# The Weak $(2, 2)$-Labelling Problem for graphs with forbidden induced structures


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## Abstract

The Weak $(2, 2)$-Conjecture is a graph labelling problem asking whether all connected graphs of at least three vertices can have their edges assigned red labels 1 and 2 and blue labels 1 and 2 so that any two adjacent vertices are distinguished either by their sums of incident red labels, or by their sums of incident blue labels. This problem emerged in a recent work aiming at proposing a general framework encapsulating several distinguishing labelling problems and notions, such as the well-known 1-2-3 Conjecture, a few of its variants, and so-called locally irregular decompositions. One further point of interest behind the Weak $(2, 2)$-Conjecture is that it is weaker than the 1-2-3 Conjecture, in the sense that the latter conjecture, if proved true, would imply the former one is true too.

In this work, we prove that the Weak $(2, 2)$-Conjecture holds for two classes of graphs defined in terms of forbidden induced structures, namely claw-free graphs and graphs with no pair of independent edges. One main point of interest for focusing on such classes of graphs is that the 1-2-3 Conjecture is not known to hold for them. Also, these two classes of graphs have unbounded chromatic number, while the 1-2-3 Conjecture is mostly understood for classes with bounded and low chromatic number.



*Keywords:* distinguishing labelling; 1-2-3 Conjecture; sum distinction.


## 1. Introduction

This work deals with several **distinguishing labelling problems**, taking part to a wide and vast area of research, as reported in several dedicated surveys on the topic, such as e.g. [7, 10]. More particularly, we focus on a subset of these problems revolving around the so-called **1-2-3 Conjecture**, which can all be defined through the following unified terminology, introduced recently in [4].

Let $G$ be a graph, and $\alpha, \beta \geq 1$ be two positive integers. An $(\alpha, \beta)$*-labelling* of $G$ is an assignment $\ell$ of labels from $\{1, \dots, \alpha\} \times \{1, \dots, \beta\}$ to the edges of $G$, where each edge $e$ gets assigned a *label* $\ell(e) = (x, y)$ with *colour* $x \in \{1, \dots, \alpha\}$ and *value* $y \in \{1, \dots, \beta\}$. Now, for every vertex $v$ of $G$ and any $i \in \{1, \dots, \alpha\}$, we denote by $\sigma_i(v)$ the *sum* of the values of the labels with colour $i$ assigned to the edges incident to $v$, which we call the $i$*-sum* of $v$. We say that $\ell$ is *distinguishing* if for every two adjacent vertices $u$ and $v$ of $G$, there is an $i \in \{1, \dots, \alpha\}$ such that the $i$-sums of $u$ and $v$ differ, that is, if $\sigma_i(u) \neq \sigma_i(v)$.

Regarding these notions, it can be noted that if $G$ is $K_2$, the complete graph of order 2, then there are no $\alpha, \beta \geq 1$ such that $G$ admits distinguishing $(\alpha, \beta)$-labellings. This peculiar case apart, it is not too complicated to prove that, for any fixed $\alpha \geq 1$, there is a $\beta \geq 1$ such that distinguishing $(\alpha, \beta)$-labellings of any graph $G$ exist. For these reasons, in the context of distinguishing labellings, we generally focus on *nice graphs*, which are those graphs with

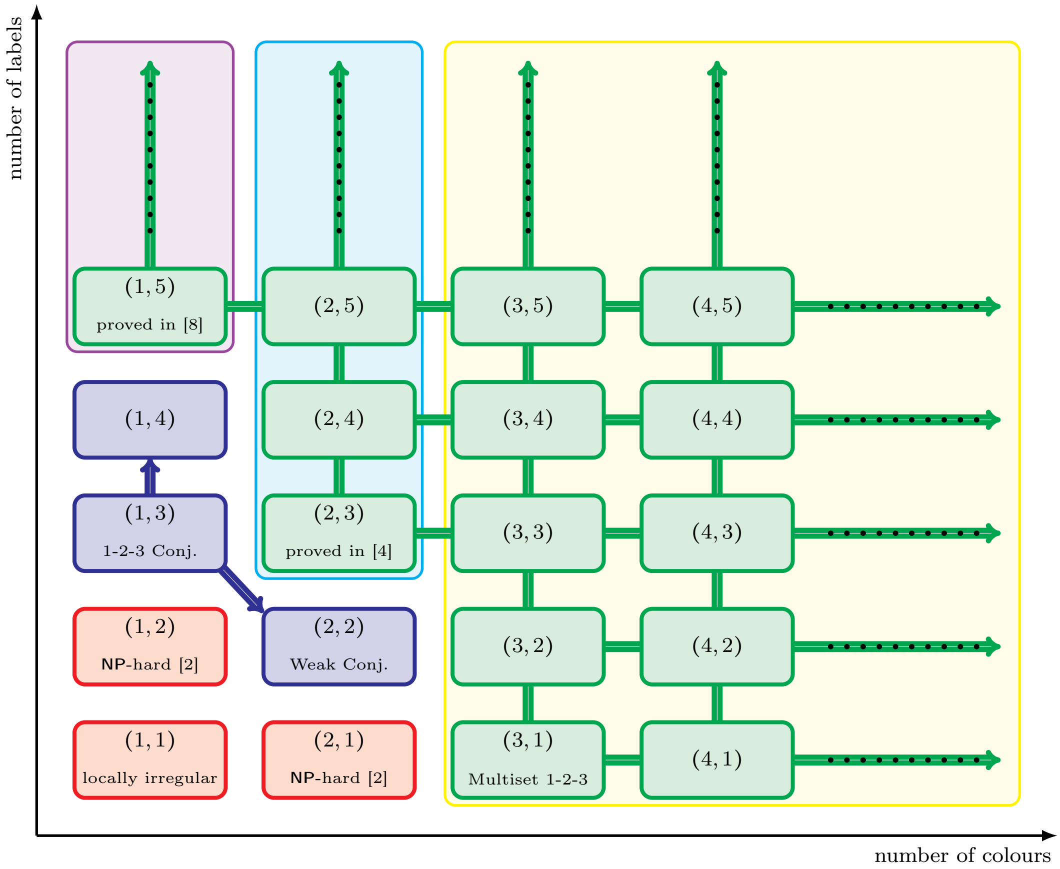



Figure 1: The current knowledge we have on whether all graphs admit distinguishing $(\alpha, \beta)$-labellings, for fixed $\alpha, \beta \geq 1$. For a pair $(\alpha, \beta)$, the associated box is green if all graphs were proved to admit the corresponding labellings, the associated box is red if it is known that not all graphs admit the corresponding labellings, while the associated box is blue if the status is currently unknown. Arrows indicate existential implications between pairs of types of labellings.

no connected component isomorphic to $K_2$. Therefore, throughout this work, every graph we consider is thus implicitly assumed nice.

A natural question, now, is whether, for some fixed $\alpha, \beta \geq 1$, every graph admits distinguishing $(\alpha, \beta)$-labellings. It turns out, as mentioned earlier, that the literature actually provides answers for several values of $\alpha$ and $\beta$. See Figure 1 for a figure depicting our current knowledge on the topic, which we make more explicit below.

- Note first that if $\alpha, \beta$ and $\alpha', \beta'$ are values such that $\alpha' \geq \alpha$, $\beta' \geq \beta$, and $(\alpha, \beta) \neq (\alpha', \beta')$, then any distinguishing $(\alpha, \beta)$-labelling is a distinguishing $(\alpha', \beta')$-labelling.

- Distinguishing $(1, \beta)$-labellings are labellings where all labels are of the same colour, and all adjacent vertices should be distinguished according to their sums of incident labels. Such labellings are exactly those behind the so-called 1-2-3 Conjecture [9] of Karoński, Łuczak, and Thomason, which asks whether all graphs admit distinguishing $(1, 3)$-labellings. To date, the best result towards this is that they all admit distinguishing $(1, 5)$-labellings, see [8].

- Distinguishing $(\alpha, 1)$-labellings can be seen as (improper) edge-colourings where, for every two adjacent vertices, there must be a colour that is not assigned the same number of times to their incident edges. These labellings are those defining the multiset version of the 1-2-3 Conjecture [1], which asks whether all graphs admit distinguishing $(3,1)$-labellings. This conjecture was proved in [11] by Vučković.

- In [4], the authors noticed that, given a distinguishing $(1,5)$-labelling of some graph, by modifying the label colours and values in a particular fashion, we can derive a distinguishing $(2,3)$-labelling of the same graph. Similarly, there is a way, from a distinguishing $(1,5)$-labelling, to derive a distinguishing $(3,2)$-labelling.
- It is not too complicated to see that, in regular graphs, distinguishing $(1,2)$-labellings and distinguishing $(2,1)$-labellings are equivalent notions. In [2], it was proved that determining whether a given cubic graph admits a distinguishing $(1,2)$-labelling is $\mathsf{NP}$-hard. This means there exist infinitely many graphs that admit neither distinguishing $(1,2)$-labellings nor distinguishing $(2,1)$-labellings.
- Graphs admitting distinguishing $(1,1)$-labellings are precisely the so-called *locally irregular graphs*, which are those graphs with no two adjacent vertices having the same degree. These graphs have been appearing frequently in the field, and have even been receiving dedicated attention, see e.g. [5].

From this all, we arrive at the conclusion that there are only three pairs $(\alpha,\beta)$ for which we are still not sure whether all graphs admit distinguishing $(\alpha,\beta)$-labellings: $(1,3)$, which corresponds to the original 1-2-3 Conjecture; $(1,4)$, which is weaker than the 1-2-3 Conjecture since more label values are available (while, similarly, all labels are of the same colour); and $(2,2)$, which is the only pair for which we have two label colours to deal with. The latter pair leads to the following conjecture:

**Weak $(2,2)$-Conjecture** (Baudon *et al.* [4])**.** *Every graph admits a distinguishing $(2,2)$-labelling.*

At first glance, the 1-2-3 Conjecture and the Weak $(2,2)$-Conjecture might seem a bit distant. It is worth emphasising, however, that the former conjecture, if true, would imply the latter [6]. For this reason, the Weak $(2,2)$-Conjecture can be perceived as a weaker version of the 1-2-3 Conjecture. Also, to get progress towards these conjectures, one can thus investigate the Weak $(2,2)$-Conjecture for classes of graphs for which the 1-2-3 Conjecture is not known to hold. To date, the 1-2-3 Conjecture was mainly proved for 3-colourable graphs[1] [10]. The weaker conjecture was mainly proved for 4-colourable graphs [6].

**Theorem 1.1** (Bensmail [6])**.** *The Weak $(2,2)$-Conjecture holds for 4-colourable graphs.*

Both conjectures were also proved for other classes of graphs, but not as significant. One reason why the chromatic number parameter appears naturally in this context is that having a proper vertex-colouring $\phi$ in hand can be helpful to design a distinguishing labelling, since $\phi$ informs on sets of vertices that are not required to be distinguished. One downside, however, is that making a labelling match $\phi$ somehow, might require lots of labels if $\phi$ itself contains lots of parts.

In this work, we prove the Weak $(2,2)$-Conjecture for two classes of graphs for which the 1-2-3 Conjecture is not known to hold. Furthermore, the two classes of graphs in question

[1]Recall that a *proper $k$-vertex-colouring* of a graph $G$ is a partition of $V(G)$ into $k$ independent sets. The *chromatic number* $\chi(G)$ of $G$ is the smallest $k$ such that $G$ admits proper $k$-vertex-colourings. We say that $G$ is *$k$-colourable* if $\chi(G) \leq k$, and *$k$-chromatic* if $\chi(G) = k$.

have unbounded chromatic number, which is significant according to the arguments above. Precisely, we prove the Weak $(2,2)$-Conjecture for $K_{1,3}$-*free graphs* (graphs with no induced claw) and $2K_2$-*free graphs* (graphs with no pair of independent edges). Both results are proved in a similar way: we first deal with the 5-colourable graphs of the class, before focusing on those with chromatic number at least 6.

This paper is organised as follows. In Section 2, we start off with some preliminaries, covering the terminology we use throughout, several lemmas, and previous results of interest. We then start by proving the Weak $(2,2)$-Conjecture for $2K_2$-free graphs in Section 3, since the proof we give serves as a good introduction to the more technical proof, in Section 4, of the same result for $K_{1,3}$-free graphs. We end this work in Section 5 with concluding words.

## 2. Preliminaries

Let $G$ be a graph, and $\ell$ be an $(\alpha,\beta)$-labelling of $G$. If $\alpha = 1$, then we will sometimes call $\ell$ a $\beta$-*labelling* for simplicity. Also, in such cases, instead of denoting the 1-sum of a vertex $v$ by $\sigma_1(v)$, we will simply denote it as $\sigma(v)$, or as $\sigma_\ell(v)$ in case we want to emphasise that we refer to the labels assigned by $\ell$. Now, in cases where we are dealing with the Weak $(2,2)$-Conjecture and, thus, $(\alpha,\beta) = (2,2)$, it will be more convenient to see the labels with colour 1 as *red labels*, and similarly those with colour 2 as *blue labels*. In this context, we will thus refer, for any vertex $v$, to the *red sum* $\sigma_{\mathrm{r}}(v)$ of $v$ (which is thus $\sigma_1(v)$), and to the *blue sum* $\sigma_{\mathrm{b}}(v)$ of $v$ (which is thus $\sigma_2(v)$).

In what follows, we point out situations where, assuming a partial labelling of a graph is given, we can extend it to some edges in such a way that some properties are preserved.

**Lemma 2.1.** *Let $G$ be a graph, $H$ be a connected bipartite subgraph of $G$, and $\ell$ be a partial 2-labelling of $G$ such that only the edges of $H$ are not labelled. For any vertex $w$ of $H$, there is a 2-labelling $\ell'$ of $H$ such that, for every two adjacent vertices $u$ and $v$ of $H$ with $w \notin \{u,v\}$, we have*

$$\sigma_\ell(u) + \sigma_{\ell'}(u) \neq \sigma_\ell(v) + \sigma_{\ell'}(v).$$

*Proof.* Let $(U,V)$ denote the bipartition of $H$. We produce a 2-labelling $\ell'$ such that, for every vertex $u \neq w$ of $H$, we have $\sigma_\ell(u) + \sigma_{\ell'}(u) \equiv 0 \bmod 2$ if $u \in U$, and $\sigma_\ell(u) + \sigma_{\ell'}(u) \equiv 1 \bmod 2$ otherwise, if $u \in V$. Note that this clearly implies what we want to prove.

Start from all edges of $H$ being assigned label 2 by $\ell'$. Now, consider any vertex $u$ of $H$ for which $\sigma_\ell(u) + \sigma_{\ell'}(u)$ does not satisfy the required condition above. Since $H$ is connected, there is a path $P$ from $u$ to $w$ that uses edges of $H$ only. Now turn all 1's assigned by $\ell'$ to the edges of $P$ into 2's, and conversely turn all 2's into 1's. As a result, note that $\sigma_\ell(v) + \sigma_{\ell'}(v)$ is not altered for every vertex $v$ of $H$ with $v \notin \{u,w\}$, while both $\sigma_\ell(u) + \sigma_{\ell'}(u)$ and $\sigma_\ell(w) + \sigma_{\ell'}(w)$ had their parity altered. So $\sigma_\ell(u) + \sigma_{\ell'}(u)$ now verifies the desired condition.

Repeating those arguments until all vertices $u \neq w$ of $H$ have $\sigma_\ell(u) + \sigma_{\ell'}(u)$ verifying the desired condition, we end up with $\ell'$ being as desired. □

Building distinguishing labellings being nothing but an algebraic problem, there are contexts in which algebraic tools come up handy naturally. Below, we recall one such useful tool, and showcase a few ways to use it.

