

Surface area and volume of excursion sets observed on point cloud based polytopic tessellations

Ryan Cotsakis, Elena Di Bernardino, Céline Duval

▶ **To cite this version:**

Ryan Cotsakis, Elena Di Bernardino, Céline Duval. Surface area and volume of excursion sets observed on point cloud based polytopic tessellations. 2022. hal-03782649

HAL Id: hal-03782649
https://hal.science/hal-03782649

Preprint submitted on 21 Sep 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Surface area and volume of excursion sets observed on point cloud based polytopic tessellations

Ryan Cotsakis*, Elena Di Bernardino* and Céline Duval†

Abstract

The excursion set of a C^2 smooth random field carries relevant information in its various geometric measures. From a computational viewpoint, one never has access to the continuous observation of the excursion set, but rather to observations at discrete points in space. It has been reported that for specific regular lattices of points in dimensions 2 and 3, the usual estimate of the surface area of the excursions remains biased even when the lattice becomes dense in the domain of observation. In the present work, under the key assumptions of stationarity and isotropy, we demonstrate that this limiting bias is invariant to the locations of the observation points. Indeed, we identify an explicit formula for the bias, showing that it only depends on the spatial dimension d. This enables us to define an unbiased estimator for the surface area of excursion sets that are approximated by general tessellations of polytopes in $\mathbb{R}^d$, including Poisson-Voronoi tessellations. We also establish a joint central limit theorem for the surface area and volume estimates of excursion sets observed over hypercubic lattices.

Key words: Bias correction, Crofton formula, Crossing probabilities, Excursion sets, Geometric inference, Joint Central Limit Theorem, Lipschitz-Killing curvatures, Surface area, Voronoi tessellations.

AMS Classification: 62R30, 62M40, 60G60, 62H11

1 Introduction

1.1 Motivations

The study of random fields through the geometry of their excursion sets has received a lot of interest in recent literature. This is mainly stimulated by their wide range of applications in domains such as cosmology, for the study of Cosmic Microwave Background radiation and the distribution of galaxies (see, *e.g.*, Casaponsa et al. (2016), Schmalzing and Górski (1998), Gott et al. (2007), Gott et al. (2008)), brain imaging (see Adler and Taylor (2011), Section 5, and the references therein), and the modelling of sea waves (see, *e.g.*, Longuet-Higgins (1957), Wschebor (2006), Lindgren (2000)). The geometric features considered are referred to as either Lipschitz-Killing curvatures, intrinsic volumes or Minkowski functionals, depending on the literature. Many studies have been dedicated to computing these objects from the observation of one excursion set on a compact domain T in $\mathbb{R}^d$ (see *e.g.* Adler and Taylor (2007)), limit results when the size of T grows to $\mathbb{R}^d$ have been established under specific conditions on the field (see *e.g.* Bulinski (2010), Bulinski et al. (2012), Kratz and Vadlamani (2018), Meschenmoser and Shashkin (2013) or Spodarev (2014)) and these estimates have been successfully used to derive testing procedures (see *e.g.* Abaach et al. (2021), Biermé et al. (2019), Di Bernardino and Duval (2022)

*Université Côte d'Azur, CNRS UMR 7351, Laboratoire J.A. Dieudonné (LJAD), France
†Université de Lille, CNRS UMR 8524, Laboratoire Paul Painlevé, Lille, France

or Berzin (2021)). In the present work, we focus on two specific additive measures of the excursion sets of a C^2 stationary random field in $\mathbb{R}^d$: the surface area and the volume (see Section 1.3 for their mathematical definition).
The main motivation of this work comes from the following ascertainment: many works rely on the assumption that "*the excursion set is observed on* $T \subset \mathbb{R}^d$" which is meant to be understood as "the field is *continuously* observed over T". This seems unrealistic as in practice, for instance in dimension 2 where excursion sets can be viewed as images, they are encoded through a matrix whose entries make a one-to-one connection with the pixels of the image. Even in cases where the resolution of the image is very high, the image of the excursion set remains a discretization of the continuous object that is the excursion set on T.
There exist estimators of the surface area and the volume taking into account the discrete nature of the observations. For the surface area, whose computation is more challenging, local counting algorithms are studied for example, in Miller (1999) and Biermé and Desolneux (2021). Both studies consider specific two-dimensional regular lattices of points (square and hexagonal) and bring to light an interesting phenomenon: for each of the considered lattices, the expected surface area (in this case, perimeter length) of the discretized excursion set does not converge to that of the continuous excursion set; there is a bias factor of $4/\pi$ (see Biermé and Desolneux (2021), Proposition 5). In dimension 3, Miller (1999) observes a similar behavior, with a bias factor of $3/2$ for the cubic lattice. Interestingly, Thäle and Yukich (2016) shows that in any dimension d, for an excursion set observed on a Poisson-Voronoi mosaic, "*the surface area asymptotics involve a universal correction factor*". However, this correction factor is not explicitly calculated in the aforementioned paper. In dimension 2, for a square lattice, a statistical estimating strategy is developed in Cotsakis et al. (2022) to avoid this asymptotic bias (see Cotsakis et al. (2022), Remark 3 for further details) at the cost of an introduced hyperparameter.
The main result of this paper offers a general picture for the mean surface area of an excursion set that is approximated by convex polytopes in the following sense: for a general tessellation of polytopes in $\mathbb{R}^d$ (in the sense of Definition 1.2), we provide an explicit formula for the limiting mean surface area of the polytopic region that approximates an excursion set as the scale of the tessellation is decreased. The limiting mean surface area of the approximated excursion set approaches a constant multiple of that of the true excursion set; surprisingly, the constant is independent of the geometry of the tessellation, and only depends on the dimension d (see Equation (22)). We compute the constant in every dimension (see Equations (22) and (14)) and identify it with the unspecified *universal correction factor* mentioned in Thäle and Yukich (2016) (see our Corollary 2.1).
Moreover, thanks to a second order expansion of this bias (see Theorem 2.1), it is possible, to derive for a *hypercubic* lattice a *joint* central limit theorem for the estimated surface area and volume (see Theorem 3.1) by imposing additional strong mixing assumptions on the underlying field. This limit result is novel among existing limit results since it gives the joint asymptotics of two *different* Lipschitz-Killing curvatures, the surface area and the volume, whereas most multivariate limit theorems hold for a single curvature at multiple levels (see for instance Corollary A.1).

The outline of the paper is the following. In Section 1.2 we define the geometric measures that we consider, as well as the Crofton formula, which is an essential tool to prove the main result (Theorem 2.1). In Section 1.3, the surface area and volume as well as their corresponding estimators on general point clouds are introduced. Since the bias of the estimated volume is a well understood deterministic quantity that is asymptotically negligible, Section 2 focuses on the study of the bias of the estimated surface area; the main results, which hold for general point clouds in any dimension (Theorems 2.1 and 2.2), are stated and proved. Section 3 restricts to the hypercubic lattice and proposes under additional strongly-mixing assumptions the joint CLT (Theorem 3.1) for the estimated surface area and volume. Sections 4 and 5

contain additional results and proofs related to Sections 2 and 3 respectively. Finally, an Appendix Section includes a refined result for the variance of the estimated volume (see Section A.1), several examples (see Section A.2) and some considerations on alternative approaches to recover the dimensional constant appearing in Theorem 2.1 and to approximate the surface area (see Section A.4).

1.2 Geometric measures and the Crofton formula

In the following, $\|\cdot\|_p$ denotes the L_p norm; $\|\cdot\|_\infty$, the supremum norm; $|\cdot|$, the absolute value; $\mathbb{1}_A$, the indicator of a set A; and ∂A, the boundary of a set A. The closed ball of radius r centered at the origin $\mathbf{0}$ in $\mathbb{R}^d$ is denoted B_r^d. Finally, recall that $(\mathbf{e}_i)_{1\leq i\leq d}$ denotes the canonical basis of $\mathbb{R}^d$.

Hausdorff measures. We first introduce the different measures considered in this article. For $k \in \{0,\dots,d\}$, let $\sigma_k(B)$ be the k-dimensional Hausdorff measure of a measurable set $B \subset \mathbb{R}^d$,

$$\sigma_k(B) := \frac{\pi^{k/2}}{2^k\Gamma\left(\frac{k}{2}+1\right)} \lim_{\delta\to 0} \sigma_k^\delta(B), \tag{1}$$

where Γ denotes the gamma function and

$$\sigma_k^\delta(B) := \inf\left\{\sum_{i\in\mathbb{N}} \mathrm{diam}(U_i)^k : \mathrm{diam}(U_i) < \delta,\ \bigcup_{i=1}^{\infty} U_i \supseteq B\right\}, \tag{2}$$

where $\mathrm{diam}(U) := \sup\{\|u-v\|_2, u, v \in U\}$ and the infimum is taken over all countable covers of B by arbitrary subsets U_i of $\mathbb{R}^d$ (see, *e.g.*, Schneider and Weil (2008) p.634). The Hausdorff dimension of B is the unique integer value d_B such that $\sigma_k(B) = 0$ if $k < d_B$ and $\sigma_k(B) = +\infty$ if $k > d_B$ (see, *e.g.*, Rogers (1998)). We have chosen to normalize $\sigma_k(B)$ in (1) such that for $k = \{0,\dots,d\}$, $\sigma_k(B)$ corresponds to the k-dimensional Lebesgue measure of B.
In the present work we focus on the measures in (1) for $k = d-1$ and $k = d$. They correspond respectively to what we call *surface area* and *volume* in dimension d. In the proofs, two other measures play an important role: σ_0, the counting measure for sets of isolated points, and σ_1, the measure of length.

Main tool: Crofton formula. In the present work, we heavily rely on the Crofton formula. For a k-dimensional smooth object embedded in $\mathbb{R}^d$, this classical result in integral geometry relates the σ_k measure of the object with the average number of times it is intersected by randomly oriented $(d-k)$-flats. We will be particularly interested in the case of $k = d-1$, in which the 1-flats correspond to randomly oriented lines.

To state the Crofton formula, we first need to introduce the affine Grassmanian $A(d,m)$, for $m \in \{1,...,d\}$, which is the set of affine m-dimensional subspaces of $\mathbb{R}^d$. Since the set $A(d,1)$ of lines in $\mathbb{R}^d$ plays a crucial role in the Crofton formula we use, we propose a particular parametrization.
It is shown in Schneider and Weil (2008) p.168 that $A(d,1)$ is equipped with a unique locally finite motion invariant measure μ_1, that is normalized such that

$$\mu_1(\{l \in A(d,1) : l \cap B_1^d \neq \emptyset\}) = \sigma_{d-1}(B_1^{d-1}).$$

For $\mathbf{s} \in \partial B_1^d$ and $\mathbf{v} \in \mathrm{vect}(\mathbf{s}^\perp)$, denote by $l_{\mathbf{s},\mathbf{v}}$ the element of $A(d,1)$ that is parallel with $\mathbf{s}$ and passes through the point $\mathbf{v}$,

$$l_{\mathbf{s},\mathbf{v}} := \{\mathbf{v} + \lambda\mathbf{s} : \lambda \in \mathbb{R}\}. \tag{3}$$

For each $\mathbf{s} \in \partial B_1^d$, there exists a unitary (rotation) operator $\theta_{\mathbf{s}}$ that maps $\mathbf{e}_1$ to $\mathbf{s}$. Therefore, we define the parametrization $\varphi : \partial B_1^d \times \mathbb{R}^{d-1} \to A(d,1)$ satisfying $\varphi(\mathbf{s},\mathbf{u}) = l_{\mathbf{s},\mathbf{v_s}(\mathbf{u})}$, with $\mathbf{v_s}(\mathbf{u}) := \theta_{\mathbf{s}} \circ \begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix} \in \mathrm{vect}(\mathbf{s}^\perp)$. Notice that for $E \subset A(d,1)$,

$$\mu_1(E) = \frac{(\sigma_{d-1} \otimes \sigma_{d-1})(\varphi^{-1}(E))}{\sigma_{d-1}(\partial B_1^d)},$$

where $\otimes$ denotes the product measure.

We recall here a particular version of the Crofton formula in Theorem 5.4.3 in Schneider and Weil (2008), which implies that for a manifold $M \subset \mathbb{R}^d$ satisfying $0 < \sigma_{d-1}(M) < \infty$, it holds that

$$\sigma_{d-1}(M) = \frac{\sqrt{\pi}\ \Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \int_{A(d,1)} \sigma_0(M \cap l)\ \mu_1(\mathrm{d}l). \tag{4}$$

By writing $A(d,1)$ in terms of the above parametrization φ, Equation (4) takes the form

$$\sigma_{d-1}(M) = \frac{\sqrt{\pi}\ \Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \int_{\mathbb{R}^{d-1}} \int_{\partial B_1^d} \frac{\sigma_0(M \cap l_{\mathbf{s},\mathbf{v_s}(\mathbf{u})})}{\sigma_{d-1}(\partial B_1^d)}\ \mathrm{d}\mathbf{s}\ \mathrm{d}\mathbf{u}. \tag{5}$$

A helpful interpretation of the Crofton formula in Equation (5) is as follows: the expected σ_{d-1} measure of the projection of M on a $(d-1)$-dimensional hyperplane with uniformly random orientation is simply a constant multiple of $\sigma_{d-1}(M)$.

The Crofton formula can also be exploited to propose algorithms to compute the surface area, *e.g.*, it is considered in Lehmann and Legland (2012) for objects in 2 and 3 dimension and recently in Aaron et al. (2020) which provides consistent estimators for the surface area of a compact domain S from the observation of *i.i.d.* random variables supported on S. The interested reader is referred to Appendix A.2 for an illustration of Equation (5) on two simple examples.

1.3 Estimated volume and surface area of excursion sets observed over a point cloud

Excursion sets and associated measures. We now apply the previous σ_d and σ_{d-1} measures to specific manifolds: the excursion sets of d-dimensional smooth random fields.

Definition 1.1 (σ_d and σ_{d-1} measures of excursion sets)**.** Let $\{X(t), t \in \mathbb{R}^d\}$, for $d \geq 2$, be a random field satisfying the following assumption

($\mathcal{A}0$) X is stationary with positive finite variance and is almost surely twice differentiable. Furthermore, the probability density of $\big(X(\mathbf{0}), \nabla X(\mathbf{0})\big)$ is bounded uniformly on $\mathbb{R}^{d+1}$.

Let $u \in \mathbb{R}$ and $T \subset \mathbb{R}^d$ be a bounded closed hypercube with non empty interior. We consider the excursion set within T above level u:

$$E_X^T(u) := \{t \in T\ :\ X(t) \geq u\} = T \cap E_X(u), \quad \text{where } E_X(u) := X^{-1}([u, +\infty)).$$

Similarly, the level curves within T are defined by

$$L_X^T(u) := \{t \in T\ :\ X(t) = u\} = T \cap \partial E_X(u),\ a.s.$$

Remark that Assumption ($\mathcal{A}0$) guarantees that X admits no critical points at the level u almost surely, which implies that $L_X^T(u)$ is a $(d-1)$-dimensional manifold possessing a σ_{d-1} measure with finite first and second moments (see, *e.g.*, Cabaña (1987), Theorem 11.2.1 and Lemma 11.2.11 in Adler and Taylor (2007)). In addition, since the considered random field X is of class C^2 *a.s.*, the random set $E_X(u)$ is a C^2 submanifold of $\mathbb{R}^d$ and its intersection with the compact, convex hypercube T provides the positive reach property (see Biermé et al. (2019)).

Define the normalized σ_d and σ_{d-1} measures of the excursion set, for $u \in \mathbb{R}$, as

$$C_{d-1}^T(u) := \frac{1}{\sigma_d(T)}\sigma_{d-1}\big(L_X^T(u)\big) = \frac{1}{\sigma_d(T)}\int_{L_X^T(u)}\sigma_{d-1}(\mathrm{d}s), \tag{6}$$

$$C_d^T(u) := \frac{1}{\sigma_d(T)}\sigma_d\big(E_X^T(u)\big) = \frac{1}{\sigma_d(T)}\int_T \mathbb{1}_{\{X(t)\geq u\}}\mathrm{d}t. \tag{7}$$

Assumption ($\mathcal{A}0$) guarantees the existence of the associated densities

$$C_k^*(u) := \mathbb{E}[C_k^T(u)], \quad \text{for } k = d, d-1, \tag{8}$$

which are independent of the size of the hypercube T.

The independence of C_d^* from the size of T is trivially verified using that X is stationary and Fubini-Tonelli theorem which give immediately that the density of the normalised volume satisfies

$$C_d^*(u) = \frac{1}{\sigma_d(T)}\,\mathbb{E}\Big[\int_T \mathbb{1}_{\{X(t)\geq u\}}\mathrm{d}t\Big] = \mathbb{P}(X(\mathbf{0}) \geq u). \tag{9}$$

Furthermore, note that we consider in (6) the σ_{d-1} measure of $L_X^T(u) = T \cap \partial E_X(u)$ and not of $\partial E_X^T(u)$. Therefore, from Definition 2.1 and Proposition 2.5 in Biermé et al. (2019), we get via kinematic formulas that $\mathbb{E}[C_{d-1}^T(u)]$ is equal to the surface area density. Indeed we do not add the artificial contribution of ∂T to the level curves in Definition 1.1. Notice that the density $C_{d-1}^*(u)$ can be explicitly obtained for certain specific random fields. Two classical examples (the isotropic Gaussian and chi-square random fields) are presented in Appendix A.2 (see Example 3).

The random quantities in (6)-(7) can only be used as estimators of $C_d^*(u)$ and $C_{d-1}^*(u)$ if we observe the excursion set $E_X^T(u)$ on the whole domain T. In practice, or at least numerically, images of excursion sets are not objects defined on all T but discretely encrypted objects, *i.e.*, for each point of a discrete grid. Then, quantities in (6)-(7) are never empirically accessible. In the remainder of this section we propose estimators of $C_d^*(u)$ and $C_{d-1}^*(u)$ from the observation of the excursion set on a general point cloud (*i.e.*, based on the knowledge of which points fall in the excursion set).

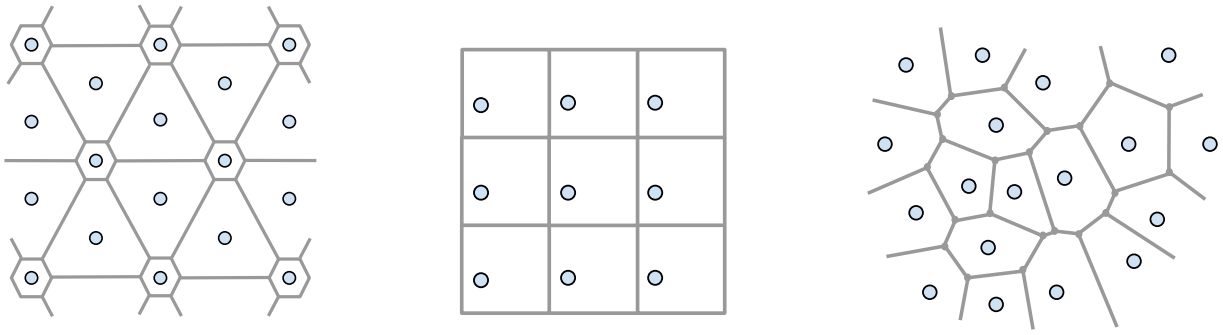
Polytopic tessellations based on point clouds. For an arbitrary point cloud, we describe the set of tesselations of $\mathbb{R}^d$ that are permissible for the construction of our estimators of $C_d^*(u)$ and $C_{d-1}^*(u)$.

Definition 1.2. Let $\mathcal{H}$ be a set of convex, closed polytopes that tessellates $\mathbb{R}^d$ in such a way that satisfies the following condition. To each $P \in \mathcal{H}$, one can assign a reference point $P^\bullet \in P$ such that for any two adjacent cells $P_1, P_2 \in \mathcal{H}$, the intersection of their boundaries is normal to the vector spanned between $P_1^\bullet$ and $P_2^\bullet$. We say that $\mathcal{H}$ is *point-referenceable*, and that the set of pairs $\dot{\mathcal{H}} = \{(P, P^\bullet) : P \in \mathcal{H}\}$ is a *point-referenced d-honeycomb*. Let $\mathfrak{T}^d$ denote the space of point-referenceable d-honeycombs, so that $\mathcal{H} \in \mathfrak{T}^d$.

Recall that the interiors of the polytopes in a tessellation do not intersect, *i.e.*, $\sigma_d(P_1 \cap P_2) = 0$, for any different $P_1, P_2 \in \mathcal{H}$. However, if two polytopes P_1 and P_2 are *adjacent* in $\mathcal{H}$, then $\sigma_{d-1}(P_1 \cap P_2) > 0$.

Remark 1. The set of point-referenceable d-honeycombs $\mathfrak{T}^d$ in Definition 1.2 contains a wide variety of tessellations. For example, the Voronoi diagram of any point cloud in $\mathbb{R}^d$ is in $\mathfrak{T}^d$. It contains also d-honeycombs that cannot be realised as a Voronoi diagram as, for example, the Archimedean tessellations or the tiling of $\mathbb{R}^2$ in Figure 1 (a). Simple examples of tessellations of $\mathbb{R}^2$ that are not in $\mathfrak{T}^2$ are Pythagorean tilings and, of course, tilings with curved tiles.

