Practical consensus tracking of homogeneous sampled-data multi-agent systems

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Abstract— **The aim of this article is to study the robustness of homogeneous sampled-data multi-agent systems including a leader whose dynamic evolution is independent of its followers. We explore the effects caused simultaneously by the sampling and the acceleration of the leader on the system, and show that practical consensus is achieved in this case. The results are illustrated in simulations.**

I. Introduction

The multi-agent consensus control has been a wide field of study in these two last decades [1], [2], [3]. A lot of articles focus on sampled-data multi-agents systems (MAS) which involve wireless communications, see for instance [4], [5], [6] and the references therein. Various frameworks have been considered in order to study consensus of MAS: considering centralized [7] or decentralized [8], [9] approach, fixed or switching topology [10], [11], [12], including communication time-delays [13], using synchronous [1], [14] or asynchronous sampling [15], [11], [16], dealing with finite-time consensus [17], [18]. With linear control laws, consensus of sampled-data MAS with synchronous sampling is ensured within a limited framework [10]: indeed, consensus is lost when a sampling time limit, called the Schur threshold, is reached. For more details on the stability of systems with aperiodic sampling and the Schur property, readers may refer to [6]. In order to overcome this difficulty, it is possible to use nonlinear homogeneous systems [19]. Homogeneity [20] is a powerful tool which allows to maintain the Lyapunov stability of systems with sampled-data inputs even with a high sampling period [21], [22]. The main drawback of such a technique is that we lose asymptotic stability and we arrive at practical stability instead. A previous study in [23] showed that practical consensus is achieved for a homogeneous sampled-data MAS with a negative homogeneity degree in a synchronous sampling framework. This study is formulated as an application of the theoretical and more general case originated from [21], which encompasses the broader field of control networks. According to [21], it turns out that, for positive degree, we can obtain local asymptotic stability; however, in practice, the convergence zone is too small to be useful and consequently are not of practical interest. In order to pursue the work made in [21], the results in [24] deal with the robustness of homogeneous sampled-data systems on which an external disturbance is applied. This study reveals that the external disturbance and the perturbation due to sampling affect the system separately, and their combined influence does not change the intrinsic behavior of the system, in contrast to the results obtained in [21]. Indeed, practical convergence is maintained, while the difference lies in the size of the practical convergence zone. This size depends on both the sampling rate and the maximum intensity of the external disturbance. The aim of this article is to apply the theoretical results obtained in [24] to the case of MAS. We consider a homogeneous sampled-data MAS with synchronous sampling constituted by agents following a leader whose dynamics is independent of them. This configuration fits exactly with the case studied in [24], since the acceleration of the leader can be considered as an external disturbance applied on the systems. We expect to have a larger attractive set than in [23], since we take into account the influence of both endogenous (due to sampling) and exogenous perturbations (due to the leader).

This paper is organized as follows. After preliminaries given in Section II, the main result is stated in Section III, which consists of applying the main theorem of [24] to MAS. Then some simulations illustrate the consensus tracking by varying the sampling rate and the acceleration of the leader in Section IV. Finally, a conclusion is reached in Section V.

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II. Preliminaries

Below let us introduce several notations used in the paper.

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $||.||_2$ the Euclidean norm on $\mathbb{R}^n$, for all $p \in \mathbb{N}^*$, $||.||_p$ the p-norm on $\mathbb{R}^n$ and $\|\cdot\|_\infty$ the infinity norm on $\mathbb{R}^n$.
- For any closed set $Z \subset \mathbb{R}^n$ and any $x \in \mathbb{R}^n$, we denote $d_Z(x) = \inf_{z \in Z} \|x - z\|$ the distance between x and the closet set Z.
- If $t \mapsto x(t)$ is a curve in $\mathbb{R}^n$, we will say that $x(t) \to Z$ as $t \to +\infty$ when $d_Z(x(t)) \to 0$ as $t \to +\infty$.
- We denote $1_n = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ and $\Delta_n = \mathrm{Span}(1_n) \subset \mathbb{R}^n$.
- $r = (r_1, r_2, \ldots, r_n)$ is called a generalized weight if its components are positive numbers.

