbol{u}_h$ in (3.6), compare the two inequalities and deduce that (with $q = p_h$)

$$\begin{cases} a(\boldsymbol{u}_h, \boldsymbol{u}_h) - b(\boldsymbol{u}_h, p_h) + j(\boldsymbol{u}_h) = \ell(\boldsymbol{u}_h) , \\ -b(\boldsymbol{u}_h, p_h) + \alpha \sum_K h_K^2 \int_K 2\mu \operatorname{div} \varepsilon(\boldsymbol{u}_h) \cdot \nabla p_h dx - \alpha \|p_h\|_h^2 = -\alpha \sum_K h_K^2 \int_K \boldsymbol{f} \cdot \nabla p_h dx . \end{cases}$$

Subtracting these equations, and using standard inequalities together with (3.1), one gets

$$\begin{aligned}
2\mu\|\varepsilon(\boldsymbol{u}_h)\|^2 + \alpha\|p_h\|_h^2 + j(\boldsymbol{u}_h) \\
=&\alpha\sum_K h_K^2\int_K 2\mu\,\mathrm{div}\,\varepsilon(\boldsymbol{u}_h)\cdot\nabla p_h dx + \ell(\boldsymbol{u}_h) + \alpha\sum_K h_K^2\int_K \boldsymbol{f}\cdot\nabla p_h dx \\
\leq&2\mu\alpha\left(\sum_K h_K^2\int_K|\,\mathrm{div}\,\varepsilon(\boldsymbol{u}_h)|^2dx\right)^{1/2}\left(\sum_K h_K^2\int_K|\nabla p_h|^2dx\right)^{1/2} + \|\boldsymbol{f}\|\|\boldsymbol{u}_h\| \\
&+\alpha\left(\sum_K h_K^2\int_K|\boldsymbol{f}|^2dx\right)^{1/2}\left(\sum_K h_K^2\int_K|\nabla p_h|^2dx\right)^{1/2} \\
\leq&2\mu\alpha c_I\|\varepsilon(\boldsymbol{u}_h)\|\|p_h\|_h + c\|\boldsymbol{f}\|\|\varepsilon(\boldsymbol{u}_h)\| + \alpha c\|\boldsymbol{f}\|\|p_h\|_h\,.
\end{aligned}$$

We apply Young's inequality with $\alpha < (2\mu c_I^2)^{-1}$ and $\alpha c_I < \gamma_1 < (2\mu c_I)^{-1}$ and obtain

$$\mu\left(1-\frac{\alpha c_I}{\gamma_1}\right)\|\varepsilon(\boldsymbol{u}_h)\|^2 + \alpha\left(\frac{1}{2}-\mu c_I\gamma_1\right)\|p_h\|_h^2 + j(\boldsymbol{u}_h) \leq \frac{c}{\mu}\|\boldsymbol{f}\|^2 + \alpha c\|\boldsymbol{f}\|^2\,.$$

The L^2 estimate on the pressure is obtained as in Proposition 4.2 and will not be repeated here.
To proof the existence of solutions of (3.6), we adopt the same strategy used to prove proposition 4.3. Thus we introduce the regularized problem associated with (3.6) which reads:

$$\begin{cases}\text{Find } (\boldsymbol{u}^\varepsilon, p^\varepsilon)\in\mathbb{V}_h^l\times M_h^l \\ \text{such that for all } (\boldsymbol{v},q)\in\mathbb{V}_h^l\times M_h^l \\ a(\boldsymbol{u}^\varepsilon,\boldsymbol{v}) - b(\boldsymbol{v},p^\varepsilon) + Dj_\varepsilon(\boldsymbol{u}^\varepsilon)\cdot\boldsymbol{v} = \ell(\boldsymbol{v})\ , \\ B(\boldsymbol{u}^\varepsilon,q) - C(q,p^\varepsilon) = \ell_2(q)\,. \end{cases} \tag{4.19}$$

We define the mapping $(\boldsymbol{u},p)\longrightarrow\mathcal{K}(\boldsymbol{u},p)$ such that

$$\langle\mathcal{K}(\boldsymbol{u},p);(\boldsymbol{v},q)\rangle = a(\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{v},p) + Dj_\varepsilon(\boldsymbol{u})\cdot\boldsymbol{v} - B(\boldsymbol{u},q) + C(q,p)\,,$$

<u>$\mathcal{K}$ is monotone</u>. Indeed, let $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v}$ elements of $\mathbb{V}_h^l$ and p_1,p_2,q elements of M_h^l; we have

$$\begin{aligned}
&(\mathcal{K}(\boldsymbol{u}_1,p_1)-\mathcal{K}(\boldsymbol{u}_2,p_2);(\boldsymbol{u}_1-\boldsymbol{u}_2,p_1-p_2)) \\
=&a(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2) + \underbrace{(Dj_\varepsilon(\boldsymbol{u}_1)-Dj_\varepsilon(\boldsymbol{u}_2))\cdot(\boldsymbol{u}_1-\boldsymbol{u}_2)}_{\geq 0} + C(p_1-p_2,p_1-p_2) \\
&-\alpha\sum_K h_K^2\int_K 2\mu\mathrm{div}\varepsilon(\boldsymbol{u}_1-\boldsymbol{u}_2)\cdot\nabla(p_1-p_2) \\
&\geq 2\mu\left\|\varepsilon(\boldsymbol{u}_1-\boldsymbol{u}_2)\right\|^2 + \alpha\left\|p_1-p_2\right\|_h^2 - 2\alpha\mu c_I\left\|\varepsilon(\boldsymbol{u}_1-\boldsymbol{u}_2)\right\|\left\|p_1-p_2\right\|_h \\
&\geq 2\mu\left(1-2\alpha\mu c_I^2\right)\left\|\varepsilon(\boldsymbol{u}_1-\boldsymbol{u}_2)\right\|^2 + \frac{\alpha}{2}\left\|p_1-p_2\right\|_h^2
\end{aligned}$$

which is non-negative as long as $\alpha < (2\mu c_I^2)^{-1}$.
<u>$\mathcal{K}$ is coercive</u>. Indeed, let $\boldsymbol{v},q$ elements of $\mathbb{V}_h^l\times M_h^l$, one has

$$\begin{aligned}
\mathcal{K}(\boldsymbol{v},q)(\boldsymbol{v},q) =&a(\boldsymbol{v},\boldsymbol{v}) + \underbrace{Dj_\varepsilon(\boldsymbol{v})\cdot\boldsymbol{v}}_{\geq 0} + C(q,q) - \alpha\sum_K h_K^2\int_K 2\mu\mathrm{div}\varepsilon(\boldsymbol{v})\cdot\nabla q \\
&\geq 2\mu\left\|\varepsilon(\boldsymbol{v})\right\|^2 + \alpha\left\|q\right\|_h^2 - 2\alpha\mu c_I\left\|\varepsilon(\boldsymbol{v})\right\|\left\|q\right\|_h \\
&\geq 2\mu\left(1-2\alpha\mu c_I^2\right)\left\|\varepsilon(\boldsymbol{v})\right\|^2 + \frac{\alpha}{2}\left\|q\right\|_h^2\,.
\end{aligned}$$