Figure 1 provides some examples of point-referenceable tessellations of $\mathbb{R}^2$. Notice that regular triangular and regular hexagonal lattices can be seen as limiting cases of the tessellation in Figure 1 (a); furthermore, as depicted, it is not a Voronoi tessellation. Figure 1 (b) represents the well-known square lattice, and panel (c) depicts the Voronoi tessellation of an arbitrary point cloud.



(a) Hexagones and truncated triangles lattice

(b) Square lattice

(c) Voronoi tessellation

Figure 1: Three examples of tilings in $\mathfrak{T}^2$ (see Definition 1.2) with a particular choice of reference points shown as light-blue circles. The resulting structures are point-referenced 2-honeycombs.

Estimators for $C_d^T(u)$ and $C_{d-1}^T(u)$. For a given point cloud in $\mathbb{R}^d$, that can be random, it is always possible to construct a point-referenced d-honeycomb (see Definition 1.2) with the point cloud as its reference points (take the Voronoi diagram, for example). Thus, for each element of $\mathfrak{T}^d$ there is at least one corresponding point cloud, and to each point cloud in $\mathbb{R}^d$ there is at least one corresponding element of $\mathfrak{T}^d$. We propose the following estimators for the quantities in (6) and (7) based on the knowledge of at which points in the point cloud the random field X exceeds the level u.

Definition 1.3. Let $T \subset \mathbb{R}^d$ be a compact domain with non empty interior. Let $\mathcal{H} \in \mathfrak{T}^d$, and let $\dot{\mathcal{H}} = \{(P, P^\bullet) : P \in \mathcal{H}\}$ be a corresponding point-referenced d-honeycomb in the sense of Definition 1.2. Let $\mathcal{H}^T \subset \mathcal{H}$ be the set of polytopes P in $\mathcal{H}$ such that $P \subseteq T$. We define an estimator of $C_{d-1}^T(u)$ in (6) as

$$\widehat{C}_{d-1}^{(\dot{\mathcal{H}},T)}(u) := \frac{1}{\sigma_d(T)} \sum_{\substack{P_1,P_2 \in \mathcal{H}^T \\ P_1 \neq P_2}} \sigma_{d-1}(P_1 \cap P_2) \mathbb{1}_{\{X(P_1^\bullet) \leq u < X(P_2^\bullet)\}} \tag{10}$$

and of $C_d^T(u)$ in (7) as

$$\widehat{C}_d^{(\dot{\mathcal{H}},T)}(u) := \frac{1}{\sigma_d(T)} \sum_{P \in \mathcal{H}^T} \sigma_d(P) \mathbb{1}_{\{X(P^\bullet) \geq u\}}. \tag{11}$$

Computing the quantities in Definition 1.3 only requires the knowledge of the excursion set $E_X^T(u)$ on the reference points of a point-referenced d-honeycomb *i.e.* a black and white image indicating at which points $P^\bullet$, for $P \in \mathcal{H}^T$, the field is above level u. In Equation (10), since the role of P_1 and P_2 is symmetric, both $\mathbb{1}_{\{X(P_1^\bullet)\leq u<X(P_2^\bullet)\}}$ and $\mathbb{1}_{\{X(P_2^\bullet)\leq u<X(P_1^\bullet)\}}$ are evaluated in the sum. Notice that it is crucial that the cells $P \in \mathcal{H}$ are closed to ensure that for adjacent cells $\sigma_{d-1}(P_1 \cap P_2) > 0$. Figure 2 provides an illustration of the behaviour of the estimator in (10) for the point-referenced 2-honeycombs in Figure 1.

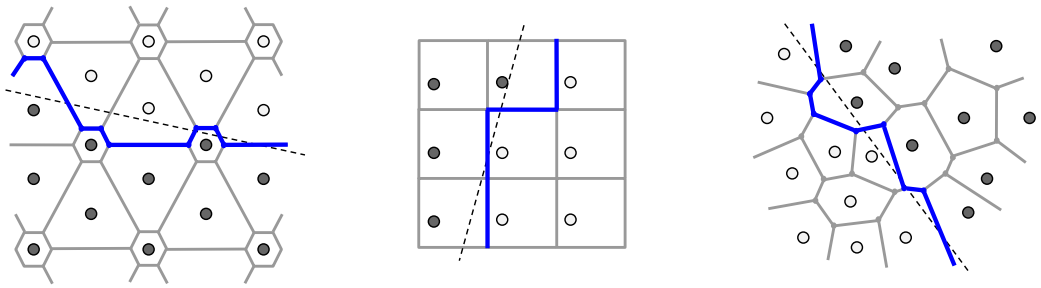


Figure 2: For each of the point-referenced 2-honeycombs in Figure 1, a portion of the level curve $\{X = u\}$ is illustrated as a dotted line. The estimator $\widehat{C}_{d-1}^{(\dot{\mathcal{H}},T)}(u)$ in (10) corresponds to the length of the resulting blue curve that separates the reference points in the same way as the curve $\{X = u\}$.

It is easy to see that $\widehat{C}_d^{(\dot{\mathcal{H}},T)}(u)$ in (11) is, in general, biased. Indeed, it follows from the stationarity in Assumption ($\mathcal{A}0$) that

$$\mathbb{E}[\widehat{C}_d^{(\dot{\mathcal{H}},T)}(u)] = \frac{\sigma_d\Big(\bigcup_{P\in\mathcal{H}^T} P\Big)}{\sigma_d(T)}\, C_d^*(u). \tag{12}$$

Nevertheless, the bias factor $\sigma_d\big(\bigcup_{P\in\mathcal{H}^T} P\big)/\sigma_d(T) \leq 1$ is deterministic and it only relies on the structure of the point-referenced d-honeycomb. It is nearly unity if $\sup\{\mathrm{diam}(P) : P \in \mathcal{H}\} \ll \mathrm{diam}(T)$ and it is exactly unity for tessellations that satisfy $\bigcup_{P\in\mathcal{H}^T} P = T$. This can be easily obtained for simple tessellations such as the hypercubic one, specifically described in Section 3.1. This is why we focus our attention to the bias in (10) for the $(d-1)$-volume of $L_X^T(u)$. In this case, the bias does not approach unity as the tile sizes become negligibly small with regards to the size of T, as previously observed *e.g.* in Biermé and Desolneux (2021) and Miller (1999) for some periodic tessellations and in Thäle and Yukich (2016) for Poisson-Voronoi mosaics. Explicit calculations of the bias have been made for square and hexagonal tilings in dimension 2 (Biermé and Desolneux (2021)) and cubic lattices in dimension 3 (Miller (1999)). The approach in Miller (1999) is adapted to the d-dimensional hypercubic setting in Appendix A.4 (see Equation (53)).
Our Theorem 2.2 provides a general formula for this limiting bias that holds in arbitrary dimension d, for a large subset of point-referenced d-honeycombs.

2 Bias of the estimated surface area

Assumptions. To control the bias of the estimated surface area we require additional sufficient assumptions related to the regularity of the derivatives of X in univariate directions.

($\mathcal{A}1$) Fix $\mathbf{w} \in \partial B_1^d$. Assume that $Y := \{X(t\mathbf{w}), t \in \mathbb{R}\}$ is such that, $Y(0)\big|\{Y'(0) = 0\}$ has a bounded density and $\|Y''\|_{\infty,[0,1]} := \sup_{t\in[0,1]} |Y''(t)| \in L^2$.

($\mathcal{A}2$) Let $L_X(u) := X^{-1}(\{u\})$, $\mathbf{w} \in \partial B_1^d$ and $\varepsilon \in (0,1)$. Assume that $\mathbb{E}[\sigma_0(L_X(u) \cap l_{\mathbf{w},\mathbf{0}} \cap [0,1]^d)^{1/\varepsilon}]$ is finite, where $l_{\mathbf{w},\mathbf{0}}$ is defined in (3).

Note that if the random field X is isotropic, the vector $\mathbf{w} \in \partial B_1^d$ appearing in ($\mathcal{A}1$) and ($\mathcal{A}2$) can be chosen arbitrarily. Assumption ($\mathcal{A}1$) permits to apply Lemma 4.1 to the unidirectional process Y. The aforementioned lemma controls the probability that a one-dimensional field crosses a level u more than once in a small time interval. Assumption ($\mathcal{A}2$) is technical and allows for a Hölder control in the proof of Theorem 2.1.

A first result on crossings. The following key result provides a first order approximation of the probability of crossings for the isotropic random field X in a given arbitrary direction. This result plays a central role in deriving a formula for the expected surface area of the limiting approximation of an excursion set by elements of a point-referenceable d-honeycomb in the sense of Definition 1.2 (see Theorem 2.2 below).

Theorem 2.1. *Consider a scalar $q \in \mathbb{R}^+$ and a fixed $\mathbf{w} \in \partial B_1^d$. Let X be an isotropic random field satisfying Assumption ($\mathcal{A}0$). Then for any fixed $u \in \mathbb{R}$,*

$$\lim_{q \to 0} \frac{1}{q} \mathbb{P}\big(X(\mathbf{0}) \leq u < X(q\mathbf{w})\big) = \frac{C^*_{d-1}(u)}{\beta_d}, \tag{13}$$

where

$$\beta_d = \frac{2\sqrt{\pi}\ \Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \tag{14}$$

is a dimensional constant and the limit in (13) is approached from below. Moreover, if X also satisfies assumptions ($\mathcal{A}1$) and ($\mathcal{A}2$) for $\mathbf{w}$ and some $\varepsilon \in (0,1)$, then there exists a constant $K \in \mathbb{R}^+$ such that for all $q \in \mathbb{R}^+$,

$$0 \leq \frac{C^*_{d-1}(u)}{\beta_d} q - \mathbb{P}\big(X(\mathbf{0}) \leq u < X(q\mathbf{w})\big) \leq K q^{2-\varepsilon}. \tag{15}$$

The proof of Theorem 2.1 is based on the Crofton formula in (5).

Proof of Theorem 2.1. Firstly, we prove Equation (13). We apply Equation (5) to the $(d-1)$-manifold $L_X(u) \cap B_1^d$,

$$\sigma_{d-1}\big(L_X(u) \cap B_1^d\big) = \frac{\sqrt{\pi}\ \Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \int_{\partial B_1^d} \int_{\mathrm{vect}(\mathbf{s}^\perp)} \frac{\sigma_0(L_X(u) \cap B_1^d \cap l_{\mathbf{s},\mathbf{v}})}{\sigma_{d-1}(\partial B_1^d)}\, \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{s}, \tag{16}$$

with $l_{\mathbf{s},\mathbf{v}}$ as in (3). The quantity in (16) is a positive random variable in L^1, and thus by taking the expectation, we get

$$\mathbb{E}\left[\sigma_{d-1}\big(L_X(u) \cap B_1^d\big)\right] = \frac{\sqrt{\pi}\ \Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \int_{\partial B_1^d} \int_{\mathrm{vect}(\mathbf{s}^\perp)} \frac{\mathbb{E}\left[\sigma_0(L_X(u) \cap B_1^d \cap l_{\mathbf{s},\mathbf{v}})\right]}{\sigma_{d-1}(\partial B_1^d)}\, \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{s}.$$

By the isotropy of X, we obtain

$$\mathbb{E}\left[\sigma_{d-1}\big(L_X(u) \cap B_1^d\big)\right] = \frac{\sqrt{\pi}\ \Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \int_{\mathrm{vect}(\mathbf{w}^\perp)} \mathbb{E}\left[\sigma_0(L_X(u) \cap B_1^d \cap l_{\mathbf{w},\mathbf{v}})\right] \mathrm{d}\mathbf{v}, \tag{17}$$

for $l_{\mathbf{w},\mathbf{v}}$ as in (3).
For $\mathbf{v} \in \text{vect}(\mathbf{w}^\perp) \cap B_1^d$, define $\overline{l_{\mathbf{w},\mathbf{v}}} := B_1^d \cap l_{\mathbf{w},\mathbf{v}}$. Then $\overline{l_{\mathbf{w},\mathbf{v}}}$ either contains a single point, or it is a line segment in $\mathbb{R}^d$ with orientation $\mathbf{w}$. Define

$$n_q = \left\lfloor \frac{\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})}{q} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Define $\mathbf{x}_0 := \mathbf{v} - \frac{\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})}{2}\mathbf{w}$ and $\mathbf{x}^0 := \mathbf{v} + \frac{\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})}{2}\mathbf{w}$ to be the two endpoints of the line segment $\overline{l_{\mathbf{w},\mathbf{v}}}$. Figure 3 provides a graphical visualisation to aid this construction.

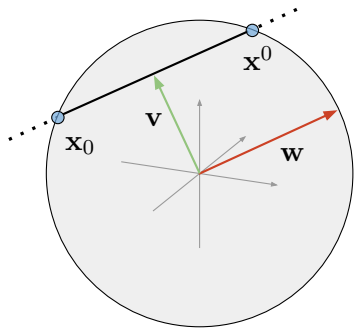


Figure 3: In dimension 3, the unit ball is represented; the red vector is a possible $\mathbf{w} \in \partial B_1^3$ and the green vector, a possible $\mathbf{v} \in \text{vect}(\mathbf{w}^\perp) \cap B_1^3$. The two endpoints $\mathbf{x}_0$ and $\mathbf{x}^0$, marked in blue, describe the segment $\overline{l_{\mathbf{w},\mathbf{v}}}$.

Now define $\mathbf{x}_j := \mathbf{x}_0 + jq\mathbf{w}$ for $j \in \{0, ..., n_q\}$, each of which being an element of $\overline{l_{\mathbf{w},\mathbf{v}}}$. With this construction, it follows that

$$\sum_{j=0}^{n_q-1} \left(\mathbb{1}_{\{X(\mathbf{x}_j) \le u < X(\mathbf{x}_{j+1})\}} + \mathbb{1}_{\{X(\mathbf{x}_j) > u \ge X(\mathbf{x}_{j+1})\}} \right) \xrightarrow{\text{a.s.}} \sigma_0(L_X(u) \cap \overline{l_{\mathbf{w},\mathbf{v}}}), \tag{18}$$

as $q \to 0$. The left-hand side of (18) is a count of the number of elements in $L_X(u) \cap \overline{l_{\mathbf{w},\mathbf{v}}}$, which approaches the exact value from below almost surely. This construction is well known in the literature as the *discretization method*. The interested reader is referred for instance to Kratz (2006) (Section 2.1) and references therein. In particular, by using the stationarity of X in Assumption ($\mathcal{A}0$), the monotonicity of the convergence in (18) implies

$$2n_q \mathbb{P}\big(X(\mathbf{0}) \le u < X(\mathbf{w}q)\big) \underset{q \to 0}{\longrightarrow} \mathbb{E}[\sigma_0(L_X(u) \cap \overline{l_{\mathbf{w},\mathbf{v}}})].$$

Moreover, this convergence is uniform for $\mathbf{v} \in \text{vect}(\mathbf{w}^\perp)$, so

$$\mathbb{P}\big(X(\mathbf{0}) \le u < X(\mathbf{w}q)\big) \int_{\text{vect}(\mathbf{w}^\perp) \cap B_1^d} 2n_q \mathrm{d}\mathbf{v} \underset{q \to 0}{\longrightarrow} \int_{\text{vect}(\mathbf{w}^\perp)} \mathbb{E}[\sigma_0(L_X(u) \cap \overline{l_{\mathbf{w},\mathbf{v}}})] \mathrm{d}\mathbf{v}.$$

By (17), this simplifies to

$$\mathbb{P}\big(X(\mathbf{0}) \le u < X(\mathbf{w}q)\big) \int_{\text{vect}(\mathbf{w}^\perp) \cap B_1^d} n_q \mathrm{d}\mathbf{v} \underset{q \to 0}{\longrightarrow} \frac{1}{\beta_d} \mathbb{E}\left[\sigma_{d-1}\big(L_X(u) \cap B_1^d\big)\right].$$

Since $0 \leq \sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})/q - n_q < 1$ and $\mathbb{P}\big(X(\mathbf{0}) \leq u < X(\mathbf{w}q)\big) \underset{q\to 0}{\longrightarrow} 0$, it holds that

$$\mathbb{P}\big(X(\mathbf{0}) \leq u < X(\mathbf{w}q)\big) \int_{\mathrm{vect}(\mathbf{w}^\perp)\cap B_1^d} \frac{\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})}{q} \mathrm{d}\mathbf{v} \underset{q\to 0}{\longrightarrow} \frac{1}{\beta_d}\, \mathbb{E}\left[\sigma_{d-1}\big(L_X(u)\cap B_1^d\big)\right].$$

By noticing that $\int_{\mathrm{vect}(\mathbf{w}^\perp)\cap B_1^d} \sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})\mathrm{d}\mathbf{v} = \sigma_d(B_1^d)$, one obtains

$$\frac{1}{q}\mathbb{P}\big(X(\mathbf{0}) \leq u < X(\mathbf{w}q)\big) \underset{q\to 0}{\longrightarrow} \frac{1}{\beta_d \sigma_d(B_1^d)}\, \mathbb{E}\left[\sigma_{d-1}\big(L_X(u)\cap B_1^d\big)\right] = \frac{C^*_{d-1}(u)}{\beta_d}.$$

This proves the first order approximation in (13).
To show that

$$\frac{C^*_{d-1}(u)}{\beta_d} q \geq \mathbb{P}\big(X(\mathbf{0}) \leq u < X(q\mathbf{w})\big) \tag{19}$$

for all $q > 0$, suppose that there exists a $t \in \mathbb{R}^+$ such that

$$\frac{C^*_{d-1}(u)}{\beta_d} t < \mathbb{P}\big(X(\mathbf{0}) \leq u < X(t\mathbf{w})\big).$$

Then, for all $n \in \mathbb{N}^+$, $\{X(\mathbf{0}) \leq u < X(t\mathbf{w})\} \subseteq \{X(\mathbf{0}) \leq u < X(\frac{t}{n}\mathbf{w})\} \cup \ldots \cup \{X(\frac{n-1}{n}t\mathbf{w}) \leq u < X(t\mathbf{w})\}$, leading under ($\mathcal{A}0$) to

$$\frac{C^*_{d-1}(u)}{\beta_d} < \frac{1}{t}\mathbb{P}\big(X(\mathbf{0}) \leq u < X(t\mathbf{w})\big) \leq \frac{n}{t}\mathbb{P}\big(X(\mathbf{0}) \leq u < X(\tfrac{t}{n}\mathbf{w})\big).$$

Keeping t fixed and having $n \to \infty$ contradicts the convergence in (13). Thus, (19) holds for all $q \in \mathbb{R}^+$. We proceed by showing (15). For fixed $q > 0$, there can only be a difference between the expression in (18) and its limit—both taking values in $\mathbb{N}_0$—if at least one of the two following conditions hold,

(i) There exists a $j \in \{0, \ldots, n_q - 1\}$ such that the line segment with endpoints $\mathbf{x}_j$ and $\mathbf{x}_{j+1}$ contains two points $\mathbf{s}_1 \neq \mathbf{s}_2$ such that $X(\mathbf{s}_1) = X(\mathbf{s}_2) = u$.

(ii) The line segment with endpoints $\mathbf{x}_{n_q}$ and $\mathbf{x}^0$ contains a point $\mathbf{s}$ such that $X(\mathbf{s}) = u$.