- $\mathrm{Diag}(r_1,\ldots,r_n)$ denotes the diagonal matrix of dimension $n\times n$ with kth diagonal entry r_k.
- For $x=(x_1,\ldots,x_n)^T$ and $\alpha>0$, we denote $\lfloor x\rceil^\alpha = (|x_1|^\alpha \mathrm{sign}(x_1),\ldots,|x_n|^\alpha \mathrm{sign}(x_n))$.
- If A is a $m\times n$ matrix and B is a $p\times q$ matrix, then the Kronecker product $A\otimes B$ is the $mp\times nq$ matrix

$$A\otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

- I_n denotes the unit matrix of size n.
- A continuous function $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0)=0$ and the function is strictly increasing. A function $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha\in\mathcal{K}$ and it is unbounded.
- A continuous function $\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot,t)\in\mathcal{K}_\infty$ for each fixed $t\in\mathbb{R}_+$ and if for each fixed $s\in\mathbb{R}_+$ the function $t\mapsto\beta(s,t)$ is decreasing to 0.

A. Graph theory

Let us recall some basic definitions in graph theory given for instance in [1, Appendix B].

A *directed graph* $\mathcal{G}_N=(V_N,\mathcal{E}_N)$ consists of a finite nonempty set of nodes $V_N=\{1,2,\ldots,N\}$ and a set of edges $\mathcal{E}_N\subset V_N\times V_N$ which is a set of ordered pairs of nodes. An edge $(i,j)\in\mathcal{E}_N$ in a directed graph $\mathcal{G}_N$ denotes that node i communicates with node j, but not conversely. An *undirected graph* $\mathcal{G}_N=(V_N,\mathcal{E}_N)$ also consists of a set of nodes $V_N=\{1,\ldots,N\}$ and a set of edges $\mathcal{E}_N\subset V_N\times V_N$ which is an unordered set of pairs of nodes. An edge $(i,j)\in\mathcal{E}_N$ in an undirected graph $\mathcal{G}_N$ denotes that nodes i and j obtain information from each other. An *undirected path* is a sequence of edges in an undirected graph of the form (i_1,i_2), (i_2,i_3),$\cdots$. An undirected graph is *connected* if there is an undirected path between every pair of distinct nodes.

The *adjacency matrix* of an undirected graph $(V_N,\mathcal{E}_N)$ is defined by $\mathcal{A}_N=[a_{ij}]\in\mathbb{R}^{N\times N}$ where $a_{ij}=a_{ji}=1$ if $(i,j)\in\mathcal{E}_N$ and $a_{ij}=0$ otherwise. The *Laplacian matrix* associated with adjacency matrix $\mathcal{A}_N$ is given as $\mathcal{L}_N=[\ell_{ij}]\in\mathbb{R}^{N\times N}$ where $\ell_{ii}=\sum\limits_{j=1,j\neq i}^{N} a_{ij}$ and $\ell_{ij}=-a_{ij}$ if $i\neq j$.

B. Lyapunov stability

Consider the following system with continuous f

$$\dot{x}=f(x),\qquad x\in\mathbb{R}^n. \tag{1}$$

Let us recall the definitions of Lyapunov set stability given for instance in [3] for compact sets.