$\mathcal{K}$ is hemi-continuous in $\mathbb{V}_h^l \times M_h^l$. We invite the reader to see similar manipulations in Proposition 4.3. We then conclude partially that (4.19) has a solution $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon) \in \mathbb{V}_h^l \times M_h^l$. But because $\mathcal{K}$ is strictly monotone, then the solution $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ is unique.
Next, the passage to the limit when ε approaches zero of $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$, solution of (4.19) is done using the bound (see how (4.18) is derived)

$$\|\boldsymbol{u}_h^\varepsilon\|_1^2 + \|p_h^\varepsilon\|_h^2 + \|p_h^\varepsilon\|^2 + j(\boldsymbol{u}_h^\varepsilon) \leq c\|\boldsymbol{f}\|^2 \ ,$$

and follow to the line the reasoning adopted in the proof of Proposition 4.3. Thus we have constructed the solutions of (3.6).
The unique solvability of (3.6) is obtained as Step 3 in Proposition 4.3. $\square$

Remark 4.2 *For $l \geq 2$, (3.5) and (3.6) are stable only for some values of α (see proposition 4.1 and proposition 4.4). But if $l = 1$, then these schemes are the same and stable for all values of α.*

5 A priori error estimates

The goal of the section is to establish the convergence by estimating the difference between the continuous solution $(\boldsymbol{u}, p)$ and the finite element solution $(\boldsymbol{u}_h, p_h)$.
We first claim that

Theorem 5.1 *Let the mesh $\mathcal{T}_h$ satisfies (3.1), and assume that for $d = 3$, the boundary $\partial\Omega$ is Lipschitz-continuous. Let $(\boldsymbol{u}, p) \in \mathbb{V}\times M$ be the solution of (2.11). Let $(\boldsymbol{u}_h, p_h) \in \mathbb{V}_h^l \times M_h^l$ the solution of (3.5). Let α be the stabilization parameter satisfying the inequality $\alpha < (2\mu c_I^2)^{-1}$, with c_I given by (4.1). Then, there is a positive constant c independent of h such that for all $(\boldsymbol{v}_h, q_h) \in \mathbb{V}_h^l \times M_h^l$*

$$\begin{aligned} \|\boldsymbol{u}_h - \boldsymbol{u}\|_1 + \|p_h - p\|_h \ &\leq \ c\|g\|_{L^\infty(S)}^{1/2} \|\boldsymbol{v}_{h,\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^1(S)}^{1/2} \\ &+ \ c\left(\|\boldsymbol{v}_h - \boldsymbol{u}\|_1 + h\|\boldsymbol{v}_h - \boldsymbol{u}\|_2 + \|q_h - p\| + h\|\nabla(q_h - p)\|\right) . \end{aligned}$$

Proof. Let $(\boldsymbol{v}_h, q_h)$ in $\mathbb{V}_h^l \times M_h^l$, having in mind $K(\cdot\,,\,\cdot)$ defined in (4.3), then from proposition 4.1 there exists c independent of h such that

$$\begin{aligned} & c\left(\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_1^2 + \|p_h - q_h\|_h^2\right) \\ \leq & K(\boldsymbol{u}_h - \boldsymbol{v}_h, p_h - q_h; \boldsymbol{u}_h - \boldsymbol{v}_h, p_h - q_h) \\ = & K(\boldsymbol{u}_h - \boldsymbol{u}, p_h - q; \boldsymbol{u}_h - \boldsymbol{v}_h, p_h - q_h) + K(\boldsymbol{u} - \boldsymbol{v}_h, p - q_h; \boldsymbol{u}_h - \boldsymbol{v}_h, p_h - q_h) \end{aligned} \tag{5.1}$$

We now estimate the first term on the right hand side of (5.1). We recall that $(\boldsymbol{u}, p)$ satisfies

$$\begin{cases} \text{for all } (\boldsymbol{v}, q) \in \mathbb{V} \times M \ , \\ K(\boldsymbol{u}, p; \boldsymbol{v} - \boldsymbol{u}, q) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq \ell_1(\boldsymbol{v} - \boldsymbol{u}) - \ell_2(q) \, . \end{cases}$$

We take successively $(\boldsymbol{v}, q) = \left(\boldsymbol{u}_h, \frac{1}{2}p_h - \frac{1}{2}q_h\right)$, and $(\boldsymbol{v}, q) = \left(2\boldsymbol{u} - \boldsymbol{v}_h, \frac{1}{2}p_h - \frac{1}{2}q_h\right)$, add the resulting equations and obtain

$$K(\boldsymbol{u}, p; \boldsymbol{u}_h - \boldsymbol{v}_h, p_h - q_h) + j(\boldsymbol{u}_h) - 2j(\boldsymbol{u}) + j(2\boldsymbol{u} - \boldsymbol{v}_h) \geq \ell_1(\boldsymbol{u}_h - \boldsymbol{v}_h) - \ell_2(p_h - q_h) \, . \tag{5.2}$$

We consider (4.3) with q_h replaced by $-p_h + q_h$, we add the resulting equation with (5.2) and obtain

$$\begin{aligned} K(\boldsymbol{u}_h - \boldsymbol{u}, p_h - p; \boldsymbol{u}_h - \boldsymbol{v}_h, p_h - q_h) \leq & j(\boldsymbol{v}_h) - j(\boldsymbol{u}) + j(2\boldsymbol{u} - \boldsymbol{v}_h) - j(\boldsymbol{u}) \\ \leq & 2j(\boldsymbol{v}_h - \boldsymbol{u}) \, . \end{aligned} \tag{5.3}$$

Returning to (5.1) with (5.3), one obtains

$$c\left(\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_1^2+\|p_h-q_h\|_h^2\right)\le 2j(\boldsymbol{v}_h-\boldsymbol{u})+K(\boldsymbol{u}-\boldsymbol{v}_h,p-q_h;\boldsymbol{u}_h-\boldsymbol{v}_h,p_h-q_h)$$
$$\le c\|g\|_{L^\infty(S)}\|\boldsymbol{v}_{h,\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^1(S)}+c\left(\|\boldsymbol{v}_h-\boldsymbol{u}\|_1^2+h^2\|\boldsymbol{v}_h-\boldsymbol{u}\|_2^2+\|q_h-p\|^2+h^2\|\nabla(q_h-p)\|^2\right)^{1/2}$$
$$\times\left(\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_1^2+\|q_h-p_h\|_h^2\right)^{1/2},$$

which by Young's inequality gives

$$\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_1^2+\|p_h-q_h\|_h^2\le c\|g\|_{L^\infty(S)}\|\boldsymbol{v}_{h,\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^1(S)}+c\|\boldsymbol{v}_h-\boldsymbol{u}\|_1^2+ch^2\|\boldsymbol{v}_h-\boldsymbol{u}\|_2^2$$
$$+c\|q_h-p\|^2+ch^2\|\nabla(q_h-p)\|^2\,. \tag{5.4}$$

The asserted result follows after application of the triangle's inequality. □

Remark 5.1 *It should be noted that the consistency argument has not been used in the proof of Theorem 5.1. Similar arguments are used in [2, 10] to derive a priori error estimates for a class of mixed problems with Dirichlet boundary condition. For Stokes equations under Dirichlet boundary condition, GLS methods were formulated and analysed in [4, 5], and convergence is obtained if consistency is required.*