The probability of event in (i) is bounded above by $n_q K_2 q^2$ for some $K_2 \in \mathbb{R}^+$, under ($\mathcal{A}1$) by applying Lemma 4.1 with $p = 2$. Furthermore, by using Markov's inequality and Theorem 3.3.1 in Beutler and Leneman (1966), the probability of event (ii) is bounded above by $K_1 q$ for some $K_1 \in \mathbb{R}^+$. Let A denote the event $\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}}) > \sum_{j=0}^{n_q-1}\left(\mathbb{1}_{\{X(\mathbf{x}_j)\leq u<X(\mathbf{x}_{j+1})\}} + \mathbb{1}_{\{X(\mathbf{x}_j)>u\geq X(\mathbf{x}_{j+1})\}}\right)$, then applying Hölder's inequality together with ($\mathcal{A}2$) implies

$$\begin{aligned}
&\mathbb{E}\left[\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right] - 2n_q\mathbb{P}\big(X(\mathbf{0}) \leq u < X(q\mathbf{w})\big)\\
&\quad= \mathbb{E}\left[\left|\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}}) - \sum_{j=0}^{n_q-1}\left(\mathbb{1}_{\{X(\mathbf{x}_j)\leq u<X(\mathbf{x}_{j+1})\}} + \mathbb{1}_{\{X(\mathbf{x}_j)>u\geq X(\mathbf{x}_{j+1})\}}\right)\right|\right]\\
&\quad\leq \mathbb{E}\left[\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\mathbb{1}_{\{A\}}\right] \leq \left(\mathbb{E}\left[\left(\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right)^{\frac{1}{\varepsilon}}\right]\right)^{\varepsilon}\mathbb{P}(A)^{1-\varepsilon}\\
&\quad\leq \left(\mathbb{E}\left[\left(\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right)^{\frac{1}{\varepsilon}}\right]\right)^{\varepsilon}\left(n_q K_2 q^2 + K_1 q\right)^{1-\varepsilon}\\
&\quad\leq \left(\mathbb{E}\left[\left(\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right)^{\frac{1}{\varepsilon}}\right]\right)^{\varepsilon}\left(\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})K_2 + K_1\right)^{1-\varepsilon}q^{1-\varepsilon}\\
&\quad\leq \left(\mathbb{E}\left[\left(\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right)^{\frac{1}{\varepsilon}}\right]\right)^{\varepsilon}\left(2K_2 + K_1\right)^{1-\varepsilon}q^{1-\varepsilon}.
\end{aligned}$$

Furthermore, note that

$$\begin{aligned}
&\Big|\mathbb{E}\left[\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right]-2\frac{\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})}{q}\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)\Big|\\
&\quad=\Big|\mathbb{E}\left[\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right]-2n_q\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)+2\Big(n_q-\frac{\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})}{q}\Big)\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)\Big|\\
&\quad\le\Big(\mathbb{E}\left[\big(\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\big)^{\frac{1}{\varepsilon}}\right]\Big)^{\varepsilon}\big(2K_2+K_1\big)^{1-\varepsilon}q^{1-\varepsilon}+2\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)\\
&\quad\le\Big(\mathbb{E}\left[\big(\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\big)^{\frac{1}{\varepsilon}}\right]\Big)^{\varepsilon}\big(2K_2+K_1\big)^{1-\varepsilon}q^{1-\varepsilon}+2K_1q.
\end{aligned}$$

Summarizing, we have shown that

$$\Big|q\,\mathbb{E}\left[\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right]-2\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)\Big|=O(q^{2-\varepsilon}),$$

for all $\mathbf{v}\in\mathrm{vect}(\mathbf{w}^{\perp})\cap B_1^d$. By integrating over $\mathbf{v}$ and taking the absolute value, one preserves

$$\Big|\int_{\mathrm{vect}(\mathbf{w}^{\perp})\cap B_1^d}\Big(q\,\mathbb{E}\left[\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right]-2\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)\Big)\,\mathrm{d}\mathbf{v}\Big|=O(q^{2-\varepsilon}).\tag{20}$$

Evaluating the integral, and again applying $\int_{\mathrm{vect}(\mathbf{w}^{\perp})\cap B_1^d}\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})\mathrm{d}\mathbf{v}=\sigma_d(B_1^d)$, one obtains

$$\begin{aligned}
&\int_{\mathrm{vect}(\mathbf{w}^{\perp})\cap B_1^d}\Big(q\,\mathbb{E}\left[\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right]-2\sigma_1(\overline{l_{\mathbf{w},\mathbf{v}}})\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)\Big)\,\mathrm{d}\mathbf{v}\\
&\qquad=q\int_{\mathrm{vect}(\mathbf{w}^{\perp})}\mathbb{E}\left[\sigma_0(L_X(u)\cap\overline{l_{\mathbf{w},\mathbf{v}}})\right]\,\mathrm{d}\mathbf{v}-2\sigma_d(B_1^d)\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)\\
&\qquad=\frac{2q}{\beta_d}\,\mathbb{E}\left[\sigma_{d-1}\big(L_X(u)\cap B_1^d\big)\right]-2\sigma_d(B_1^d)\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)\\
&\qquad=2\sigma_d(B_1^d)\Big(\frac{C^*_{d-1}(u)}{\beta_d}q-\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)\Big),
\end{aligned}$$

where the second equality follows from (17). Thus, by (20), we obtain the desired result

$$\Big|\frac{C^*_{d-1}(u)}{\beta_d}q-\mathbb{P}\big(X(\mathbf{0})\le u<X(q\mathbf{w})\big)\Big|=O(q^{2-\varepsilon}).$$

□

Remark 2. Theorem 2.1 can also be very useful for sample simulations. To have a rapid evaluation of the surface area of excursion sets of a given random field satisfying above assumptions, it is not necessary to generate the whole isotropic random field on $\mathbb{R}^d$ but only *i.i.d.* r.v. with the same bivariate distribution as $\big(X(\mathbf{0}),X(q\,\mathbf{e}_1)\big)$, for q small enough. Then, the numerical first order approximation of the surface area would be

$$\beta_d\frac{\widehat{\mathbb{P}}\big(X(\mathbf{0})\le u<X(q\mathbf{e}_1)\big)}{q}.$$

This induces an error of the order $q^{1-\varepsilon}$ according to Equation (15). This approach can be compared for instance with the tedious computations encountered when using tube formulas (see, *e.g.*, Adler and Taylor (2007)) to obtain an exact value of the expected surface area.

A related result. Notice that Theorem 2.1 fully identifies the limit of the level crossing and relies on classical assumptions. Another result on level crossings can be obtained by adapting a result in Leadbetter et al. (1983) to dimension d (see Proposition 2.1 below). For clarity, the proof of this adaptation in general dimension d is given in Section 4.

Proposition 2.1 (A $d-$dimensional formulation of Leadbetter et al. (1983), Theorem 7.2.4)**.** *Let $X : \mathbb{R}^d \to \mathbb{R}$ be a continuous stationary random field such that $X(\mathbf{0})$ and $X_q := \frac{1}{q}(X(q\mathbf{e}_1) - X(\mathbf{0}))$ have a joint density denoted $g_q(u,z)$ that is continuous in u for all z and all sufficiently small $q > 0$, and that there exists a function $p(u,z)$ such that $g_q(u,z) \to p(u,z)$ uniformly in u for fixed z as $q \to 0$. Assume furthermore that there is a function $h(z)$ such that $\int_0^\infty z h(z) \mathrm{d}z < \infty$ and $g_q(u,z) \leq h(z)$ uniformly for all u, q. Then, it holds that*

$$\lim_{q\to 0} \frac{\mathbb{P}\left(X(\mathbf{0}) \leq u < X(q\mathbf{e}_1)\right)}{q} = \int_0^\infty z\, p(u,z) \mathrm{d}z.$$

Remark that it is difficult to apply Proposition 2.1 in practice, since computing the asymptotic joint density p is not easy outside of specific cases such as the Gaussian one. Moreover, it requires that the convergence of the density of X_q as $q \to 0$ is established. Even if it is clear that X_q converges almost surely to $\partial_1 X(\mathbf{0})$, where $\partial_1 X$ denotes the partial derivative of X in the direction $\mathbf{e}_1$, having the convergence of the corresponding densities is much more delicate to establish (see, *e.g.*, Boos (1985) or Sweeting (1986)). If $d = 1$, Leadbetter et al. (1983) explains that "*In many cases, the limit $p(u,z)$ is simply the joint density of $(X(\mathbf{0}), \partial_1 X(\mathbf{0}))$.*" This holds for example, for Gaussian processes. Translating this in dimension $d \geq 1$, if we write $p_{X(\mathbf{0})}(u)$ for the density of $X(\mathbf{0})$, we see that

$$\lim_{q\to 0} \frac{\mathbb{P}\left(X(\mathbf{0}) \leq u < X(q\mathbf{e}_1)\right)}{q} = p_{X(\mathbf{0})}(u)\, \mathbb{E}\left[\partial_1 X(\mathbf{0}) \mathbb{1}_{\{\partial_1 X(\mathbf{0}) > 0\}} | X(\mathbf{0}) = u\right]. \tag{21}$$

Bias control for the estimated surface area measure. The following theorem provides the explicit formula for the bias of the surface area in any dimension d and for arbitrary point-referenced d-honeycombs in Definition 1.2 whose polytopes do not grow too large near ∞. In this way it generalizes and unifies existing results (see, *e.g.* Proposition 5 in Biermé and Desolneux (2021) for the square and hexagonal tiling in dimension 2 and Miller (1999) for the triangular tiling in dimension 2 and hypercubic lattice in dimension 3). Furthermore, Theorem 2.2 applies to Poisson-Voronoi tessellations, as shown in Corollary 2.1.

Theorem 2.2. *Let $T \subset \mathbb{R}^d$ be a compact domain with non empty interior containing the origin. Let X be an isotropic random field on $\mathbb{R}^d$ that satisfies Assumption (A0).*

i) Let $\mathcal{H} \in \mathfrak{T}^d$ (see Definition 1.2) and let $\dot{\mathcal{H}}$ be a corresponding point-referenced d-honeycomb. Define $D^{(\mathcal{H})} := \sup\{\mathrm{diam}(P \cap T) : P \in \mathcal{H}\}$ and $\delta\mathcal{H} := \{\delta P : P \in \mathcal{H}\}$ for $\delta \in \mathbb{R}^+$. Let $\widehat{C}_{d-1}^{(\dot{\mathcal{H}},T)}(u)$ be as in Definition 1.3. Suppose that $\lim_{\delta\to 0} D^{(\delta\mathcal{H})} = 0$. Then, it holds

$$\mathbb{E}\left[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{H}},T)}(u)\right] \underset{\delta\to 0}{\longrightarrow} \frac{2d}{\beta_d} C_{d-1}^*(u), \tag{22}$$

where $\delta\dot{\mathcal{H}} := \{(\delta P, \delta P^\bullet) : (P, P^\bullet) \in \dot{\mathcal{H}}\}$ is $\dot{\mathcal{H}}$ linearly rescaled by δ, β_d is as in (14), and $C_{d-1}^*(u)$ is as in (8).

ii) Furthermore, if ξ is a point process on $\mathbb{R}^d$ that is independent of X, let $\mathcal{V}_\xi$ be the Voronoi diagram in $\mathfrak{T}^d$ generated from the points in ξ. Reference the polytopes in $\mathcal{V}_\xi$ by their corresponding points in ξ and denote the resulting point-referenced d-honeycomb by $\dot{\mathcal{V}}_\xi$. If $D^{(\delta\mathcal{V}_\xi)} \xrightarrow{\mathbb{P}} 0$ as $\delta \to 0$, then

$$\mathbb{E}\left[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{V}}_\xi,T)}(u)\right] \underset{\delta\to 0}{\longrightarrow} \frac{2d}{\beta_d} C_{d-1}^*(u).$$

Proof of Theorem 2.2. First, we establish Equation (22) where the point-referenced d-honeycomb $\dot{\mathcal{H}}$ is deterministic. The estimator $\widehat{C}_{d-1}^{(\delta\dot{\mathcal{H}},T)}(u)$ in (10) is written

$$\widehat{C}_{d-1}^{(\delta\dot{\mathcal{H}},T)}(u) = \frac{1}{\sigma_d(T)} \sum_{\substack{P_1,P_2\in\mathcal{H}\\(P_1\neq P_2),(\delta P_1,\delta P_2\subseteq T)}} \sigma_{d-1}(\delta P_1 \cap \delta P_2)\mathbb{1}_{\{X(\delta P_1^\bullet)\leq u< X(\delta P_2^\bullet)\}}.$$

By the linearity of the expectation, for $\delta > 0$,

$$\mathbb{E}[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{H}},T)}(u)] = \frac{1}{\sigma_d(T)} \sum_{\substack{P_1,P_2\in\mathcal{H}\\(P_1\neq P_2),(\delta P_1,\delta P_2\subseteq T)}} \sigma_{d-1}(\delta P_1 \cap \delta P_2)\mathbb{P}\big(X(\delta P_1^\bullet) \leq u < X(\delta P_2^\bullet)\big). \tag{23}$$

Let $P_1, P_2 \in \mathcal{H}$ be such that $P_1 \neq P_2$ and $\sigma_{d-1}(P_1 \cap P_2) > 0$, *i.e.*, such that P_1 and P_2 are adjacent in $\mathcal{H}$. Then as $\delta \to 0$, clearly $||\delta P_2^\bullet - \delta P_1^\bullet||_2 \to 0$, and so by Equation (13) in Theorem 2.1,

$$\frac{\mathbb{P}\big(X(\delta P_1^\bullet) \leq u < X(\delta P_2^\bullet)\big)}{||\delta P_2^\bullet - \delta P_1^\bullet||_2} \underset{\delta\to 0}{\longrightarrow} \frac{C_{d-1}^*(u)}{\beta_d}. \tag{24}$$

Moreover, if one restricts to neighbouring polytopes P_1 and P_2 in $\mathcal{H}$ such that $\delta P_1, \delta P_2 \subseteq T$, then $D^{(\delta\mathcal{H})} \to 0$ implies that the convergence in (24) is uniform as T remains fixed, *i.e.*

$$\sup\left\{\left|\frac{\mathbb{P}\big(X(\delta P_1^\bullet) \leq u < X(\delta P_2^\bullet)\big)}{||\delta P_2^\bullet - \delta P_1^\bullet||_2} - \frac{C_{d-1}^*(u)}{\beta_d}\right| : P_1 \text{ neighbours } P_2, \text{ and } \delta P_1, \delta P_2 \subseteq T\right\} \underset{\delta\to 0}{\longrightarrow} 0.$$

Therefore, Equation (23) can be written

$$\mathbb{E}[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{H}},T)}(u)] - \frac{C_{d-1}^*(u)}{\beta_d\sigma_d(T)} \sum_{\substack{P_1,P_2\in\mathcal{H}\\(P_1\neq P_2),(\delta P_1,\delta P_2\subseteq T)}} \sigma_{d-1}(\delta P_1 \cap \delta P_2)||\delta P_2^\bullet - \delta P_1^\bullet||_2 \underset{\delta\to 0}{\longrightarrow} 0.$$

With the change of variables $Q_1 = \delta P_1$ and $Q_2 = \delta P_2$, we have

$$\mathbb{E}[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{H}},T)}(u)] - \frac{C_{d-1}^*(u)}{\beta_d\sigma_d(T)} \sum_{\substack{Q_1,Q_1\in(\delta\mathcal{H})^T\\Q_1\neq Q_2}} \sigma_{d-1}(Q_1 \cap Q_2)||Q_2^\bullet - Q_1^\bullet||_2 \underset{\delta\to 0}{\longrightarrow} 0. \tag{25}$$

For adjacent $Q_1, Q_2 \in (\delta\mathcal{H})^T$, denote by Λ_{Q_1,Q_2} the d-dimensional hyperpyramid with base $Q_1 \cap Q_2$ and vertex $Q_1^\bullet$ (see Figure 4).
The d dimensional analogue of the area of a triangle implies that $\sigma_d(\Lambda_{Q_1,Q_2}) = \frac{1}{d}\sigma_{d-1}(Q_1 \cap Q_2)h_1$, where h_1 is the distance from $Q_1^\bullet$ to the hyperplane containing $Q_1 \cap Q_2$. Now, $\sigma_d(\Lambda_{Q_1,Q_2} \cup \Lambda_{Q_2,Q_1}) =$

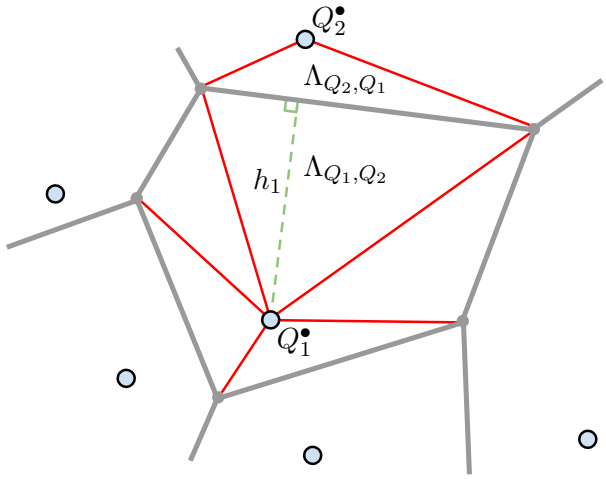


Figure 4: A pentagonal cell Q_1 in a point-referenced 2-honeycomb is partitioned into five pyramids (triangles in the case $d = 2$) each with summit $Q_1^\bullet$. A sixth pyramid Λ_{Q_2,Q_1} with summit $Q_2^\bullet$ is drawn in the cell Q_2, which shares its base $Q_1 \cap Q_2$ with the pyramid Λ_{Q_1,Q_2} contained in Q_1. The height of the pyramid Λ_{Q_1,Q_2} is marked h_1.

$\frac{1}{d}\sigma_{d-1}(Q_1 \cap Q_2)||Q_2^\bullet - Q_1^\bullet||_2$, and so

$$\sum_{\substack{Q_1,Q_1\in(\delta\mathcal{H})^T\\ Q_1\neq Q_2}} \sigma_{d-1}(Q_1\cap Q_2)||Q_2^\bullet - Q_1^\bullet||_2 = \sum_{\substack{Q_1,Q_2\in(\delta\mathcal{H})^T\\ Q_1 \text{ adjacent to } Q_2}} d\,\sigma_d(\Lambda_{Q_1,Q_2}\cup\Lambda_{Q_2,Q_1}) = 2d \sum_{\substack{Q_1,Q_2\in(\delta\mathcal{H})^T\\ Q_1 \text{ adjacent to } Q_2}} \sigma_d(\Lambda_{Q_1,Q_2})$$

$$= 2d\,\sigma_d\Bigg(\bigcup_{\substack{Q_1,Q_2\in(\delta\mathcal{H})^T\\ Q_1 \text{ adjacent to } Q_2}} \Lambda_{Q_1,Q_2}\Bigg).$$

Since every point in T at distance of at least $2D^{(\delta\mathcal{H})}$ from the boundary of T is contained in a Λ_{Q_1,Q_2} for some $Q_1, Q_2 \in (\delta\mathcal{H})^T$, and Λ_{Q_1,Q_2} is contained in $Q_1 \subseteq T$,

$$\lim_{\delta\to 0} \bigcup_{\substack{Q_1,Q_2\in(\delta\mathcal{H})^T\\ Q_1 \text{ adjacent to } Q_2}} \Lambda_{Q_1,Q_2} = T.$$

Thus, by the continuity of the measure σ_d,

$$\sum_{\substack{Q_1,Q_1\in(\delta\mathcal{H})^T\\ Q_1\neq Q_2}} \sigma_{d-1}(Q_1\cap Q_2)||Q_2^\bullet - Q_1^\bullet||_2 \xrightarrow[\delta\to 0]{} 2d\,\sigma_d(T), \tag{26}$$

and by (25),

$$\mathbb{E}[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{H}},T)}(u)] \xrightarrow[\delta\to 0]{} \frac{2d}{\beta_d} C_{d-1}^*(u).$$

This proves (22). Now if the tessellation is produced from a point process ξ and if $D^{(\delta\mathcal{V}_\xi)} \xrightarrow{\mathbb{P}} 0$ as $\delta \to 0$, then (22) implies that

$$\mathbb{E}[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{V}}_\xi,T)}(u) \mid \xi] \xrightarrow{\mathbb{P}} \frac{2d}{\beta_d} C_{d-1}^*(u), \qquad \delta \to 0. \tag{27}$$

What remains to be shown is that the convergence in (27) holds in L^1. First, remark that the sequence in (24) converges to its limit from below by Theorem 2.1. Remark also that the sequence in (26) converges to its limit from below. These two remarks, together with the convergence in probability in (27), imply that

$$\sup_{\delta>0} \mathbb{E}[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{V}}_\xi,T)}(u) \mid \xi] = \frac{2d}{\beta_d} C_{d-1}^*(u), \qquad a.s.$$

Therefore, the convergence in (27) holds in L^1 and $\mathbb{E}\left[\mathbb{E}[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{V}}_\xi,T)}(u) \mid \xi]\right] \underset{\delta\to 0}{\longrightarrow} \frac{2d}{\beta_d} C_{d-1}^*(u)$ as desired. □

Remark 3. In the second part of Theorem 2.2, the assumption that the point process ξ—that generates the point cloud of sample locations—is independent of X is crucial. Otherwise, for any realization X, it would be possible to build a dependent point-referenced d-honeycomb adapted to the realized $L_X^T(u)$ such that $\mathbb{E}\left[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{V}}_\xi,T)}(u)\right] \underset{\delta\to 0}{\longrightarrow} C_{d-1}^*(u)$ and no corrective constant is required (see Cotsakis et al. (2022), for $d = 2$). On the contrary, with a deterministic point-referenced d-honeycomb, the dimensional constant $2d/\beta_d$ is unavoidable.

In the following corollary, we apply Theorem 2.2 to Poisson-Voronoi tessellations. The interested reader is also referred to Theorem 1.1 in Thäle and Yukich (2016).