Definition 1: A compact set $C\subset\mathbb{R}^n$ is:

- *stable* w.r.t. the system (1) if for any $\varepsilon>0$ there exists $\eta>0$ such that for any maximal solution $x(t)$ of (1), if there exists t_0 such that $\mathrm{d}_C(x(t_0))<\eta$, then $x(t)$ is defined for all $t\geq t_0$ and $\mathrm{d}_C(x(t))<\varepsilon$ for all $t\geq t_0$;
- *locally attractive* w.r.t. the system (1) if there exists $\varepsilon>0$ such that for any maximal solution $x(t)$ of (1), if there exists t_0 such that $\mathrm{d}_C(x(t_0))<\varepsilon$, then $x(t)$ is defined for all $t\geq t_0$ and $\mathrm{d}_C(x(t))\to 0$ when $t\to+\infty$;
- *globally attractive* w.r.t. the system (1) if it is locally attractive and if the previous point holds for any $\varepsilon>0$;
- *locally* (resp. *globally*) *asymptotically stable* w.r.t. the system (1) if it is stable and locally (resp. globally) attractive w.r.t. the system (1);
- *unstable* if it is not stable.

C. Homogeneity

The most common notion of homogeneity is the *weighted homogeneity* introduced in [25], based on a particular choice of the coordinates, while the most generic one is the *geometric homogeneity*, which is coordinate free [19]. Here we place ourselves within the framework of weighted homogeneity.

Definition 2: Let $\mathbf{A}:\mathbb{R}^n\to\mathbb{R}^n$ an endomorphism such that $\mathbf{A}$ is anti-Hurwitz. We say that a function $V:\mathbb{R}^n\to\mathbb{R}$ is $\mathbf{A}$-*homogeneous of degree* d if $V(\exp(\mathbf{A}s)x)=e^{ds}V(x)$, for all $s\in\mathbb{R}$ and $x\in\mathbb{R}^n$. A vector field f defined on $\mathbb{R}^n$ is $\mathbf{A}$-*homogeneous of degree* d if $f(\exp(\mathbf{A}s)x)=e^{ds}\exp(\mathbf{A}s)f(x)$, for all $s\in\mathbb{R}$ and $x\in\mathbb{R}^n$.

Let us now give the definition of a homogeneous norm detailed for instance in [21].

Definition 3: A $\mathbf{A}$-*homogeneous norm* is a positive definite and continuous mapping $\mathcal{N}:\mathbb{R}^n\to\mathbb{R}$ that is $\mathbf{A}$-homogeneous of degree 1.

Lemma 1: [22] Let $\mathbf{A}$ be an anti-Hurwitz matrix and $\mathcal{N}$ any $\mathbf{A}$-homogeneous norm. Then there exist $\mathcal{K}_\infty$ functions α_1 and α_2 such that for all $x\in\mathbb{R}^n$, we have $\alpha_1(\mathcal{N}(x))\leqslant ||x||_2\leqslant\alpha_2(\mathcal{N}(x))$.

D. Technical result

Let us recall Theorem 1 of [24] used in the following for proving the main result of the article.

We consider the following nonlinear system:

$$\dot{x}=f(x,u,d) \tag{2}$$

where $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}^m$ is the control, $d: t\mapsto d(t)\in\mathbb{R}^p$ indicates the perturbation, which is supposed to be essentially bounded and $f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p\mapsto\mathbb{R}^n$ a continuous function. We consider that a continuous static feedback law $u(x)$ is known for which the following two assumptions hold.

Assumption 1: The origin is a globally asymptotically stable equilibrium of the closed-loop system $\dot{x}=f(x,u(x),0)$.

Assumption 2: There exist a matrix $\mathbf{A}\in\mathbb{R}^{n\times n}$ and a matrix $\overline{\mathbf{A}}\in\mathbb{R}^{p\times p}$ such that $\mathbf{A}$ and $\overline{\mathbf{A}}$ are anti-Hurwitz (that is, $-\mathbf{A}$ and $-\overline{\mathbf{A}}$ are Hurwitz) and a degree $\kappa<0$ such that

$$f(e^{\mathbf{A}s}x,u(e^{\mathbf{A}s}z),e^{\overline{\mathbf{A}}s}d)=e^{\kappa s}e^{\mathbf{A}s}f(x,u(z),d)$$

for all $x,z\in\mathbb{R}^n, d\in\mathbb{R}^p$ and all $s\in\mathbb{R}$.