Remark 5.2 *If $(\boldsymbol{u}_h,p_h)$ are approximated by piecewise linear functions, then for all values of α, the error estimate becomes*

$$\begin{aligned}\|\boldsymbol{u}_h-\boldsymbol{u}\|_1+\|p_h-p\|_h &\le c\|g\|_{L^\infty(S)}^{1/2}\|\boldsymbol{v}_{h,\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^1(S)}^{1/2}+ch\|\boldsymbol{u}\|_2\\ &+ c\left(\|\boldsymbol{v}_h-\boldsymbol{u}\|_1+\|q_h-p\|+h\|\nabla(q_h-p)\|\right)\,.\end{aligned}$$

Using interpolation operators constructed by V. Girault and F. Hecht [40, Chap 5] to take into account the slip boundary condition, we have:

- *If the solution is such that $\boldsymbol{u}_{\boldsymbol{\tau}}|_S\in H^2(S)$, and $(\boldsymbol{u},p)\in\boldsymbol{H}^2(\Omega)\times H^1(\Omega))$, then*

$$\|\boldsymbol{v}_{h,\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^1(S)}\le c\|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{v}_{h,\boldsymbol{\tau}}\|_S\le ch^2\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{2,S}\,.$$

Thus

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1+\|p-p_h\|_h\le c\,h\,.$$

- *If the solution $(\boldsymbol{u},p)\in\boldsymbol{H}^2(\Omega)\times H^1(\Omega)$, then from [41, p.39] and $1\le p<\infty$, there exists c such that*

$$\|v\|_{L^p(\partial\Omega)}\le c\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W^{1,p}(\Omega)}^{1/p}\ ,\ \textit{for all } v\in W^{1,p}(\Omega)\,.$$

Hence

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1+\|p-p_h\|_h\le ch^{3/4}\,.$$

It should be noted that these results are not new, and confirmed results obtained before (see [6, 7, 8, 9]). We can also conclude at this level that the stabilization procedure do not change the convergence rate, but rather we have the possibility to use a larger class of finite element approximations.

Remark 5.3 *Let $(\boldsymbol{u}_h,p_h)\in\mathbb{V}_h^l\times M_h^l$ the solution of (3.6). Given the assumptions of theorem 5.1, then there is a positive constant c independent of h such that for all $(\boldsymbol{v}_h,q_h)\in\mathbb{V}_h^l\times M_h^l$*

$$\begin{aligned}\|\boldsymbol{u}_h-\boldsymbol{u}\|_1+\|p_h-p\|_h &\le c\|g\|_{L^\infty(S)}^{1/2}\|\boldsymbol{v}_{h,\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^1(S)}^{1/2}\\ &+ c\left(\|\boldsymbol{v}_h-\boldsymbol{u}\|_1+\|q_h-p\|+h\|\nabla(q_h-p)\|\right)\,.\end{aligned}$$

Theorem 5.1 is concerned with the mesh dependent norm on the pressure and the question we answer next is to know whether it is possible to have a control on the pressure with the L^2 norm. For that purpose, we claim that

Theorem 5.2 *Assume that the mesh $\mathcal{T}_h$ satisfies (3.1), and assume that for $d = 3$, the boundary $\partial\Omega$ is Lipschitz-continuous. Let $(\boldsymbol{u}, p) \in \mathbb{V} \times M$ be the solution of (2.11). Let $(\boldsymbol{u}_h, p_h) \in \mathbb{V}_h^l \times M_h^l$ the solution of (3.5). Let α be the stabilization parameter satisfying the relation $\alpha < (2\mu c_I^2)^{-1}$ with with c_I given by (4.1). Then there is a positive constant c independent of h such that for all $(\boldsymbol{v}_h, q_h) \in \mathbb{V}_h^l \times M_h^l$*

$$\begin{aligned}\|p_h - p\| \quad \leq \quad & c\|g\|_{L^\infty(S)}^{1/2}\|\boldsymbol{v}_{h,\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^1(S)}^{1/2} + c\|\boldsymbol{v}_h - \boldsymbol{u}\|_1 + ch\|\boldsymbol{v}_h - \boldsymbol{u}\|_2 + c\|q_h - p\| \\ & + ch\|\nabla(q_h - p)\| + ch\|q_h - p\| \,.\end{aligned}$$

Proof.

We recall that $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ satisfy for all $(\boldsymbol{v}, \boldsymbol{v}_h) \in \mathbb{V} \times \mathbb{V}_h^l$

$$\begin{cases} A(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + B(\boldsymbol{v} - \boldsymbol{u}, p) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq \ell_1(\boldsymbol{v} - \boldsymbol{u}) \\ A(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + B(\boldsymbol{v}_h - \boldsymbol{u}_h, p_h) + j(\boldsymbol{v}_h) - j(\boldsymbol{u}_h) \geq \ell_1(\boldsymbol{v}_h - \boldsymbol{u}_h) \,. \end{cases}$$

Let $\boldsymbol{w} \in H_0^1(\Omega)^d$, and $H_{0h}^1(\Omega)^d$ the conforming finite element space approximating $H_0^1(\Omega)^d$. We take $\boldsymbol{v} - \boldsymbol{u} = \pm\boldsymbol{w}$ and $\boldsymbol{v}_h - \boldsymbol{u}_h = \pm\boldsymbol{w}_h$. Thus one obtains

$$\begin{cases} A(\boldsymbol{u}, \boldsymbol{w}) + B(\boldsymbol{w}, p) = \ell_1(\boldsymbol{w}) \\ A(\boldsymbol{u}_h, \boldsymbol{w}_h) + B(\boldsymbol{w}_h, p_h) = \ell_1(\boldsymbol{w}_h) \,, \end{cases}$$

which implies that ($\boldsymbol{w}_h \in H_{0h}^1(\Omega)^d \subset H_0^1(\Omega)^d$)

$$A(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{w}_h) = B(\boldsymbol{w}_h, p - p_h)$$

which by linearity gives

$$B(\boldsymbol{w}_h, q_h - p_h) = B(\boldsymbol{w}_h, q_h - p) + A(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{w}_h) \,. \tag{5.5}$$

From [4] (see Lemma 3.2), there exists c_1, c_1 independent of h such that

$$c_1\|p_h - q_h\| \leq c_2\|p_h - q_h\|_h + \sup_{0 \neq \boldsymbol{w}_h \in H_{0h}^1(\Omega)^d} \frac{(\operatorname{div} \boldsymbol{w}_h, p_h - q_h)}{\|\boldsymbol{w}_h\|_1}$$

which together with (5.5), the definition of $B(\cdot,\cdot)$, standard inequalities and (5.4) yields

$$\begin{aligned} & \|p_h - q_h\| \\ \leq & c\|p_h - q_h\|_h + c\|q_h - p\| + ch\|q_h - p\| + c\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_1 + c\|\boldsymbol{v}_h - \boldsymbol{u}\|_1 + ch\|\boldsymbol{v}_h - \boldsymbol{u}\|_2 \\ \leq & c\|g\|_{L^\infty(S)}^{1/2}\|\boldsymbol{v}_{h,\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^1(S)}^{1/2} + c\|\boldsymbol{v}_h - \boldsymbol{u}\|_1 + ch\|\boldsymbol{v}_h - \boldsymbol{u}\|_2 + c\|q_h - p\| + ch\|\nabla(q_h - p)\| \\ & + ch\|q_h - p\| \,. \end{aligned}$$

The asserted result is obtained after application of the inequality of the triangle. □

The next sections are concerned with the solution strategy for (3.5) and (3.6), and their numerical simulations. We assume that both the velocity and pressure are approximated by linear piecewise functions. Hence problem (3.5) and (3.6) coincide. The problem (3.5) is a mixed elliptic variational inequalities of second kind for which several approaches are available in the literature (see [27, 28], or more recently [30, 42]) for its resolution. But in this work we propose to solve it with the following strategies

(i) Projection-like method based on the introduction of a Lagrange multiplier field.