Corollary 2.1. *Let $T \subset \mathbb{R}^d$ be a compact domain with non empty interior containing the origin. Let X be an isotropic random field on $\mathbb{R}^d$ satisfying Assumption (A0), and let ξ be a homogeneous Poisson point process on $\mathbb{R}^d$ with unit rate, independent of X. Let $\mathcal{V}_\xi$ be the Poisson-Voronoi tessellation constructed from ξ, thus making $\dot{\mathcal{V}}_\xi$ a random point-referenced d-honeycomb. With $\delta > 0$ and $\widehat{C}_{d-1}^{(\delta\dot{\mathcal{V}}_\xi,T)}(u)$ defined as in Definition 1.3, it holds that*

$$\mathbb{E}\left[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{V}}_\xi,T)}(u)\right] \overset{\delta\to 0}{\longrightarrow} \frac{2d}{\beta_d} C_{d-1}^*(u).$$

Proof. By Theorem 2.2, it suffices to show that $D^{(\delta\mathcal{V}_\xi)} \xrightarrow{\mathbb{P}} 0$ as $\delta \to 0$, where $D^{(\delta\mathcal{V}_\xi)} = \sup\{\mathrm{diam}(P \cap T) : P \in \delta\mathcal{V}_\xi\}$. Let $1 \geq \epsilon > 0$, denote by A_ϵ the event $\{D^{(\delta\mathcal{V}_\xi)} > \epsilon\}$. Note that

$$\begin{aligned} A_\epsilon = \{\exists P \in \delta\mathcal{V}_\xi : \mathrm{diam}(P \cap T) > \epsilon\} &\subset \left\{\exists r \in \frac{1}{\delta}(T+1), (r + B^d_{\frac{\varepsilon}{2\delta}}) \cap \xi = \emptyset\right\} \\ &\subset \bigcup_{r \in \frac{\varepsilon}{4}\mathbb{Z} \cap \frac{T+1}{\delta}} \left\{(r + B^d_{\frac{\varepsilon}{4\delta}}) \cap \xi = \emptyset\right\}, \end{aligned}$$

where $T + 1 = \{r \in \mathbb{R}^d,\ d(r, T) \leq 1\}$. It follows from the stationarity of ξ and the compactness of T that

$$\mathbb{P}(A_\xi) \leq \frac{\sigma_d(T+1)4^d}{(\varepsilon\delta)^d} \mathbb{P}\left(B^d_{\frac{\varepsilon}{4\delta}} \cap \xi = \emptyset\right) = \frac{\sigma_d(T+1)4^d}{(\varepsilon\delta)^d} e^{-\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}\left(\frac{\varepsilon}{4\delta}\right)^d} \to 0 \text{ as } \delta \to 0.$$

□

Remark 4 (Convergence of the bias factor)**.** Theorem 2.2 *i*) offers insight about the limiting value of $\mathbb{E}\left[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{H}},T)}(u)\right]$ as $\delta \to 0$. However, this result does not imply the convergence of the bias factor to $2d/\beta_d$. It can be shown by slightly modifying the proofs of Theorems 2.1 and 2.2 that

$$E\left[\frac{\widehat{C}_{d-1}^{(\delta\dot{\mathcal{H}},T)}(u)}{C_{d-1}^T(u)} \;\middle|\; C_{d-1}^T(u) > 0\right] \underset{\delta\to 0}{\longrightarrow} \frac{2d}{\beta_d}, \tag{28}$$

with X an isotropic random field on $\mathbb{R}^d$ satisfying Assumption ($\mathcal{A}0$) and $\dot{\mathcal{H}}$ a point-referenced d-honeycomb as in Theorem 2.2 i). A sketch of the proof of the convergence in (28) is provided in Appendix A.3.

3 Joint central limit theorem for the hypercubic lattice

In this section we prove a joint central limit theorem for the estimated volume and surface area of excursion sets of $d-$dimensional isotropic smooth random fields observed over a hypercubic lattice. Firstly, we rewrite estimators in (10)-(11) in this specific setting.

3.1 Estimators for the hypercubic lattice

For $\mathbf{i} \in \mathbb{Z}^d$ and $\delta \in \mathbb{R}^+$, let $V_{\mathbf{i}}(\delta) := \delta\mathbf{i} + [0,\delta]^d$. Define $\delta\mathcal{G} := \{V_{\mathbf{i}}(\delta) : \mathbf{i} \in \mathbb{Z}^d\} \in \mathfrak{T}^d$ (see Definition 1.2) and the point-referenced d-honeycomb

$$\delta\dot{\mathcal{G}} := \{(V_{\mathbf{i}}(\delta), \delta\mathbf{i}) : \mathbf{i} \in \mathbb{Z}^d\}.$$

Define

$$\mathbb{G}(\delta, T) := \{\delta\mathbf{i} : \mathbf{i} \in \mathbb{Z}^d, V_{\mathbf{i}}(\delta) \subseteq T\} \subset \delta\mathbb{Z}^d. \tag{29}$$

Then, referring to Definition 1.3, we write

$$\widehat{C}_d^{(\delta\dot{\mathcal{G}},T)}(u) = \frac{\delta^d}{\sigma_d(T)} \sum_{t\in\mathbb{G}(\delta,T)} \mathbb{1}_{\{X(t)\geq u\}}, \tag{30}$$

and

$$\widehat{C}_{d-1}^{(\delta\dot{\mathcal{G}},T)}(u) = \frac{\delta^{d-1}}{\sigma_d(T)} \sum_{j=1}^{d} \sum_{\substack{t\in\mathbb{G}(\delta,T)\\ t+\delta\mathbf{e}_j\in\mathbb{G}(\delta,T)}} \left|\mathbb{1}_{\{X(t)\geq u\}} - \mathbb{1}_{\{X(t+\delta\mathbf{e}_j)\geq u\}}\right|. \tag{31}$$

In the following, without loss of generality, we assume that T is centered at the origin. Furthermore, we suppose that $0 < \delta \leq 1$ is chosen such that $T = [-\delta N, \delta N]^d$ for some $N \in \mathbb{N}^+$, which implies that $\bigcup_{P\in(\delta\mathcal{G})^T} P = T$, and that the $\mathrm{Card}(\mathbb{G}(\delta,T)) = (2N)^d$ and $\sigma_d(T) = (2N\delta)^d$. Remark that this constriction implies that the deterministic bias ratio in Equation (12) is exactly unity.

In the sequel we simplify the notation of Equations (30)-(31) by writing $\widehat{C}_d^{(\delta,T)}(u)$ and $\widehat{C}_{d-1}^{(\delta,T)}(u)$, respectively. Furthermore, we write $T = T_N$ when the dependence in N needs to be explicitly specified.

3.2 The dominant role of the L_1-norm for the hypercubic lattice

Note that, $C_{d-1}^T(u)$ in (6) can be rewritten, by the Crofton formula in (5), as

$$C_{d-1}^T(u) = \frac{1}{\sigma_d(T)} \int_{L_X^T(u)} \sigma_{d-1}(\mathrm{d}s) = \frac{\sqrt{\pi}\,\Gamma(\frac{d+1}{2})}{\sigma_d(T)\Gamma(\frac{d}{2})} \int_{\mathbb{R}^{d-1}} \int_{\partial B_1^d} \frac{\sigma_0(L_X^T(u)\cap l_{\mathbf{s},\mathbf{v_s}(\mathbf{u})})}{\sigma_{d-1}(\partial B_1^d)}\,\mathrm{d}\mathbf{s}\,\mathrm{d}\mathbf{u}, \quad a.s. \tag{32}$$

A first intuition is that when δ gets small, the estimated surface area (31) gets close to (32). However, this is incorrect and we show in the following result that when δ gets small our estimate is close in the L^1 sense to the following random variable

$$\widetilde{C}_{d-1}^T(u) := \frac{1}{\sigma_d(T)} \int_{L_X^T(u)} \frac{\|\nabla X(s)\|_1}{\|\nabla X(s)\|_2}\,\sigma_{d-1}(\mathrm{d}s) = \frac{1}{\sigma_d(T)} \int_T \delta_0\big(X(t)-u\big)\|\nabla X(t)\|_1\,\mathrm{d}t, \tag{33}$$

where $\delta_0(\cdot)$ denotes the Dirac delta distribution. The last equality is obtained by the well-known Coarea formula (see, *e.g.*, Equation (7.4.14) in Adler and Taylor (2007)). The difference between (32) and (33), that induces the asymptotic bias, is the ratio $\|\nabla X(s)\|_1/\|\nabla X(s)\|_2$. A similar competing behavior between the L^1 and L^2 norms was already visible in Equation (21) which leads to

$$\lim_{\delta\to 0}\mathbb{E}\left[\widehat{C}_{d-1}^{(\delta,T_N)}(u)\right] = p_{X(\mathbf{0})}(u)\,\mathbb{E}\left[\|\nabla X(\mathbf{0})\|_1 | X(\mathbf{0})=u\right], \tag{34}$$

with $\widehat{C}_{d-1}^{(\delta,T_N)}(u)$ as in (31). In addition, we see from the last equality in (33) that

$$\begin{aligned}\mathbb{E}[\widetilde{C}_{d-1}^{T}(u)] &= \frac{1}{\sigma_d(T)}\int_T\int_{-\infty}^{\infty}\mathbb{E}\left[\delta_0\big(x-u\big)||\nabla X(t)||_1 \;\middle|\; X(t)=x\right]p_{X(\mathbf{0})}(x)\,\mathrm{d}x\,\mathrm{d}t\\ &= p_{X(\mathbf{0})}(u)\,\mathbb{E}\left[\|\nabla X(\mathbf{0})\|_1 | X(\mathbf{0})=u\right].\end{aligned}$$

Equation (34) should be put in parallel with the desired limit given by Rice's formula (see, *e.g.*, Equation (6.27) in Azaïs and Wschebor (2009) or Proposition 2.2.1 in Berzin et al. (2017) (with $j=1$) written for a stationary process), *i.e.*,

$$\mathbb{E}[C_{d-1}^{T}(u)] = p_{X(\mathbf{0})}(u)\,\mathbb{E}[\|\nabla X(\mathbf{0})\|_2 | X(\mathbf{0})=u], \tag{35}$$

with $C_{d-1}^{T}(u)$ as in (6). This difference of norms in the limits in (34) and (35) motivates the presence of the dimensional constant β_d relating the L_1 and L_2 norms (see main Theorem 3.1 and Equation (14)).

Taking inspiration from the relationship between (6) and (32), the following proposition provides a similar result for $\widetilde{C}_{d-1}^{T}$ in (33) (first item) and a L^1 control between the estimator $\widehat{C}_{d-1}^{(\delta,T)}$ and $\widetilde{C}_{d-1}^{T}$ (second item) as $\delta\to 0$. To this end, we introduce the following technical assumption.

($\mathcal{A}3$) Fix $j\in\{1,\dots,d\}$. Let $f(t) := (X(t), \partial_j X(t))$ for $t\in\mathbb{R}^d$, where $\partial_j X$ denotes the partial derivative of X in the direction $\mathbf{e}_j$. Fix $u\in\mathbb{R}$, and let $U\subset\mathbb{R}^2$ be a neighbourhood containing the point $(u,0)$. Suppose that the density of $f(\mathbf{0})$ is bounded uniformly on U. Define

$$W := \sup_{t\in[0,1]^d}\max_{1\le k\le d}\left\{|\partial_k X(t)|, |\partial_k\partial_j X(t)|\right\}$$

and suppose that $\mathbb{E}[W^2 \mid f(\mathbf{0})=s]<\infty$ for all $s\in U$.

Similarly, suppose that for some interval $I\subset\mathbb{R}$ containing u, the marginal density of $X(\mathbf{0})$ is bounded uniformly on I, and that $\mathbb{E}[\sup_{t\in[0,1]^d}||\nabla X(t)||_2 \mid X(\mathbf{0})=s]<\infty$, for all $s\in I$.

This assumption is central in the proof of the key Lemma 4.3 in Section 4. It allows for the use of techniques similar to the ones used in Leadbetter et al. (1983) (see the proof of Proposition 2.1 in Section 4). Lemma 4.3 is used in the proof of the following proposition.

Proposition 3.1. *Let X be a random field on $\mathbb{R}^d$ as in Assumption ($\mathcal{A}0$). It holds that*

i) $\widetilde{C}_{d-1}^{T}(u) = \frac{1}{\sigma_d(T)}\sum_{j=1}^{d}\int_{\mathrm{vect}(\mathbf{e}_j^{\perp})}\sigma_0(L_X^T(u)\cap l_{\mathbf{e}_j,\mathbf{v}})\,\mathrm{d}\mathbf{v}$, *a.s.*

ii) If, furthermore, X satisfies Assumptions ($\mathcal{A}1$) and ($\mathcal{A}2$) for $\mathbf{w}=\mathbf{e}_j$, for each $1\le j\le d$, and for some $\varepsilon\in(0,1)$, and ($\mathcal{A}3$), then there exists a constant K such that for all $\delta\in(0,1)$

$$\mathbb{E}\left[\left|\widehat{C}_{d-1}^{(\delta,T)}(u) - \widetilde{C}_{d-1}^{T}(u)\right|\right] \le K\delta^{1-\varepsilon}. \tag{36}$$

The proof of Proposition 3.1 in postponed to Section 5.1. Remark that isotropy is not required in Proposition 3.1.

3.3 Strong alpha mixing random fields

We present and discuss sufficient hypotheses to prove the asymptotic normality in Theorem 3.1, below. We impose some spatial asymptotic independence conditions for the random field X which also apply to integrals of continuous functions over the level-curves of X (such as the surface area in (6)). To this end, mixing conditions are particularly appropriate (see for instance Cabaña (1987), Iribarren (1989)).

Definition 3.1 (Strongly mixing random field)**.** Let $X := \{X(t) : t \in \mathbb{R}^d\}$ be a random field satisfying Assumption ($\mathcal{A}0$), and let $\sigma_U := \sigma\{X(t) : t \in U\}$ for a subset $U \subset \mathbb{R}^d$, *i.e.* the $\sigma-$field generated by $\{X(t) : t \in U\}$. We define the following mixing coefficient for Borelian subsets $U, V \subset \mathbb{R}^d$,

$$\alpha(U, V) := \sup_{A \in \sigma_U,\, B \in \sigma_V} \{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|\}. \tag{37}$$

Further we define

$$\alpha(s) := \sup\{\alpha(U, V) : d_2(U, V) > s\},$$

where $d_2(U, V) := \inf\{||u - v||_2 : u \in U, v \in V\}$. A random field X is said to be strongly mixing if $\alpha(s) \to 0$ for $s \to \infty$.

Existing results that establish asymptotic normality of geometric quantities classically rely on the continuous observation of X on T and on a quasi-association notion of dependence (see Bulinski and Shashkin (2007), Meschenmoser and Shashkin (2011), Bulinski et al. (2012), Spodarev (2014)) or are specific to Gaussian random fields (see Meschenmoser and Shashkin (2013), Müller (2017), Kratz and Vadlamani (2018)). There are also results for fields observed on the fixed lattice grid $\mathbb{Z}^d$ in Reddy et al. (2018), where the notion of clustering spin model is introduced and is implied by either mixing assumptions or quasi-association. Finally, it is worth noting that Bulinski (2010) proposes a limit theorem for the empirical mean of X observed on a grid with vanishing size and large T, *i.e.* the same grid of observation $\mathbb{G}(\delta, T)$ as in the present work.
Corollary 3 in Dedecker (1998) (see also Bolthausen (1982)) proposes minimal α-mixing conditions to get central limit results for stationary random fields observed on the fixed lattice $\mathbb{Z}^d$. However, we can not rely on such results as we are interested in non-trivial functionals of the field and we aim at imposing the conditions on the underlying field X and not on the observed sequence on the varying lattice $\delta\mathbb{Z}^d$. Instead, to establish Theorem 3.1 below, we heavily rely on the inheritance properties of mixing sequences: if a random field X is strongly alpha-mixing (see Definition 3.1), so is any measurable transformation of it. The latter property does not hold for quasi-association properties. Examples of mixing random fields include Gaussian random fields (see Doukhan (1994) Section 2.1 and Corollary 2 for an explicit control of (37)), and therefore any transformation of Gaussian random fields such as Student or chi-square fields, or Max-infinitely divisible random fields (see Dombry and Eyi-Minko (2012)). We refer the reader to Bradley (2005) for other examples of processes satisfying various mixing conditions.

3.4 Joint Central Limit Theorem

In the following, $\xrightarrow{\mathcal{L}}$ denotes the convergence in law. Taking advantage of the results of Section 2 and adapting the results of Iribarren (1989), we state the following joint limit result for the estimated volume and estimated surface area.

Theorem 3.1 (Joint central limit theorem for $\widehat{C}_d^{(\delta,T_N)}$ and $\widehat{C}_{d-1}^{(\delta,T_N)}$)**.** *Let X be an isotropic random field satisfying Assumptions ($\mathcal{A}0$), ($\mathcal{A}1$), ($\mathcal{A}2$) for some $\varepsilon \in (0, 1)$, and ($\mathcal{A}3$). Assume that X is strongly*

mixing as in Definition 3.1 and that for some $\eta > 0$, the mixing coefficients satisfy the rate condition

$$\sum_{r=1}^{+\infty} r^{3d-1}\alpha(r)^{\frac{\eta}{2+\eta}} < +\infty,$$

and $\mathbb{E}[\sigma_{d-1}\big(L_X(u) \cap [0,1]^d\big)^{2+\eta}] < +\infty$. *Let* $(T_N)_{N\geq 1}$ *be a sequence of hypercubes in* $\mathbb{R}^d$ *such that* $\sigma_d(T_N) = (2N\delta)^d$. *Define*

$$\widehat{C}^{(\delta,T_N)}(u) := \Big(\widehat{C}_d^{(\delta,T_N)}(u), \widehat{C}_{d-1}^{(\delta,T_N)}(u)\Big)^t \quad \text{and} \quad C^*(u) := \Big(C_d^*(u), \frac{2d}{\beta_d}C_{d-1}^*(u)\Big)^t,$$

with $\widehat{C}_d$ *(resp.* $\widehat{C}_{d-1}$*) as in (30) (resp. in (31)) on the hypercubic lattice* $\mathbb{G}(\delta, T)$ *in (29),* $C_k^*(u)$ *as in (8), where* $(\cdot)^t$ *denotes the matrix transposition and* β_d *is as in (14). Then, there exists a finite covariance matrix* $\Sigma(u)$ *such that, if* $\Sigma_{(i,j)}(u) > 0$, $\forall\, 1 \leq i, j \leq 2$, *it holds*

$$\sqrt{\sigma_d(T_N)}\left(\widehat{C}^{(\delta,T_N)}(u) - C^*(u)\right) \xrightarrow{\mathcal{L}} \mathcal{N}_2\big(0, \Sigma(u)\big),$$

as $\delta \to 0$, $N\delta \to \infty$, *and* $(N\delta)^d\delta^{1-\varepsilon} \to 0$, *with* ε *as in Assumption (A2), with variances*

$$\begin{aligned}
\Sigma_{(1,1)}(u) &= \int_{\mathbb{R}^d} \mathrm{Cov}(\mathbb{1}_{\{X(\mathbf{0})\geq u\}}, \mathbb{1}_{\{X(t)\geq u\}})\mathrm{d}t,\\
\Sigma_{(2,2)}(u) &= \int_{\mathbb{R}^d} \Big[p_{X(\mathbf{0}),X(t)}(u,u)\,\mathbb{E}\left[||\nabla X(\mathbf{0})||_1||\nabla X(t)||_1 \;\middle|\; X(\mathbf{0}) = X(t) = u\right]\\
&\qquad - \Big(p_{X(\mathbf{0})}(u)\,\mathbb{E}\left[||\nabla X(\mathbf{0})||_1 \;\middle|\; X(\mathbf{0}) = u\right]\Big)^2\Big]\mathrm{d}t.
\end{aligned}$$

and covariances $\Sigma_{(1,2)}(u) = \lim_{N\to\infty} \sigma_d(T_N)\mathrm{Cov}(C_d^{T_N}(u), C_{d-1}^{T_N}(u))$.

Proof of Theorem 3.1. Let $C^{T_N}(u) := \Big(C_d^{T_N}(u), \widetilde{C}_{d-1}^{T_N}(u)\Big)^t$, with $C_d^{T_N}(u)$ as in (7) and $\widetilde{C}_{d-1}^{T_N}(u)$ as in (33). We decompose

$$\begin{aligned}
\sqrt{\sigma_d(T_N)}\Big(\widehat{C}^{(\delta,T_N)}(u) - C^*(u)\Big) &= \sqrt{\sigma_d(T_N)}\left(\widehat{C}^{(\delta,T_N)}(u) - C^{T_N}(u)\right) + \sqrt{\sigma_d(T_N)}\left(C^{T_N}(u) - \mathbb{E}[C^{T_N}(u)]\right)\\
&\quad + \sqrt{\sigma_d(T_N)}\left(\mathbb{E}[C^{T_N}(u)] - \mathbb{E}[\widehat{C}^{(\delta,T_N)}(u)]\right) + \sqrt{\sigma_d(T_N)}\left(\mathbb{E}[\widehat{C}^{(\delta,T_N)}(u)] - C^*(u)\right)\\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

From Proposition 5.1, we get that first coordinates of I_1 and I_3 go in probability to zero. The second coordinates are handled with Proposition 3.1. It follows that $I_1 \xrightarrow{\mathbb{P}} 0$ and $I_3 \xrightarrow{\mathbb{P}} 0$, for $N\delta \to \infty$, $\delta \to 0$ and $(N\delta)^d\delta^{1-\varepsilon} \to 0$ as $N \to \infty$, with ε as in Assumption (A2).
The joint central limit theorem under mixing conditions (see Theorem 5.1, for $n = N\delta$) for the random vector $C^{T_N}(u)$ gives that $I_2 \xrightarrow{\mathcal{L}} \mathcal{N}_2\big(0, \Sigma(u)\big)$. The given asymptotic variances come from Cotsakis et al. (2022) (Equation (11)), Equation (33) and Equation (50). Notice that $\widehat{C}_d^{(\delta,T)}(u)$ in (30) does not generate any bias in the estimation of the volume measure σ_d (see (9)), so the first coordinate of I_4 is trivially equal to zero. The second coordinate of I_4 is obtained via Corollary 5.1. Then, by Slutsky's Theorem, we obtain the result. □

All auxiliary results necessary for the proof of the joint central limit theorem for $(\widehat{C}_d^{(\delta,T_N)}, \widehat{C}_{d-1}^{(\delta,T_N)})$ are provided in Section 5.2.