**Theorem 2.2** (Combinatorial Nullstellensatz [3]). *Let $\mathbb{F}$ be an arbitrary field, and $P = P(Z_1,\dots,Z_p)$ be a polynomial in $\mathbb{F}[Z_1,\dots,Z_p]$. Suppose that the coefficient of a monomial*

$Z_1^{k_1}\dots Z_p^{k_p}$, *where every* $k_i$ *is a non-negative integer, is non-zero in* $P$ *and the degree of* $P$ *equals* $\sum_{i=1}^{p} k_i$*. If* $S_1,\dots,S_p$ *are subsets of* $\mathbb{F}$ *with* $|S_i| > k_i$ *for every* $i \in \{1,\dots,p\}$*, then there are* $z_1 \in S_1,\dots,z_p \in S_p$ *so that* $P(z_1,\dots,z_p) \neq 0$*.*

**Lemma 2.3.** *Let* $G$ *be a graph,* $H$ *be a subgraph of* $G$*, and* $\ell$ *be a partial* 2*-labelling of* $G$ *such that only the edges of* $H$ *are not labelled. Then, there is a* 2*-labelling* $\ell'$ *of* $H$ *such that, for every two adjacent vertices* $u$ *and* $v$ *of* $H$*, we have*

$$\sigma_\ell(u) + \sigma_{\ell'}(u) \neq \sigma_\ell(v) + \sigma_{\ell'}(v),$$

*for* $H$ *being any of:*

- *a path of length at least* 2 *not* 3*;*
- *a cycle with length multiple of* 4*.*

*Proof.* Regarding the first case, assume $H$ is a path $v_1\dots v_p$ of length $p-1 \geq 2$ different from 3. For every $i \in \{1,\dots,p\}$, set $n_i = \sigma_\ell(v_i)$. Now, for every $i \in \{1,\dots,p-1\}$, define $e_i$ as the edge $v_iv_{i+1}$, and let $Z_i$ be a variable belonging to $\{1,2\}$ and representing any label assignment to $e_i$. We consider $P$, the polynomial defined as

$$P(Z_1,\dots,Z_{p-1}) = (n_1 - Z_2 - n_2)\cdot \prod_{i=2}^{p-2}(Z_{i-1}+n_i - Z_{i+1} - n_{i+1})\cdot(Z_{p-2}+n_{p-1}-n_p).$$

Note that the degree of $P$ is $p-1$, and that the monomial $M = Z_1\dots Z_{p-1}$ is thus of maximum degree. Note also that, since the $n_i$'s are fixed, the coefficient of $M$ in the expansion of $P$ is the same as the coefficient of the same monomial in the expansion of

$$P'(Z_1,\dots,Z_{p-1}) = (-Z_2)\cdot(Z_1-Z_3)\cdot(Z_2-Z_4)\cdot(Z_3-Z_5)\cdots(Z_{p-3}-Z_{p-1})\cdot(Z_{p-2}).$$

If $p \neq 4$, then $p-2 \neq 2$. In this case, it can then be noted that there is only one way to form $M$ by expending $P'$ (due to the fact that the first factor contains $Z_2$ only, and that only the second one contains $Z_1$), and thus its coefficient is $\pm 1$. Thus $M$ has non-zero coefficient. So the Combinatorial Nullstellensatz applies, implying we can assign labels from $\{1,2\}$ to the edges of $H$ so that, together with the labels by $\ell$, the adjacent vertices of $H$ are distinguished as desired.

Now consider the second case, where $H$ is a cycle $v_0\dots v_{p-1}v_0$ of length $p \equiv 0 \bmod 4$. Again, for every $i \in \{0,\dots,p-1\}$, set $n_i = \sigma_\ell(v_i)$, define $e_i$ as the edge $v_iv_{i+1}$ (where, here and further, the operations over subscripts are modulo $p$), and let $Z_i$ be a variable belonging to $\{1,2\}$ associated to $e_i$. We consider $P$, the polynomial

$$P(Z_0,\dots,Z_{p-1}) = \prod_{i=0}^{p-1}(Z_{i-1}+n_i-Z_{i+1}-n_{i+1}).$$

Since the $n_i$'s are constant, the coefficient of $M = Z_0\dots Z_{p-1}$ in the expansion of $P$ is the same as in that of

$$P'(Z_0,\dots,Z_{p-1}) = (Z_{p-1}-Z_1)\cdot(Z_0-Z_2)\cdot(Z_1-Z_3)\cdot(Z_2-Z_4)\cdots(Z_{p-2}-Z_0).$$

Note that $P'$ can be seen as

$$\prod_{\substack{i \text{ even}\\ 0\leq i\leq p-2}}(Z_i - Z_{i+2})\prod_{\substack{i \text{ odd}\\ 1\leq i\leq p-3}}(Z_i - Z_{i+2}),$$

where the two involved products contain an even number of factors each ($p/2$), since $p \equiv 0 \bmod 4$. From this, it is easy to see that the coefficient of $Z_0Z_2Z_4 \dots Z_{p-2}$ in the first product is 2, and similarly for the coefficient of $Z_1Z_3Z_5 \dots Z_{p-1}$ in the second product. Thus, the coefficient of $M$ in $P$ is 4, hence non-zero. Since $M$ is of maximum degree, from the Combinatorial Nullstellensatz we get our conclusion here as well. □

**Lemma 2.4.** *Let $G$ be a graph, $H$ be a subgraph of $G$ isomorphic to a path $p_1p_2p_3p_4$ of length 3, and $\ell$ be a partial $(2,2)$-labelling of $G$ such that only the edges of $H$ are not labelled. Assume also that the red sum of $p_1$ or $p_4$ by $\ell$ is at most 1, while it is 0 for $p_2$ and $p_3$. Then, there is a $(2,2)$-labelling $\ell'$ of $H$ such that every two adjacent vertices of $H$ are distinguished by their red sums by $\ell$ and $\ell'$, or similarly by their blue sums by $\ell$ and $\ell'$. Also, we can make sure that the red sum of any of $p_1$, $p_2$, $p_3$, or $p_4$ is at most 1.*

*Proof.* If the red sum of both $p_1$ and $p_4$ by $\ell$ is 0, then, by $\ell'$, we first assign red label 1 to $p_2p_3$ by $\ell'$, before assigning blue label 1 to $p_1p_2$, and blue label 2 to $p_3p_4$. This way, $p_1$ and $p_2$, and similarly $p_4$ and $p_3$, are distinguished since the former vertex has red sum 0 while the latter has red sum 1. Also, $p_2$ and $p_3$ are distinguished since $p_2$ has blue sum 1 while $p_3$ has blue sum 2. We also have red sum at most 1 for all $p_i$'s.

If, say, $p_1$ has red sum 1 by $\ell$ while $p_4$ has red sum 0, then note that, upon assigning blue labels by $\ell'$ to the edges of $H$, we cannot get any conflict between $p_1$ and $p_2$, since they are distinguished by their red sums. In this case, a similar application of the Combinatorial Nullstellensatz as in the proof of Lemma 2.3 can be invoked to conclude that we can assign blue labels 1 and 2 by $\ell'$ to the edges of $H$ to get the desired labelling. Denoting, for every $i \in \{1,2\}$, by $Z_i$ a variable in $\{1,2\}$ corresponding to a blue label assigned to $p_ip_{i+1}$, note here that, by the previous remark, we can indeed restrict our attention to the polynomial $(Z_1 - Z_3)(Z_2)$, and more particularly to the monomial $Z_1Z_2$, to get our conclusion.

Now, if both $p_1$ and $p_4$ have red sum 1 by $\ell$, then, again, upon assigning blue labels to the edges of $H$ by $\ell'$, we cannot get any conflict between $p_1$ and $p_2$, and similarly between $p_4$ and $p_3$, since the former vertices have red sum 1 while the latter ones have red sum 0. So only $p_2$ and $p_3$ need to be distinguished, which can be done by assigning blue label 1 to $p_1p_2$, blue label 2 to $p_3p_4$, and any blue label to $p_2p_3$. □

**Lemma 2.5.** *Let $G$ be a graph, $H$ be a subgraph of $G$ isomorphic to a cycle of even length, and $\ell$ be a partial $(2,2)$-labelling of $G$ such that only the edges of $H$ are not labelled and all the edges of $E(G) \setminus E(H)$ are assigned red labels. Then, there is a $(2,2)$-labelling $\ell'$ of $H$ such that every two adjacent vertices of $H$ are distinguished by their red sums by $\ell$ and $\ell'$, or similarly by their blue sums by $\ell$ and $\ell'$. Also, we can make sure that the blue sum of every vertex of $H$ is at most 1.*

*Proof.* Assume $H$ is a cycle of even length $k \geq 4$. We denote the consecutive vertices of $H$ by $v_0v_1 \dots v_{k-1}v_0$, and set $e_i = v_iv_{i+1}$ for every $i \in \{0, \dots, k-1\}$ (where all operations over the subscripts in this proof are modulo $k$).

Consider $B$, the subset of edges of $H$ obtained as follows. We add $e_1$ to $B$, and, from here, we add every three edges of $H$, namely $e_4$, $e_7$, and so on, to $B$, so that we add as many such edges to $B$ as possible, but every two edges added to $B$ are at distance at least 3 from each other in $H$. In particular, since $e_1 \in B$, neither $e_0$ nor $e_{k-1}$ belongs to $B$. In particular, for every $e_i \in B$, we have $e_{i-1} \notin B$ and $e_{i+1} \notin B$, and for every $e_i, e_j \in B$ with $i \neq j$, we have $\{e_{i-1}, e_{i+1}\} \cap \{e_{j-1}, e_{j+1}\} = \varnothing$. Also, $H - B$, by how $B$ was constructed, consists of paths $P_1, \dots, P_p$, all of which have length 2, but maybe one of them (the one containing $v_0$, say it is $P_p$), which might be of length 2, 3, or 4.

By $\ell'$, we start by assigning blue label 1 to all edges of $B$. In what follows, the edges of $E(H) \setminus B$ will all be assigned red labels. Note that these edges are precisely the edges of the $P_i$'s. Also, if $P_i = v_i v_{i+1} v_{i+2}$ is of length 2, then $v_i$ and $v_{i+2}$ are both incident to an edge of $B$, and thus are of blue sum 1, while $v_{i+1}$ is of blue sum 0. Thus, when assigning red labels to the edges of the $P_i$'s, we only need to make sure to distinguish adjacent vertices $v_i$ and $v_{i+1}$ such that $v_i v_{i+1} \in B$, or $v_i$ and $v_{i+1}$ are inner vertices of $P_p$. Last, remark that if $e_i \in B$, then, so that $v_i$ and $v_{i+1}$ are distinguished by their red sums, it suffices to make sure we assign red labels to $e_{i-1}$ and $e_{i+1}$ so that, when taking into account the contribution by $\ell$, the red sums of $v_i$ and $v_{i+1}$ are of different parity.

We consider three distinct cases, involving the possible lengths of $P_p$:

- If $P_p$ is of length 2, we are thus done when considering every $e_i \in B$ in turn, and assigning, by $\ell'$, a red label to $e_{i-1}$ and $e_{i+1}$ so that, when taking into account the contribution by $\ell$, the red sums of $v_i$ and $v_{i+1}$ are of different parity.
- Assume $P_p$ is of length 3, *i.e.*, $P_p = v_{k-3} v_{k-2} v_{k-1} v_0$. In this case, we need to make sure that the red sums of $v_{k-2}$ and $v_{k-1}$ get different. To that end, we proceed as in the previous case, except that, when labelling $e_0$ and $e_2$ (to deal with $e_1 \in B$) and $e_{k-5}$ and $e_{k-3}$ (to deal with $e_{k-4}$), we do so so that the red sum of $v_{k-2}$, when taking into account the contribution by $\ell$, becomes even, while that of $v_{k-1}$ becomes odd.
- Similarly, if $P_p = v_{k-4} v_{k-3} v_{k-2} v_{k-1} v_0$ is of length 4, then we need to make sure that the red sums of $v_{k-3}$ and $v_{k-2}$, and similarly of $v_{k-2}$ and $v_{k-1}$, are different. This can be done by labelling $e_{k-4}$, $e_{k-3}$, $e_{k-2}$, $e_{k-1}$, and $e_0$ first, following that order, so that the desired pairs of adjacent vertices are distinguished due to their red sums having different parity. From here, we can then again consider the edges in $B$ and treat them at previously, taking into account, when dealing with $e_1$ and $e_{k-5}$, that $e_0$ and $e_{k-4}$ have already been labelled.

This concludes the proof. □

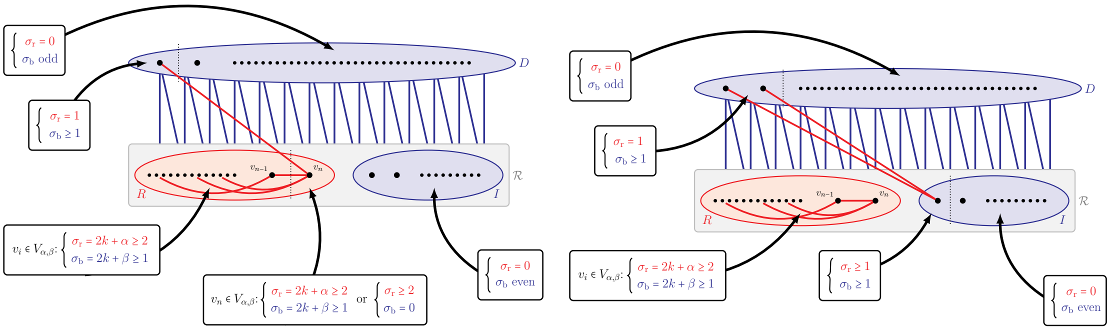
To finish off, we recall a nice tool that proved to be very useful towards proving the multiset version of the 1-2-3 Conjecture from [1]. Let $G$ be a graph. A *balanced tripartition* of $G$ is a partition $V_0, V_1, V_2$ of $V(G)$ fulfilling, for every vertex $v \in V_i$ for any $i \in \{0, 1, 2\}$, that $d_{V_{i+1}}(v) \geq \max\{1, d_{V_i}(v)\}$ (note that all operations over the subscripts are modulo 3). That is, $v$ has at least one neighbour in the next part $V_{i+1}$, and it actually has more neighbours in $V_{i+1}$ than in $V_i$. It turns out that graphs with sufficiently large chromatic number admit such a balanced tripartition.

**Theorem 2.6** (Addario-Berry *et al.* [1]). *Every graph $G$ with $\chi(G) > 3$ admits a balanced tripartition.*

## 3. Graphs with no induced pair of independent edges

As mentioned earlier, we prove the Weak $(2,2)$-Conjecture for $2K_2$-free graphs by first proving it for the 5-chromatic ones, and then for those with chromatic number at least 6. This implies the result, since the conjecture also holds for the 4-colourable ones, by Theorem 1.1. In what follows, we thus consider the two cases separately.

**Theorem 3.1.** *Every $2K_2$-free graph with chromatic number 5 admits a distinguishing $(2,2)$-labelling.*




(a) An upward edge of $R$ is assigned red label 1. (b) All upward edges of $R$ are assigned blue labels.

Figure 2: Terminology used in the proof of Theorem 3.1, and the red sums and blue sums we aim at getting for the vertices by the designed $(2,2)$-labelling. (a) and (b) depict the two main cases we consider.

*Proof.* Let $G$ be a $2K_2$-free graph with chromatic number 5. We construct a distinguishing $(2,2)$-labelling of $G$ assigning red labels 1 and 2 and blue labels 1 and 2. We can assume $G$ is connected, since each of its 5-chromatic connected components can be handled through the arguments below, while Theorem 1.1 applies for its 4-colourable connected components.