Remark that Assumption 2 is a homogeneity assumption, meaning that the system must have a specific structure. In practice, this assumption can be ensured by a careful selection of the control law, as we will see later.

Since in networked communication, the state information is only updated at discrete time instants, we consider a sequence of sampling times $(t_k)_{k\in\mathbb{N}}$ such that $t_0 = 0$, and a maximum sampling period $h > 0$ and

$$0 < t_{k+1} - t_k \leqslant h. \tag{3}$$

Due to the sampling, the control is now $u_{SD}(t) = u(x(t_k))$ for all $t \in [t_k, t_{k+1})$ (sample and hold). The system can therefore be rewritten under the following form

$$\dot{x}(t) = f(x(t), u(x(t_k)), d(t)), \quad t \in [t_k, t_{k+1}). \tag{4}$$

Theorem 1 ([24]): Assume that the sampled system (4) is such that the sampling times satisfy (3) and Assumptions 1 and 2 hold. Consider $\mathcal{N}$ as any $\mathbf{A}$-homogeneous norm and $\overline{\mathcal{N}}$ as any $\overline{\mathbf{A}}$-homogeneous norm. Then there exist constants $c_1 > 0$, $c_2 > 0$ such that the set

$$\left\{x \in \mathbb{R}^n : \mathcal{N}(x) \leqslant c_1 h^{-\frac{1}{\kappa}} + c_2 d_{max}\right\}$$

is globally asymptotically stable w.r.t. the system (2), where $d_{max} = \operatorname*{ess\,sup}_{t\in\mathbb{R}^+} \overline{\mathcal{N}}(d(t))$. Hence, there exists a class $\mathcal{KL}$ function β such that

$$\mathcal{N}(x) \leq \beta\left(\mathcal{N}(x_0), t\right) + c_1 h^{-\frac{1}{\kappa}} + c_2 d_{max}.$$

III. Main result

We consider a MAS constituted by a leader and N followers whose dynamics are described by a double integrator. The goal of the followers is to reach the leader, but the leader follows a trajectory independent of the followers. We assume that the leader sends its position and velocity data to at least one follower. We also assume that a special control law for the followers is already known and proven to achieve consensus tracking in continuous time. However, due to networked communication, the control laws for the N followers must be sampled. Hence, the agents send position and velocity data to the neighbors at some discrete instants according to a connected and static graph. Our aim is to show that the sampled MAS achieves practical consensus tracking under suitable assumptions. For this purpose, we shall apply Theorem 1.

We denote the leader's dynamics by

$$\begin{cases} \dot{q}_0 = p_0 \\ \dot{p}_0 = u_0 \end{cases} \tag{5}$$

where $q_0 \in \mathbb{R}^n$ indicates its position, $p_0 \in \mathbb{R}^n$ its velocity, and $u_0 \in \mathbb{R}^n$ its acceleration, which is supposed to be essentially bounded, that is $\operatorname*{ess\,sup}_{t\in\mathbb{R}^+}||u_0(t)||_2 < +\infty$. For $i \in \{1, \ldots, N\}$, we denote the ith agent's dynamics by:

$$\begin{cases} \dot{q}_i = p_i \\ \dot{p}_i = u_i \end{cases} \tag{6}$$

where $q_i \in \mathbb{R}^n$ indicates the position of agent i, $p_i \in \mathbb{R}^n$ its velocity, u_i its control law, $q = (q_1^T, \ldots, q_N^T)^T$, and $p = (p_1^T, \ldots, p_N^T)^T$.