Remark 5. The regime restriction $(N\delta)^d\delta^{1-\varepsilon} \to 0$ (where ε can be small) is imposed by Proposition 3.1 and Proposition 5.1. It can be improved by requiring more stringent assumptions: *e.g.*, if X is quasi-associated (see Bulinski and Shashkin (2007) and Bulinski et al. (2012)) and under a decay assumption for its correlation function (see Appendix A.1). However, improving the rate in Proposition 3.1 seems much more delicate and is out of the scope of the present work.
Generalizing Theorem 3.1 to point clouds other than hypercubic lattices requires that one first identifies the associated random variable when δ gets small, *i.e.* an analog to (33), and that one proves an analog of (36). In this sense, this asymptotic result is lattice-dependent and generalizing it to general tessellations is also an interesting open point.

4 Additional results and proofs associated to Section 2

4.1 Auxiliary lemmas on crossings

The following technical lemma controls the probability that a one-dimensional random process crosses the level u more than once on a small interval of length t.

Lemma 4.1 (Level crossings of random processes)**.** *Let $Y = \{Y(s) : s \in \mathbb{R}\}$ be a one-dimensional stationary random process with twice continuously differentiable sample paths. Suppose that the probability density function of $Y(0)\big|\{Y'(0) = 0\}$ is uniformly bounded by $M < \infty$ and that $\|Y''\|_{\infty,[0,1]} \in L^p$ for some $p > \frac{1}{2}$. For $u \in \mathbb{R}$, let $Y^{-1}(u) := \{s \in \mathbb{R} : Y(s) = u\}$. Then, there exists a constant $K \in \mathbb{R}^+$ such that*

$$\mathbb{P}\big(\mathrm{Card}\,\big(Y^{-1}(u) \cap [0,t]\big) \geq 2\big) \leq K t^{\frac{3p}{p+1}},$$

for all $t \in [0,1]$.

Proof of Lemma 4.1. If there are to exist two points $s_1, s_2 \in [0,t]$, $s_1 < s_2$ such that $Y(s_1) = Y(s_2) = u$, then by Rolle's theorem, there exists a $c \in [s_1, s_2]$ such that $Y'(c) = 0$. Moreover, by Taylor's theorem,

$$Y(s_2) = u = Y(c) + \frac{(s_2 - c)^2}{2} Y''(\eta),$$

for some $\eta \in [c, s_2]$, which implies $2|Y(c) - u| \leq t^2 \|Y''\|_{\infty,[0,1]}$, where as Y is twice continuously differentiable, $\|Y''\|_{\infty,[0,1]} < \infty$ almost surely. It implies that for any $x \in (0,1)$,

$$\{\mathrm{Card}\,\big(Y^{-1}(u) \cap [0,t]\big) \geq 2\} \subseteq \{\|Y''\|_{\infty,[0,1]} \geq (2/t^2)^x\} \cup \{\exists c \in [0,t] : Y'(c) = 0, |Y(c) - u| \leq (t^2/2)^{1-x}\}.$$

To control the probability of the first event, Markov's inequality gives

$$\mathbb{P}\big(\|Y''\|_{\infty,[0,1]} \geq (2/t^2)^x\big) = \mathbb{P}\big(\|Y''\|^p_{\infty,[0,1]} \geq (2/t^2)^{px}\big) \leq \Big(\frac{t^2}{2}\Big)^{px} \mathbb{E}[\|Y''\|^p_{\infty,[0,1]}].$$

As for the second event, we write

$$\begin{aligned}
&\mathbb{P}\big(\exists c \in [0,t] : Y'(c) = 0, |Y(c) - u| \leq (t^2/2)^{1-x}\big) \\
&\qquad = \mathbb{P}\big(\mathrm{Card}\,\big(\{c \in [0,t] : Y'(c) = 0\}\big) \geq 1\big) \mathbb{P}\big(|Y(s) - u| \leq (t^2/2)^{1-x} \mid Y'(s) = 0, s \in [0,t]\big) \\
&\qquad \leq \mathbb{E}\left[\mathrm{Card}\,\big(\{c \in [0,t] : Y'(c) = 0\}\big)\right] \mathbb{P}\big(|Y(s) - u| \leq (t^2/2)^{1-x} \mid Y'(s) = 0\big).
\end{aligned}$$

By Theorem 3.3.1 in Beutler and Leneman (1966), $\mathbb{E}[\mathrm{Card}(\{c \in [0,t] : Y'(c) = 0\})] = t\,\mathbb{E}[\mathrm{Card}(\{c \in [0,1] : Y'(c) = 0\})]$. In addition, $\mathbb{P}\big(|Y(s)-u| \le (t^2/2)^{1-x} \mid Y'(s) = 0\big) \le 2M(t^2/2)^{1-x}$ as Y is stationary and $Y(0)\big|\{Y'(0) = 0\}$ has bounded density. In total, we have shown that

$$\mathbb{P}\big(\mathrm{Card}\left(Y^{-1}(u) \cap [0,t]\right) \ge 2\big) \le \Big(\frac{t^2}{2}\Big)^{px} \mathbb{E}[\|Y''\|^p_{\infty,[0,1]}] + 2M\,\mathbb{E}[\mathrm{Card}(\{c \in [0,1] : Y'(c) = 0\})]t\Big(\frac{t^2}{2}\Big)^{1-x}.$$

Thus, optimizing in x leads to consider $x := 3/(2p+2) \in (0,1)$, if $p > \frac{1}{2}$, and to the result. □

The following lemma provides a bound on the crossing probability for general, possibly anisotropic, random fields.

Lemma 4.2. *Let X be a random field on $\mathbb{R}^d$ satisfying (A0). Fix $t \in \mathbb{R}^d$ and $u \in \mathbb{R}$. Then,*

$$\mathbb{P}\left(X(\mathbf{0}) \le u < X(t)\right) \le C^*_{d-1}(u)\|t\|_2.$$

Proof of Lemma 4.2. Let $\bar{t}$ denote the line segment in $\mathbb{R}^d$ whose endpoints are $\mathbf{0}$ and t. Define $K_X(u) := \{L_X(u) - \lambda t : \lambda \in (0,1)\}$ to be the set of points that are at a distance of at most $||t||_2$ from $L_X(u)$ in the direction of $-t$ (see Figure 5 for an illustration of the set $K_X(u)$ in $\mathbb{R}^2$). By the assumption of stationarity it holds that

$$\mathbb{P}\left(X(\mathbf{0}) \le u < X(t)\right) \le \mathbb{P}\left(\bar{t} \cap L_X(u) \ne \emptyset\right) = \mathbb{P}\left(\mathbf{0} \in K_X(u)\right) = \mathbb{E}\left[\frac{\sigma_d\left(K_X(u) \cap B^d_r\right)}{\sigma_d(B^d_r)}\right],$$

for all $r \in \mathbb{R}^+$. If we denote $r^+ := r + ||t||_2$ for $r \in \mathbb{R}^+$, then

$$\begin{aligned}\sigma_d\left(K_X(u) \cap B^d_r\right) &\le \int_{L_X(u)\cap B^d_{r^+}} \Big\langle t, \frac{\nabla X(s)}{||\nabla X(s)||_2}\Big\rangle\, \sigma_{d-1}(\mathrm{d}s) \le \int_{L_X(u)\cap B^d_{r^+}} ||t||_2\, \sigma_{d-1}(\mathrm{d}s) \\ &= \|t\|_2 \sigma_{d-1}\left(L_X(u) \cap B^d_{r^+}\right), \quad a.s.\end{aligned}$$

Taking the expectation and sending $r \to \infty$ yields the result, since $\sigma_d(B^d_r)/\sigma_d(B^d_{r^+}) \to 1$. □

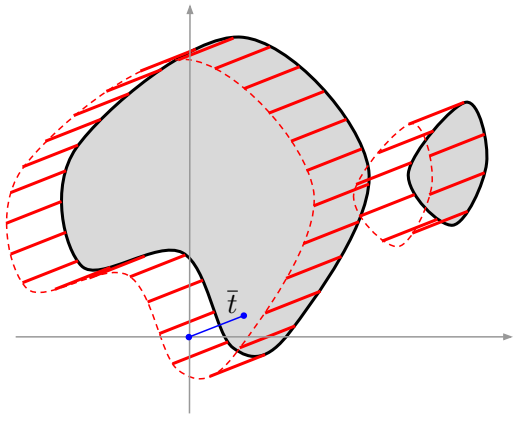


Figure 5: In dimension 2, for the level curve $L_X(u)$ in black, $K_X(u)$ is the area covered by solid red line segments. Note that the line segment $\bar{t}$ (in blue) crosses $L_X(u)$ (in black) if and only if $\mathbf{0} \in K_X(u)$.

Definition 4.1. Define the map $\pi_j : \mathbb{R}^d \to \mathbb{R}^d$ to be the orthogonal projection onto the $(d-1)$-dimensional subspace $\mathrm{vect}(\mathbf{e}_j^\perp)$. That is, for $S \subseteq \mathbb{R}^d$,

$$\pi_j(S) = \Big\{ \sum_{i \neq j} s_i \mathbf{e}_i : \mathbf{s} \in S \Big\} = \{\mathbf{v} \in \mathrm{vect}(\mathbf{e}_j^\perp) : l_{\mathbf{e}_j,\mathbf{v}} \cap S \neq \emptyset\},$$

with $l_{\mathbf{e}_j,\mathbf{v}}$ as in (3).

The following lemma allows to obtain the rate of convergence of a Riemann sum used in Proposition 3.1. The techniques used in the proof can be used to bound the probability that a random manifold intersects a small region of space.

Lemma 4.3. *Let X be a stationary random field on $\mathbb{R}^d$ satisfying Assumption (A0) and (A3). Fix $j \in \{1, \dots, d\}$. For $\mathbf{v} \in \mathrm{vect}(\mathbf{e}_j^\perp)$, denote $n_{\mathbf{v}} := \sigma_0(L_X(u) \cap l_{\mathbf{e}_j,\mathbf{v}} \cap [0,1]^d)$, with $l_{\mathbf{e}_j,\mathbf{v}}$ as in (3). Then, there exists a constant $K \in \mathbb{R}^+$ such that for all $\delta \in [0,1)$,*

$$\mathbb{P}\Big(\sup_{\mathbf{v}_1,\mathbf{v}_2 \in \pi_j([0,\delta]^d)} \big| n_{\mathbf{v}_1} - n_{\mathbf{v}_2} \big| > 0 \Big) \leq K\delta.$$

Proof of Lemma 4.3. Note that $n_{\mathbf{v}}$ seen as a function that maps $\mathbf{v} \in \mathrm{vect}(\mathbf{e}_j^\perp)$ into $\mathbb{N}_0$ is almost surely piecewise constant on $\pi_j([0,1]^d)$. The discontinuities occur at points in $\pi_j(H_1 \cup H_2 \cup H_3)$, where $H_1 := f^{-1}((u,0)) \cap [0,1]^d$—recall from Assumption (A3) that $f = (X, \partial_j X)$, $H_2 := \{\mathbf{s} \in L_X(u) : s_j = 0\}$, and $H_3 := \{\mathbf{s} \in L_X(u) : s_j = 1\}$ (see Figure 6 for an example with $d = 2$).

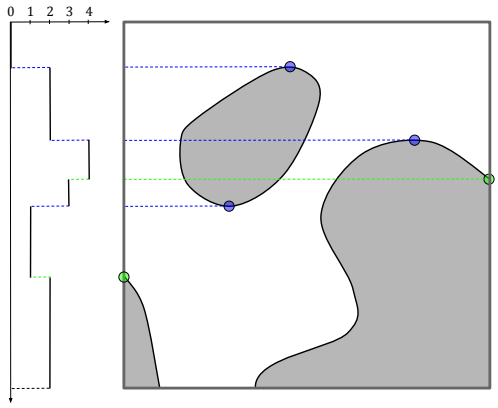


Figure 6: On the left a representation of the piecewise constant function $n_{\mathbf{v}}$ of $\mathbf{v} \in \mathrm{vect}(\mathbf{e}_1^\perp)$ computed for the excursion set represented in grey on the right. The points in blue are in H_1 and the points in green are in H_2 and H_3.

For $i \in \{1,2,3\}$, we aim to show that $\mathbb{P}\big(\pi_j([0,\delta]^d) \cap \pi_j(H_i) \neq \emptyset\big)$ is of the order of δ, which implies the desired result.
We begin with the case of H_1. By stationarity, we apply the union bound,

$$\mathbb{P}\big(\pi_j([0,\delta]^d) \cap \pi_j(H_1) \neq \emptyset\big) \leq \lceil 1/\delta \rceil \mathbb{P}\big([0,\delta]^d \cap H_1 \neq \emptyset\big), \tag{38}$$

where $\lceil \cdot \rceil$ denotes the ceiling function.

By Taylor's theorem, for each $t \in [0,\delta]^d$, there exists an $s \in [0,\delta]^d$ such that $f(t) = f(\mathbf{0}) + \langle t, \nabla f(s)\rangle$. If $[0,\delta]^d \cap H_1 \neq \emptyset$, then $(u,0) - f(\mathbf{0}) = \langle t, \nabla f(s)\rangle$, for some $s,t \in [0,\delta]^d$.

Let us denote by $M_1 \in \mathbb{R}^+$ the constant which uniformly bounds the density of $f(\mathbf{0})$ on U, and M_2 the constant which bounds the marginal density of $X(\mathbf{0})$ on I (see Assumption ($\mathcal{A}3$)).
For $\delta \leq 1$, we bound the variation of $f(\mathbf{0})$ by writing

$$||(u,0) - f(\mathbf{0})||_\infty \leq ||t||_1 W \leq d\delta W.$$

Therefore, we obtain

$$\mathbb{P}\big([0,\delta]^d \cap H_1 \neq \emptyset\big) \leq \mathbb{P}\big(||(u,0) - f(\mathbf{0})||_\infty \leq d\delta W\big).$$

let $g(w,x,\dot{x})$ be the joint probability distribution of W, $X(\mathbf{0})$, and $\partial_j X(\mathbf{0})$. Note that

$$\begin{aligned}\mathbb{P}\big(||(u,0) - f(\mathbf{0})||_\infty \leq d\delta W\big) &= \int_0^\infty \int_{u-\delta w d}^{u+\delta w d} \int_{-\delta w d}^{\delta w d} g(w,x,\dot{x})\,\mathrm{d}\dot{x}\mathrm{d}x\mathrm{d}w \\ &= (2d\delta)^2 \int_0^\infty \int_{-w/2}^{w/2} \int_{-w/2}^{w/2} g(w,u+2d\delta z, 2d\delta\dot{z})\,\mathrm{d}\dot{z}\mathrm{d}z\mathrm{d}w,\end{aligned}$$

where a change of the variables $x = u + 2d\delta z$ and $\dot{x} = 2d\delta\dot{z}$ was used. Now, by the dominated convergence theorem,

$$\begin{aligned}\lim_{\delta\to 0} \frac{1}{\delta^2}\mathbb{P}\big(||(u,0) - f(\mathbf{0})||_\infty \leq d\delta W\big) &= \lim_{\delta\to 0} 4d^2 \int_0^\infty \int_{-w/2}^{w/2} \int_{-w/2}^{w/2} g(w,u+2d\delta z, 2d\delta\dot{z})\,\mathrm{d}\dot{z}\mathrm{d}z\mathrm{d}w \\ &= 4d^2 \int_0^\infty w^2 g(w,u,0)\,\mathrm{d}w \leq 4d^2 M_1\, \mathbb{E}[W^2 \mid f(\mathbf{0}) = (u,0)],\end{aligned}$$

which is finite by ($\mathcal{A}3$). Thus, $\mathbb{P}\big([0,\delta]^d \cap H_1 \neq \emptyset\big) = O(\delta^2)$, and by (38), $\mathbb{P}\big(\pi_j([0,\delta]^d) \cap \pi_j(H_1) \neq \emptyset\big) = O(\delta)$, as desired.
Now, we consider the case of H_2 (the case of H_3 being identical). Using similar arguments as in the case of H_1, we readily obtain

$$\begin{aligned}\lim_{\delta\to 0} \frac{1}{\delta}\mathbb{P}\big(\pi_j([0,\delta]^d) \cap \pi_j(H_2) \neq \emptyset\big) &= \lim_{\delta\to 0} \frac{1}{\delta}\mathbb{P}\big([0,\delta]^d \cap H_2 \neq \emptyset\big) \\ &\leq \lim_{\delta\to 0} \frac{1}{\delta}\mathbb{P}\Big(|u - X(\mathbf{0})| \leq \sqrt{d}\delta \sup_{t\in[0,1]^d} ||\nabla X(t)||_2\Big) \\ &\leq 2\sqrt{d} M_2\, \mathbb{E}\Big[\sup_{t\in[0,1]^d} ||\nabla X(t)||_2 \;\Big|\; X(\mathbf{0}) = u\Big] < \infty.\end{aligned}$$

Thus, $\mathbb{P}\big(\pi_j([0,\delta]^d) \cap \pi_j(H_2) \neq \emptyset\big) = O(\delta)$ as desired. □

4.2 Proof of Proposition 2.1

First note that

$$\{X(\mathbf{0}) \leq u < X(q\mathbf{e}_1)\} = \{X(\mathbf{0}) \leq u < X(\mathbf{0}) + qX_q\} = \{X(\mathbf{0}) \leq u\} \cap \left\{\frac{1}{q}(u - X(\mathbf{0})) < X_q\right\}.$$

It follows that

$$\begin{aligned}\frac{1}{q}\mathbb{P}\left(X(\mathbf{0}) \leq u < X(q)\right) &= \frac{1}{q}\mathbb{P}\left(X(\mathbf{0}) \leq u,\ X_q > q^{-1}(u - X(\mathbf{0}))\right) \\ &= \frac{1}{q}\int_{-\infty}^{u}\int_{\frac{1}{q}(u-x)}^{\infty} g_q(x,y)\mathrm{d}y\mathrm{d}x = \frac{1}{q}\int_{0}^{\infty}\int_{u-qy}^{u} g_q(x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^\infty z\int_0^1 g_q(u - qzv, z)\mathrm{d}v\mathrm{d}z,\end{aligned}$$

using the change of variable $v = \frac{u-x}{qy}$, $z = y$ ($x = u - qzv$, $y = z$). From the assumptions, we derive that $g_q(u - qzv, z) \to p(u, z)$ pointwise as $q \to 0$, and the dominated convergence theorem applies and leads to the result.