Let $D$ be a maximal independent set of $G$, and set $\mathcal{R} = G - D$. Note that every vertex $v$ in $\mathcal{R}$ is incident to at least one *upward edge* $vu$, *i.e.*, going to $D$ (so, $u \in D$). We say that a connected component of $\mathcal{R}$ is *empty* if it contains no edges, while it is *non-empty* otherwise. Since $G$ is $2K_2$-free, note that $\mathcal{R}$ contains at most one non-empty connected component. Actually, $\mathcal{R}$ must contain exactly one non-empty connected component $R$ as otherwise $G$ would be bipartite, contradicting that its chromatic number is 5. Let now $I$ denote the vertices from the empty connected components of $\mathcal{R}$, and let $\mathcal{H}$ be the subgraph of $G$ induced by the edges incident to the vertices of $I$. Then $\mathcal{H}$ is bipartite, and, again, because $G$ is $2K_2$-free, it must be that $\mathcal{H}$ consists of only one connected component.

Since $G$ is 5-chromatic, note that $R$ is 4-chromatic; let thus $V_{0,0}, V_{0,1}, V_{1,0}, V_{1,1}$ be parts forming a proper 4-vertex-colouring $\phi$ of $R$. We modify $\phi$, if needed, so that if $v$ is a vertex of $R$ with $d_R(v) = 1$, then $v$ belongs to $V_{0,0}$ or $V_{0,1}$ (note that this is clearly possible, since $v$ has exactly one neighbour in $R$, thus at most one neighbour in $V_{0,0} \cup V_{0,1}$). Now order the vertices $v_1, \dots, v_n$ of $R$ in any way satisfying that, for every $i \in \{1, \dots, n-1\}$, vertex $v_i$ is incident to at least one *forward edge* $v_i v_j$ (*i.e.*, with $j > i$, which is a *backward edge* from $v_j$'s point of view). Such an ordering can be obtained e.g. by reversing the ordering in which vertices are encountered while performing a breadth-first search algorithm from any vertex (standing as the last vertex $v_n$).

We are now ready to start labelling the edges of $G$. We begin with all edges incident to the vertices of $R$. We consider the $v_i$'s one by one, following the ordering above, and for every vertex $v_i$ considered in that course, we assign a label to all upward edges (assigning them blue labels, except in one peculiar case) and forward edges (assigning them red labels only) incident to $v_i$ so that some desired red sum and blue sum are realised at $v_i$. When proceeding that way, note that, whenever considering a new vertex as $v_i$, only its backward edges can be assumed to be labelled, with red labels. The procedure goes as follows:

- If $i \neq n$, then $v_i$ is incident to forward edges. We start by assigning blue label 2 to all upward edges incident to $v_i$, and red label 2 to all forward edges incident to $v_i$. Assume $v_i \in V_{\alpha,\beta}$. If $\sigma_b(v_i) \not\equiv \beta \bmod 2$, then we change to blue label 1 the label assigned to any one upward edge incident to $v_i$. Likewise, if $\sigma_r(v_i) \not\equiv \alpha \bmod 2$, then we change to red label 1 the label assigned to any one forward edge incident to $v_i$.

This way, we get $\sigma_{\mathrm{r}}(v_i) \equiv \alpha \bmod 2$ and $\sigma_{\mathrm{b}}(v_i) \equiv \beta \bmod 2$. In particular, by how we modified $\phi$ earlier, note that we must have $\sigma_{\mathrm{r}}(v_i) \geq 2$ (either $d_R(v_i) \geq 2$ in which case this condition clearly holds; or $d_R(v_i) = 1$, in which case $\alpha = 0$ and thus the only inner edge incident to $v_i$ is assigned red label 2, implying the condition).

- If $i = n$, then the only edges incident to $v_n$ that remain to be labelled are upward edges. Recall, in particular, that all backward edges incident to $v_n$ are assigned red labels. We consider two cases, assuming $v_n \in V_{\alpha,\beta}$:
  - If $\sigma_{\mathrm{r}}(v_n) \equiv \alpha \bmod 0$, then we assign blue labels to all upward edges incident to $v_n$, their values being chosen so that $\sigma_{\mathrm{b}}(v_n) \equiv \beta \bmod 0$. In that case, we thus have $\sigma_{\mathrm{r}}(v_n) \equiv \alpha \bmod 2$ and $\sigma_{\mathrm{b}}(v_n) \equiv \beta \bmod 2$. Again, by how $\phi$ was modified earlier, we must have $\sigma_{\mathrm{r}}(v_n) \geq 2$.
  - If $\sigma_{\mathrm{r}}(v_n) \not\equiv \alpha \bmod 0$, then we assign red label 1 to any one upward edge incident to $v_n$, while we assign blue labels to the other upward edges (if any) so that $\sigma_{\mathrm{b}}(v_n) \equiv \beta \bmod 2$. In this case, either $\sigma_{\mathrm{b}}(v_n) \neq 0$ in which case $\sigma_{\mathrm{r}}(v_n) \equiv \alpha \bmod 2$ and $\sigma_{\mathrm{b}}(v_n) \equiv \beta \bmod 2$; or $\sigma_{\mathrm{b}}(v_n) = 0$ in which case all edges incident to $v_n$ are assigned red labels (implying that $\sigma_{\mathrm{r}}(v_n) \geq 2$).

Note that, in all cases above, for all vertices $v_i \in V_{\alpha,\beta}$, we guarantee $2 \leq \sigma_{\mathrm{r}}(v_i) \equiv \alpha \bmod 2$. Also, except maybe for $v_n$, we also guarantee $0 < \sigma_{\mathrm{b}}(v_i) \equiv \beta \bmod 2$. Regarding $v_n$, either $\sigma_{\mathrm{b}}(v_n) = 0$, in which case $v_n$ is distinguished from all its neighbours in $R$ through its blue sum, or $0 < \sigma_{\mathrm{b}}(v_n) \equiv \beta \bmod 2$, in which case $v_n$ is distinguished from its neighbours in $R$ through its red sum and/or blue sum. Regarding the vertices of $D$, only one of them can currently be incident to an edge being assigned a red label, and, if this is the case, then it is incident to exactly one such edge, being assigned red label 1. So, for every $u \in D$, we currently have $\sigma_{\mathrm{r}}(u) \leq 1$, while $\sigma_{\mathrm{r}}(v) \geq 2$ for every $v \in R$. Thus, currently, vertices of $R$ are distinguished from their neighbours in $D$. If $\mathcal{H}$ has no edges (*i.e.*, $I = \varnothing$), then all edges of $G$ are actually labelled, and we end up with a distinguishing $(2,2)$-labelling. So, in what follows, we can assume $\mathcal{H}$ has edges.

We are now left with labelling the edges of $\mathcal{H}$, which, recall, consists of exactly one connected component. We consider two main cases (illustrated in Figure 2):

- Assume there is some vertex $w \in \mathcal{H}$ with $\sigma_{\mathrm{r}}(w) = 1$. Recall that there can be only one such vertex, which belongs to $D$ and must be a neighbour of $v_n$. Recall also that the vertices of $D \cap V(\mathcal{H})$ can be incident to edges being currently assigned blue labels (being upward edges incident to vertices of $R$). Taking these labels into account, by Lemma 2.1 we can assign blue labels 1 and 2 to the edges of $\mathcal{H}$ so that any two of its adjacent vertices $u$ and $v$ with $w \notin \{u, v\}$ are distinguished by their blue sums.

  Since we did not modify labels assigned to edges incident to the vertices in $R$, and the edges of $\mathcal{H}$ are assigned blue labels only, the vertices of $R$ remain distinguished from their neighbours due to arguments above. Regarding adjacent vertices of $\mathcal{H}$, they are either distinguished by their blue sums (if $w$ is not involved), or because one of them has red sum 1 (if $w$ is involved). So, here as well, we do not have conflicts.

- Assume no vertex of $\mathcal{H}$ currently has red sum 1. In this case, let $w$ be any vertex of $I$. By Lemma 2.1, we can assign blue labels 1 and 2 to the edges of $\mathcal{H}$ so that, taking into account the other edges of $G$ that are currently already assigned blue labels, and omitting $w$, any two adjacent vertices of $\mathcal{H}$ are distinguished by their blue sums. In

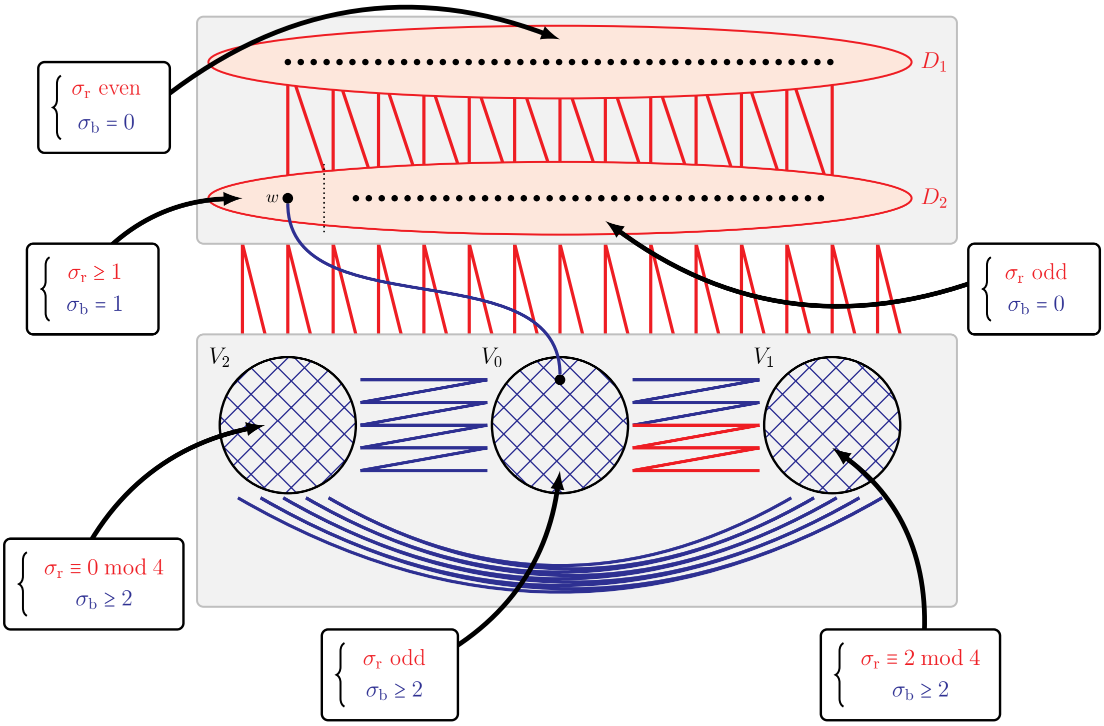



Figure 3: Terminology used in the proof of Theorem 3.2, and the red sums and blue sums we aim at getting for the vertices by the designed $(2,2)$-labelling.

case $w$ has $d \geq 2$ neighbours $x_1, \dots, x_d$ (which lie in $D$), then we further modify the labelling by changing to red label 1 the label assigned to $wx_1, \dots, wx_d$.

Again, we did not modify the red sums and blue sums of the vertices in $R$. Also, the only vertex of $D \cup I$ that might have red sum at least 2 is $w$ (note that the $x_i$'s, if they exist, have red sum 1), which lies in $I$, the set of isolated vertices of $\mathcal{R}$, and thus cannot be adjacent to the vertices of $R$. Since the vertices of $R$ have red sum at least 2, they thus cannot be involved in conflicts. Now, if $d_G(w) = 1$, then, because $G$ is not just an edge, the unique neighbour of $w$ must have degree at least 2, meaning that $w$ is necessarily distinguished from its unique neighbour. Otherwise, *i.e.*, $w$ has $d \geq 2$ neighbours $x_1, \dots, x_d \in D$, then $\sigma_r(w) = d \geq 2$ while the $x_i$'s have red sum 1, and thus $w$ cannot be involved in conflicts. Regarding the $x_i$'s, they have red sum 1, so they cannot be in conflict with their neighbours of $\mathcal{H}$ different from $w$, since they have red sum 0. Finally, for every vertex of $\mathcal{H}$ not in $\{w, x_1, \dots, x_d\}$, note that we did not modify its blue sum when introducing red labels. Then we still have that any two such adjacent vertices are distinguished by their blue sums, due to how we applied Lemma 2.1. So, no conflicts exist in $G$.

In both cases, the resulting $(2,2)$-labelling of $G$ is thus distinguishing, as desired. □

**Theorem 3.2.** *Every $2K_2$-free graph with chromatic number at least 6 admits a distinguishing $(2,2)$-labelling.*

*Proof.* Let $G$ be a $2K_2$-free graph with chromatic number at least 6. We construct a distinguishing labelling of $G$ assigning red labels 1 and 2 and blue labels 1 and 2. Note that we may assume that $G$ is connected, due to Theorems 1.1 and 3.1, and the arguments below.

Let $D_1$ be a maximal independent set of $G$. Note that every vertex of $G - D_1$ has at least one neighbour in $D_1$. Now let $D_2$ be a maximal independent set of $G - D_1$. Similarly, every vertex of $G - D_1 - D_2$ has at least one neighbour in $D_2$. Since $\chi(G) \geq 6$, note that $\chi(G - D_1 - D_2) \geq 4$. According to Lemma 2.6, there is thus a balanced tripartition $V_0, V_1, V_2$ of $G - D_1 - D_2$. Note that $D_1$, $D_2$, $V_0$, $V_1$, and $V_2$ form a partition of $V(G)$. An *upward edge* of $G$ is an edge with one end in $V_0 \cup V_1 \cup V_2$ and the other in $D_1 \cup D_2$. An *inner edge* of $G$ is an edge with both ends in some $V_i$. If $u \in V_i$ and $u' \in V_{i+1}$ (where, throughout this proof, operations over the subscripts of the $V_i$'s are modulo 3) are adjacent for some $i \in \{0, 1, 2\}$, then $uu'$ is a *forward edge* from $u$'s perspective, and a *backward edge* from that of $u'$. Because $G$ is $2K_2$-free, note that all three of $G[V_0]$, $G[V_1]$, and $G[V_2]$ contain at most one connected component with edges each.

We denote by $\mathcal{H}$ the set of the connected components of $G[D_1 \cup D_2]$. Since every vertex of $D_2$ has neighbours in $D_1$, note that $\mathcal{H}$ has edges. Actually, since $G$ is $2K_2$-free, there is exactly one connected component $H$ of $\mathcal{H}$ that is *non-empty*, *i.e.*, that contains edges. $\mathcal{H}$ can also contain *empty* connected components, which consist in a single vertex of $D_1$.

We design the desired $(2,2)$-labelling of $G$ following four steps. First, we label all inner, upward, and forward edges incident to the vertices of $V_0$ so that they fulfil certain properties on $\sigma_{\mathrm{r}}$ and $\sigma_{\mathrm{b}}$. Second and third, we then achieve the same for the vertices of $V_1$ and $V_2$. During a fourth and last step, we label the edges of $\mathcal{H}$. The reader, throughout what follows, can refer to Figure 3, which summarises the sum conditions we aim at reaching.