Definition 4: We say that the MAS defined by (5) and (6) achieves practical consensus tracking if there exists a class $\mathcal{KL}$ function β and a constant $C \geq 0$ such that $\|x\| \leq \beta(\|x_0\|, t) + C$, where $x = ((q_1 - q_0)^T, \ldots, (q_N - q_0)^T, (p_1 - p_0)^T, \ldots, (p_N - p_0)^T)^T$. If $C = 0$, we say that stable consensus tracking is achieved.

Remark 1: Consensus tracking is usually formulated as saying that, for any $i \in \{1, \ldots, N\}$, $q_i - q_0 \to 0$ and $p_i - p_0 \to 0$ when $t \to +\infty$. Definition 4 can be seen as a strengthened version of this classical formulation where the followers not only must asymptotically reach the leader but also have to do it in a stable way. Indeed, it is necessary that $\|q_i - q_0\| \leq \|x\| \to 0$ when $t \to \infty$ but not sufficient.

We assume that the control laws u_i in (6) are sampled at times $(t_k)_{k\in\mathbb{N}}$ such that (3) holds. We aim to show that consensus tracking is achieved for the MAS if the control laws u_i satisfy the following assumptions:

Assumption 3: For any $i \in \{1, \ldots, N\}$, the control law u_i depends only on $q_i - q_j$ and on $p_i - p_j$ for all $j \in \{0, \ldots, N\}$ such that agent i receives data from agent j.

Denoting $\tilde{q}_j = q_j - q_0$ and $\tilde{p}_j = p_j - p_0$, Assumption 3 implies in particular that each control law u_i only depends on $\tilde{q}_1, \ldots, \tilde{q}_N, \tilde{p}_1, \ldots, \tilde{p}_N$. Further on, we will denote $\tilde{q} = \left(\tilde{q}_1^T, \ldots, \tilde{q}_N^T\right) \in \mathbb{R}^{nN}$ and $\tilde{p} = \left(\tilde{p}_1^T \ldots, \tilde{p}_N^T\right)^T \in \mathbb{R}^{nN}$ and we will write $u_i = u_i(\tilde{q}, \tilde{p})$ for the sake of clarity.

Assumption 4: In continuous time, if $u_0 = 0$ then the control laws u_i with $i \in \{1, \ldots, N\}$ applied to (5) and (6) achieve stable consensus tracking.

Assumption 5: There exists $\frac{1}{2} < r < 1$ such that for all $s \in \mathbb{R}$ and all $i \in \{1, \ldots, N\}$, we have

$$u_i(e^s \tilde{q}, e^{rs} \tilde{p}) = e^{(2r-1)s} u_i(\tilde{q}, \tilde{p}) \tag{7}$$

Again, Assumption 5 is a homogeneity assumption.Let us now state the main result of this paper.

Theorem 2: We consider the MAS system given by (5) and (6). We assume that Assumptions 3–5 hold. We consider that the control laws u_i are sampled at times $(t_k)_{k\in\mathbb{N}}$ verifying (3). Let us denote $x = \left(\tilde{q}^T, \tilde{p}^T\right)^T$ and

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \otimes I_{nN}, \qquad \overline{\mathbf{A}} = (2r - 1) I_{nN}.$$

$\mathbf{A}$ and $\overline{\mathbf{A}}$ are both anti-Hurwitz. For all $\mathbf{A}$-homogeneous norm $\mathcal{N}$ and $\overline{\mathbf{A}}$-homogeneous norm $\overline{\mathcal{N}}$, there exist constants $c_1 > 0$ and $c_2 > 0$ such that the compact set

$$\mathcal{K} = \left\{x \in \mathbb{R}^{2nN} : \mathcal{N}(x) \leqslant c_1 h^{-\frac{1}{\kappa}} + c_2 d_{max}\right\}$$

is globally asymptotically stable where h denotes the sampling time and $d_{\max} = \operatorname*{ess\,sup}_{t\in\mathbb{R}^+} \overline{\mathcal{N}}(u_0 \otimes 1_N)$.