5 Additional results and proofs associated to Section 3

5.1 Proof of Proposition 3.1

Firstly, we prove the first item in Proposition 3.1. As $L_X^T(u)$ is a C^1 smooth manifold, integrals over $L_X^T(u)$ can be expressed via sequences of polygonal approximations. Let $(A_1, \ldots, A_k)$ be a finite sequence of $(d-1)$-dimensional hyperplanes residing in $\mathbb{R}^d$. For $i \in \{1, \ldots, k\}$, let $\mathbf{n}_i \in \mathbb{R}^d$ denote the unit normal vector to A_i. We aim to show

$$\sum_{i=1}^{k}\sum_{j=1}^{d}|\langle \mathbf{n}_i, \mathbf{e}_j\rangle|\sigma_{d-1}(A_i) = \sum_{j=1}^{d}\int_{\mathrm{vect}(\mathbf{e}_j^\perp)} \sigma_0\left(\left(\bigcup_{i=1}^{k} A_i\right)\cap l_{\mathbf{e}_j,\mathbf{v}}\right)\mathrm{d}\mathbf{v}, \tag{39}$$

with $l_{\mathbf{e}_j,\mathbf{v}}$ as in (3). By noticing that for all $\mathbf{v} \in \mathrm{vect}(\mathbf{e}_j^\perp)$,

$$\sigma_0\left(\left(\bigcup_{i=1}^{k} A_i\right)\cap l_{\mathbf{e}_j,\mathbf{v}}\right) = \sum_{i=1}^{k}\mathbb{1}_{\{A_i\cap l_{\mathbf{e}_j,\mathbf{v}}\neq\emptyset\}},$$

we see that it suffices to show that for each pair of indices $(i, j) \in \{1, \ldots, k\} \times \{1, \ldots, d\}$,

$$|\langle \mathbf{n}_i, \mathbf{e}_j\rangle|\sigma_{d-1}(A_i) = \int_{\mathrm{vect}(\mathbf{e}_j^\perp)} \mathbb{1}_{\{A_i\cap l_{\mathbf{e}_j,\mathbf{v}}\neq\emptyset\}}\ \mathrm{d}\mathbf{v}. \tag{40}$$

Equation (40) is evident, since both sides of the equality describe the σ_{d-1} measure of the projection of A_i onto the hyperplane $\mathrm{vect}(\mathbf{e}_j^\perp)$. Note that $\sum_{j=1}^{d}|\langle \mathbf{n}_i, \mathbf{e}_j\rangle| = ||\mathbf{n}_i||_1$, and that for $t \in L_X(u)$, $\nabla X(t)/||\nabla X(t)||_2$ is the unit normal vector to $L_X(u)$ at t, so the left-hand side of Equation (39) is a Riemann sum that approximates the integral expression for $\widetilde{C}_{d-1}^T(u)$ in (33) when k goes to infinity.
Now, we prove the second item in Proposition 3.1. For $\mathbf{g} \in \mathbb{G}(\delta, T)$ in (29), let $Z_{\mathbf{g}} := \mathbb{1}_{\{X(\mathbf{g})\geq u\}}$. Equations (30)-(31) can be rewritten as

$$\widehat{C}_d^{(\delta,T)}(u) = \frac{\delta^d}{\sigma_d(T)}\sum_{\mathbf{g}\in\mathbb{G}(\delta,T)} Z_{\mathbf{g}} = \frac{1}{(2N)^d}\sum_{\mathbf{g}\in\mathbb{G}(\delta,T)} Z_{\mathbf{g}},$$

$$\widehat{C}_{d-1}^{(\delta,T)}(u) = \frac{\delta^{d-1}}{\sigma_d(T)}\sum_{j=1}^{d}\sum_{\substack{\mathbf{g}\in\mathbb{G}(\delta,T)\\ \mathbf{g}+\delta\mathbf{e}_j\in\mathbb{G}(\delta,T)}} \mathbb{1}_{\{Z_{\mathbf{g}}\neq Z_{\mathbf{g}+\delta\mathbf{e}_j}\}}.$$

Set $G := \mathbb{G}(\delta, T) \cap [0,1]^d$ and $n_{\mathbf{v}} := \sigma_0(L_X(u) \cap l_{\mathbf{e}_j,\mathbf{v}} \cap [0,1]^d)$, suppose that we have established

$$\mathbb{E}\left[\left|\delta^{d-1} \sum_{\mathbf{g}\in G} \mathbb{1}_{\{Z_{\mathbf{g}} \neq Z_{\mathbf{g}+\delta\mathbf{e}_j}\}} - \int_{\mathrm{vect}(\mathbf{e}_j^{\perp})\cap[0,1]^d} n_{\mathbf{v}} \mathrm{d}\mathbf{v}\right|\right] = O\big(\delta^{1-\varepsilon}\big), \tag{41}$$

for all $j \in \{1, \ldots, d\}$. Then, by stationarity and the triangle inequality,

$$\mathbb{E}\left[\left|\delta^{d-1} \sum_{\mathbf{g}\in \mathbb{G}(\delta,T)} \mathbb{1}_{\{Z_{\mathbf{g}} \neq Z_{\mathbf{g}+\delta\mathbf{e}_j}\}} - \int_{\mathrm{vect}(\mathbf{e}_j^{\perp})} \sigma_0\big(L_X^T(u) \cap l_{\mathbf{e}_j,\mathbf{v}}\big)\mathrm{d}\mathbf{v}\right|\right] = O\Big((N\delta)^d \delta^{1-\varepsilon}\Big)$$

so the desired result holds by summing over the d dimensions and again applying the triangle inequality together with the first item. Therefore, it suffices to show (41). This is done in two steps. First, we show that (recall Definition 4.1)

$$\mathbb{E}\left[\left|\delta^{d-1} \sum_{\mathbf{g}\in G} \mathbb{1}_{\{Z_{\mathbf{g}} \neq Z_{\mathbf{g}+\delta\mathbf{e}_j}\}} - \delta^{d-1} \sum_{\mathbf{v}\in\pi_j(G)} n_{\mathbf{v}}\right|\right] = O(\delta^{1-\varepsilon}). \tag{42}$$

Second, we show that

$$\mathbb{E}\left[\left|\delta^{d-1} \sum_{\mathbf{v}\in\pi_j(G)} n_{\mathbf{v}} - \int_{\mathrm{vect}(\mathbf{e}_j^{\perp})\cap[0,1]^d} n_{\mathbf{s}} \mathrm{d}\mathbf{s}\right|\right] = O(\delta^{1-\varepsilon}). \tag{43}$$

Let $M^{(\delta)} := \big(\sigma_0(G)\big)^{1/d} = \lfloor \delta^{-1} \rfloor + 1$ be the number of rows in the grid G that is contained in $[0,1]^d$. By construction, $M^{(\delta)}\delta \leq 1 + \delta$ for all $\delta \in \mathbb{R}^+$.
To see that (42) holds, we use the triangle inequality to write

$$\mathbb{E}\left[\left|\delta^{d-1} \sum_{\mathbf{g}\in G} \mathbb{1}_{\{Z_{\mathbf{g}} \neq Z_{\mathbf{g}+\delta\mathbf{e}_j}\}} - \delta^{d-1} \sum_{\mathbf{v}\in\pi_j(G)} n_{\mathbf{v}}\right|\right] \leq \delta^{d-1} \sum_{\mathbf{v}\in\pi_j(G)} \mathbb{E}\left[\left| \sum_{\mathbf{g}\in G\cap l_{\mathbf{e}_j,\mathbf{v}}} \mathbb{1}_{\{Z_{\mathbf{g}} \neq Z_{\mathbf{g}+\delta\mathbf{e}_j}\}} - n_{\mathbf{v}}\right|\right]. \tag{44}$$

Moreover, for fixed $\mathbf{v} \in \pi_j(G)$, $\sum_{\mathbf{g}\in G\cap l_{\mathbf{e}_j,\mathbf{v}}} \mathbb{1}_{\{Z_{\mathbf{g}} \neq Z_{\mathbf{g}+\delta\mathbf{e}_j}\}}$ approaches $n_{\mathbf{v}}$ from below, and both quantities take values in $\mathbb{N}_0$. If the two quantities are distinct, then there must be two elements of $L_X(u) \cap [0,1]^d \cap l_{\mathbf{e}_j,\mathbf{v}}$ with a spacing of less than δ. Let $A_1^{(\mathbf{v})}$ denote the event $\big\{\sum_{\mathbf{g}\in G\cap l_{\mathbf{e}_j,\mathbf{v}}} \mathbb{1}_{\{Z_{\mathbf{g}} \neq Z_{\mathbf{g}+\delta\mathbf{e}_j}\}} \neq n_{\mathbf{v}}\big\}$. With $(\cdot)_j$ denoting the projection onto the j^{th} component, note that

$$A_1^{(\mathbf{v})} \subset \bigcup_{i=0}^{M^{(\delta)}-1} \Big\{ \mathrm{Card}\Big([i\delta, (i+2)\delta] \cap \big(L_X(u) \cap l_{\mathbf{e}_j,\mathbf{v}}\big)_j\Big) \geq 2 \Big\},$$

since any two points in $\mathbb{R}$ with a spacing of less than δ will be contained in an interval $[i\delta, (i+2)\delta]$ for some $i \in \mathbb{Z}$. It follows from Assumption ($\mathcal{A}1$), Lemma 4.1 with $p = 2$, and the definition of $M^{(\delta)}$ that $\mathbb{P}(A_1^{(\mathbf{v})}) = O(\delta)$. This fact, with Hölder's inequality yields for the ε defined in ($\mathcal{A}2$),

$$\begin{aligned}
\mathbb{E}\left[\left| \sum_{\mathbf{g}\in G\cap l_{\mathbf{e}_j,\mathbf{v}}} \mathbb{1}_{\{Z_{\mathbf{g}} \neq Z_{\mathbf{g}+\delta\mathbf{e}_j}\}} - n_{\mathbf{v}}\right|\right] &= \mathbb{E}\left[\left| \sum_{\mathbf{g}\in G\cap l_{\mathbf{e}_j,\mathbf{v}}} \mathbb{1}_{\{Z_{\mathbf{g}} \neq Z_{\mathbf{g}+\delta\mathbf{e}_j}\}} - n_{\mathbf{v}}\right| \mathbb{1}_{A_1^{(\mathbf{v})}}\right] \\
&\leq \mathbb{E}[\, n_{\mathbf{v}}\ \mathbb{1}_{A_1^{(\mathbf{v})}}\,] \leq \big(\mathbb{E}[n_{\mathbf{v}}^{1/\varepsilon}]\big)^{\varepsilon}\ \mathbb{P}(A_1^{(\mathbf{v})})^{1-\varepsilon} = O(\delta^{1-\varepsilon}).
\end{aligned}$$

By the stationarity of X, each of the $(M^{(\delta)})^{d-1}$ terms in (44) is identical, and as $M^{(\delta)}\delta \leq 1+\delta$, (42) follows immediately. We continue by showing (43). For $\mathbf{v} \in \mathrm{vect}(\mathbf{e}_j^{\perp})$, define the event $A_2^{(\mathbf{v})} := \Big\{ \sup_{\mathbf{s}\in \mathbf{v}+\pi_j([0,\delta]^d)} |n_{\mathbf{v}} - n_{\mathbf{s}}| > 0 \Big\}$. Then, the triangle inequality gives

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\delta^{d-1}\sum_{\mathbf{v}\in\pi_j(G)} n_{\mathbf{v}} - \int_{\mathrm{vect}(\mathbf{e}_j^{\perp})\cap[0,1]^d} n_{\mathbf{s}}\mathrm{d}\mathbf{s}\Big|\Big] \leq \delta^{d-1}\sum_{\mathbf{v}\in\pi_j(G)} \mathbb{E}\Big[\Big|n_{\mathbf{v}} - \delta^{1-d}\int_{\mathbf{v}+\pi_j([0,\delta]^d)} n_{\mathbf{s}}\ \mathrm{d}\mathbf{s}\Big|\Big] \\
&= \delta^{d-1}\sum_{\mathbf{v}\in\pi_j(G)} \mathbb{E}\Big[\Big|n_{\mathbf{v}} - \delta^{1-d}\int_{\mathbf{v}+\pi_j([0,\delta]^d)} n_{\mathbf{s}}\ \mathrm{d}\mathbf{s}\Big|\mathbb{1}_{A_2^{(\mathbf{v})}}\Big] \\
&\leq \delta^{d-1}\sum_{\mathbf{v}\in\pi_j(G)} \Big(\mathbb{E}\Big[\Big|n_{\mathbf{v}} - \delta^{1-d}\int_{\mathbf{v}+\pi_j([0,\delta]^d)} n_{\mathbf{s}}\ \mathrm{d}\mathbf{s}\Big|^{\frac{1}{\varepsilon}}\Big]\Big)^{\varepsilon}\mathbb{P}(A_2^{(\mathbf{v})})^{1-\varepsilon} \\
&\leq \delta^{d-1}\sum_{\mathbf{v}\in\pi_j(G)} \Big(\big(\mathbb{E}\big[n_{\mathbf{0}}^{\frac{1}{\varepsilon}}\big]\big)^{\varepsilon} + \Big(\mathbb{E}\Big[\Big|\delta^{1-d}\int_{\mathbf{v}+\pi_j([0,\delta]^d)} n_{\mathbf{s}}\ \mathrm{d}\mathbf{s}\Big|^{\frac{1}{\varepsilon}}\Big]\Big)^{\varepsilon}\Big)\mathbb{P}(A_2^{(\mathbf{v})})^{1-\varepsilon} \\
&\leq \delta^{d-1}\sum_{\mathbf{v}\in\pi_j(G)} \Big(\big(\mathbb{E}\big[n_{\mathbf{0}}^{\frac{1}{\varepsilon}}\big]\big)^{\varepsilon} + \Big(\delta^{1-d}\int_{\mathbf{v}+\pi_j([0,\delta]^d)} \mathbb{E}\big[n_{\mathbf{s}}^{\frac{1}{\varepsilon}}\big]\ \mathrm{d}\mathbf{s}\Big)^{\varepsilon}\Big)\mathbb{P}(A_2^{(\mathbf{v})})^{1-\varepsilon} \\
&\leq \delta^{d-1}\sum_{\mathbf{v}\in\pi_j(G)} 2\big(\mathbb{E}\big[n_{\mathbf{0}}^{\frac{1}{\varepsilon}}\big]\big)^{\varepsilon}\mathbb{P}(A_2^{(\mathbf{v})})^{1-\varepsilon},
\end{aligned}
$$

where we used Jensen inequality for the penultimate inequality and the fact that $\sigma_{d-1}(\pi_j([0,\delta]^d)) = \delta^{d-1}$ for the last inequality. Of the $(M^{(\delta)})^{d-1}$ terms in the sum over $\pi_j(G)$, there are $(M^{(\delta)}-1)^{d-1}$ terms such that $\mathbf{v}+\pi_j([0,\delta]^d) \subset \pi_j([0,1]^d)$. For these terms, by stationarity, $\mathbb{P}(A_2^{(\mathbf{v})}) = \mathbb{P}(A_2^{(\mathbf{0})}) \leq K_1\delta$ for some $K_1 \in \mathbb{R}^+$ by Lemma 4.3. For the remaining ${M^{(\delta)}}^{d-1} - (M^{(\delta)}-1)^{d-1}$ terms in the sum, the bound $\mathbb{P}(A_2^{(\mathbf{v})}) \leq 1$ suffices. Thus,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\delta^{d-1}\sum_{\mathbf{v}\in\pi_j(G)} n_{\mathbf{v}} - \int_{\mathrm{vect}(\mathbf{e}_j^{\perp})\cap[0,1]^d} n_{\mathbf{s}}\mathrm{d}\mathbf{s}\Big|\Big] \\
&\qquad \leq 2\delta^{d-1}\Big((M^{(\delta)}-1)^{d-1}(K_1\delta)^{1-\varepsilon} + \big({M^{(\delta)}}^{d-1} - (M^{(\delta)}-1)^{d-1}\big)\Big)\big(\mathbb{E}\big[|n_{\mathbf{0}}|^{\frac{1}{\varepsilon}}\big]\big)^{\varepsilon} = O\big(\delta^{1-\varepsilon}\big)
\end{aligned}
$$

and (43) holds which completes the proof.

5.2 Auxiliary results for the proof of the joint central limit theorem

Analogously to (36), the following result provides an L^1 control of the approximation error of the first coordinate of I_1 (*i.e.*, the estimated volume) in the proof of Theorem 3.1. To this end, notice that we cannot directly apply an approximation inequality such as Proposition 4 in Biermé and Desolneux (2021) as the function $y \mapsto \mathbb{1}_{\{y\geq u\}}$ appearing in the definition of the volume in (7) is not Lipschitz.

Proposition 5.1. *Let X be an isotropic random field satisfying Assumption (A0). Let $C_d^T(u)$ as in (7) and $\widehat{C}_d^{(\delta,T)}(u)$ as in (30). It holds that*

$$
\mathbb{E}\Big[\Big|\widehat{C}_d^{(\delta,T)}(u) - C_d^T(u)\Big|\Big] \leq K\delta, \tag{45}
$$

where $K \in \mathbb{R}^+$ is a constant independent of δ and $\sigma_d(T)$.

Remark 6. Proposition 5.1 provides a control of the discretization error in the computation of the volume. This error is of the same order as in Proposition 3.1 (item *ii*). This control can be greatly improved by asking more stringent assumptions: *e.g.*, if X is quasi-associated (see Bulinski and Shashkin (2007) and Bulinski et al. (2012)) and under a decay assumption for its correlation function. In this setting, the bound in (45) can be improved to get an upper bound in L^2 with a faster rate in $\delta(\sigma_d(T))^{-1}$ instead of δ. The interested reader is referred to the Appendix A.1.

Proof of Proposition 5.1. Remark that

$$\begin{aligned}\left|\widehat{C}_d^{(\delta,T)}(u) - C_d^T(u)\right| &= \left|\frac{\delta^d}{\sigma_d(T)} \sum_{t\in\mathbb{G}(\delta,T)} \left(\mathbb{1}_{\{X(t)\geq u\}} - \frac{1}{\delta^d}\int_{t+[0,\delta)^d} \mathbb{1}_{\{X(s)\geq u\}}\,\mathrm{d}s\right)\right| \\ &= \left|\frac{1}{\sigma_d(T)} \sum_{t\in\mathbb{G}(\delta,T)} \int_{t+[0,\delta)^d} \left(\mathbb{1}_{\{X(t)\geq u\}} - \mathbb{1}_{\{X(s)\geq u\}}\right)\mathrm{d}s\right| \\ &\leq \frac{1}{\sigma_d(T)} \sum_{t\in\mathbb{G}(\delta,T)} \int_{t+[0,\delta)^d} \left|\mathbb{1}_{\{X(t)\geq u\}} - \mathbb{1}_{\{X(s)\geq u\}}\right| \mathrm{d}s.\end{aligned}$$

Therefore, by Lemma 4.2, it holds that

$$\begin{aligned}\mathbb{E}\left[\left|\widehat{C}_d^{(\delta,T)}(u) - C_d^T(u)\right|\right] &\leq \frac{1}{\sigma_d(T)} \sum_{t\in\mathbb{G}(\delta,T)} \int_{t+[0,\delta)^d} \Big(\mathbb{P}\left(X(t) < u \leq X(s)\right) + \mathbb{P}\left(X(t) \geq u > X(s)\right)\Big)\,\mathrm{d}s \\ &\leq \frac{1}{\sigma_d(T)} \sum_{t\in\mathbb{G}(\delta,T)} \int_{t+[0,\delta)^d} 2C_{d-1}^*(u)\|t-s\|_2\,\mathrm{d}s \leq 2\sqrt{d}C_{d-1}^*(u)\delta.\end{aligned}$$

□

Corollary 5.1 below describes the behaviour of the second coordinate of the term I_4 in our main Theorem 3.1, by using the second order approximation in Theorem 2.1 for the specific hypercubic point-referenced d-honeycomb $\delta\dot{\mathcal{G}}$ in Section 3.1.

Corollary 5.1. *Consider a discrete regular hypercubic grid $\mathbb{G}(\delta,T)$ as in (29). Let X be an isotropic random field satisfying Assumptions (A0), (A1) and (A2) for some $\varepsilon \in (0,1)$. Let $C_{d-1}^*(u)$ be as in (8) and $\widehat{C}_{d-1}^{(\delta,T_N)}(u)$ as in (31). Then, it holds that*

$$\sqrt{\sigma_d(T_N)}\left(\mathbb{E}[\widehat{C}_{d-1}^{(\delta,T_N)}(u)] - \frac{2d}{\beta_d}C_{d-1}^*(u)\right) \to 0$$

for $N\delta \to \infty$, $(N\delta)^{d/2}\delta^{1-\varepsilon} \to 0$ with β_d as in (14) and ε as in Assumption (A2).

Proof. It is a direct consequence of (31) the stationarity and isotropy of X, Equation (15) in Theorem 2.1, and the fact that by construction $\mathrm{Card}(\mathbb{G}(\delta,T)) = (2N)^d$ and $\sigma_d(T_N) = (2N\delta)^d$. □

Finally, we focus on the term I_2 in the proof of Theorem 3.1. The following theorem establishes a joint central limit theorem for the vector $\left(C_d^{T_n}(u), \widetilde{C}_{d-1}^{T_n}(u)\right)$ in the case where T_n is a sequence of hypercubes in $\mathbb{R}^d$ such that $T_n \nearrow \mathbb{R}^d$ as $n \to \infty$. Our result is based on techniques used in Iribarren (1989).

Theorem 5.1. *Let u be a fixed level in $\mathbb{R}$. Assume that X is strongly mixing as in Definition 3.1 such that for some $\eta > 0$, the mixing satisfies the rate condition*

$$\sum_{r=1}^{+\infty} r^{3d-1} \alpha(r)^{\frac{\eta}{2+\eta}} < +\infty,$$

and $\mathbb{E}[\sigma_{d-1}\big(L_X(u) \cap [0,1]^d\big)^{2+\eta}] < +\infty$. Let $C^{T_n}(u) := \Big(C_d^{T_n}(u), \widetilde{C}_{d-1}^{T_n}(u)\Big)^t$. Let $(T_n)_{n\geq 1}$ be a sequence of hypercubes in $\mathbb{R}^d$ such that $\sigma_d(T_n) = (2n)^d$. Then there exists a finite covariance matrix $\Sigma(u)$ such that, if $\Sigma(u) > 0$,

$$\sqrt{\sigma_d(T_n)}\,\big(C^{T_n}(u) - \mathbb{E}[C^{T_n}(u)]\big) \xrightarrow{\mathcal{L}} \mathcal{N}_2\big(0, \Sigma(u)\big),$$

as $n \to \infty$.

Proof. Let $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$ and $V_{\mathbf{i}}(1)$ be as in Section 3.1. Write

$$\xi_{\mathbf{i}} := t_1 \int_{L_X(u)\cap V_{\mathbf{i}}(1)} \frac{\|\nabla X(s)\|_1}{\|\nabla X(s)\|_2}\, \sigma_{d-1}(\mathrm{d}s) + t_2 \sigma_d\big(E_X(u) \cap V_{\mathbf{i}}(1)\big),$$

for $\mathbf{i} \in \mathbb{Z}^d$, which makes $\xi := \{\xi_{\mathbf{i}} : \mathbf{i} \in \mathbb{Z}^d\}$ a stationary random field on $\mathbb{Z}^d$. It is straightforward to see that ξ inherits the mixing property of the random field X and that the mixing coefficients of ξ satisfy $\alpha_\xi(n + \lceil\sqrt{d}\rceil) \leq \alpha_X(n)$ for all $n \in \mathbb{N}^+$, since the diameter of $V_{\mathbf{i}}(1)$ is $\sqrt{d}$. Notice also that $|\xi_{\mathbf{i}}| \leq ||\mathbf{t}||_2\big(\sqrt{d}\sigma_{d-1}\big(L_X(u) \cap V_{\mathbf{i}}(1)\big) + \sigma_d\big(E_X(u) \cap V_{\mathbf{i}}(1)\big)\big) \leq ||\mathbf{t}||_2\big(\sqrt{d}\sigma_{d-1}\big(L_X(u) \cap V_{\mathbf{i}}(1)\big) + 1\big)$ almost surely, so the $2+\eta$ moment of $\xi_{\mathbf{i}}$ is finite. Finally, since

$$\langle \mathbf{t}, C^{T_n}(u)\rangle = \sum_{\mathbf{i}\in\mathbb{Z}^d} \xi_{\mathbf{i}}\, \mathbb{1}_{\{V_{\mathbf{i}}(1) \subset T_n\}},$$

the proof is completed with an application of Proposition 1 and Lemma 1 in Iribarren (1989) and the Cramér-Wold device. □

Acknowledgments: This work has been supported by the French government, through the 3IA Côte d'Azur Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR-19-P3IA-0002. This work has been partially supported by the project ANR MISTIC (ANR-19-CE40-0005).