(ii) Alternative direction method of multiplier associated with the augmented Lagrangian method based on the introduction of a new variable aiming to decouple the velocity $\boldsymbol{u}$ from its tangential part $\boldsymbol{u}_{\boldsymbol{\tau}}$, and a Lagrange multiplier field aiming to enforce the relation $\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{\phi} = 0$.

(iii) Active set approach associated with the augmented Lagrangian method based on the introduction of Lagrange multipliers link to the constraints $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ and $\operatorname{div} \boldsymbol{u} = 0$.

6 Dual approximation methods

6.1 Projection-like algorithm

This approach relies on the equivalence between (3.5) and the following one: there exists a vector value $\boldsymbol{\lambda}_h \in \Lambda$ such that

$$\begin{cases} \text{for all } \ (\boldsymbol{v}, q) \in \mathbb{V}_h^1 \times M_h^1, \\ a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) + \displaystyle\int_S g\boldsymbol{\lambda}_h \cdot \boldsymbol{v}_{\boldsymbol{\tau}} = \ell(\boldsymbol{v}), \\ b(\boldsymbol{u}_h, q) - C(q, p_h) = \ell_2(q)\,, \\ \boldsymbol{u}_{\boldsymbol{\tau},h} \cdot \boldsymbol{\lambda}_h = |\boldsymbol{u}_{\boldsymbol{\tau},h}| \ \text{ a.e. in } S, \end{cases} \tag{6.1}$$

with

$$\Lambda = \{\boldsymbol{\alpha} | \boldsymbol{\alpha} \in \boldsymbol{L}^\infty(S)\ , \quad |\boldsymbol{\alpha}| \leq 1 \text{ a.e. in } S\}\ .$$

At this step, it is worth noting that one of the difficulties in implementing (6.1) is to enforce the relation $\boldsymbol{\lambda}_h \cdot \boldsymbol{u}_{\boldsymbol{\tau},h} = |\boldsymbol{u}_{\boldsymbol{\tau},h}|$ a.e. in S. We provide next an equivalent characterization of that relation for a better derivation of iterative schemes. We claim that

Lemma 6.1 *[27] Given that g is non-negative, the following problems are equivalent*

(a) *Find $\boldsymbol{\lambda} \in \Lambda$ such that $\boldsymbol{\lambda} \cdot \boldsymbol{u}_{\boldsymbol{\tau}} = |\boldsymbol{u}_{\boldsymbol{\tau}}|$ a.e. in S,*

(b) *Find $\boldsymbol{\lambda} \in \Lambda$ such that $\displaystyle\int_S g\boldsymbol{u}_{\boldsymbol{\tau}} \cdot (\boldsymbol{\mu} - \boldsymbol{\lambda}) d\sigma \leq 0$ for all $\boldsymbol{\mu} \in \Lambda$.*

(c) *$\boldsymbol{\lambda} = \mathcal{P}_\Lambda(\boldsymbol{\lambda} + \gamma g \boldsymbol{u}_{\boldsymbol{\tau}})$ for all $\gamma > 0$,*

with

$$\mathcal{P}_\Lambda : \boldsymbol{L}^2(S) \longrightarrow \Lambda \quad , \quad \mathcal{P}_\Lambda(\boldsymbol{\alpha})(x) = \frac{\boldsymbol{\alpha}(x)}{\max\left(1, |\boldsymbol{\alpha}(x)|\right)}\,.$$

Using Lemma 6.1, we formulate the following equivalent problem more suitable for the derivation of iterative methods

$$\begin{cases} \text{Find } \ (\boldsymbol{u}_h, p_h, \lambda_h) \in \mathbb{V}_h^1 \times M_h^1 \times \Lambda \ \text{ such that}\,, \\ \text{for all } \ (\boldsymbol{v}, q, \rho) \in \mathbb{V}_h^1 \times M_h^1 \ \text{ and all } \gamma > 0 \\ a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) + \displaystyle\int_S g\boldsymbol{\lambda}_h \cdot \boldsymbol{v}_{\boldsymbol{\tau}} = \ell(\boldsymbol{v}), \\ b(\boldsymbol{u}_h, q) - C(q, p_h) = \ell_2(q)\,, \\ \lambda_h = \mathcal{P}_\Lambda(\lambda_h + \gamma g \boldsymbol{u}_{\boldsymbol{\tau},h}) \ \text{ a.e. in } S\,. \end{cases} \tag{6.2}$$

From (6.2), we consider the Algorithm 1 based on Uzawa iteration
clearpage

Remark 6.1 *Proving the convergence of Algorithm 1 (for $\gamma > 0$ and sufficiently small) is a classical exercise and we refer the interested reader to [27, 28]. Note that the equations for GLS 2 and GLS 1 are identical when using piecewise linear elements because* $\operatorname{div} \varepsilon(\boldsymbol{v}) = \boldsymbol{0}$.

6.2 Alternating Direction Method of Multiplier (ADMM)

With piecewise linear polynomial approximation, i.e. with the finite element pair $\mathcal{P}_1/\mathcal{P}_1$, Lagrangian functional (3.3) can be simplified. Indeed, with $\mathcal{P}_1$ finite element, the additional stabilization terms involving derivatives of order greater than one vanish and we obtain

$$J_\alpha(\boldsymbol{v}, q) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - \ell(\boldsymbol{v}) + j(\boldsymbol{v}) - b(\boldsymbol{v}, q) - C(q, q) - \ell_2(q)\,. \tag{6.3}$$

To derive ADMM algorithm for the numerical approximation of (6.3), we introduce an auxiliary variable ϕ on S and we replace J_α by the following augmented Lagrangian functional

Algorithm 1 : Uzawa iterative algorithm

Initialization: Given $\boldsymbol{\lambda}_h^0 = (0,1) \in \Lambda$, we compute $(\boldsymbol{u}_h^0, p_h^0)$ such that

$$\begin{cases} \text{for all} \quad (\boldsymbol{v}, q) \in \mathbb{V}_h \times M_h, \\ a(\boldsymbol{u}_h^0, \boldsymbol{v}) + b(\boldsymbol{v}, p_h^0) = \ell(\boldsymbol{v}) - \displaystyle\int_S g \boldsymbol{\lambda}_h^0 \cdot \boldsymbol{v}_{\boldsymbol{\tau}} \ , \\ b(\boldsymbol{u}_h^0, q) - C(q, p_h^0) = \ell_2(q) \ . \end{cases}$$

Iteration $k \geq 0$ By induction, knowing $\{\boldsymbol{u}_h^k, p_h^k, \boldsymbol{\lambda}_h^k\}$, we compute $\{\boldsymbol{u}_h^{k+1}, p_h^{k+1}, \boldsymbol{\lambda}_h^{k+1}\}$ iteratively as follows.