**Step 1: Labelling the inner, upward, and forward edges of $V_0$.**

We start by labelling the following edges of $G$:

1. We first assign blue label 2 to all inner edges incident to vertices of $V_0$.
2. We then consider every vertex $u$ of $V_0$ in turn, assign red label 2 to all upward edges incident to $u$, and eventually change to red label 1 one of these red labels so that the red sum of $u$ becomes odd.
3. We now distinguish two cases, through which we get to defining a special vertex $w \in D_2$ that will be useful later on, by the last step of the proof.
   - $|V_0| = 1$, *i.e.*, $G[V_0]$ is a single vertex $u$. We here assign blue label 2 to all forward edges incident to $u$. We also modify the labelling further as follows. Set $w$ as any neighbour of $u$ in $D_2$. Note that, by swapping the red labels assigned to $uw$ and another upward edge incident to $u$, we can, if necessary, assume $uw$ is assigned red label 2. We then change the label assigned to $uw$ to blue label 1.
   - Otherwise, *i.e.* $|V_0| \geq 2$. Here, let $u_1, \dots, u_n$ be an arbitrary ordering over the vertices of $V_0$, and consider the $u_i$'s one by one in order. Since extra modifications must be made around $u_1$, let us consider that vertex specifically before describing the general case. Just as in the previous case, let $w$ be any neighbour of $u_1$ in $D_2$. Again, we can swap labels assigned to upward edges, if necessary, so that $u_1w$ is assigned red label 2. Then we change the label assigned to $u_1w$ to blue label 1, before assigning blue label 2 to all forward edges incident to $u_1$. Now, for every subsequent $u_i$ with $i \geq 2$, denote by $u_{i_1}, \dots, u_{i_d}$ the $d \geq 0$ neighbours of $u_i$ in $V_0$ that precede $u_i$ in the ordering. If $d = 0$, then assign blue label 2 to all forward edges incident to $u_i$. Now, if $d \geq 1$, then recall that $u_i$ is

incident to $d_{V_1}(u_i) \geq d$ forward edges. By assigning red label 2 to none, one, two, etc., or all of these edges, and blue label 2 to all others, we can increase the red sum of $u_i$ by any amount in $\{0, 2, \dots, 2d_{V_1}(u_i)\}$, which set contains $d_{V_1}(u_i) + 1 \geq d + 1$ elements. There is thus a way to assign red label 2 to at most $d$ forward edges incident to $u_i$, and blue label 2 to the rest, so that the red sum of $u_i$ is different from the red sums of $u_{i_1}, \dots, u_{i_d}$.

Once the steps above have been performed fully, note that all inner, upward, and forward edges incident to the vertices of $V_0$ are assigned a label. Also, for every vertex $u \in V_0$, we currently have $\sigma_{\mathrm{r}}(u) \equiv 1 \bmod 2$, and it can be checked that also $\sigma_{\mathrm{b}}(u) \geq 2$. Furthermore, every two adjacent vertices of $V_0$ currently have their red sums being different. Remark last that all upward edges incident to the vertices of $V_0$ are assigned red labels, except for exactly one upward edge incident to $w$, which is assigned blue label 1.

**Step 2: Labelling the inner, upward, and forward edges of $V_1$.**

Due to the previous step, note also that all backward edges incident to the vertices in $V_1$ are labelled with red label 2 and blue label 2. So, one should keep in mind that, currently, $\sigma_{\mathrm{r}}(u)$ is even for every $u \in V_1$.

We now label more edges as follows:

1. First, we assign blue label 2 to all inner edges incident to vertices of $V_1$.

2. Second, we consider every vertex $u$ of $V_1$ in turn. Recall that $u$ is incident to at least two upward edges. We assign red label 2 to all these edges. If necessary, we change the label assigned to two of these edges to red label 1, so that $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$.

3. Third, let $u_1, \dots, u_n$ be an arbitrary ordering over the vertices of $V_1$, and consider the $u_i$'s one by one in turn. For every $u_i$ considered that way, denote by $u_{i_1}, \dots, u_{i_d}$ the $d \geq 0$ neighbours of $u_i$ in $V_1$ that precede $u_i$ in the ordering. If $d = 0$, then assign blue label 2 to all forward edges incident to $u_i$. Now, if $d \geq 1$, then recall that $u_i$ is incident to $d_{V_2}(u_i) \geq d$ forward edges. Thus, through assigning blue labels to these edges, we can make the blue sum of $u_i$ vary by any amount in the set $\{d_{V_2}(u_i), \dots, 2d_{V_2}(u_i)\}$, which contains $d_{V_2}(u_i) + 1 \geq d + 1$ elements. Thus, it is possible to assign blue labels to the forward edges incident to $u_i$ so that its resulting blue sum is different from that of $u_{i_1}, \dots, u_{i_d}$.

After completing the previous steps, all edges incident to the vertices in $V_1$ are labelled. For every vertex $u \in V_1$, we get $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$, and also $\sigma_{\mathrm{b}}(u) \geq 2$, because either $d_{V_1}(u) = 0$ and at least one forward edge incident to $u$ is assigned blue label 2, or $d_{V_1}(u) > 0$ and at least one inner edge incident to $u$ is assigned blue label 2. Also, every two adjacent vertices of $V_1$ are distinguished by their blue sums. Note last that all upward edges incident to the vertices of $V_1$ are assigned red labels.

**Step 3: Labelling the inner, upward, and forward edges of $V_2$.**

Note that after performing the previous step, all backward edges incident to the vertices of $V_2$ are assigned blue labels, meaning that their red sum is currently 0.

We now perform the following:

1. We assign blue label 2 to all inner edges incident to vertices in $V_2$.

2. We then consider every vertex $u$ of $V_2$ in turn, which, recall, is incident to at least two upward edges. We assign red label 2 to all these edges before, if necessary, changing the label assigned to two of these edges to red label 1, so that $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$.

3. We finish off this step similarly as the previous one. let $u_1, \dots, u_n$ be any ordering over the vertices of $V_2$, and consider the $u_i$'s one after the other. For every $u_i$, let $u_{i_1}, \dots, u_{i_d}$ be the $d \geq 0$ neighbours of $u_i$ in $V_2$ that precede $u_i$ in the ordering. If $d = 0$, then assign blue label 2 to all forward edges incident to $u_i$. Otherwise, if $d \geq 1$, then recall that $u_i$ is incident to $d_{V_0}(u_i) \geq d$ forward edges. Via assigning blue labels to these edges, we can thus make the blue sum of $u_i$ increase by any value in $\{d_{V_0}(u_i), \dots, 2d_{V_0}(u_i)\}$, which set contains $d_{V_0}(u_i) + 1 \geq d + 1$ elements. Thus, we can assign blue labels to the forward edges incident to $u_i$ so that its blue sum is different from that of $u_{i_1}, \dots, u_{i_d}$.

Once this step achieves, all edges incident to vertices in $V_0 \cup V_1 \cup V_2$ are labelled. For every vertex $u \in V_2$, we have $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$ and $\sigma_{\mathrm{b}}(u) \geq 2$. Every two adjacent vertices of $V_2$ are distinguished by their blue sums, while all upward edges incident to the vertices in $V_2$ are assigned red labels. It is important to emphasise also that assigning blue labels to the edges joining vertices of $V_2$ and $V_0$ altered the blue sums of the vertices in $V_0$, which is not an issue since the adjacent vertices of $V_0$ are distinguished by their red sums, which were not altered. So, any two adjacent vertices in $V_0$ remain distinguished, and similarly for any two adjacent vertices in $V_1$. Finally, note that any two adjacent vertices in distinct $V_i$'s are distinguished by their red sums having different values modulo 4.

**Step 4: Labelling the edges of $\mathcal{H}$.**

Recall that, at this point, we have $\sigma_{\mathrm{b}}(v) = 0$ for every vertex $v \in D_1 \cup D_2 \setminus \{w\}$ and $\sigma_{\mathrm{b}}(w) = 1$, while $\sigma_{\mathrm{b}}(u) \geq 2$ for every vertex $u \in V_0 \cup V_1 \cup V_2$. In particular, if $v \in D_1$ belongs to an empty connected component of $\mathcal{H}$, then all edges incident to $v$ are already labelled, and $v$ is distinguished from its neighbours due to its blue sum.

Recall that $H$ denotes the unique non-empty connected component of $\mathcal{H}$, and that $H$ actually contains all edges of $G$ that remain to be labelled. Recall also that $H$ contains $w$, a special vertex we defined in the first labelling step, which is the only vertex of $H$ having non-zero blue sum. According to Lemma 2.1, we can assign red labels 1 and 2 to the edges of $H$ so that, even when taking into account the red labels assigned to the upward edges incident to the vertices in $V_0 \cup V_1 \cup V_2$, any two adjacent vertices of $H$ different from $w$ are distinguished by their red sums. Since $\sigma_{\mathrm{b}}(w) = 1$ while $\sigma_{\mathrm{b}}(v) = 0$ for every $v \in V(\mathcal{H}) \setminus \{w\}$, vertex $w$ is also distinguished from its neighbours in $\mathcal{H}$. These conditions guarantee we have not introduced any conflicts involving vertices of $D_1 \cup D_2$ and vertices of $V_0 \cup V_1 \cup V_2$.

All these arguments imply that the resulting $(2,2)$-labelling of $G$ is distinguishing. $\square$

## 4. Graphs with no induced claw

We now prove the Weak $(2,2)$-Conjecture for $K_{1,3}$-free graphs. Again, we do so by first focusing on the 5-chromatic ones, before focusing on those with chromatic number at least 6. Again, we consider the two cases separately.

**Theorem 4.1.** *Every claw-free graph with chromatic number 5 admits a distinguishing $(2,2)$-labelling.*

*Proof.* The proof starts similarly as that of Theorem 3.1. We can assume $G$ is a connected 5-chromatic claw-free graph. We define $D$ and $\mathcal{R}$ as previously, as well as the 4-vertex-colouring $\phi$ of $\mathcal{R}$ with parts $V_{0,0}$, $V_{0,1}$, $V_{1,0}$, and $V_{1,1}$. The notions of empty and non-empty connected components of $\mathcal{R}$ are also defined similarly, as well as the classification of the edges of $G$ into upward and inner edges. The set $I$ and the subgraph $\mathcal{H}$ are also defined.

Some differences here, however, are because $G$ is claw-free. Note in particular that $\mathcal{R}$ might contain several non-empty connected components. However, any vertex $v$ of $\mathcal{R}$ has at most two neighbours in $D$, and conversely any vertex $u \in D$ can have neighbours in at most two connected components of $\mathcal{R}$. Also, $\mathcal{H}$ can now have several connected components containing edges. As will be pointed out, further strong assumptions on $\mathcal{H}$ can be made.

Similarly as in the proof of Theorem 3.1, we start by considering every non-empty connected component $R$ of $\mathcal{R}$, and defining a particular ordering over its vertices. In some cases, we also modify the parts of $\phi$ by a bit.

- If $R$ has a vertex $v$ with $d_R(v) = 2$, then we denote the vertices of $R$ by $v_1, \dots, v_n$ in reverse order as they are encountered during a breadth-first search algorithm performed from $v$. So, $v = v_n$, and every $v_i \neq v_n$ is incident to a forward edge, which is a backward edge from the other vertex's point of view.

  Regarding $\phi$, denoting by $v_i$ and $v_j$ the two neighbours of $v_n$, we need to make sure that we do not have $v_i$ in $V_{0,0}$ and $v_j$ in $V_{0,1}$ (or *vice versa*), or $v_i$ in $V_{1,0}$ and $v_j$ in $V_{1,1}$ (or *vice versa*). That is, if $v_i \in V_{\alpha,\beta}$ and $v_j \in V_{\alpha',\beta'}$, we need $\alpha \neq \alpha'$. Assume this is not verified, and that we have, w.l.o.g., $v_i$ in $V_{0,0}$ and $v_j$ in $V_{0,1}$. Then, since $\phi$ is proper, $v_n$ belongs to $V_{1,0}$ or $V_{1,1}$. Assume $v_n$ belongs to $V_{1,0}$, w.l.o.g. We modify $\phi$ by swapping the parts $V_{0,1}$ and $V_{1,0}$. Note that the resulting $\phi$ remains proper, and that, now, $v_i$ still lies in $V_{0,0}$, while $v_j$ lies in $V_{1,0}$, as desired.

  Finally, if $R$ has a vertex $v_i$ with $d_R(v_i) = 1$, then, keeping $\phi$ proper, we make sure that $v_i$ lies in $V_{0,0}$ or $V_{0,1}$. This is clearly possible, since $v_i$ has exactly one neighbour in $R$, and thus at most one neighbour in $V_{0,0} \cup V_{0,1}$. So we can freely guarantee this for all the degree-1 vertices of $R$.

- If $R$ has no degree-2 vertex but has a vertex $v$ with degree 1, *i.e.*, $d_R(v) = 1$, then we denote by $v_1, \dots, v_n$ the vertices of $R$ as in the previous case, *i.e.*, from a breadth-first search algorithm performed from $v = v_n$. In this case as well, we also modify $\phi$, if needed, so that all the degree-1 vertices of $R$ belong to $V_{0,0} \cup V_{0,1}$.

- If $R$ has minimum degree 3, then we consider any vertex $v$ of $R$, and denote by $v_1, \dots, v_n$ the vertices of $R$ as in the precious cases (by reversing a breadth-first search algorithm performed from $v$), so that $v_n = v$. Here, $\phi$ is not modified further.

We are now ready to start designing the $(2,2)$-labelling of $G$. Just as in the proof of Theorem 3.1, we start by labelling all edges incident to vertices in the non-empty connected components of $\mathcal{R}$, so that every two of their adjacent vertices are distinguished either by their red sums or by their blue sums. To achieve this, we will assign red labels to all inner edges and blue labels to most upward edges, so that the red sums and blue sums obtained for the vertices in $\mathcal{R}$ match $\phi$. By that, we mean that for every vertex $v$ in $V_{\alpha,\beta}$, we aim at getting $\sigma_{\mathrm{r}}(v) \equiv \alpha \bmod 2$ and $\sigma_{\mathrm{b}}(v) \equiv \beta \bmod 2$, except in a few cases (such as for some last vertices of some non-empty connected components).

Consider every non-empty connected component $R \in \mathcal{R}$ in turn. Recall that $v_1, \dots, v_n$ is an ordering over the vertices of $R$ with specific properties we described earlier. We consider the $v_i$'s one by one following the ordering, and, whenever considering a $v_i$ in this way, we assign a label to all its incident inner edges and upward edges. This way, note that, whenever starting treating a $v_i$, only its incident backward edges are labelled.

Now, for every $v_i \in V_{\alpha,\beta}$ to be considered:

- If $i \neq n$, then $v_i$ is incident to forward edges. We first assign blue label 2 to all upward edges incident to $v_i$, and red label 2 to all incident forward edges. Note that all edges incident to $v_i$ are now assigned a label. Now, if $\sigma_{\mathrm{b}}(v_i) \not\equiv \beta \bmod 2$, then we change to blue label 1 the label assigned to any upward edge incident to $v_i$. Similarly, if $\sigma_{\mathrm{r}}(v_i) \not\equiv \alpha \bmod 2$, then we change to red label 1 the label assigned to any forward edge incident to $v_i$. As a result, $\sigma_{\mathrm{r}}(v_i) \equiv \alpha \bmod 2$ and $\sigma_{\mathrm{b}}(v_i) \equiv \beta \bmod 2$. Recall also that if $d_R(v_i) = 1$, then $\alpha = 0$, and thus $\sigma_{\mathrm{r}}(v_i) \geq 2$. Since all inner edges incident to $v_i$ are assigned red labels, we also have $\sigma_{\mathrm{r}}(v_i) \geq 2$ whenever $d_R(v_i) \geq 2$. Thus, $\sigma_{\mathrm{r}}(v_i) \geq 2$ regardless of $d_R(v_i)$. Also, $\sigma_{\mathrm{b}}(v_i) \geq 1$.