References

Aaron, C., Cholaquidis, A., and Fraiman, R. (2020). Surface and length estimation based on Crofton's formula. *Preprint arXiv 2007.08484*.

Abaach, M., Biermé, H., and Di Bernardino, E. (2021). Testing marginal symmetry of digital noise images through the perimeter of excursion sets. *Electronic Journal of Statistics*, 15(2):6429–6460.

Adler, R. J. and Taylor, J. E. (2007). *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York.

Adler, R. J. and Taylor, J. E. (2011). *Topological complexity of smooth random functions*, volume 2019 of *Lecture Notes in Mathematics*. Springer, Heidelberg. Lectures from the 39th Probability Summer School held in Saint-Flour, 2009, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

Azaïs, J. M. and Wschebor, M. (2009). *Level sets and extrema of random processes and fields*. John Wiley & Sons.

Berzin, C. (2021). Estimation of local anisotropy based on level sets. *Electronic Journal of Probability*, 26:1–72.

Berzin, C., Latour, A., and León, J. R. (2017). *Kac-Rice formulas for random fields and theirs applications in: random geometry, roots of random polynomials and some engineering problems*. Ediciones IVIC.

Beutler, F. J. and Leneman, O. A. Z. (1966). The theory of stationary point processes. *Acta Mathematica*, 116:159 – 190.

Biermé, H. and Desolneux, A. (2021). The effect of discretization on the mean geometry of a 2D random field. *Annales Henri Lebesgue*, 4:1295–1345.

Biermé, H., Di Bernardino, E., Duval, C., and Estrade, A. (2019). Lipschitz-Killing curvatures of excursion sets for two-dimensional random fields. *Electronic Journal of Statistics*, 13(1):536–581.

Bolthausen, E. (1982). On the Central Limit Theorem for Stationary Mixing Random Fields. *The Annals of Probability*, 10(4):1047 – 1050.

Boos, D. D. (1985). A converse to Scheffe's theorem. *The Annals of Statistics*, pages 423–427.

Bradley, R. C. (2005). Basic properties of strong mixing conditions. A survey and some open questions. *Probability surveys*, 2:107–144.

Bulinski, A. (2010). Central limit theorem for random fields and applications. In *Advances in Data Analysis*, pages 141–150. Springer.

Bulinski, A. and Shashkin, A. (2007). *Limit theorems for Associated Random Fields and Related Systems*, volume 10. World Scientific.

Bulinski, A., Spodarev, E., and Timmermann, F. (2012). Central limit theorems for the excursion set volumes of weakly dependent random fields. *Bernoulli*, 18(1):100–118.

Bulinski, A. V. and Shabanovich, E. (1998). Asymptotical behaviour for some functionals of positively and negatively dependent random fields. *Fundamentalnaya i Prikladnaya Matematika*, 4(2):479–492.

Cabaña, E. M. (1987). Affine processes: a test of isotropy based on level sets. *SIAM Journal on Applied Mathematics*, 47(4):886–891.

Casaponsa, B., Crill, B., Colombo, L., Danese, L., Bock, J., Catalano, A., Bonaldi, A., Basak, S., Bonavera, L., Coulais, A., et al. (2016). Planck 2015 results: Xvi. Isotropy and statistics of the CMB.

Cotsakis, R., Di Bernardino, E., and Opitz, T. (2022). On the perimeter estimation of pixelated excursion sets of 2D anisotropic random fields. *Preprint hal-03582844v2*.

Dedecker, J. (1998). A central limit theorem for stationary random fields. *Probability Theory and Related Fields*, 110(3):397–426.

Di Bernardino, E. and Duval, C. (2022). Statistics for Gaussian random fields with unknown location and scale using Lipschitz-Killing curvatures. *Scandinavian Journal of Statistics*, 49(1):143–184.

Dombry, C. and Eyi-Minko, F. (2012). Strong mixing properties of max-infinitely divisible random fields. *Stochastic Processes and their Applications*, 122(11):3790–3811.

Doukhan, P. (1994). Mixing. In *Mixing*, pages 15–23. Springer.

Gott, J. R., Colley, W. N., Park, C.-G., Park, C., and Mugnolo, C. (2007). Genus topology of the cosmic microwave background from the WMAP 3-year data. *Monthly Notices of the Royal Astronomical Society*, 377(4):1668–1678.

Gott, J. R., Hambrick, D. C., Vogeley, M. S., Kim, J., Park, C., Choi, Y.-Y., Cen, R., Ostriker, J. P., and Nagamine, K. (2008). Genus topology of structure in the sloan digital sky survey: Model testing. *The Astrophysical Journal*, 675(1):16.

Iribarren, I. (1989). Asymptotic behaviour of the integral of a function on the level set of a mixing random field. *Probability and Mathematical Statistics*, 10(1):45 – 56. MR-990398.

Kratz, M. and Vadlamani, S. (2018). Central limit theorem for Lipschitz–Killing curvatures of excursion sets of Gaussian random fields. *Journal of Theoretical Probability*, 31(3):1729–1758.

Kratz, M. F. (2006). Level crossings and other level functionals of stationary Gaussian processes. *Probability Surveys*, 3:230–288.

Leadbetter, M., Lindgren, G., and Rootzén, H. (1983). *Extremes and related properties of random sequences and processes*. Springer Series in Statistics. Springer, Germany.

Lehmann, G. and Legland, D. (2012). Efficient N-Dimensional surface estimation using Crofton formula and run-length encoding. *The Insight Journal, INRA*.

Lindgren, G. (2000). Wave analysis by Slepian models. *Probabilistic engineering mechanics*, 15(1):49–57.

Longuet-Higgins, M. S. (1957). The statistical analysis of a random, moving surface. *Philosophical Transactions of the Royal Society A*, 249(966):321–387.

Meschenmoser, D. and Shashkin, A. (2011). Functional central limit theorem for the volume of excursion sets generated by associated random fields. *Statistics & Probability Letters*, 81(6):642–646.

Meschenmoser, D. and Shashkin, A. (2013). Functional central limit theorem for the measures of level surfaces of the Gaussian random field. *Theory of Probability & Its Applications*, 57(1):162–172.

Miller, E. (1999). Alternative Tilings for Improved Surface Area Estimates by Local Counting Algorithms. *Computer Vision and Image Understanding*, 74(3):193–211.

Müller, D. (2017). A central limit theorem for Lipschitz–Killing curvatures of Gaussian excursions. *Journal of Mathematical Analysis and Applications*, 452(2):1040–1081.

Reddy, T. R., Vadlamani, S., and Yogeshwaran, D. (2018). Central Limit Theorem for Exponentially Quasi-local Statistics of Spin Models on Cayley Graphs. *Journal of Statistical Physics*, (173):941–984.

Rogers, C. A. (1998). *Hausdorff measures*. Cambridge University Press.

Schmalzing, J. and Górski, K. M. (1998). Minkowski functionals used in the morphological analysis of cosmic microwave background anisotropy maps. *Monthly Notices of the Royal Astronomical Society*, 297(2):355–365.

Schneider, R. and Weil, W. (2008). *Stochastic and integral geometry*. Probability and its Applications. Springer-Verlag, Berlin.

Shashkin, A. P. (2002). Quasi-associatedness of a Gaussian system of random vectors. *Russian Mathematical Surveys*, 57(6):1243–1244.

Spodarev, E. (2014). Limit theorems for excursion sets of stationary random fields. In *Modern stochastics and applications*, volume 90 of *Springer Optimization and Its Applications*, pages 221–241. Springer, Cham.

Sweeting, T. (1986). On a converse to Scheffé's theorem. *The Annals of Statistics*, 14(3):1252–1256.

Thäle, C. and Yukich, J. E. (2016). Asymptotic theory for statistics of the Poisson–Voronoi approximation. *Bernoulli*, 22(4):2372–2400.

Wschebor, M. (2006). *Surfaces aléatoires: mesure géométrique des ensembles de niveau*, volume 1147. Springer.

A Supplementary material

A.1 Proposition 5.1 improved under the assumption of quasi-associativity

The obtained bound in Proposition 5.1 comes without any assumption on the correlation function of the field X. Indeed the proof Proposition 5.1 only focuses on the control of the pixelisation error in a fixed domain. The growing domain T_N is considered *a posteriori* without taking into account the asymptotic spatial independency between pixels with respect to their Euclidean distance. Obviously the bound in (36) only depends on δ and not $\sigma_d(T)$.
In this supplementary material we aim to improve the bound in Proposition 5.1 by adding dependency conditions on the field X and considering a L^2 control. The proposed L^2 control in Proposition A.1 below can be used to transfer for instance a multilevel central limit theorem provided for $C_d^T(u)$ in (7) to $\widehat{C}_d^{(\delta,T)}(u)$ in (30) (see Corollary A.1 below) under to the sole assumptions $N\delta \to \infty$ and $\delta \to 0$. To improve the result of Proposition 5.1 we will use that distant pixels have a small correlation.

Proposition A.1. *Let X a random field satisfying Assumption (A0) with $\mathbb{V}(X(\mathbf{0})) = \upsilon^2$, $\upsilon > 0$. Suppose moreover that $X(\mathbf{0})$ has density bounded by M and that*

(H1) X is quasi-associated, i.e.,

$$|\mathrm{Cov}(f(X_I), g(X_J)) \leq |\mathrm{Lip}(f)\mathrm{Lip}(g)| \sum_{s\in I}\sum_{t\in J} |\mathrm{Cov}(X(s), X(t))|,$$

for all disjoint finite sets $I, J \subset T$ and any Lipschitz functions $f : \mathbb{R}^I \mapsto \mathbb{R}$ and $g : \mathbb{R}^J \mapsto \mathbb{R}$.
Assume the following decay condition for the correlation function $\rho(t) = \mathrm{corr}(X(\mathbf{0}), X(t))$, $t \in \mathbb{R}^d$:

(H2) $|\rho(t)| \leq (1+\|t\|)^{-\gamma}$, $\gamma > 3d$.

Then, it holds that

$$\mathbb{V}\left(C_d^T(u) - \widehat{C}_d^{(\delta,T)}(u)\right) \leq \kappa \frac{\delta}{\sigma_d(T)}, \tag{46}$$

where κ *is a constant only depending on* d, M, υ^2, $C_{d-1}^*(u)$ *in* (8) *and* γ.

Remark 7 (Quasi-associativity property)**.** Assumption (H1) is called *quasi-associativity property* and is classically used to get the asymptotic normality of $C_d^T(u)$ in (7) (see for instance Bulinski et al. (2012), Spodarev (2014)). Bulinski and Shabanovich (1998) shows that positively associated random fields are quasi-associated, Shashkin (2002) ensures that Gaussian fields are quasi-associated and Bulinski and Shashkin (2007) that shot noise fields are also quasi-associated (see Theorem 1.3.8). Notice that if X is a stationary Gaussian random field on $\mathbb{R}^d$, then Bulinski et al. (2012) shows that Assumption (H2) on the covariance function can be softened as $\gamma > d$. The interested reader is referred to Lemma 2 and Equation (34) in Bulinski et al. (2012) together with Lemma A.1, postponed and proved below.

Remark 8. The control (46) on the variance provides information about the amplitude of the error made when replacing the uncomputable quantity $C_d^T(u)$ in (7) with its accessible version $\widehat{C}_d^{(\delta,T)}(u)$ in (30). Two regimes can be considered for this variance to be small: *i)* $T_N \nearrow \mathbb{R}^d$, *i.e.* $N\delta \to \infty$, and $\delta \to \ell \geq 0$ or *ii)* $\delta \to 0$ with the domain T held constant. Both situations are of interest, but if one wishes to transfer a central limit result for growing domain T from the true volume $C_d^T(u)$ to $\widehat{C}_d^{(\delta,T)}(u)$ one should require both $T_N \nearrow \mathbb{R}^d$ and $\delta \to 0$ (see Corollary A.1 below).

Proof of Proposition A.1. Let $\mathbf{i} \in \mathbb{Z}^d$. Recall that the reference point of $\delta\dot{\mathcal{G}}$ (see Section (3.1)) is denoted $t_{\mathbf{i},\delta} := (\delta i_1, \ldots, \delta i_d)$. For $\mathbf{i}$ and $\mathbf{j}$ in $\left[|-N, N-1|\right]^d$, $\mathbf{i} \neq \mathbf{j}$ stands for $\exists\, k \in \{1,\ldots,d\}$, $i_k \neq j_k$. Similarly, $\mathbf{i} > \mathbf{j}$ (resp. $\mathbf{i} < \mathbf{j}$) means that $\forall\, k \in \{1,\ldots,d\}$ $i_k > j_k$ (resp. $i_k < j_k$), $\mathbf{i} = \mathbf{i}'$ means $i_1 = i_1', \ldots, i_d = i_d'$ and $\sum_{\mathbf{i}=m}^M$ stands for $\sum_{i_1=m}^M \ldots \sum_{i_d=m}^M$. We study the remainder term $R_\delta := \sigma_d(T)\left(\widehat{C}_d^{(\delta,T)}(u) - C_d^T(u)\right)$ given by

$$R_\delta = \int_T \mathbb{1}_{\{X(t)\geq u\}} \mathrm{d}t - \sum_{t\in\mathbb{G}(\delta,T)} \mathbb{1}_{\{X(t)\geq u\}} \delta^d = \sum_{\mathbf{i}=-N}^{N-1} \int_{V_\mathbf{i}(\delta)} \left(\mathbb{1}_{\{X(t)\geq u\}} - \mathbb{1}_{\{X(t_{\mathbf{i},\delta})\geq u\}}\right) \mathrm{d}t,$$

from Equation (30). Denote $d_\mathbf{i}(t) := \mathbb{1}_{\{X(t)\geq u\}} - \mathbb{1}_{\{X(t_{\mathbf{i},\delta})\geq u\}}$, for $t \in V_\mathbf{i}(\delta)$. Note that by stationarity this term is centered and $\mathbb{V}(R_\delta) = \mathbb{E}[R_\delta^2]$. We write,

$$\mathbb{E}[R_\delta^2] = \sum_{\mathbf{i},\mathbf{i}'=-N}^{N-1} \mathbb{E}\left[\int_{V_\mathbf{i}(\delta)} \int_{V_{\mathbf{i}'}(\delta)} d_\mathbf{i}(t) d_{\mathbf{i}'}(t') \mathrm{d}t\, \mathrm{d}t'\right],$$

where $d_\mathbf{i}(t) := \mathbb{1}_{\{X(t)\geq u\}} - \mathbb{1}_{\{X(t_{\mathbf{i},\delta})\geq u\}}$, for $t \in V_\mathbf{i}(\delta)$. We now study the quantities

$$\begin{aligned}
\mathbb{E}[d_\mathbf{i}(t) d_{\mathbf{i}'}(t')] &= \mathbb{E}\left[(\mathbb{1}_{\{X(t)\geq u\}} - \mathbb{1}_{\{X(t_{\mathbf{i},\delta})\geq u\}})(\mathbb{1}_{\{X(t')\geq u\}} - \mathbb{1}_{\{X(t_{\mathbf{i}',\delta})\geq u\}})\right] \\
&= \mathrm{Cov}(\mathbb{1}_{\{X(t-t')\geq u\}}, \mathbb{1}_{\{X(\mathbf{0})\geq u\}}) + \mathrm{Cov}(\mathbb{1}_{\{X(t_{\mathbf{i},\delta}-t_{\mathbf{i}',\delta})\geq u\}}, \mathbb{1}_{\{X(\mathbf{0})\geq u\}}) \\
&\quad - \mathrm{Cov}(\mathbb{1}_{\{X(t-t_{\mathbf{i}',\delta})\geq u\}}, \mathbb{1}_{\{X(\mathbf{0})\geq u\}}) - \mathrm{Cov}(\mathbb{1}_{\{X(t_{\mathbf{i},\delta}-t')\geq u\}}, \mathbb{1}_{\{X(\mathbf{0})\geq u\}}),
\end{aligned}$$

for $t \in V_{\mathbf{i}}(\delta)$, $t' \in V_{\mathbf{i}'}(\delta)$. To connect the later quantity to the covariance function of X we rely on Lemma 7.3.5. in Bulinski and Shashkin (2007) (see Lemma A.1 where a revised proof is provided for sake of clarity). Then, we obtain,

$$|\,\mathbb{E}[d_{\mathbf{i}}(t)d_{\mathbf{i}'}(t')]| \leq K_M\big(|r(t-t')|^{1/3} + |r(t_{\mathbf{i},\delta} - t_{\mathbf{i}',\delta})|^{1/3} + |r(t - t_{\mathbf{i}',\delta})|^{1/3} + |r(t' - t_{\mathbf{i},\delta})|^{1/3}\big).$$

Under Assumption $(\mathcal{H}2)$ and using the structure of the approximating grid, it follows that

$$\mathbb{E}[d_{\mathbf{i}}(t)d_{\mathbf{i}'}(t')] \leq 4\,\sigma^{2/3} K_M \big(1 + \|t_{\mathbf{i}',\delta} - t_{\mathbf{i}+\mathbf{1},\delta}\|\big)^{-\gamma/3} \quad \text{if } \mathbf{i}' > \mathbf{i}. \tag{47}$$

The latter bound gets small whenever $\mathrm{dist}(t_{\mathbf{i}+\mathbf{1},\delta}, t_{\mathbf{i}'+\mathbf{1}})$ is large. Hereafter, we propose another control of this term when this distance is small. It holds that

$$\text{for } t \in V_{\mathbf{i}}(\delta), \quad d_{\mathbf{i}}(t) = \begin{cases} 1 & \text{if } X(t_{\mathbf{i},\delta}) \leq u < X(t), \\ -1 & \text{if } X(t) \leq u < X(t_{\mathbf{i},\delta}), \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the hypothesis of stationarity that $\mathbb{E}[d_{\mathbf{i}}(t)] \leq \mathbb{P}(|d_{\mathbf{i}}(t)| = 1) = \mathbb{P}\big(X(t_{\mathbf{i},\delta}) \leq u < X(t)\big) + \mathbb{P}\big(X(t) \leq u < X(t_{\mathbf{i},\delta})\big)$. By Lemma 4.2 we obtain that

$$\mathbb{P}\big(X(\mathbf{0}) \leq u < X(t)\big) \leq C^*_{d-1}(u)\|t\|_2 \leq \sqrt{d}\delta C^*_{d-1}(u),$$

Combining with the definition of $d_{\mathbf{i}}(t)$ and the Cauchy-Schwarz inequality provides

$$\mathbb{E}[d_{\mathbf{i}}(t)d_{\mathbf{i}'}(t)] \leq \sqrt{\mathbb{E}[d_{\mathbf{i}}(t)^2]\,\mathbb{E}[d_{\mathbf{i}'}(t')^2]} \leq 4\sqrt{d}\delta C^*_{d-1}(u). \tag{48}$$

Therefore, using $(\mathcal{A}0)$ we derive that

$$\mathbb{V}(R_\delta) = \left(\sum_{\mathbf{i}=\mathbf{i}'=-N}^{N-1} + 2^d \sum_{\substack{\mathbf{i}'\geq\mathbf{i}=-N \\ \mathbf{i}\neq\mathbf{i}'}}^{N-1}\right) \int_{V_{\mathbf{i}}(\delta)} \int_{V_{\mathbf{i}'}(\delta)} \mathbb{E}\left[d_{\mathbf{i}}(t)d_{\mathbf{i}'}(s)\right] \mathrm{d}t\,\mathrm{d}s,$$

Recall that $\sigma_d(T_N) = (2\delta N)^d$ and that $|d_{\mathbf{i}}| \leq 1$ *a.s.*. Combining with (47) and (48), we obtain

$$\begin{aligned}
\mathbb{V}(R_\delta) &\leq \sum_{\mathbf{i}=-N}^{N-1} \delta^{2d} + 2^d\delta^{2d} \sum_{\substack{\mathbf{i}'\geq\mathbf{i}=-N \\ \mathbf{i}\neq\mathbf{i}'}}^{N-1} \frac{4\upsilon^{2/3}K_M}{\big(1 + \|t_{\mathbf{i}',\delta} - t_{\mathbf{i}+\mathbf{1},\delta}\|\big)^{\gamma/3}} \wedge \left\{4\sqrt{d}\delta C^*_{d-1}(u)\right\} \\
&= \delta^d\sigma_d(T_N) + \kappa\delta^d \frac{\sigma_d(T_N)}{N^d} \sum_{\substack{\mathbf{i}'\geq\mathbf{i}=-N \\ \mathbf{i}\neq\mathbf{i}'}}^{N-1} \{(1 + \|t_{\mathbf{i}',\delta} - t_{\mathbf{i}+\mathbf{1},\delta}\|)^{-\gamma/3} \wedge \delta\},
\end{aligned}$$

where κ is a constant depending on $(d, M, \upsilon^2, C^*_{d-1}(u))$. As for all $(\mathbf{i}, \mathbf{i}')$ it holds $\|t_{\mathbf{i}',\delta} - t_{\mathbf{i},\delta}\|^2 = \delta_N^2 \sum_{k=1}^d (i'_k - i_k)^2$, we get

$$\begin{aligned}
\frac{1}{N^d} \sum_{\substack{\mathbf{i}'\geq\mathbf{i}=-N \\ \mathbf{i}\neq\mathbf{i}'}}^{N-1} \left\{(1 + \|t_{\mathbf{i}',\delta} - t_{\mathbf{i}+\mathbf{1},\delta}\|)^{-\gamma/3} \wedge \delta_N\right\} &= \frac{1}{N^d} \sum_{\substack{\mathbf{i}'\geq\mathbf{i}=-N \\ \mathbf{i}\neq\mathbf{i}'}}^{N-1} \left\{\left(1 + \delta\sqrt{\sum_{k=1}^d (i'_k - i_k - 1)^2}\right)^{-\gamma/3} \wedge \delta\right\} \\
&= \sum_{\ell_1,\ldots,\ell_d=1}^{N-1} \left\{\left(1 + \delta\sqrt{\sum_{k=1}^d \ell_k^2}\right)^{-\gamma/3} \wedge \delta\right\}.
\end{aligned}$$