Step 1: For all $(\boldsymbol{v}, q) \in \mathbb{V}_h^1 \times M_h^1$, solve

$$\begin{aligned} &a(\boldsymbol{u}_h^{k+1}, \boldsymbol{v}) + b(\boldsymbol{v}, p_h^{k+1}) = \ell(\boldsymbol{v}) - \int_S g \boldsymbol{\lambda}_h^k \cdot \boldsymbol{v}_{\boldsymbol{\tau}} d\sigma \, , \\ &b(\boldsymbol{u}_h^{k+1}, q) - C(q, p_h^{k+1}) = \ell_2(q) \ . \end{aligned}$$

Step 2: For $\gamma > 0$, compute the Lagrange multiplier

$$\boldsymbol{\lambda}_h^{k+1} = \mathcal{P}_\Lambda(\boldsymbol{\lambda}_h^k + \gamma g \boldsymbol{u}_{\boldsymbol{\tau},h}^{k+1}) \ .$$

$$\begin{aligned} \mathcal{L}_{r\alpha}(\boldsymbol{v}, q, \boldsymbol{\phi}, \boldsymbol{\mu}) =& \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - \ell(\boldsymbol{v}) + j(\boldsymbol{\phi}) - b(\boldsymbol{v}, q) - C(q, q) - \ell_2(q) \\ &+ (\boldsymbol{\mu}, \boldsymbol{v}_{\boldsymbol{\tau}})_S + \frac{r}{2} \parallel \boldsymbol{\phi} - \boldsymbol{v}_{\boldsymbol{\tau}} \parallel_S^2 \ . \end{aligned}$$

The idea is to separate the non-differentiable part of the problem (i.e. j) from the differentiable part and to use block relaxation scheme as follows

$$(\boldsymbol{u}^{k+1}, p^{p+1}) = \arg\min_{\boldsymbol{v}} \max_q \mathcal{L}_{r\alpha}(\boldsymbol{v}, q, \boldsymbol{\phi}^k, \boldsymbol{\lambda}^k) \, , \tag{6.4}$$

$$\boldsymbol{\phi}^{k+1} = \arg\min_{\psi} \mathcal{L}_{r\alpha}(\boldsymbol{u}^{k+1}, p^{k+1}, \psi, \lambda^k) \, , \tag{6.5}$$

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + r(\boldsymbol{\phi}^{k+1} - \boldsymbol{u}_{\boldsymbol{\tau}}^{k+1}) \, .$$

Subproblem (6.4) is equivalent to the Stokes problem with tangential traction on S, i.e.

$$\begin{aligned} a_r(\boldsymbol{u}^{k+1}, \boldsymbol{v}) - b(\boldsymbol{v}, p^{k+1}) &= \ell(\boldsymbol{v}) + (r\boldsymbol{\phi}^k - \boldsymbol{\lambda}^k, \boldsymbol{v}_{\boldsymbol{\tau}})_S, \quad \forall \boldsymbol{v} \\ -b(\boldsymbol{u}^{k+1}, q) - C(p^{k+1}, q) &= \ell_2(q), \quad \forall q \end{aligned}$$

where

$$a_r(\boldsymbol{u}^{k+1}, \boldsymbol{v}) = a(\boldsymbol{u}^{k+1}, \boldsymbol{v}) + r(\boldsymbol{u}_{\boldsymbol{\tau}}^{k+1}, \boldsymbol{v}_{\boldsymbol{\tau}})_S.$$

Subproblem (6.5) can be solved analytically using Fenchel duality theory, and we get (see, e.g., [42])

$$\boldsymbol{\phi}^{k+1} = \frac{1}{r} \max(0, \parallel \boldsymbol{\lambda}^k - r\boldsymbol{\phi}^k \parallel - g) \frac{\boldsymbol{\lambda}^k - r\boldsymbol{\phi}^k}{\parallel \boldsymbol{\lambda}^k - r\boldsymbol{\phi}^k \parallel}.$$

Gathering the results above, we obtain Algorithm 2. We iterate until the relative error in $(\boldsymbol{u}^k, p^k, \boldsymbol{\phi}^k, \boldsymbol{\lambda}^k)$ becomes sufficiently small.

Algorithm 2 : Alternating Direction Method of Multiplier

Initialization $k = 0$ $(\boldsymbol{\phi}^0, \boldsymbol{\lambda}^0)$ and $r > 0$ are given.

$k \geq 0$ Compute successively $(\boldsymbol{u}^{k+1}, p^{k+1})$, $\boldsymbol{\phi}^{k+1}$ and $\boldsymbol{\lambda}^{k+1}$ as follows.

Step 1. Find $(\boldsymbol{u}^{n+1}, p^{n+1}) \in \boldsymbol{V} \times L_0^2(\Omega)$ such that for all $(\boldsymbol{v}, q) \in \boldsymbol{V} \times L^2(\Omega)$

$$\begin{aligned} a(\boldsymbol{u}^{n+1}, \boldsymbol{v}) + r(\boldsymbol{u}_{\boldsymbol{T}}^{n+1}, \boldsymbol{v}_{\boldsymbol{T}})_S - b(\boldsymbol{v}, p^{n+1}) &= \ell_1(\boldsymbol{v}) + (r\boldsymbol{\phi}^k - \boldsymbol{\lambda}^k, \boldsymbol{v}_{\boldsymbol{T}})_S \ , \\ -b(\boldsymbol{u}^{n+1}, q) - C(p^{k+1}, q) &= \ell_2(q). \end{aligned}$$

Step 2. Compute the auxiliary unknown

$$\boldsymbol{\phi}^{k+1} = \frac{1}{r} \max(0, \parallel \boldsymbol{\lambda}^k - r\boldsymbol{\phi}^k \parallel - g) \frac{\boldsymbol{\lambda}^k - r\boldsymbol{\phi}^k}{\parallel \boldsymbol{\lambda}^k - r\boldsymbol{\phi}^k \parallel}.$$

Step 3. Multiplier update $\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + r(\boldsymbol{\phi}^{k+1} - \boldsymbol{u}_{\boldsymbol{T}}^{k+1})$

7 A Primal-Dual approximation method

As in the previous section, we consider only piecewise linear elements. The primal dual method we formulate in this section derives from the one proposed by [29]. Just like the ADMM method, this strategy is based on the introduction of a functional for which the saddle point plays a crucial role. We recall the basic steps, proceed to the algebraic formalism which leads to the formulation of the algorithm.

7.1 Active set strategy

We first regularize the non-differentiable term $j(\boldsymbol{v})$ by using the equality (obtained using Fenchel duality)

$$\inf_{\boldsymbol{v}} j(\boldsymbol{v}) = \inf_{\boldsymbol{v}} \sup_{|\boldsymbol{\lambda}| \leq g} (\boldsymbol{\lambda}, \boldsymbol{v}_{\boldsymbol{T}})_S.$$

Let us introduce the set of admissible Lagrange multiplier

$$\Lambda_g = \left\{ \boldsymbol{\lambda} \in L^2(S) \mid |\boldsymbol{\lambda}| \leq g \right\}$$

and the new Lagrangian functional

$$\mathcal{L}_\alpha(\boldsymbol{v}, q, \boldsymbol{\mu}) = J(\boldsymbol{v}) - b(\boldsymbol{v}, q) + (\boldsymbol{v}_{\boldsymbol{T}}, \boldsymbol{\mu})_S - \sum_{K \in \mathcal{T}_h} \alpha \frac{h_K^2}{2} \int_K |2\mu \operatorname{div} \varepsilon(\boldsymbol{v}) - \nabla q - \boldsymbol{f}|^2 \mathrm{d}x.$$