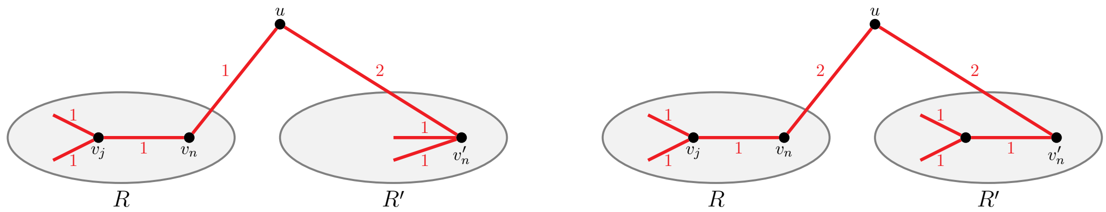
- If $i = n$, then all inner edges incident to $v_n$ are currently assigned red labels.

  - If $d_R(v_n) = 2$, then recall that, due to how we ordered the vertices of $R$, the two neighbours $v_j$ and $v'_j$ of $v_n$ in $R$ have their red sums being of distinct parity. Assume that, currently, $\sigma_{\mathrm{r}}(v_n) \equiv \sigma_{\mathrm{r}}(v_j) \bmod 2$ and $\sigma_{\mathrm{r}}(v_n) \not\equiv \sigma_{\mathrm{r}}(v'_j) \bmod 2$. We here assign blue labels to all upward edges incident to $v_n$, their values being chosen so that $\sigma_{\mathrm{b}}(v_n) \not\equiv \sigma_{\mathrm{b}}(v_j) \bmod 2$. For the sake of formality, we also change, if needed, the part of $\phi$ that contains $v_n$, so that the part it belongs to matches the resulting $\sigma_{\mathrm{r}}(v_n)$ and $\sigma_{\mathrm{b}}(v_n)$.
  - If $d_R(v_n) = 1$ and $R$ is just an edge $v_1v_2$ (thus with $v_2 = v_n$), then, by how $\phi$ was modified earlier ($v_1$ and $v_2$ belong to $V_{0,0} \cup V_{0,1}$), recall that $v_1v_2$ must be assigned red label 2. We here assign blue labels to the upward edges incident to $v_n$ so that $\sigma_{\mathrm{b}}(v_1) \not\equiv \sigma_{\mathrm{b}}(v_2) \bmod 2$.
  - Otherwise, *i.e.*, $d_R(v_n) = 1$ and $R$ is not just an edge, or $d_R(v_n) \geq 3$, then we assign red label 1 to all upward edges incident to $v_n$.

Once the process above is led for all $v_i$'s, all edges incident to the $v_i$'s are labelled. Also, if $v_i \in V_{\alpha,\beta}$ for some $i < n$, then $\sigma_{\mathrm{r}}(v_i) \equiv \alpha \bmod 2$ and $\sigma_{\mathrm{b}}(v_i) \equiv \beta \bmod 2$ with $\sigma_{\mathrm{r}}(v_i), \sigma_{\mathrm{b}}(v_i) \geq 1$ (actually, even $\sigma_{\mathrm{r}}(v_i) \geq 2$ in this case). Since $\phi$ is a proper vertex-colouring, for every two adjacent vertices $v_i$ and $v_j$ of $R$ with $i, j \neq n$, we thus have $\sigma_{\mathrm{r}}(v_i) \neq \sigma_{\mathrm{r}}(v_j)$ or $\sigma_{\mathrm{b}}(v_i) \neq \sigma_{\mathrm{b}}(v_j)$. Regarding $v_n$, either $v_n$ is not in conflict with any of its neighbours in $R$ (with respect to $\sigma_{\mathrm{r}}$ or $\sigma_{\mathrm{b}}$) and none of its incident upward edges is assigned a red label (when $d_R(v_n) = 2$, or $d_R(v_n) = 1$ with $R$ being an edge), or all its incident edges are assigned red labels and thus $\sigma_{\mathrm{b}}(v_n) = 0$ (while all neighbours $v_j$ of $v_n$ in $R$ have $\sigma_{\mathrm{b}}(v_j) \geq 1$). Also, $\sigma_{\mathrm{r}}(v_n) \geq 2$.

At this point, only edges incident to the vertices in $I$ remain to be labelled. Later on, these edges will be assigned blue labels only. This means that, through labelling these edges, the red sums of the vertices in $D$ will not be modified. Recall that the vertices in $D$ might be incident to edges assigned red labels. We need to make sure that such vertices will not be in conflict with the vertices from the non-empty connected components of $\mathcal{R}$.

Let $u$ be any vertex in $D$. Note that, by how we labelled the upward edges earlier, if $v_iu$ is an edge assigned a red label, then $v_iu$ is assigned red label 1, and $i = n$, *i.e.*, $v_i$ is the last vertex of its non-empty connected component of $\mathcal{R}$. Since $G$ is claw-free, this means $u$ must be incident to at most two edges assigned a red label. Thus, currently, $\sigma_{\mathrm{r}}(u) \leq 2$. Meanwhile, for every vertex $v$ in a non-empty connected component of $\mathcal{R}$, we have $\sigma_{\mathrm{r}}(v) \geq 2$. Hence, if $\sigma_{\mathrm{r}}(u) = \sigma_{\mathrm{r}}(v)$, then $\sigma_{\mathrm{r}}(u) = 2$.




(a) Case where $\sigma_r(v'_n) > \sigma_r(u)$.

(b) Case where $\sigma_r(v'_n) = \sigma_r(u) = 2$.

Figure 4: Local adjustments made in the proof of Theorem 4.1.

In what follows, we modify the current labelling, if needed, so that there are no two adjacent vertices $u \in D$ and $v \in V(\mathcal{R})$ with $\sigma_r(u) = \sigma_r(v) = 2$, without introducing new conflicts between adjacent vertices of $\mathcal{R}$. To achieve this, we perform label modifications to make the number of such conflicts decrease, until no such conflict remains. We perform this so that, for every $v \in V(\mathcal{R})$, we preserve, except in very peculiar cases, $\sigma_r(v) \geq 2$ while, for every $u \in D$, we have $\sigma_r(u) \leq 2$. This way, no conflicts between the vertices of $D$ and $V(\mathcal{R})$ will remain.

Assume there is a $u \in D$ with $\sigma_r(u) = 2$. As mentioned earlier, there are thus exactly two edges $uv_n$ and $uv'_n$ incident to $u$ assigned red label 1, where $v_n$ is the last vertex of some non-empty connected component $R$ of $\mathcal{R}$, and $v'_n$ is the last vertex of another non-empty connected component $R' \neq R$ of $\mathcal{R}$. Assume that $u$ is adjacent to a vertex $v_i$ from a non-empty connected component of $\mathcal{R}$ with the same red sum (possibly, $v_i \in \{v_n, v'_n\}$). Then $\sigma_r(v_i) = 2$, and, since all edges of $\mathcal{R}$ are assigned red labels, $d_R(v_i) \leq 2$.

- Assume $\sigma_r(u) = \sigma_r(v_i)$ for some $v_i \notin \{v_n, v'_n\}$. Recall that $v_iu$ is assigned a blue label. Also, because $G$ is claw-free, $v_i$ must belong to the same (non-empty) connected component of $\mathcal{R}$ as one of $v_n$ and $v'_n$. Assume $v_i$ belongs to $R$. Then, $v_iv_n$ is an edge.
  - If $d_R(v_i) = 1$, then, by how the vertices of $R$ were ordered, $d_R(v_n) \leq 2$.
    * If $d_R(v_n) = 1$, then $R$ is actually just the edge $v_iv_n = v_1v_2$. By how we treated $v_n$ earlier, recall that all upward edges incident to $v_n$ are assigned blue labels. So, this case cannot occur.
    * If $d_R(v_n) = 2$, then, by how we treated $v_n$ earlier, it cannot be that $v_nu$ is assigned a red label. Thus, this case cannot occur as well.
  - Assume now that $d_R(v_i) = 2$. Then, again, by how the vertices of $R$ were ordered, it must be that $d_R(v_n) = 2$, and no upward edge incident to $v_n$ is actually assigned a red label. So, again, this case cannot occur.

- Assume now that $\sigma_r(u) = \sigma_r(v_n)$, w.l.o.g., and that there is no $v_i \in V(R) \cup V(R') \setminus \{v_n, v'_n\}$ such that $\sigma_r(u) = \sigma_r(v_i)$ (that is, the previous case does not apply). Since $v_nu$ is assigned red label 1, note that, in order to have $\sigma_r(v_n) = 2$ with upward edges incident to $v_n$ being assigned red labels, it must be that $v_n$ is incident to exactly one inner edge $v_jv_n$ (that is, $d_R(v_n) = 1$) and to the one upward edge $v_nu$. So, $d_G(v_n) = 2$. Furthermore, $v_jv_n$ is assigned red label 1, while we also assigned red label 1 to $v_nu$. Also, by our choice of $v_n$ and by how we treated $R$, we have $d_R(v_j) \geq 3$
  - If $d_G(u) \geq 3$, then note that, regardless of how the edges incident to $u$ that are not the two assigned red label 1 are labelled, we will eventually not have any conflict between $u$ and $v_n$, and can thus leave things as is.

– Assume now $d_G(u) = 2$. Regarding $v'_n$, the fact that we had to assign red label 1 to $v'_n u$ means (by previous arguments) that $d_{R'}(v'_n) \neq 2$, and that if $d_{R'}(v'_n) = 1$ then $R'$ is not just an edge. Actually, all edges incident to $v'_n$ are assigned red labels. If $\sigma_{\mathrm{r}}(v'_n) \neq \sigma_{\mathrm{r}}(u)$ (that is, if $\sigma_{\mathrm{r}}(v'_n) > \sigma_{\mathrm{r}}(u)$), then we change the label assigned to $v'_n u$ to red label 2 (see Figure 4(a)). This way, we get $\sigma_{\mathrm{r}}(u) = 3 > 2 = \sigma_{\mathrm{r}}(v_n)$, and we thus got rid of the conflict between $v_n$ and $u$. Meanwhile, we still have $\sigma_{\mathrm{r}}(v'_n) > \sigma_{\mathrm{r}}(u)$ while $\sigma_{\mathrm{r}}(v'_n) \geq 3$ and $\sigma_{\mathrm{b}}(v'_n) = 0$, while all neighbours of $v'_n$ in $R'$ have blue sum at least 1. So, $v'_n$ cannot be involved in a conflict.

  The last case is thus when also $\sigma_{\mathrm{r}}(v'_n) = \sigma_{\mathrm{r}}(u) = 2$, which, for similar reasons as for $v_n$, occurs when $d_{R'}(v'_n) = 1$, the only inner edge incident to $v'_n$ is assigned red label 1, and $v'_n u$ is the only upward edge incident to $v'_n$, which is assigned red label 1. So, $d_G(v'_n) = 2$. In this case, we are done when changing the label assigned to $v_n u$ and $v'_n u$ to red label 2 (see Figure 4(b)). As a result, $\sigma_{\mathrm{r}}(v_n) = \sigma_{\mathrm{r}}(v'_n) = 3$ while $\sigma_{\mathrm{r}}(u) = 4$. Meanwhile, we still have $\sigma_{\mathrm{b}}(v_n) = \sigma_{\mathrm{b}}(v'_n) = 0$, while $u$ is the only neighbour of $v_n$ and $v'_n$ with blue sum 0.

It now remains to label edges incident to the vertices in $I$. Recall that $\mathcal{H}$ is the subgraph of $G$ induced by these edges. Then $\mathcal{H}$ is bipartite. In particular, since $G$ is claw-free, in every connected component $H$ of $\mathcal{H}$, every vertex must be of degree at most 2. So $H$ must be a path, or an even-length cycle. Actually, if $H$ is an even-length cycle $u_1 v_1 \dots u_k v_k u_1$ (where the $u_i$'s belong to $D$ and the $v_i$'s belong to $I$), then note that every $u_i$ cannot have another neighbour in $G$, *i.e.*, in a non-empty connected component of $\mathcal{R}$, because, since $G$ is claw-free, this would imply that one of its neighbours in $H$ must be adjacent to a vertex from a non-empty connected component of $\mathcal{R}$, a contradiction. So, all connected components of $\mathcal{H}$ must be paths. Besides, if $H$ is a path of $\mathcal{H}$, then, due to the claw-freeness of $G$, every degree-2 vertex of $H$ must also be a degree-2 vertex in $G$.

Now let $H$ be a connected component of $\mathcal{H}$, *i.e.*, a path. If $H$ has length 1, then $H = u_1 v_1$ where $u_1 \in D$ and $v_1 \in I$, meaning that $d_G(v_1) = 1$, and, because $G$ is connected and is not a one-edge graph, whatever labelling we consider, it must be that $u_1$ and $v_1$ are distinguished either by their red sums or by their blue sums. So assume now $H$ has length more than 1. Set $H = w_1 \dots w_k$ with $k \geq 3$. Now, for every $i \in \{1, \dots, k\}$, denote by $n_i$ the current value of $\sigma_{\mathrm{b}}(w_i)$. Possibly, $n_i = 0$. Actually, recall that only $n_1$ and $n_k$ can be non-zero. According to Lemma 2.3 or 2.4, it is possible to assign blue labels 1 and 2 (and, if needed, red label 1 to independent edges) to the edges of $H$ so that its adjacent vertices are distinguished, even with the blue contribution from the upward edges.

Since we have not altered the red sums of the vertices of $\mathcal{R}$, every two adjacent vertices of $\mathcal{R}$ remain distinguished, and similarly for any two adjacent vertices from $\mathcal{R}$ and $D$ (in particular, the only vertices of $D$ which had their red sums modified have red sum 1, while the vertices of $\mathcal{R}$ still have red sum at least 2). Regarding the adjacent vertices of $\mathcal{H}$, the application of Lemma 2.3 or 2.4 guarantees that they are distinguished by their blue sums, or by their red sums in certain cases. So, the resulting labelling of $G$ is distinguishing. □

**Theorem 4.2.** *Every claw-free graph with chromatic number at least* 6 *admits a distinguishing* $(2,2)$*-labelling.*

*Proof.* The proof starts similarly as that of Theorem 3.2. Again, we can assume $G$ is a connected claw-free graph with chromatic number at least 6. We again start from two maximal independent sets $D_1$ and $D_2$, chosen consecutively, and define $\mathcal{H}$ as $G[D_1 \cup D_2]$. For the current proof, we classify the connected components of $\mathcal{H}$ into three groups. That is, a connected component $H \in \mathcal{H}$ is *empty* if it contains no edges, *bad* if it consists of one edge only, and *nice* otherwise, *i.e.*, if it contains at least two edges. Since all vertices in $V(G) \setminus D_1$ have at least one neighbour in $D_1$, note that if $H$ is empty, then its only vertex belongs to $D_1$. Meanwhile, if $H$ is bad, then it consists of one vertex in $D_1$ and one in $D_2$.