To bound this term, we use the following upper bound,

$$\left(1+\delta\sqrt{\sum_{k=1}^{d}\ell_k^2}\right)^{-\gamma/3}\wedge\delta\leq\begin{cases}\delta & \text{if } \sqrt{\sum_{k=1}^{d}\ell_k^2}\leq\left\lfloor\frac{1}{\delta}\right\rfloor,\\ \left(1+\delta\sqrt{\sum_{k=1}^{d}\ell_k^2}\right)^{-\gamma/3} & \text{otherwise.}\end{cases}$$

We derive that

$$\sum_{\ell_1,\dots,\ell_d=1}^{N-1}\left\{\left(1+\delta\sqrt{\sum_{k=1}^{d}\ell_k^2}\right)^{-\gamma/3}\wedge\delta\right\}\leq\delta\mathrm{Card}\Big\{\ell_1,\dots,\ell_d\in\{1,\dots,N-1\}^d,\ \sqrt{\sum_{k=1}^{d}\ell_k^2}\leq\left\lfloor\frac{1}{\delta}\right\rfloor\Big\}$$
$$+\sum_{\substack{\ell_1,\dots,\ell_d=1\\ \sqrt{\sum_{k=1}^{d}\ell_k^2}>\lfloor 1/\delta\rfloor}}^{N-1}\left(1+\delta\sqrt{\sum_{k=1}^{d}\ell_k^2}\right)^{-\gamma/3}=:S_1+S_2.$$

Furthermore, $S_1\leq\delta\lfloor\delta^{-1}\rfloor^d\leq\delta^{1-d}$ and

$$S_2=\sum_{\substack{\ell_1,\dots,\ell_d=1\\ \sqrt{\sum_{k=1}^{d}\ell_k^2}>\lfloor 1/\delta_N\rfloor}}^{N-1}\left(1+\delta\sqrt{\sum_{k=1}^{d}\ell_k^2}\right)^{-\gamma/3}=\sum_{\substack{p_1,\dots,p_d=0\\ \|\mathbf{p}\|>1}}^{\lfloor(N-1)\delta\rfloor}(1+\|\mathbf{p}\|)^{-\gamma/3},$$

where $S_2=O(1)$[1] whenever $\gamma>3d$ if $N\delta\to\infty$ and for every γ if $N\delta$ remains bounded. Gathering all these elements, we obtain (assuming additionally that $\gamma>3d$ if $N\delta\to\infty$),

$$\mathbb{V}(R_\delta)\leq\delta^d\sigma_d(T_{_N})+\kappa\,\delta^d\sigma_d(T_N)\big(\delta^{1-d}+S_2\big)=\kappa\,\delta\sigma_d(T_N)+O\left(\delta^d\sigma_d(T_N)\right).$$

It follows that, $\mathbb{V}(R_\delta)\leq\kappa\delta\sigma_d(T_N)$, where κ is a constant depending on d, M, v^2, $C^*_{d-1}(u)$ and γ only. Dividing by $\sigma_d(T)^2$ completes the proof using that $\mathbb{V}(R_\delta)=\sigma_d(T)^2\mathbb{V}\big(C_d^T(u)-\widehat{C}_d^{(\delta,T)}(u)\big)$. □

Lemma A.1. *For a quasi-associated random vector (U,V) such that U and V are square integrable and with densities bounded by $M>0$. Then, for all $(u,v)\in\mathbb{R}$ it holds*

$$|\mathrm{Cov}(\mathbb{1}_{U\geq u},\mathbb{1}_{V\geq v})|\leq 3\,a^{2/3}\,|\mathrm{Cov}(U,V)|^{1/3}:=K_M|\mathrm{Cov}(U,V)|^{1/3}.$$

Lemma A.1 is a slight modification of Lemma 7.3.5. in Bulinski and Shashkin (2007) and for sake of clarity, we provide below a proof.

[1]Indeed, using that in $\mathbb{R}^d$ there exists $0<c_d<C_d$ such that $c_d\|p\|_1\leq\|p\|_2\leq C_d\|p\|_1$ we obtain

$$\sum_{\mathbf{p}\geq 0}\frac{1}{(1+\|\mathbf{p}\|)^\alpha}\leq\sum_{\mathbf{p}\geq 0}\frac{1}{(1+c_d\|\mathbf{p}\|_1)^\alpha}\frac{1}{\sum_{r\geq 0}\ \sum_{\mathbf{p}\geq 0,\|\mathbf{p}\|_1=r}\frac{1}{(1+r)^\alpha}}=\sum_{r\geq 0}\frac{K_d(r)}{(1+r)^\alpha},$$

where $K_d(r):=\mathrm{Card}\{\mathbf{p}\geq 0,\|\mathbf{p}\|_1=r\}$. An immediate upper bound is given by $K_d(r)\leq r^{d-1}$, which lead to the sufficient condition $\alpha>d$ to get the convergence of the sum. Note that this condition should be necessary, to prove this one should use similar computations, $\|p\|_2\leq C_d\|p\|_1$ and show (most delicate part) that $K_d(r)\geq cr^{d-1}$ for some $1>c>0$.

Proof of Lemma A.1. If $\mathrm{Cov}(U,V)=0$, by Corollary 1.5.5 in Bulinski and Shashkin (2007) implies that U and V are independent and the bound is trivial. Else for any $\delta>0$ and the function $h_{\delta,u}$ for $u\in\mathbb{R}$ such that

$$h_{\delta,u}(t)=\begin{cases}0 & \text{if } t\le u-\delta,\\ \frac{1}{\delta}\big(t-(u-\delta)\big) & \text{if } t\in(u-\delta,u),\\ 1 & \text{if } t\ge u.\end{cases}$$

It follows from the triangle inequality that

$$\begin{aligned}\big|\mathrm{Cov}(\mathbb{1}_{U\ge u},\mathbb{1}_{V\ge v})\big| &\le \big|\mathrm{Cov}(h_{\delta,u}(U),h_{\delta,v}(V))\big|+\big|\mathrm{Cov}(\mathbb{1}_{U\ge u}-h_{\delta,u}(U),h_{\delta,v}(V))\big|\\ &\quad+\big|\mathrm{Cov}(\mathbb{1}_{U\ge u},\mathbb{1}_{V\ge v}-h_{\delta,v}(V))\big|\\ &\le \delta^{-2}|\mathrm{Cov}(U,V)|+4M\delta,\end{aligned}$$

where the last inequality follows from Theorem 1.5.3 of Bulinski and Shashkin (2007) to control the first term, the last two terms are controlled using that $\mathbb{1}_{U\ge u}-h_{\delta,u}(U)$ is nonzero for $U\in(u,u-\delta)$ and that the density of U is bounded by M. Minimizing in δ this last term, we find $\delta^*=\big(|\mathrm{Cov}(U,V)|/M\big)^{1/3}$. Then

$$\begin{aligned}\big|\mathrm{Cov}(\mathbb{1}_{U\ge u},\mathbb{1}_{V\ge v})\big| &\le \big(|\mathrm{Cov}(U,V)|/M\big)^{-2/3}|\mathrm{Cov}(U,V)|+4\,M\,\big(|\mathrm{Cov}(U,V)|/M\big)^{1/3}\\ &=3\,M^{2/3}\,|\mathrm{Cov}(U,V)|^{1/3}.\end{aligned}$$

□

Notice that in Lemma A.1, the factor 4 in Lemma 7.3.5. in Bulinski and Shashkin (2007) is improved to 2 as $-1\le\mathbb{1}_{U\ge u}-h_{\delta,u}(U)\le 1$ *a.s.*

Proposition A.1 can be used to transfer for instance multi-level central limit theorem provided for $C_d^T(u)$ in (7) to $\widehat{C}_d^{(\delta,T)}(u)$ in (30) (see Corollary A.1 below).

Corollary A.1. *Let X a random field satisfying assumptions in Proposition A.1. Let $\mathbf{u}=(u_1,\dots,u_m)^t\in\mathbb{R}^m$, $C_d^*(\mathbf{u})=(\mathbb{P}(X(\mathbf{0})\ge u_1),\dots,\mathbb{P}(X(\mathbf{0})\ge u_m))^t$ (see (9)) and $\widehat{C}_d^{(\delta,T)}(\mathbf{u})=\Big(\widehat{C}_d^{(\delta,T)}(u_1),\dots,\widehat{C}_d^{(\delta,T)}(u_m)\Big)^t$ (see Equation (30)). Then it holds that*

$$\sigma_d(T)^{1/2}\left(\widehat{C}_d^{(\delta,T)}(\mathbf{u})-C_d^*(\mathbf{u})\right)\xrightarrow{\mathcal{L}}\mathcal{N}(0,\Sigma(\mathbf{u})),\tag{49}$$

as $N\delta\to\infty$ and $\delta\to 0$, where $\Sigma(\mathbf{u})=(\upsilon_{ij}(\mathbf{u}))_{i,j=1}^m$ is an $(m\times m)$-matrix having the elements

$$\upsilon_{ij}(\mathbf{u})=\int_{\mathbb{R}^d}\mathrm{Cov}(\mathbb{1}_{\{X(\mathbf{0})\ge u_i\}},\mathbb{1}_{\{X(t)\ge u_j\}})\mathrm{d}t.\tag{50}$$

Proof of Corollary A.1. Recall that $\sigma_d(T_N)=(2N\delta)^d$ and $C_d^T(\mathbf{u})=\widehat{C}_d^{(\delta,T)}(\mathbf{u})+R_\delta(\mathbf{u})$, where $R_\delta(\mathbf{u}):=(R_\delta(u_1),\dots,R_\delta(u_m))$ with

$$R_\delta(u_k):=\frac{1}{\sigma_d(T_N)}\sum_{\mathbf{i}=-N}^{N-1}\int_{V_{\mathbf{i}}(\delta)}\Big(\mathbb{1}_{\{X(t)\ge u_k\}}-\mathbb{1}_{\{X(t_{\mathbf{i},\delta})\ge u_k\}}\Big)\mathrm{d}t,\quad\text{for }\ k\in\{1,\dots,m\}.$$

Then,

$$\sqrt{\sigma_d(T_N)}\left(\widehat{C}_d^{(\delta,T)}(\mathbf{u}) - C_d^*(\mathbf{u})\right) = \sqrt{\sigma_d(T_N)}\left(C_d^T(\mathbf{u}) - C_d^*(\mathbf{u})\right) - \sqrt{\sigma_d(T_N)}R_\delta(\mathbf{u})$$
$$:= \mathbf{I}_{1,N}(\mathbf{u}) - \mathbf{I}_{2,N}(\mathbf{u}).$$

Notice that, since $N\delta_N \to \infty$, T_N is a VH-growing sequence (see Definition 6 in Bulinski et al. (2012)). Then, using Theorem 2 in Bulinski et al. (2012), it holds that $\mathbf{I}_{1,N}(\mathbf{u}) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(0, \Sigma(\mathbf{u}))$, with $\xrightarrow{\mathcal{L}}$ the convergence in law and where $\Sigma(\mathbf{u}) = (\upsilon_{ij}(\mathbf{u}))_{i,j=1}^m$ is an $(m \times m)$-matrix having the elements $\upsilon_{ij}(\mathbf{u})$ as in (50). As for $k \in \{1, \ldots, m\}$, $\mathbb{E}[R_\delta(u_k)] = 0$, $\mathbb{V}(R_\delta(u_k)) = o(\sigma_d(T_N)^{-1})$, from Proposition A.1, then $\mathbf{I}_{2,N}(\mathbf{u}) \xrightarrow[N\to\infty]{\mathbb{P}} 0$. By Slutsky's Theorem, we get the result in (49). □

A.2 Examples

We begin with two simple examples that illustrate the use of the Crofton formula in Equation (5).

Example 1. Let $d = 2$, Equation (5) takes the form:

$$\sigma_1(M) = \frac{1}{4}\int_{-\infty}^{\infty}\int_0^{2\pi} \sigma_0(M \cap l_{\mathbf{s}(\theta),v})\ \mathrm{d}\theta\ \mathrm{d}v, \tag{51}$$

where $\mathbf{s}(\theta) := (\cos(\theta), \sin(\theta))$. Let M be a circle of radius R in $\mathbb{R}^2$. For all $\theta \in [0, 2\pi)$, the function $f_\theta(v) := \sigma_0(M \cap l_{\mathbf{s}(\theta),v})$ is equal to 2 on an interval of length $2R$, and 0 elsewhere. Therefore, we easily recover

$$\sigma_1(M) = \frac{1}{4}\int_0^{2\pi}\int_{-\infty}^{\infty} f_\theta(v)\ \mathrm{d}v\ \mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi} 4R\ \mathrm{d}\theta = 2\pi R.$$

Example 2. Let M be the boundary of a a square K with side-length a. As $\partial K = \bigcup_{i=1}^4 \mathrm{side}_i$, using the additivity of (51), it suffices to consider only a single side of the square. Without loss of generality, let us suppose that side_1 is horizontal. Then, the function $f_\theta(v) := \sigma_0(\mathrm{side}_1 \cap l_{\mathbf{s}(\theta),v})$ is equal to 1 on an interval of length $a|\sin\theta|$, and 0 elsewhere. Thus, we easily recover that

$$\sigma_1(\mathrm{side}_1) = \frac{1}{4}\int_0^{2\pi}\int_{-\infty}^{\infty} f_\theta(v)\ \mathrm{d}v\ \mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi} a|\sin\theta|\ \mathrm{d}\theta = \frac{a}{2}\int_0^{\pi}\sin\theta\ \mathrm{d}\theta = a.$$

We now present two classical examples where the density $C_{d-1}^*(u)$ in (8) can be explicitly obtained. The interested reader is referred to Exercises 6.2.c and 6.3 in Azaïs and Wschebor (2009) and to Biermé et al. (2019).

Example 3. Let $X = \{X(t), t \in \mathbb{R}^d\}$, be an isotropic Gaussian field, with zero mean, unit variance and second spectral moment λ finite satisfying Assumption ($\mathcal{A}0$). Then we get

$$C_{d-1}^*(u) = \sqrt{\frac{\lambda}{\pi}} e^{-\frac{u^2}{2}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.$$

Let $X^{(K)} = \{X^{(K)}(t), t \in \mathbb{R}^d\}$ be an isotropic chi-square field satisfying Assumption ($\mathcal{A}0$) with $K \in \mathbb{N}^+$ degree of freedom such that $X^{(K)}(t) = X_1(t)^2 + \ldots + X_K(t)^2$ where $X_1(t), \ldots, X_K(t)$ are K *i.i.d.* stationary isotropic Gaussian random fields defined as above. Then one can get

$$C_{d-1}^*(u) = \sqrt{\lambda}\left(\frac{u}{2}\right)^{\frac{K-1}{2}} e^{-\frac{u}{2}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{K}{2}\right)\Gamma\left(\frac{d}{2}\right)}.$$

A.3 Convergence of the bias factor

In this appendix, we provide a short justification for Equation (28). The Crofton formula in (5) applied to a random manifold M can be written in terms of the conditional expectation,

$$\sigma_{d-1}(M) = \frac{\sqrt{\pi}\ \Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \int_{\mathbb{R}^{d-1}} \int_{\partial B_1^d} \frac{\mathbb{E}\left[\sigma_0(M \cap l_{\mathbf{s},\mathbf{v_s(u)}}) \mid \sigma_{d-1}(M)\right]}{\sigma_{d-1}(\partial B_1^d)}\, \mathrm{d}\mathbf{s}\, \mathrm{d}\mathbf{u}, \qquad a.s.. \tag{52}$$

By slightly modifying the proof of Theorem 2.1 so as to leverage (52), one obtains

$$\frac{1}{q}\mathbb{P}\big(X(\mathbf{0}) \le u < X(q\mathbf{e}_1) \mid C_{d-1}^T(u)\big) \xrightarrow[q\to 0]{L^1} \frac{C_{d-1}^T(u)}{\beta_d}.$$

Likewise, by slightly adjusting the proof of Theorem 2.2 accordingly, one writes

$$\mathbb{E}\left[\widehat{C}_{d-1}^{(\delta\dot{\mathcal{H}},T)}(u) \mid C_{d-1}^T(u)\right] \xrightarrow[\delta\to 0]{L^1} \frac{2d}{\beta_d} C_{d-1}^T(u).$$

Equation (28) follows.

A.4 Alternative approaches

Further elements on the dimensional constant β_d

Here, we provide an alternative justification for the value of the bias factor $\frac{2d}{\beta_d}$ in (22) for the simple case of the hypercubic lattice by adapting in any dimension d the methodology proposed for instance by Miller (1999).

Remark that the $(d-1)$-dimensional surface $L_X(u)$ is C^1, and so it can be approximated arbitrarily well by a union of $(d-1)$-dimensional hyperplanar surfaces. Then, the expectation of $\widehat{C}_{d-1}^{(\delta\dot{\mathcal{G}},T)}$ in (31) is linear over these hyperpanar surfaces. Thus, it suffices to consider the bias in the estimate of the σ_{d-1} measure of a single hyperplanar surface with orientation vector distributed uniformly on ∂B_1^d. It is not difficult to see that the contribution to $\widehat{C}_{d-1}^{(\delta\dot{\mathcal{G}},T)}$ of a hyperplanar surface with orientation vector $\mathbf{r} \in \partial B_1^d$ is $||\mathbf{r}||_1$ times its σ_{d-1} measure. Thus, the expected bias factor should be the average value of $||\mathbf{r}||_1$, when $\mathbf{r}$ is distributed uniformly on ∂B_1^d. Indeed,

$$\frac{1}{\sigma_{d-1}(\partial B_1^d)} \int_{\partial B_1^d} ||\mathbf{r}||_1\, \sigma_{d-1}(\mathrm{d}\mathbf{r}) = \frac{2d}{\beta_d}. \tag{53}$$

Building another approximation for the σ_{d-1} measure

We discuss shortly an alternative approach for the estimation of the considered surface area. However it is not well adapted to our framework since it is not accessible from the sole knowledge of the field X on the grid $\mathbb{G}(\delta, T)$. The interested reader is also referred to Biermé and Desolneux (2021) (see in particular Equation (4.5) and Figure 4.2) in the case $d = 2$.

A first idea, following definition of the $d-1$ dimensional Haussdorff measure in (2), and the fact the considered lattice is a grid, is to consider the hypercubic cover of T induced by the grid $\mathbb{G}$ in (29) (see Section 3.1). To this end, define $J^{(\delta)}(u) := \left\{\mathbf{i} \in \mathbb{Z}^d : V_{\mathbf{i}}(\delta) \subseteq T,\ V_{\mathbf{i}}(\delta) \cap L_X^T(u) \neq \emptyset\right\}$ and introduce

$$\check{C}_{d-1}^T(u) := \frac{a(\delta)}{\sigma_d(T)}(\sqrt{d}\delta)^{d-1}, \quad \text{where} \quad a(\delta) := \mathrm{Card}\left(J^{(\delta)}(u)\right). \tag{54}$$

Indeed, $L_X^T(u) \subset \bigcup_{\mathbf{i}\in J^{(\delta)}(u)} V_{\mathbf{i}}(\delta)$ and $\mathrm{diam}(V_{\mathbf{i}}(\delta)) = \sqrt{d}\delta$ for all $\mathbf{i} \in J^{(\delta)}(u)$. So $\bigcup_{\mathbf{i}\in J^{(\delta)}(u)} V_{\mathbf{i}}(\delta)$ is a $(\sqrt{d}\delta)$-cover of $L_X^T(u)$ in the sense of (2). The random quantity $a(\delta)$ in (54) represents the number of cells of the grid whose intersection with the level set $L_X^T(u)$ is non empty and it can be rewritten as follows

$$a(\delta) = \sum_{\mathbf{i}=-N}^{N-1} \mathbb{1}_{\left\{ \min_{s\in V_{\mathbf{i}}(\delta)} X(s) \le u < \max_{s\in V_{\mathbf{i}}(\delta)} X(s) \right\}}.$$

This object is not empirically accessible from the discretized excursion set as it requires the knowledge of the field on T. For these reasons, this quantity is not a good candidate to estimate C_{d-1} in our sparse information setting, *i.e.*, when we have the sole observation of X on $\mathbb{G}(\delta, T)$.