The saddle-point problem becomes

Find $(\boldsymbol{u}, p, \boldsymbol{\lambda}) \in \mathbb{V}_h^1 \times M_h^1 \times \Lambda_g$ such that

$$\mathcal{L}_\alpha(\boldsymbol{u}, q, \boldsymbol{\mu}) \leq \mathcal{L}_\alpha(\boldsymbol{u}, p, \boldsymbol{\lambda}) \leq \mathcal{L}_\alpha(\boldsymbol{v}, q, \boldsymbol{\mu}), \quad \forall (\boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathbb{V}_h^1 \times M_h^1 \times \Lambda_g \tag{7.1}$$

Our aim is to design a primal-dual active set strategy for the numerical approximation of (7.1). Our primal-dual active set strategy derives from [29] and based on the following facts:

- If $|\boldsymbol{\lambda}| < g$ then $\boldsymbol{u}_{\boldsymbol{T}} = 0$. We can therefore eliminate the corresponding nodal values of $\boldsymbol{u}_{\boldsymbol{T}}$ (and $\boldsymbol{\lambda}$) from the global system.
- If $|\boldsymbol{\lambda}| = g$, then the multiplier is known and acts as a tangential traction.

7.2 Algebraic formulation

We need the algebraic formulation for the active set strategy. We use the same discrete formulation as in [31]. Assuming that $\mathbf{u} \in \mathbb{R}^{dn}$ is the unknown vector of nodal values of the velocity fields on Ω_h, $\mathrm{p} \in \mathbb{R}^n$ the unknown vector of nodal values of the pressure and $\boldsymbol{\lambda} \in \mathbb{R}^m$ the multiplier vector, we introduce the following matrices and vectors:

- $\mathbf{A}$ the stiffness matrix ($dn \times dn$ symmetric positive definite), $\mathbf{C}$ the pressure stiffness matrix ($n \times n$ symmetric positive semi-definite);
- $\mathbf{B}$ the divergence matrix, $n \times dn$.
- $\mathbf{f}$, the volume forces (vector of $\mathbb{R}^{dn}$),
- $\mathbf{T}$, the tangential matrix on S, i.e. $\mathbf{Tu} = \mathbf{u}_{\boldsymbol{\tau}}$,
- $\mathbf{M}$, the mass matrix on S,
- $\mathbf{g}$, the vector of slip threshold.

The discrete formulation of the Lagrangian functional (6.3) is therefore

$$\mathcal{L}_\alpha(\mathbf{v}, \mathrm{q}, \boldsymbol{\mu}) = \frac{1}{2}\mathbf{v}^\top \mathbf{A}\mathbf{v} - \mathbf{f}^\top + \boldsymbol{\mu}^\top \mathbf{M}\mathbf{v} - \mathrm{q}^\top \mathbf{B}\mathbf{v} - \alpha \mathrm{q}^\top \mathbf{C}\mathrm{q} - \mathrm{q}^\top \mathbf{B}\mathbf{f}.$$

The (tangential) Lagrange multiplier is such that $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m)^\top$ in two-dimensional problems, and $\boldsymbol{\mu} = ((\boldsymbol{\mu}_{1,1}, \boldsymbol{\mu}_{1,2}), \ldots, (\boldsymbol{\mu}_{m,1}, \boldsymbol{\mu}_{m,2}))^\top$ for three-dimensional problems. Then

$$\begin{aligned} |\boldsymbol{\mu}| &= (|\boldsymbol{\mu}_1|, \ldots, |\boldsymbol{\mu}_m|)^\top \quad \text{for 2D problems} \\ |\boldsymbol{\mu}| &= ((\boldsymbol{\mu}_{1,1}^2 + \boldsymbol{\mu}_{1,2}^2)^{1/2}, \ldots, (\boldsymbol{\mu}_{m,1}^2 + \boldsymbol{\mu}_{m,2}^2)^{1/2})^\top \quad \text{for 3D problems} \end{aligned}$$

Then writing $|\boldsymbol{\mu} \leq \mathbf{g}$ means $|\boldsymbol{\mu}_i| \leq \mathbf{g}_i$ in 2D, or $(\boldsymbol{\mu}_{i,1}^2 + \boldsymbol{\mu}_{i,2}^2)^{1/2} \leq \mathbf{g}$ in 3D.

Gathering the notations above, our primal dual active set method is described in Algorithm 3.

Algorithm 3 : First primal dual active set method

Initialization $(\mathbf{u}^0, \mathrm{p}^0, \boldsymbol{\lambda}^0)$ given, set $k = 0$.

Step 1. Set $\mathcal{I}^k = \{i;\ |\boldsymbol{\lambda}_i^k| < \mathbf{g}_i\}$, $\mathcal{A}^k = \{i;\ |\boldsymbol{\lambda}_i^k| \geq \mathbf{g}_i\}$

Step 2. Set $\boldsymbol{\lambda}_i^{k+1} = \mathbf{g}_i \boldsymbol{\lambda}_i^k / |\boldsymbol{\lambda}_i^k|$ for $i \in \mathcal{A}^k$

Step 3. Compute $(\mathbf{u}^{k+1}, \mathrm{p}^{k+1})$ with $\mathbf{u}_{\boldsymbol{\tau}}^{k+1} = 0$ on $\mathcal{I}^k$ such that

$$\begin{aligned} \mathbf{A}\mathbf{u}^{k+1} - \mathbf{B}^\top \mathrm{p}^{k+1} &= \mathbf{f} - \mathbf{T}^\top \mathbf{M}\boldsymbol{\lambda}^{k+1}, \\ \mathbf{B}\mathbf{u}^{k+1} - \alpha \mathbf{C}\mathrm{p}^{k+1} &= \mathbf{B}\mathbf{f} \end{aligned}$$

Step 4. Compute $\boldsymbol{\lambda}^{k+1}$ on $\mathcal{I}^k$ as a reaction of $\boldsymbol{u}_{\boldsymbol{\tau}}^{k+1} = 0$, i.e.

$$\boldsymbol{\lambda}^{k+1} = \mathbf{M}^{-1}(\mathbf{f} - \mathbf{A}\mathbf{u}^{k+1} + \mathbf{B}^\top \mathrm{p}^{k+1})$$

Step 5. Stop if the relative error on $(\mathbf{u}^{k+1}, \mathrm{p}^{k+1}, \boldsymbol{\lambda}^{k+1})$ becomes sufficiently small and $\mathcal{A}^{k+1} = \mathcal{A}^k$, else set $k = k + 1$ and got to Step 1.

---

8 Numerical experiments

We now study the numerical behavior of the algorithms described in the previous sections. We have implemented Algorithms 1,2 and 3 in MATLAB (R2018a), using vectored assembling functions and the mesh generator provided in [43, 44], on a computer running Linux

(Ubuntu 16.04) with 3.00GHz clock frequency and 32GB RAM. The test problems used are designed to illustrate the numerical behavior of the algorithms and validate the theoretical findings in the paper.