Before going on, we need to add a last constraint on the choice of $D_1$ and $D_2$. Namely, among all possible choices as $D_1$ and $D_2$, we choose one that minimises the number of empty connected components in $\mathcal{H}$. Under this hypothesis, we derive the following property:

**Claim 4.3.** *If $u \in V(G) \setminus (D_1 \cup D_2)$ is adjacent to an isolated vertex $v_1 \in D_1$, then $u$ must be adjacent to two vertices $v_1' \in D_1$ and $v_2 \in D_2$ such that $v_1'v_2$ is an edge of $\mathcal{H}$.*

*Proof of the claim.* Assume $u \in V(G) \setminus (D_1 \cup D_2)$ is adjacent to some $v_1 \in D_1$ that forms an empty connected component of $\mathcal{H}$. Let $v_2 \in D_2$ be any neighbour of $u$. If $v_1$ is the only neighbour of $u$ in $D_1$, then note that, due to the edge $uv_2$, by removing $v_1$ from $D_1$ and adding $u$ to $D_1$, we would end up with two new independent sets as $D_1$ and $D_2$ inducing one less empty connected component in $\mathcal{H}$, a contradiction to our choice of $D_1$ and $D_2$. So, $v_1$ cannot be the only neighbour of $u$ in $D_1$. Let thus $v_1' \in D_1$ be another neighbour of $u$. Now, since $D_1$ is independent, and $v_1$ is isolated in $\mathcal{H}$, the fact that $G$ is claw-free implies that $v_1'v_2$ must be an edge of $\mathcal{H}$. $\diamond$

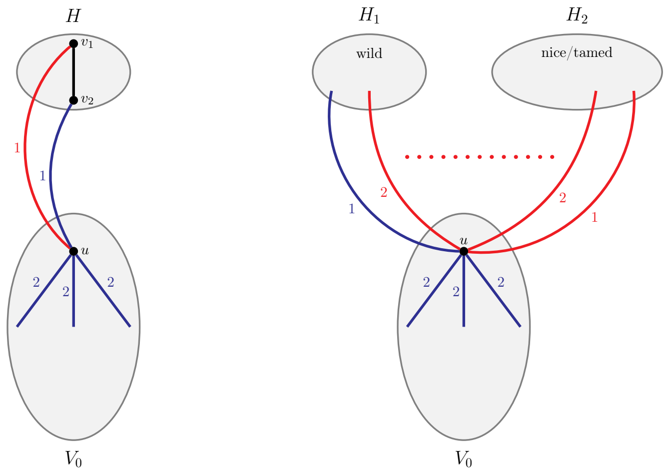
As in the proof of Theorem 3.2, we also partition $V(G) \setminus (D_1 \cup D_2)$ into $V_0$, $V_1$, and $V_2$ forming a balanced tripartition of $G - D_1 - D_2$. We also reuse the notions of inner, upward, forward, and backward edges.

The distinguishing $(2,2)$-labelling of $G$ we construct below will again be obtained through four main labelling steps, followed to produce a labelling which is very reminiscent[2] to that we aimed to produce in the proof of Theorem 3.2. However, the structure of claw-free graphs is less permissive than that of $2K_2$-free graphs, so, in several occasions, our distinguishing and labelling rules will have to be tweaked a bit.

In particular, the most troublesome point is the possible presence, in $\mathcal{H}$, of bad connected components. Note indeed that if $v_1v_2$ is a bad connected component, then the fact that $v_1$ and $v_2$ are eventually distinguished does not rely at all on the choice of the label assigned to $v_1v_2$. This means that, throughout the proof, whenever labelling an upward edge $uv_i$ (with $u \in V_0 \cup V_1 \cup V_2$ and $i \in \{1,2\}$), we have to wonder whether assigning a certain label to $uv_i$ might result in $v_1$ and $v_2$ being impossible to distinguish later on. To guarantee $v_1$ and $v_2$ can be distinguished, we will, here, sometimes have to assign blue labels to upward edges. One problem, however, is that blue sums, in the proof of Theorem 3.2, were the main way to guarantee that vertices in $D_1 \cup D_2$ can be distinguished from vertices in $V_0 \cup V_1 \cup V_2$. To counter this, we will need to guarantee that vertices in $V_0 \cup V_1 \cup V_2$ have "large" blue sums, while those in $D_1 \cup D_2$ have "small' blue sums.

With respect to these considerations, we introduce a bit more terminology for the bad connected components. Let $H = v_1v_2$ be a bad connected component of $\mathcal{H}$. At any time of our labelling steps below, we say that $H$ is *tamed* if exactly one of $v_1$ and $v_2$ is incident to an edge assigned blue label 1, while it is *wild* otherwise. The point is that, once $H$ gets tamed, then $v_1$ and $v_2$ will necessarily be distinguishable at any time as long as all of their other incident edges (different from $v_1v_2$) are assigned red labels. Now, if $H$ is a wild connected component of $\mathcal{H}$, then $H$ is said *dangerous* if, omitting $v_1v_2$, all edges incident to $v_1$ and $v_2$ that remain to be labelled are incident to the same vertex $u \in V_0 \cup V_1 \cup V_2$.

[2]An important difference we should highlight, is that, for technical reasons, we will here require the vertices in $V_1$ to have red sum 0 modulo 4, and the vertices in $V_2$ to have red sum 2 modulo 4. Note that we required the contrary in the proof of Theorem 3.2.




(a) Wild $H$.

(b) Wild $H_1$, nice/tamed $H_2$.

Figure 5: Two cases from the proof of Theorem 4.2, when labelling upward edges of $V_0$.

Those conditions mean that all upward edges incident to $v_1$ and $v_2$ have been labelled, except for at most two of them, being incident to $u$. So, the task of making sure $v_1$ and $v_2$ are distinguished will need to be handled when labelling the upward edges incident to $u$.

**Step 1: Labelling the inner, upward, and forward edges of $V_0$.**

During this step, we perform the following three substeps:

1. We start by assigning blue label 2 to all inner edges incident to vertices of $V_0$.

2. We next consider every vertex $u \in V_0$ in turn, and assign a label to all its incident upward edges in the following way. Note that, because $G$ is claw-free, the upward edges incident to $u$ go to at most two connected components of $\mathcal{H}$.

   - Assume all upward edges incident to $u$ go to only one connected component $H \in \mathcal{H}$. Since $u$ is incident to at least two upward edges, $H$ cannot be empty.
     - Assume $H$ is bad and wild (see Figure 5(a)). Then $u$ is incident to exactly two upward edges $uv_1$ and $uv_2$, where $H = v_1v_2$. Here, we assign red label 1 to $uv_1$ and blue label 1 to $uv_2$, thereby taming $H$.
     - Assume $H$ is nice or tamed. Let $uv$ be any upward edge incident to $H$. We here assign red label 1 to $uv$, and red label 2 to all other upward edges incident to $u$.
   - Assume now all upward edges incident to $u$ go to two connected components $H_1, H_2 \in \mathcal{H}$.
     - If, say, $H_1$ is empty, then, by Claim 4.3, it cannot be that $H_2$ is also empty. Denote by $v$ the unique vertex of $H_1$. If $H_2$ is nice or tamed, then we assign red label 1 to $uv$ and red label 2 to all upward edges incident to $u$ going to $H_2$. Otherwise, $H_2$ is wild, in which case we assign red label 1 to $uv$, blue label 1 to any one upward edge incident to $u$ going to $H_2$ (taming $H_2$), and red label 2 to the other upward edge to $H_2$ (which exists by Claim 4.3).
     - If $H_1$ and $H_2$ are both nice or tamed, then we assign red label 1 to any one upward edge incident to $u$ going to $H_1$ or $H_2$, and red label 2 to all others.

  - If, say, $H_1$ is wild and $H_2$ is nice or tamed (see Figure 5(b)), then we assign blue label 1 to any upward edge incident to $u$ going to $H_1$ (thereby taming $H_1$), red label 1 to any upward edge going to $H_2$, and red label 2 to all other upward edges.
  - If $H_1$ and $H_2$ are both wild, then we claim they cannot be both dangerous. Indeed, let $x \in V_1$ be any neighbour of $u$. Since $G$ is claw-free, note that a neighbour of $u$ in $H_1$, one in $H_2$, and $x$ must induce at least one edge, contradicting the fact that $H_1$ and $H_2$ are bad and dangerous. Thus, we may assume, w.l.o.g., that $H_1$ is not dangerous. Then we assign red label 1 to an upward edge incident to $u$ going to $H_1$, blue label 1 to any upward edge going to $H_2$ (taming $H_2$), and red label 2 to all other upward edges (except if $u$ is also adjacent to the second vertex of $H_1$, in which case we take this opportunity to tame $H_1$, by assigning blue label 1 to the edge).

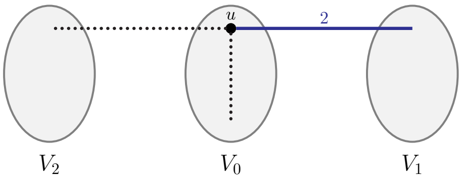
   As a result, note that, after any of the cases above, $\sigma_{\mathrm{r}}(u)$ is necessarily odd. Also, the only situations where a wild connected component adjacent to $u$ was not tamed, are when that connected component is not dangerous, because it is adjacent to a vertex in $V_1$. For every tamed connected component of $\mathcal{H}$, note that only one of its two vertices is incident to an upward edge assigned a blue label, with value 1.

3. Last, let $u_1, \dots, u_n$ be the vertices of $V_0$ ordered in increasing order over their degrees (in $V_0$), and consider the $u_i$'s one by one in order. For every $u_i$ considered that way, denote by $u_{i_1}, \dots, u_{i_d}$ the $d \geq 0$ neighbours of $u_i$ in $V_0$ that precede $u_i$ in the ordering. If $d = 0$, then assign blue label 2 to all edges incident to $u_i$ going to $V_1$. Now, if $d \geq 1$, then recall that $u_i$ is incident to $d_{V_1}(u_i) \geq d$ edges going to $V_1$. By assigning red label 2 to none, one, two, etc., or all of these edges, and blue label 2 to all others, we can increase the red sum of $u_i$ by any amount in $\{0, 2, \dots, 2d_{V_1}(u_i)\}$, which is a set of $2d_{V_1}(u_i) + 1 \geq d + 1$ elements. There is thus a way to assign red label 2 to at most $d$ edges incident to $u_i$ going to $V_1$, and blue label 2 to the rest, so that the red sum of $u_i$ is different from the red sums of $u_{i_1}, \dots, u_{i_d}$. We assign such labels so that we maximise the number of forward edges incident to $u_i$ assigned blue label 2.

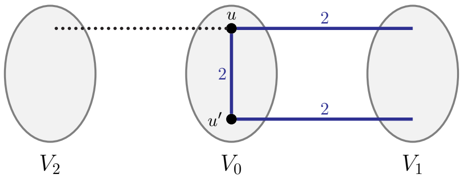
Once the labelling process above is achieved, note that all vertices of $V_0$ have their red sum being odd, while every two adjacent vertices of $V_0$ are distinguished by their red sums. Also, every vertex of $V_0$ has blue sum at least 2, due either to an incident inner edge, or to an incident forward edge. The only edges incident to the vertices of $V_0$ that are not labelled yet are backward edges, which will be assigned blue labels during later Step 3. Also, all forward edges incident to the vertices in $V_0$ were labelled, assigned red label 2 or blue label 2. Finally, recall that we tamed the bad connected components of $\mathcal{H}$ adjacent to vertices in $V_0$ whenever possible (as described above).

In later Step 3, the forward edges incident to the vertices in $V_2$ (thus going to $V_0$) will all be assigned blue labels. Thus, already at this point, we can predict that, in most cases, actually the vertices of $V_0$ will have blue sum at least 3. There are a few peculiar cases, however, where this could not be the case, which might cause eventual problems. For this reason, we need, right away, to possibly modify the current labelling a bit, to guarantee the vertices of $V_0$ will eventually either have odd red sum and blue sum at least 3, or verify other sum conditions.

Since we have assigned blue label 2 to all inner edges incident to the vertices in $V_0$, any vertex of $V_0$ has blue sum at least 4 provided it is incident to at least two inner edges. Likewise, since, in Step 3, all backward edges incident to the vertices of $V_0$ will be assigned



(a) First case.



(b) Second case.

Figure 6: Cases, in the proof of Theorem 4.2, where a vertex of $V_0$ does not get red sum at least 3. Dashed edges indicate that $u$ has no incident edges to certain sets.

blue labels, any vertex of $V_0$ will have blue sum at least 3 provided it is incident to at least one inner edge and at least one backward edge. Furthermore:

- If a vertex $u \in V_0$ is incident to no inner edge (so, $d_{V_0}(u) = 0$), then, since we assigned blue label 2 to all forward edges incident to $u$, eventually we will have $\sigma_{\mathrm{b}}(u) \geq 3$ as soon as $u$ is incident to at least two forward edges, or to only one forward edge and at least one backward edge. Similarly, in that case, we will have $\sigma_{\mathrm{b}}(u) \geq 3$ if we assigned a blue label to any upward edge incident to $u$.

- Consider now the case of a vertex $u \in V_0$ incident to a single inner edge (so, $d_{V_0}(u) = 1$). Again, eventually $\sigma_{\mathrm{b}}(u) \geq 3$ will be achieved, provided $u$ is incident to at least one backward edge. Recall also that, in the third substep above (when we labelled all forward edges), we considered the vertices with the lowest degrees first, and, through the procedure, we did our best to assign blue label 2 to the forward edges as much as possible. In particular, the only reason why we were perhaps not able to assign blue label 2 to any of the forward edges incident to $u$, is because $u$ is incident to only one forward edge, and the only neighbour $u' \in V_0$ of $u$ in $V_0$ was treated earlier in the process, and thus also verifies $d_{V_0}(u') = 1$. In that case, however, a forward edge incident to $u'$ must have been assigned blue label 2, and thus eventually we will have $\sigma_{\mathrm{b}}(u') \geq 4$.

So, the vertices $u$ of $V_0$ for which we might end up with $\sigma_{\mathrm{b}}(u) = 2$ are those depicted in Figure 6, that is:

- $d_{V_2}(u) = d_{V_0}(u) = 0$ and $d_{V_1}(u) = 1$.
- $d_{V_2}(u) = 0$ and $d_{V_0}(u) = d_{V_1}(u) = 1$, and the unique neighbour $u'$ of $u$ in $V_0$ verifies $d_{V_0}(u') = 1$ and $\sigma_{\mathrm{b}}(u') \geq 4$.

We now perform label modifications around any such $u$. So, $u$ is incident to no backward edge, to exactly one forward edge, and to at most one inner edge (which is assigned blue label 2). We unlabel all upward and forward edges incident to $u$. Then, the red sum of $u$ becomes precisely 0. We relabel all these edges in the following way:

- Assume the upward edges incident to $u$ go to only one connected component $H$ of $\mathcal{H}$. Then, $H$ is not empty. If $H$ is nice or tamed, then we assign red label 2 to all but at most two upward edges incident to $u$, assign red label 1 or red label 2 to the remaining two edges so that the red sum of $u$ becomes congruent to 2 modulo 4, and finally assign blue label 2 to the only forward edge incident to $u$. Now, if $H$ is wild, then we assign blue label 1 to any upward edge incident to $u$ (so that we tame $H$), red label 2 to the second upward edge going to $H$, and blue label 2 to the forward edge incident to $u$.

- Now assume the upward edges go to two connected components $H_1, H_2 \in \mathcal{H}$.
  - If, say, $H_1$ is empty, then $H_2$ is not empty, and $u$ is incident to at least two upward edges going to $H_2$ (by Claim 4.3). If $H_2$ is wild, then we start by assigning blue label 1 to any upward edge incident to $u$ going to $H_2$ (so that $H_2$ is tamed). There now remain at least two upward edges to be labelled. By assigning red labels to them, we can make sure the red sum of $u$ becomes congruent to 2 modulo 4. Eventually, we assign blue label 2 to the forward edge incident to $u$. Now, if $H_2$ is nice, then we assign red labels to the at least three upward edges incident to $u$ so that its red sum, again, becomes congruent to 2 modulo 4, before assigning blue label 2 to the forward edge incident to $u$.
  - If $H_1$ and $H_2$ are both nice, then we assign red labels to the upward edges incident to $u$ so that its red sum becomes congruent to 2 modulo 4, before assigning blue label 2 to its incident forward edge.
  - If, say, $H_1$ is wild while $H_2$ is nice or tamed, then we assign blue label 1 to any upward edge going to $H_1$, so that $H_1$ is tamed. There now remains at least one upward edge to be labelled. Since, currently, the red sum of $u$ is 0, we can assign red labels to these edges so that the red sum of $u$ becomes congruent to 2 modulo 4. Lastly, we assign blue label 2 to the forward edge incident to $u$.
  - If $H_1$ and $H_2$ are both wild, then, because $u$ has a neighbour in $V_1$, since $G$ is claw-free, they cannot be both dangerous. Assume w.l.o.g. that $H_1$ is not dangerous. We assign blue label 1 to any upward edge incident to $u$ going to $H_2$, thereby taming $H_2$. Now, we assign red labels to the other upward edges so that the red sum of $u$ becomes congruent to 2 modulo 4 (where, again, if $u$ is adjacent to the two vertices of $H_1$, then we can also assign blue label 1 to another upward edge to tame $H_1$, without preventing $u$ from getting red sum 2 modulo 4). Then, we assign blue label 2 to the forward edge incident to $u$.