Proof: We denote $u(\tilde{q}, \tilde{p}) = (u_1(\tilde{q}, \tilde{p})^T, \ldots, u_N(\tilde{q}, \tilde{p})^T)^T \in \mathbb{R}^{nN}$. By subtracting system (5) to the N systems (6), we have:

$$\begin{cases} \dot{\tilde{q}} = \tilde{p} \\ \dot{\tilde{p}} = u(\tilde{q}, \tilde{p}) - u_0 \otimes 1_N \end{cases} \tag{8}$$

Let us denote f the right-hand side of (8):

$$f(x, u(x), d) = \begin{pmatrix} \tilde{p} \\ u(x) + d \end{pmatrix}$$

with $d = -u_0 \otimes 1_N \in \mathbb{R}^{nN}$. u_0 refers to the acceleration of the leader, its influence on the system will be assimilated to an external disturbance. In order to apply Theorem 1, we need to check the two required assumptions.

First, we show that the origin is a globally asymptotically stable equilibrium for the closed-loop system $\dot{x} = f(x, u(x), 0)$. This assertion is exactly equivalent to Assumption 4.

Second, we check the homogeneity condition (Assumption 2) for f. Denoting $z = (\zeta^T, \xi^T)^T \in \mathbb{R}^{2nN}$, we have:

$$\begin{aligned}
&f(e^{As}x, u(e^{\mathbf{A}s}z), e^{\overline{\mathbf{A}}s}d)\\
&= \begin{pmatrix} e^{rs}\tilde{p} \\ u(e^{s}\zeta, e^{rs}\xi) - e^{(2r-1)s}u_0 \otimes 1_N \end{pmatrix}\\
&= \begin{pmatrix} e^{rs}\tilde{p} \\ e^{(2r-1)s}u(\zeta, \xi) - e^{(2r-1)s}u_0 \otimes 1_N \end{pmatrix} \text{ by Assumption 5}\\
&= e^{(r-1)s}\left[\begin{pmatrix} e^{s} & 0 \\ 0 & e^{rs} \end{pmatrix} \otimes I_{nN}\right]\begin{pmatrix} \tilde{p} \\ u(\zeta, \xi) - u_0 \otimes 1_N \end{pmatrix}\\
&= e^{(r-1)s}\left[\begin{pmatrix} e^{s} & 0 \\ 0 & e^{rs} \end{pmatrix} \otimes I_{nN}\right] f(x, u(z), d).
\end{aligned}$$

This shows that Assumption 2 holds for f with $\kappa = r-1 < 0$.

According to Theorem 1, we can conclude that there exists a class $\mathcal{K}L$ function β and constants $c_1, c_2 > 0$ such that

$$\mathcal{N}(x) \leq \beta(\mathcal{N}(x_0, t)) + c_1 h^{-\frac{1}{\kappa}} + c_2 d_{\max}.$$

Consequently, from Lemma 1, there exist class $\mathcal{K}_\infty$ functions α_1, α_2 such that

$$\|x\| \leq \alpha_2(\beta(\alpha_1^{-1}(\|x_0\|), t) + c_1 h^{-\frac{1}{\kappa}} + c_2 d_{\max}).$$

By usual operations on class $\mathcal{K}$ functions, we finally obtain practical consensus tracking. ■

Let us stress that this result holds for any value of the maximum allowable sampling time $h > 0$.

IV. Simulation Results

In this section,we illustrate the performance of the proposed homogeneous control law through numerical simulations. We consider a second-order MAS in the form of (5) and (6), constituted by 5 followers and a leader, evolving on the plane, whose communication topology is given by Figure 1.