8.1 Driven cavity problem

We consider a classical driven cavity example with stick/slip boundary conditions (see e.g. [30, 45]). We set $\Omega = (0,1)^2$ and we assume that its boundary consists of two portions Γ_D and S defined as follows

$$\begin{aligned} \Gamma_D &= \{0\} \times (0,\ 1) \cup (0,\ 1) \times \{0\} \\ S &= S_1 \cup S_2, \quad S_1 = (0,\ 1) \times \{1\}, \quad S_2 = \{1\} \times (0,\ 1). \end{aligned}$$

The right-hand side

$$\boldsymbol{f} = -2\mu \operatorname{div} \varepsilon(\boldsymbol{u}) + \nabla p$$

where $\mu = 0.1$ and $(\boldsymbol{u}, p)$ is

$$u_1(x,y) = -x^2 y(x-1)(3y-2), \tag{8.1}$$

$$u_2(x,y) = xy^2(y-1)(3x-2), \tag{8.2}$$

$$p(x,y) = (2x-1)(2y-1),$$

from which we deduce that (see [30]);

$$\begin{aligned} (\boldsymbol{Tn})_{\boldsymbol{\tau}} &= -4\mu x^2(x-1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ on } S_1\,, \\ (\boldsymbol{Tn})_{\boldsymbol{\tau}} &= -4\mu y^2(y-1) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ on } S_2\,. \end{aligned}$$

For $\mu = 0.1$, a direct calculation reveals that

$$\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| = 4\mu \max_{x\in S_1}\{x^2(x-1)\} = 4\mu \max_{x\in S_2}\{y^2(y-1)\} = 0.059\,.$$

Then it follows that if $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| < g$, no slip occurs on S and $\boldsymbol{u}_{\boldsymbol{\tau}}|_S = \boldsymbol{0}$. If $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| = g$ a non-trivial slip occurs.
Figure 1 shows the streamlines obtained, using Algorithm 3, with two values of the slip bound g. We can notice that for $g = 0.059 = \max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}|$ a non-trivial slip occurs, while for $\max_S |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| < g = 0.075$, the no slip and $\boldsymbol{u}_{\boldsymbol{\tau}} = 0$. Figure 2 shows the tangential component of the velocity on S. Hence there is a nonlinear slip on S.

8.2 Convergence

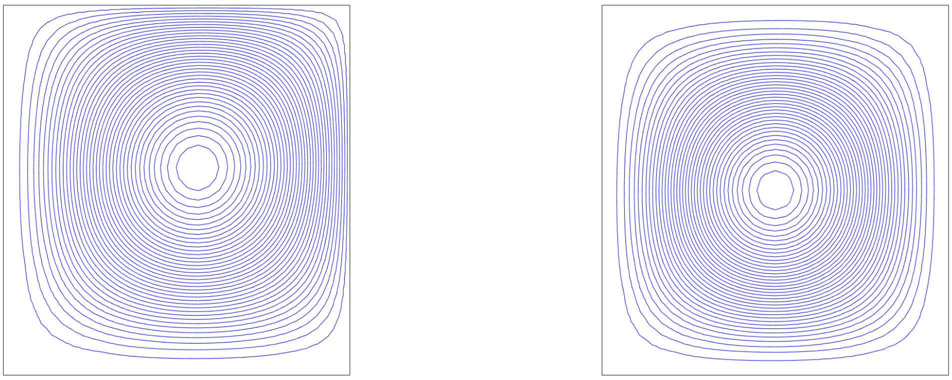
We evaluate the convergence rate of the GLS FEM by deriving a priori error estimate between $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$. It is noted that because we do not know the exact solution explicitly, we use an approximate solution on a finer mesh as the reference solution. The convergence errors are computed as follows

$$e_h(\boldsymbol{u}) = \| \boldsymbol{u}_h - \boldsymbol{u}_* \|_{L^2} \tag{8.3}$$

$$e_h(\boldsymbol{u}, p) = \| \boldsymbol{u}_h - \boldsymbol{u}_* \|_{H^1} + \| p_h - p_* \|_{L^2} \tag{8.4}$$

where $(\boldsymbol{u}_*, p_*)$ is the reference solution obtained on a mesh with $h = 1/1024$.

The convergence rates of the GLS finite element plotted in Figure 3. For the stick case ($g = 0.075$) the convergence rates are better than expected and correspond to convergence rate of the GLS stabilization for the standard Stokes equation. For the slip case ($g = 0.059$) the convergence rate is in line with the expected value (Remark 5.2) since , from (8.1)–(8.2), $\boldsymbol{u}|_S \in H^2(S)$.



(a) (b)

Figure 1: Streamlines for the driven cavity; (a): $g = 0.059$, (b): $g = 0.075$

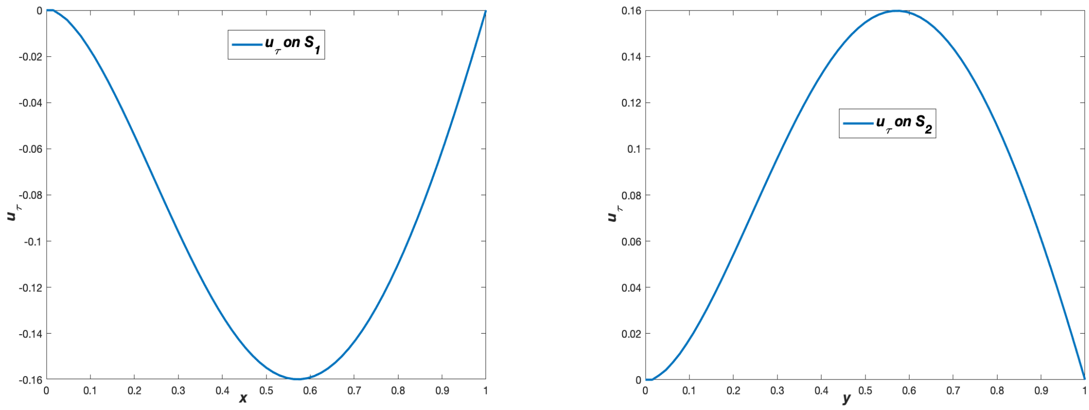


Figure 2: Tangential component of the velocity on S for $g = 0.059$

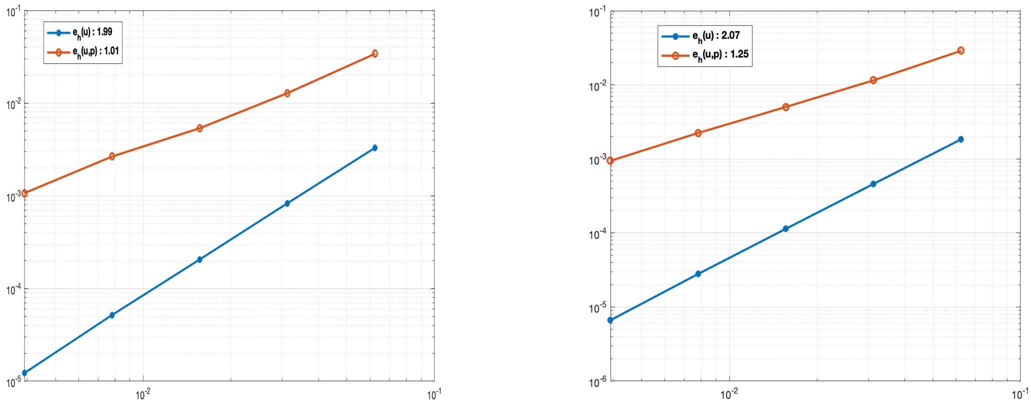


Figure 3: Convergence errors $e_h(\boldsymbol{u})$ and $e_h(\boldsymbol{u}, p)$ and estimation of the convergence rates; left: $g = 0.059$, right: $g = 0.075$

8.3 Algorithms comparison

We report in Tables 1-2 the performances of Algorithms 1-3 on the driven cavity problem. We first notice the poor convergence properties of the Uzawa iteration Algorithm 1, especially for the slip case. For the largest problem, Uzawa iteration requires almost 10 times more CPU times than Algorithm 2, and 50 times more iterations than Algorithm 3. In terms of number of iterations required for convergence, the primal dual Algorithm 3 has the best property. For the solely sticking case, the convergence is reached after only two iterations. In this case, the problem is equivalent to the standard Stokes problem with Dirichlet boundary conditions. The algorithm proposed needs only two iterations to confirm that the active set is empty. For the slip case ($g = 0.059$) the number of iterations required for convergence is asymptotically bounded. In terms of CPU time, Algorithm 2 outperforms the primal dual active set strategy, even with a higher number of iterations. This is due to the fact that in Algorithm 2, the matrix is constant during the iterative process. A factorization can be performed once and for all in the initialization step. The solution of the linear systems during the iterative process is then reduced to backward-forward substitutions.