In all cases, note that we end up with $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$ and $\sigma_{\mathrm{b}}(u) \geq 2$. Meanwhile, the other vertices of $V_0$, to which the extra care above was not applied, still have odd red sum and blue sum at least 3. Furthermore, we still have that all edges joining vertices of $V_0$ and $V_1$ are assigned red label 2 or blue label 2.

**Step 2: Labelling the inner, upward, and forward edges of $V_1$.**

We now deal with the vertices in $V_1$, for which we need some additional terminology. For every connected component $C$ of $G[V_1]$, choose $r$ a vertex of maximum degree in $C$, *i.e.*, with $d_C(r)$ as large as possible. Now let $T$ be any spanning tree of $C$ having $r$ as its root. This defines a natural orientation of $T$, from which we can infer notions of vertices being more or less deep in $T$, w.r.t. $r$. In particular, any edge $uu' \in T$, assuming $u$ is closer to $r$ than $u'$ is, is a *parent edge* incident to $u'$, and a *child edge* incident to $u$. Note that $r$ is incident to no parent edge, the leaves of $T$ are incident to no child edges, while every other vertex is incident to exactly one parent edge and at least one child edge.

We now label edges incident to the vertices in $V_1$ as follows:

1. We start by assigning blue label 2 to all edges of $E(C) \setminus E(T)$, for every connected component $C$ of $G[V_1]$ (where $T$ is the spanning tree of $C$ described earlier).

2. We now consider every connected component $C$ of $G[V_1]$ in turn, and treat the vertices of $C$ one by one, considering them according to their decreasing distance

to $r$ in $T$ (where $T$ is the spanning tree of $C$ chosen earlier, w.r.t. $C$). Whenever considering a vertex $u$ this way, we will label its incident parent edge (if it exists) and its incident upward edges. This way, note that, whenever considering a new $u \in V(C)$, all its child edges can be assumed to be labelled. Also, through what follows we will always assign a blue label or red label 2 to any incident parent edge. Since all edges joining vertices in $V_0$ and $V_1$ have been assigned blue labels and red label 2, this implies that, when starting considering a new $u \in V(C)$, currently its red sum can be assumed to be even.

We consider two main cases, treating $r$ in a particular way. Note that it is possible that $C$ consists of $r$ only, and thus that $r$ is a root with no neighbours.

- Assume we are currently considering a non-root vertex $u \in V(C)$. As mentioned earlier, the only edge of $C$ incident to $u$ that remains to be labelled is the parent edge $uu'$. Also, as will be apparent later on, even though we might have already labelled several child edges incident to $u$ (when treating deeper vertices), we can assume that, currently, $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 2$ . Since $G$ is claw-free, recall that the upward edges incident to $u$ go to at most two connected components of $\mathcal{H}$.
  - Assume first all upward edges incident to $u$ go to a single connected component $H$ of $\mathcal{H}$. Then $H$ cannot be empty, since $u$ is incident to at least two upward edges.
    * If $H$ is bad and wild, then $H = v_1v_2$ with $uv_1$ and $uv_2$ being the exact two upward edges incident to $u$. If $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$, then we assign red 2 to $uv_1$ and $uu'$, and blue label 1 to $uv_2$ (thereby taming $H$). Otherwise, *i.e.*, if $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$, then we assign blue label 1 to $uv_1$ (taming $H$), red label 2 to $uv_2$, and blue label 2 to $uu'$.
    * Otherwise, $H$ is nice or tamed. Here, we assign red label 2 to all upward edges incident to $u$ but at most two of them, to which we both assign either red label 1 or red label 2, so that the red sum of $u$ becomes a multiple of 4. We then assign blue label 2 to $uu'$.
  - Now assume the upward edges incident to $u$ go to two connected components $H_1, H_2 \in \mathcal{H}$.
    * Again, if, say, $H_1$ is empty, then $H_2$ cannot be empty by Claim 4.3. If $H_2$ is nice or tamed, then we assign red label 2 to all upward edges incident to $u$ but at most two of them, to which we both assign either red label 1 or red label 2 so that the red sum of $u$ becomes a multiple of 4, before assigning blue label 2 to $uu'$. Otherwise, if $H_2$ is wild, then we assign blue label 1 to any upward edge incident to $u$ going to $H_2$, so that $H_2$ is tamed. Now, recall that, by Claim 4.3, there must remain two upward edges incident to $u$ to be labelled, and thus we can proceed as previously to reach the same conclusions.
    * If $H_1$ and $H_2$ are both nice or tamed, then we assign red label 2 to all upward edges incident to $u$ but at most two of them, to which we both assign either red label 1 or red label 2 so that the red sum of $u$ becomes a multiple of 4. We then assign blue label 2 to $uu'$.
    * If, say, $H_1$ is wild while $H_2$ is nice or tamed, then we assign blue label 1 to any upward edge incident to $u$ going to $H_1$ (so that $H_1$ is currently tamed). If there remain at least two upward edges incident to $u$ to be labelled, then, as previously, we assign red label 1 or 2 to these edges

so that the red sum of $u$ becomes a multiple of 4, before assigning blue label 2 to $uu'$. Otherwise, only one upward edge $uv$ (with $v \in V(H_2)$) remains to be labelled, which means that there is only one upward edge $uv'$ incident to $u$ going to $H_1$. If, currently, we have $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$, then we assign red label 2 to $uv$, and blue label 2 to $uu'$. Otherwise, if $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$, then we assign red label 2 to both $uv$ and $uu'$.

  * If $H_1$ and $H_2$ are both wild, then, just as in the similar case in Step 1, due to the existence of a forward edge incident to $u$, we deduce that at least one of $H_1$ and $H_2$ must be not dangerous. Assume w.l.o.g. that $H_1$ is not dangerous. We start by assigning blue label 1 to any upward edge incident to $u$ going to $H_2$, so that $H_2$ is tamed. Again, if there remain at least two upward edges incident to $u$ to be labelled, then, by assigning red labels to these edges, we can make sure the red sum of $u$ becomes a multiple of 4, before eventually assigning blue label 2 to $uu'$. Otherwise, $u$ is incident to exactly two upward edges $uv$ and $uv'$, with $v \in V(H_1)$ and $v' \in V(H_2)$. Depending on whether $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$ or $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$, we can either assign blue label 1 or red label 2 to $uv$ (taming $H_1$ in the former case), so that, together with assigning blue label 2 to $uu'$, the red sum of $u$ becomes a multiple of 4.

In all cases above, note that, after treating $u$, we get $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$, and also $\sigma_{\mathrm{b}}(u) \geq 1$. Also, the parent edge incident to $u$ is always assigned blue label 2 or red label 2, as desired. Finally, the only situation where we did not tame a bad component adjacent to $u$, is when that bad component was not dangerous yet (because it is adjacent to a vertex in $V_2$).

- Now consider the case where $u = r$. Due to the order in which we considered the vertices of $C$, we have that all edges incident to $u$ in $C$ are currently labelled, with blue label 2 or red label 2, which is also the case for the edges joining vertices of $V_0$ and $V_1$. So, $\sigma_{\mathrm{r}}(u)$ is currently even.

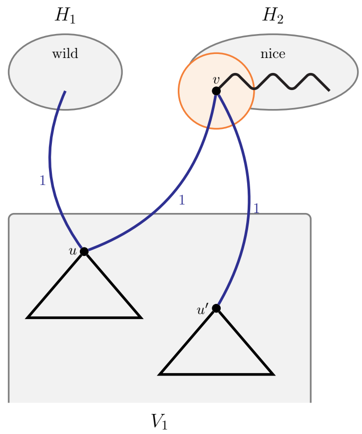
  So, we focus on labelling the upward edges incident to $u$. Recall that they go to at most two connected components of $\mathcal{H}$. Note that, below, we mark with some "$\star$" symbols two technical places of the proof for which extra explanations and care are needed. These places are discussed right after the case distinction.

  - Assume first all upward edges incident to $u$ go to exactly one connected component $H$ of $\mathcal{H}$. Then, again, $H$ cannot be empty.
    * If $H$ is bad and wild, then $H = v_1v_2$ and $uv_1$ and $uv_2$ are the exact two upward edges incident to $u$. If $H$ is not dangerous, then we assign red label 1 or red label 2 to both $uv_1$ and $uv_2$, so that the red sum of $u$ becomes a multiple of 4. Now assume $H$ is dangerous. If $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$, then we assign blue label 1 to $uv_1$ (thereby taming $H$) and red label 2 to $uv_2$. Now, if $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$, then we assign blue label 1 to $uv_1$ and blue label 2 to $uv_2$.$\star$
    * Otherwise, $H$ is tamed or nice. Here, we assign red label 2 to all but at most two upward edges incident to $u$, to both of which we either assign red label 1, or assign red label 2, so that the red sum of $u$ becomes a multiple of 4.
  - Assume second that the upward edges incident to $u$ go to two connected components $H_1, H_2 \in \mathcal{H}$.

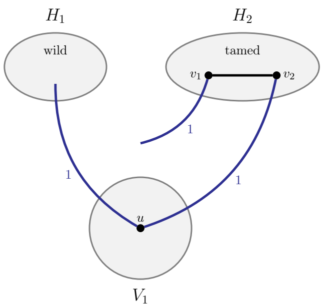
* If, say, $H_1$ is empty, then, by Claim 4.3, $H_2$ cannot be empty and $u$ is incident to at least two upward edges going to $H_2$. If $H_2$ is nice or tamed, then we first assign red label 2 to all but at most two upward edges incident to $u$. To the remaining two upward edges, we then either assign red label 1, or assign red label 2, so that the red sum of $u$ becomes a multiple of 4. Now, consider the case where $H_2$ is wild. Here, we assign blue label 1 to an upward edge going to $H_2$ (so that $H_2$ is tamed), before assigning red labels to the remaining ones so that the red sum of $u$ becomes a multiple of 4. This is indeed possible, since there are exactly two such edges.
* If $H_1$ and $H_2$ are both nice or tamed, then we assign red label 2 to all upward edges incident to $u$ but at most two of them, to which both we assign red label 1 or red label 2 so that $\sigma_{\mathrm{r}}(u)$ becomes a multiple of 4.
* Assume here that, say, $H_1$ is wild and $H_2$ is nice or tamed. If $u$ is incident to at least three upward edges, then we assign blue label 1 to any edge going to $H_1$ (thereby taming $H_1$), before assigning red labels to the remaining upward edges so that the red sum of $u$ becomes a multiple of 4. Now, assume $u$ is incident to only one upward edge $uv_1$ going to $H_1$, and only one upward edge $uv_2$ going to $H_2$. If $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$, then we assign blue label 1 to $uv_1$ (so that $H_1$ is tamed) and red label 2 to $uv_2$. Lastly, suppose $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$. If we cannot assign red label 2 to $uv_1$ (which would allow us to also assign red label 2 to $uv_2$, to make the red sum of $u$ become a multiple of 4), then it means that $H_1$ is dangerous. Regarding $v_2$, since $G$ is claw-free and $H_2$ is thus a path or a cycle (for reasons we mentioned in the proof of Theorem 4.1), it must be that $d_{H_2}(v_2) = 1$ (as otherwise $u$ would be adjacent to another vertex of $H_2$, and there would be at least three upward edges incident to $u$, a case we handled earlier). In this case, we assign blue label 1 to both $uv_1$ (taming $H_1$) and $uv_2$, which preserves $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$.⋆⋆
* If $H_1$ and $H_2$ are both wild, then, again, because $G$ is claw-free and $u$ has at least one neighbour in $V_2$, we deduce that $H_1$ and $H_2$ cannot both be dangerous. Assume $H_1$ is not dangerous. We start by assigning blue label 1 to an upward edge $uv'$ incident to $u$ going to $H_2$, so that $H_2$ is tamed. If there remain at least two upward edges to be labelled, then, once more, by assigning red labels to these edges we can make sure the red sum of $u$ becomes a multiple of 4. Otherwise, there remains only one such edge $uv$, going to $H_1$. If $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$, then we assign red label 2 to $uv$. Otherwise, we have $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$, in which case we assign blue label 1 to $uv$, thereby taming $H_1$.

In all these cases, we, again, always end up with $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$. Note that, this time, there are also cases where we end up with $\sigma_{\mathrm{b}}(u) = 0$. We also need to discuss technical points related to the places we marked with "⋆" symbols.

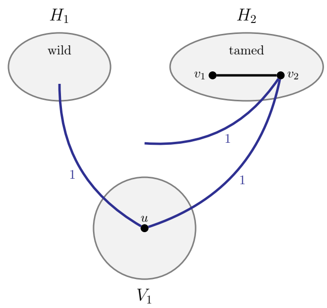
⋆ This place of the proof is the only one (up to this point) where we label an upward edge with blue label 2. This upward edge assigned label 2 goes to a wild connected component $H = v_1v_2$ of $\mathcal{H}$ that is dangerous. This means that, later on in the process, no further upward edge going to $H$ can be considered, and thus that the blue sums of $v_1$ and $v_2$, provided we eventually assign a red label to $v_1v_2$, will remain 1 and 2 (thus distinguishing these two vertices).




(a) Wild $H_1$, nice $H_2$.




(b) Wild $H_1$, nice $H_2$ (case 1).




(c) Wild $H_1$, nice $H_2$ (case 2).

Figure 7: Some problematic cases when dealing with the vertices of $V_1$, in the proof of Theorem 4.2. In (a), the orange circle highlights an induced claw.

⋆⋆ This place of the proof is the only one where an upward edge $uv$ going to a nice or tamed connected component $H$ is assigned a blue label (with value 1). Recall that we must have $d_H(v) = 1$.

- If $H$ is nice, then note that this place of the proof is actually the only one where an upward edge incident to $v$ can be assigned a blue label, with value 1. More precisely, an upward edge $uv$ is assigned blue label 1 only if $u$ is the root we have chosen for some connected component of $G[V_1]$, and $v$ is the only neighbour of $u$ in $H$. This means, see Figure 7(a), that $v$ cannot be incident to two such edges, as otherwise there would be two vertices $u$ and $u'$ belonging to distinct connected components of $G[V_1]$ sharing $v$ as a common neighbour, which is not possible as, because $G$ is claw-free, we would deduce either that $u$ and $u'$ are adjacent (thereby being part of a single connected component of $G[V_1]$), or that one of $u$ and $u'$ is also adjacent to the unique neighbour of $v$ in $H$ (leading to a different, non-exceptional case when treating $u$ or $u'$). This means that $v$, throughout Step 2, must remain of blue sum 1.
- If $H = v_1v_2$ (where $v \in \{v_1, v_2\}$) is tamed, then a technical point is that, although one of $v_1$ and $v_2$, say $v_1$, was already incident to an upward edge assigned blue label 1 (which earlier led to $H$ being tamed), we are here creating another upward edge $uv$ incident to $v_1$ or $v_2$ assigned blue label 1. This can lead to two peculiar situations:
  - ∗ If $v = v_2$, then note that this makes $v_1$ and $v_2$ both have blue sum 1 (see

Figure 7(b)), which makes them not distinguishable through their blue sums. In this case, we instead assign blue label 2 to $uv_2$.