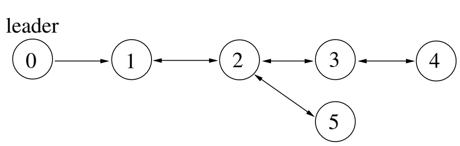


Fig. 1: Communication graph

We choose constant sampling periods $t_k = kh$ with $h > 0$, $k \in \mathbb{N}$. We consider the following linear controller:

$$\begin{aligned} u_i \;=\; & -\left(\sum_{j=1}^{5} a_{ij}(q_i - q_j) + d_{ii}(q_i - q_0)\right) \\ & -\left(\sum_{j=1}^{5} a_{ij}(p_i - p_j) + d_{ii}(p_i - p_0)\right) \end{aligned} \tag{9}$$

with $i \in \{1, \ldots, 5\}$, where $d_{ii} = 1$ if the follower i has a direct link with the leader and $d_{ii} = 0$ otherwise. Note that (9) allows the consensus of the MAS if and only if the leader acceleration is equal to zero. Otherwise, there is only practical consensus tracking. The Schur threshold h_{schur} corresponding to control law (9) is equal to 0.48s. Let us now consider the homogeneous controllers from [26] given by

$$\begin{aligned} u_i \;=\; & -\left\lfloor\left(\sum_{j=1}^{5} a_{ij}(q_i - q_j) + d_{ii}(q_i - q_0)\right)\right\rceil^{\alpha} \\ & -\left\lfloor\left(\sum_{j=1}^{5} a_{ij}(p_i - p_j) + d_{ii}(p_i - p_0)\right)\right\rceil^{\frac{2\alpha}{1+\alpha}} \end{aligned} \tag{10}$$

with $\alpha = 0.5$. We can check that Assumptions 2–5 are satisfied with a degree of homogeneity equal to -0.5. We denote $q_i = (q_{ix}, q_{iy})$ and $p_i = (p_{ix}, p_{iy})$ the positions and the velocities of the agents for all $i \in \{1, \ldots, 5\}$. We suppose that the initial positions of the agents $(q_{ix}(0), q_{iy}(0))$ are contained in the square $[-1, 1] \times [-1, 1]$ and all the initial velocities of the agents $(p_{ix}(0), p_{iy}(0))$ are equal to zero. Initial conditions for the leader are $q_0 = (0, 0)$ and $p_0 = (0, 0)$. Let us denote $u_0 = (u_{0x}, u_{0y})$ the acceleration of the leader. Figure 2 shows the reference acceleration $u_0(t)$ of the leader. The corresponding velocities never exceed 1.5 in absolute value.

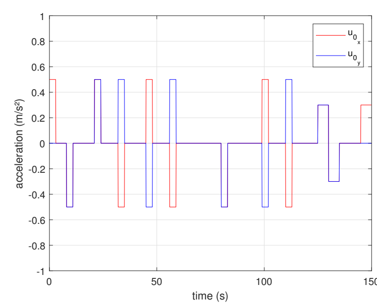
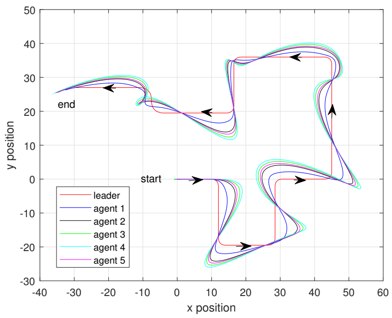


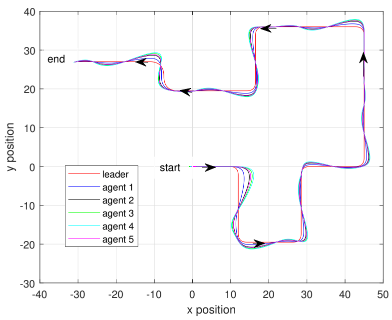
Fig. 2: Linear accelerations $u_0 = (u_{0_x}, u_{0_y})$ of the leader

Case 1: the sampling time h is fixed to 0.01s. Figure 3a shows trajectories of each agent on the plane when the linear control law (9) is applied. The trajectories plotted in Figure 3b are obtained via the homogeneous control law (10). The coordinate units are given in meters. Figures 4a and 4b represent the agents' positions versus time for the reference acceleration given in Figure 2 for both control laws. For the sampling time $h = 0.01$s, the effects of the discretization of the control are negligible. The practical consensus tracking is reached for both control laws. Only the perturbations due to the leader's accelerations impact the consensus tracking. Homogeneous control gives better results than linear control, in the sense that the trajectories of the agents are closer to the leader one.