	Uzawa		ADMM		PDAS 1	
h	Iter.	CPU	Iter.	CPU	Iter.	CPU
1/16	595	0.583	59	0.056	5	0.042
1/32	513	2.341	58	0.236	7	0.185
1/64	512	12.843	58	1.379	8	0.992
1/128	513	83.106	58	5.379	10	7.963
1/256	515	465.077	58	50.050	13	67.176

Table 1: Performances of Algorithms 1,2 and 3 on the driven cavity for $g = 0.059$

	Uzawa		ADMM		PDAS 1	
h	Iter.	CPU	Iter.	CPU	Iter.	CPU
1/16	418	0.089	27	0.026	2	0.069
1/32	369	0.365	16	0.065	2	0.052
1/64	361	1.592	9	0.213	2	0.235
1/128	360	9.096	9	1.139	2	1.434
1/256	359	58.542	9	7.760	2	13.088

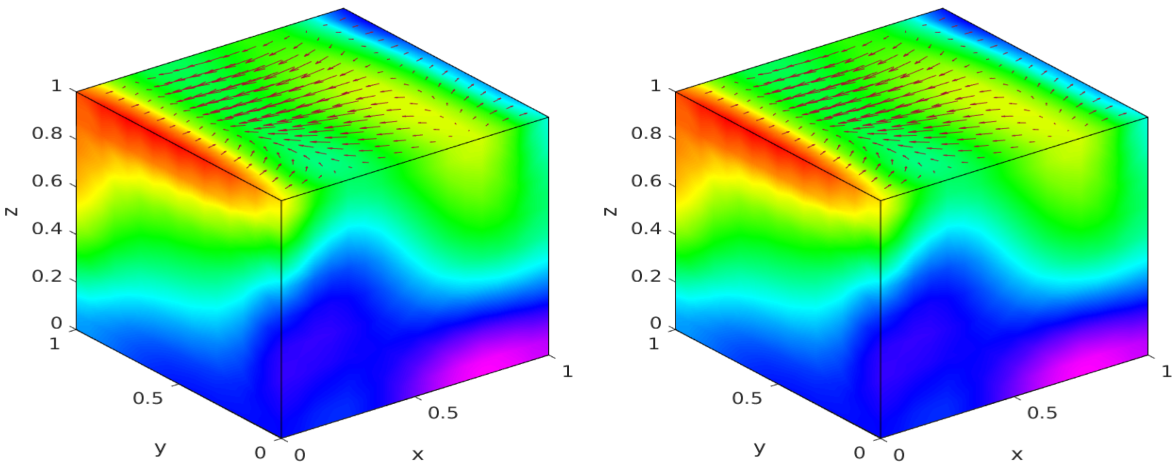
Table 2: Performances of Algorithms 1,2 and 3 on the driven cavity for $g = 0.075$

8.4 3D driven cavity

We consider the cubic cavity $\Omega = (0,\ 1)^3$ with $\Gamma = \{x = 0\} \cup \{x = 1\} \cup \{y = 0\} \cup \{y = 1\}$ and $\boldsymbol{u}|_\Gamma = 0$. The remaining part of the boundary is $S = \{z = 0\} \cup \{z = 1\}$ where the slip takes places. We set $\mu = 1$ and, for the right-hand side,

$$\begin{aligned} f_1(x,y,z) &= 80x^2(1-x)^2 - 20(2+12x^2-12x)z(1-2z) + 2(2z-1), \\ f_2(x,y,z) &= 20(12x-6)z^2(1-z)^2 + 20x(1-2x)(1-x)(2+12z^2-12z) + 2(2x-1), \\ f_3(x,y,z) &= -20y(1-y). \end{aligned}$$

We use ADMM (Algorithm 2) for the numerical simulation. Figure 4 shows the velocity field and pressure at the boundary for $g = 0.5$ and $g = 5$. One can notice that the friction occurs for both values of g. In [42], numerical experiments show that, for the 3D lid-driven cavity,



(a) (b)

Figure 4: Velocity field and pressure for the 3D driven cavity (a): $g = 0.5$, (b): $g = 5$

	$g = 0.5$		$g = 1$		$g = 5$	
h	Iter.	CPU	Iter.	CPU	Iter.	CPU
1/4	3	0. 149	3	0.179	2	0.018
1/8	2	0. 303	2	0.305	2	0. 147
1/16	2	2.761	2	2.828	2	2.820
1/32	2	70.352	2	81. 006	2	73.793

Table 3: Performances of ADMM on the 3D driven cavity with various values of g

the friction always occurs. The performances of the ADMM algorithm are summarized in Table 3 for various values of g.

As in the 2D case, we evaluate the convergence of the GLS FEM using (8.3)–(8.4), where the reference solution is obtained with $h = 1/64$. The reference solution requires solving a linear system of size 1098500 from a 3D mesh. Table 4 summarize the convergence errors and rate for the 3D driven cavity problem.

	$g = 0.5$				$g = 5$			
h	$e_h(\boldsymbol{u})$	Rate	$e_h(\boldsymbol{u}, p)$	Rate	$e_h(\boldsymbol{u})$	Rate	$e_h(\boldsymbol{u}, p)$	Rate
1/4	4.4576e-02		1.2385e+00		4.4576e-02		1.2384e+00	
1/8	1.9463e-02	1.19	5.9782e-01	1.05	1.9461e-02	1.195	5.9777e-01	1.050
1/16	6.2795e-03	1.63	2.3115e-01	1.37	6.2623e-03	1.635	2.3084e-01	1.372
1/32	1.4692e-03	2.05	7.0530e-02	1.73	1.3508e-03	2.212	6.8552e-02	1.751

Table 4: Convergence errors and rates for the 3D driven cavity with various values of g

9 Conclusion

We studied theoretically and numerically the GLS methods for the numerical approximation of the Stokes equations under Tresca's boundary condition allowing the use of equal order approximation for both the velocity and the pressure. The resulting variational system of equations of the model is a nonlinear set of partial differential equations that are effectively solved by exploiting the rich structure of the formulation. Existence of solutions, a priori error estimates and convergence of finite element approximations are thoroughly examined. The numerical experiments exhibited confirms the predictions of the theory. From the simulations, it appears that the alternating direction method of multiplier (ADMM) and the primal-dual active set method (PDAS) are the best numerical approximation methods, but this need to be demonstrated mathematically and it is object of a future research.

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conflict of interest statement. None

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