* If $v = v_1$, then note that this makes the blue sum of $v_1$ be equal to 2 (see Figure 7(c)).

A problem is that, when dealing with a later connected component $C' \neq C$ of $G[V_1]$, this exact situation can occur again, with the root $r'$ of $C'$ also requiring to have an incident upward edge going to $H$ to be assigned blue label 1. Fortunately, this exact situation with $H$ cannot occur for three different roots $r$, $r'$, and $r''$ being in different connected components $C$, $C'$, and $C''$ of $G[V_1]$, as we would have at least two of $r$, $r'$, and $r''$, say $r$ and $r'$, sharing a neighbour in $\{v_1, v_2\}$, say $v_1$, which because of the claw with center $v_1$ and leaves $r$, $r'$, and $v_2$, would imply, since $G$ is claw-free, that either $r$ and $r'$ are adjacent (and thus $C$ and $C'$ should be part of the same connected component), or that $r$ or $r'$ neighbours both $v_1$ and $v_2$ (and thus we would fall into a different case of the case distinction).

In case, say, $r$ and $r'$ both need to assign a blue label to an incident upward edge going to $H$, by these arguments we must have, say, that $r$ neighbours $v_1$ only, while $r'$ neighbours $v_2$ only. In that case, we assign blue label 1 to these two edges, $rv_1$ and $r'v_2$, to make sure $\{\sigma_{\mathrm{b}}(v_1), \sigma_{\mathrm{b}}(v_2)\} = \{1, 2\}$. In particular, $v_1$ and $v_2$ remain distinguished by their blue sums.

By these arguments, in this place of the proof, in cases where roots need to have an incident upward edge going to a nice or tamed connected component $H$ being assigned a blue label, this can be done in such a way that both vertices of $H$ are distinguished by their blue sums, and that these blue sums have value at most 2. In particular, recall that the situation we marked with $\star$ deals with a dangerous connected component, which, thus, cannot be considered in the present case as $H$.

3. Last, we consider the vertices of $V_1$ one by one, following any ordering $u_1, \dots, u_n$ where the roots of the connected components of $G[V_1]$ appear first (in any order). For every $u_i$ considered that way, let $u_{i_1}, \dots, u_{i_d}$ denote the $d \geq 0$ neighbours of $u_i$ in $V_1$ that have already been treated during this step. If $d = 0$, then we assign blue label 2 to all forward edges (going to $V_2$) incident to $u_i$. Otherwise, $u_i$ is incident to $d_{V_2}(u_i) \geq d$ forward edges, and by assigning blue labels to these edges we can increase the blue sum of $u_i$ by any amount in $\{d_{V_2}(u_i), \dots, 2d_{V_2}(u_i)\}$, a set of $d_{V_2}(u_i)+1 \geq d+1$ values. So we can assign blue labels to the forward edges incident to $u_i$ so that its eventual blue sum is different from those of $u_{i_1}, \dots, u_{i_d}$. We do this so that the blue sum of $u_i$ is always as large as possible.

Once every vertex $u$ of $V_1$ has been treated that way, note that it must verify $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$. We claim it must also verify $\sigma_{\mathrm{b}}(u) \geq 2$. Indeed, if $d_{V_1}(u) = 0$, then all forward edges incident to $u$ are assigned blue label 2, and the claim holds. Otherwise, if $d_{V_1}(u) \geq 1$, then either $u$ is not the root of its connected component $C$ of $G[V_1]$, in which case, as mentioned earlier, at least one upward edge or inner edge incident to $u$ is assigned a blue label, which, together with an incident forward edge, yields $\sigma_{\mathrm{b}}(u) \geq 2$; or $u$ is the root of $C$, in which case we treated $r$ early in the third step above, which means, since we maximised the resulting blue sums, that all its incident forward edges are assigned blue label 2, yielding $\sigma_{\mathrm{b}}(u) \geq 2$.

Note also that all edges joining vertices of $V_1$ and $V_2$ have been assigned blue labels. Also, as pointed out earlier, the only vertices $v$ of some $H \in \mathcal{H}$ that currently have non-zero blue sum verify $\sigma_{\mathrm{b}}(v) \leq 2$. Also, for such vertices $v$, we have $d_H(v) = 1$. Last, as pointed out above in the remarks marked with "$\star$" symbols, if $H$ is bad and some of its vertices have non-zero blue sum, then its two vertices are distinguished by their blue sums.

**Step 3: Labelling the inner, upward, and forward edges of $V_2$.**

Now, we deal with the vertices of $V_2$. Recall that only the backward edges incident to these vertices have been labelled at this point, and they were assigned blue labels. We label their remaining incident edges in the following way.

1. We first assign blue label 2 to all inner edges incident to the vertices of $V_2$.
2. Next, we consider every vertex $u \in V_2$ in turn, and label its incident upward edges. Again, since $G$ is claw-free, the upward edges incident to $u$ go to at most two connected components of $\mathcal{H}$. Also, currently $\sigma_{\mathrm{r}}(u) = 0$.
    - Assume first all upward edges incident to $u$ go to a single connected component $H$ of $\mathcal{H}$. Again, $H$ cannot be empty.
        - If $H$ is bad and wild, then $u$ is incident to exactly two upward edges $uv_1$ and $uv_2$ with $H = v_1v_2$. In this case, we assign blue label 1 to $uv_1$ and red label 2 to $uv_2$, so that we get $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$, and we tame $H$.
        - Otherwise, *i.e.*, $H$ is nice or tamed, then we assign red label 2 to all upward edges incident to $u$ but at most two of them, to both of which we assign either red label 1 or red label 2 so that we get $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$.
    - Second, assume all upward edges incident to $u$ go to two connected components $H_1, H_2 \in \mathcal{H}$.
        - If, say, $H_1$ is empty, then, by Claim 4.3, $H_2$ is not empty, and $u$ is incident to at least two upward edges going to $H_2$. If $H_2$ is wild, then we assign blue label 1 to any upward edge incident to $u$ going to $H_2$, so that $H_2$ is tamed; there then remain at least two upward edges to be labelled, to which we assign red labels so that we get $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$. Otherwise, $H_2$ is nice or tamed, in which case we assign red label 2 to all but at most two upward edges incident to $u$, to both of which we either assign red label 1 or red label 2 so that we get $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$.
        - If $H_1$ and $H_2$ are both nice or tamed, then we assign red label 2 to all but at most two upward edges incident to $u$, to both of which we assign either red label 1 or red label 2 so that we obtain $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$.
        - If $H_1$ is wild and $H_2$ is nice or tamed, then we first assign blue label 1 to any upward edge incident to $u$ going to $H_1$ (thereby taming $H_1$). There then remain at least one upward edge to be labelled. If there is only one such edge, then we assign red label 2 to it. Otherwise, we assign red label 2 to all but at most two remaining upward edges incident to $u$, to both of which we either assign red label 1 or red label 2. In both cases, we obtain $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$.
        - Now assume both $H_1$ and $H_2$ are wild, and let $v$ be any vertex of $H_1$ adjacent to $u$, and $v'$ be any vertex of $H_2$ adjacent to $u$. We start by assigning blue label 1 to both $uv$ and $uv'$, thereby taming $H_1$ and $H_2$. If there remain at least one upward edge incident to $u$ to be labelled, then we assign red labels to those edges so that we get $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$. Otherwise, it means

that $uv$ and $uv'$ are the only two upward edges incident to $u$. If, say, $H_1$ is not dangerous, then we assign red label 2 to $uv$ and blue label 1 to $uv'$ (thereby taming $H_2$). Now, assume both $H_1$ and $H_2$ are dangerous.

* If assigning red label 1 to both $uv$ and $uv'$ guarantees that both vertices of $H_1$, and similarly both vertices of $H_2$, are not in conflict, then we do assign labels this way. Note that we get $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$ as a result.
* If both $v$ and $v'$ get in conflict with their unique neighbour in $H_1$ and $H_2$, respectively, upon assigning red label 1 to $uv$ and $uv'$, then we label $uv$ with blue label 1 and $uv'$ with red label 2. This way, note that $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$, while $v$ and $v'$ are not in conflict with their respective neighbour in $H_1$ and $H_2$.
* Last, if, when having $uv$ and $uv'$ being assigned red label 1, only, say, $v$ is in conflict with its neighbour in $H_1$ (while $v'$ is not with its neighbour in $H_2$), then we label $uv$ with red label 2 and $uv'$ with blue label 1. We then reach the same conclusions as in the previous case.

After performing this labelling substep, we get $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$ in all cases. All bad connected components adjacent to $u$ have been either tamed, or their incident edges have been labelled so that its two adjacent vertices cannot be in conflict. Also, when taming a wild connected component, we did so, in this case, by assigning blue label 1 to an incident edge.

3. Now, as previously, let $u_1, \dots, u_n$ be an arbitrary ordering over the vertices of $V_2$, and consider the $u_i$'s one by one in any order. For every $u_i$ considered like this, let $u_{i_1}, \dots, u_{i_d}$ be the $d \geq 0$ neighbours of $u_i$ in $V_2$ preceding $u_i$ in the ordering. If $d = 0$, then assign blue label 2 to all forward edges incident to $u_i$, going to $V_0$. Now, if $d \geq 1$, then recall that $u_i$ is incident to $d_{V_0}(u_i) \geq d$ forward edges. By assigning blue labels to these edges, we can thus make the blue sum of $u_i$ increase by any amount in $\{d_{V_0}(u), \dots, 2d_{V_0}(u)\}$, thus in $d_{V_0} + 1 \geq d + 1$ possible ways. So we can assign blue labels to the forward edges incident to $u_i$ so that the blue sum of $u_i$ is different from the blue sums of $u_{i_1}, \dots, u_{i_d}$.

At this point of the proof, note that all edges incident to the vertices in $V_0$, $V_1$, and $V_2$ have been labelled. For all vertices $u \in V_0$, we either have $\sigma_{\mathrm{r}}(u) \equiv 1 \bmod 2$ and $\sigma_{\mathrm{b}}(u) \geq 3$, or $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$, $\sigma_{\mathrm{b}}(u) \geq 2$, and $u$ is not adjacent to any vertex in $V_2$ and has no neighbour in $V_0$ having its red sum verifying the same properties. For all vertices $u \in V_1$, we have $\sigma_{\mathrm{r}}(u) \equiv 0 \bmod 4$ and $\sigma_{\mathrm{b}}(u) \geq 2$. For all vertices $u \in V_2$, we have $\sigma_{\mathrm{r}}(u) \equiv 2 \bmod 4$ and $\sigma_{\mathrm{b}}(u) \geq 2$. Furthermore, forward edges were labelled so that adjacent vertices with odd red sum in $V_0$ are distinguished w.r.t. their red sums, adjacent vertices in $V_1$ are distinguished w.r.t. their blue sums, and similarly for adjacent vertices in $V_2$. So, any two adjacent vertices in $V_0 \cup V_1 \cup V_2$ are distinguished by the current partial labelling.

Now, regarding any connected component $H \in \mathcal{H}$, in general its vertices should have blue sum 0. Precisely, the only vertices $v \in V(H)$ with $\sigma_{\mathrm{b}}(v) > 0$ verify $\sigma_{\mathrm{b}}(v) \leq 2$. Those with $\sigma_{\mathrm{b}}(v) = 2$ verify $d_H(v) = 1$. The typical cases in which this occurs, is when $H$ is bad, in which case its only two vertices have blue sum 1 and 2. Otherwise, if $\sigma_{\mathrm{b}}(v) = 1$, then most of the times $H$ is bad, in which case only one edge incident to the two vertices of $H$ was assigned blue label 1 (in order to tame $H$). It is also possible to have $\sigma_{\mathrm{b}}(v) = 1$ when $H$ is a path of length at least 2, in which cases $v$ must be an end of that path.

**Step 4: Labelling the edges of $\mathcal{H}$.**

We now consider the edges of every connected component $H \in \mathcal{H}$. Recall that $H$ can be of three main types, which we treat as follows:

- If $H$ is bad, then $H$ has only one edge $v_1v_2$. By how we labelled the upward edges through Steps 1 to 3, recall that $v_1$ and $v_2$ are already distinguished, by either their red sums or their blue sums. In particular, if we do not have $\sigma_{\mathrm{b}}(v_1) = \sigma_{\mathrm{b}}(v_2) = 0$, then $v_1$ and $v_2$ cannot have the same blue sum. Also, we have $\sigma_{\mathrm{b}}(v_1), \sigma_{\mathrm{b}}(v_2) \leq 2$, while, in $V_0$, all vertices $u$ with $\sigma_{\mathrm{r}}(u) \equiv 1 \bmod 2$ verify $\sigma_{\mathrm{b}}(u) \geq 3$. Also, all other vertices $u$ in $V_0 \cup V_1 \cup V_2$ verify $\sigma_{\mathrm{b}}(u) \geq 2$.

  By all these arguments, it can be noted that, by assigning a red label to $v_1v_2$ so that, assuming $\sigma_{\mathrm{b}}(v_1) \geq \sigma_{\mathrm{b}}(v_2)$, we get $\sigma_{\mathrm{r}}(v_1) \equiv 1 \bmod 2$, then we cannot get any conflict involving a vertex of $H$ and one of $V_0 \cup V_1 \cup V_2$.

- If $H$ is a path $p_1 \dots p_k$ of length $k-1$ at least 2, then recall that $\sigma_{\mathrm{b}}(p_1), \sigma_{\mathrm{b}}(p_k) \leq 1$ while $\sigma_{\mathrm{b}}(p_2) = \sigma_{\mathrm{b}}(p_{k-1}) = 0$, while $\sigma_{\mathrm{b}}(u) \geq 2$ for every $u \in V_0 \cup V_1 \cup V_2$. So, upon assigning only red labels to the edges of $H$, we cannot get a conflict between vertices of $H$ and vertices in $V_0 \cup V_1 \cup V_2$. Now, by Lemma 2.3 or 2.4, we can assign red and blue labels to the edges of $H$ so that its adjacent vertices, when taking into account how we labelled the upward edges, are distinguished by their red sums or blue sums, while maintaining $\sigma_{\mathrm{b}}(p_1), \sigma_{\mathrm{b}}(p_k) \leq 1$.

- If $H$ is a cycle $v_1 \dots v_k v_1$ of even length, then recall that $\sigma_{\mathrm{b}}(v_i) = 0$ for every $i \in \{1, \dots, k\}$, while, again, $\sigma_{\mathrm{b}}(u) \geq 2$ for every $u \in V_0 \cup V_1 \cup V_2$. So, provided we assign blue label 1 to edges forming a matching of $H$ and red labels to the rest, we cannot get conflicts involving vertices of $H$ and vertices of $V_0 \cup V_1 \cup V_2$. Here, Lemma 2.5 tells us we can label the edges of $H$ this way, so that any two of its adjacent vertices are distinguished.

By all these arguments, we end up with a distinguishing $(2,2)$-labelling of $G$. □

## 5. Conclusion

In this work, we proved the Weak $(2,2)$-Conjecture for $2K_2$-free graphs and $K_{1,3}$-free graphs, two classes of graphs for which the 1-2-3 Conjecture is not known to hold. Another source of interest for those graphs is that they have unbounded chromatic number.

Proving the Weak $(2,2)$-Conjecture in all cases, or even the 1-2-3 Conjecture itself, would of course be the main achievement that one could hope for in this field. Towards this, one could also, for similar reasons as the ones that motivated us, first focus on proving the Weak $(2,2)$-Conjecture for more classes of graphs, such as other graph classes defined in terms of forbidden induced structures. As such, we believe it would be interesting to wonder about triangle-free graphs, or only graphs with large girth in general. Conversely, one could wonder about graphs in which many short cycles are present, such as chordal graphs. Another class of graphs could be e.g. that of $P_4$-free graphs (a.k.a. cographs).

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