Case 2: we consider now a sampling time $h = 0.45$s closer

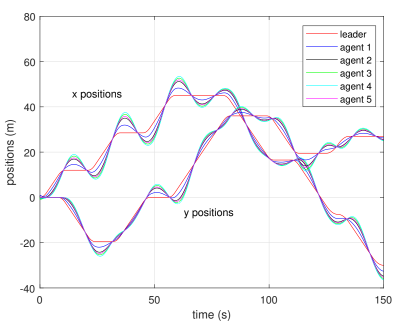


(a) Linear control law (9)

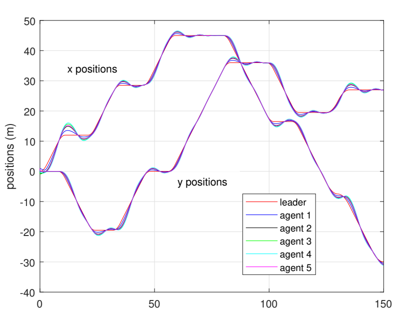


(b) Homogeneous control law (10)

Fig. 3: Trajectories of leader and agents for $h = 0.01$s



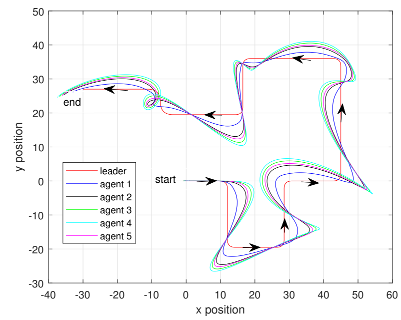
(a) Linear control law (9)



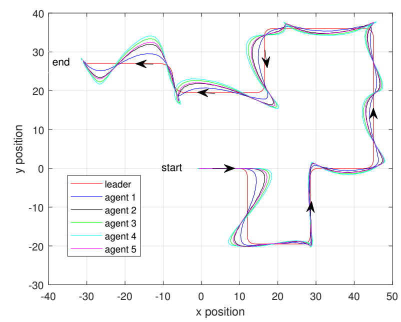
(b) Homogeneous control law (10)

Fig. 4: Positions of leader and agents for $h = 0.01$s

to $h_{schur} = 0.48$s to highlight the effects of the discretization the control laws. As expected, the error between the trajectories of the agents and the leader increases as the sampling period increases. This error remains bounded in accordance with Theorem 2 when the homogeneous control law is applied. For the linear case, the system leans toward instability as h approaches h_{schur}, see Figures 5a, 5b, 6a, and 6b.



(a) Linear control law (9)



(b) Homogeneous control law (10)

Fig. 5: Trajectories of leader and agents for $h = 0.45$s

Case 3: the sampling period $h = 0.6$s is chosen greater than $h_{schur} = 0.48$s. In Figures 7a and 7b, the homogeneous control law is applied. It can be noted that the practical consensus tracking is achieved, although the tracking is less precise. The differences between the trajectories of the agents and the leader remain bounded whereas with the linear control law, the closed-loop system becomes unstable.

V. Conclusion

This article dealt with the consensus tracking of homogeneous sampled-data multi-agent systems. By using theoretical results dedicated to homogeneous sampled data systems, we proved a practical stability theorem for homogeneous sampled data multi-agent systems. Homogeneity ensures stability and convergence in the vicinity of the origin even for large sampling periods. The main result was applied to a second-order multi-agent system to show its effectiveness.

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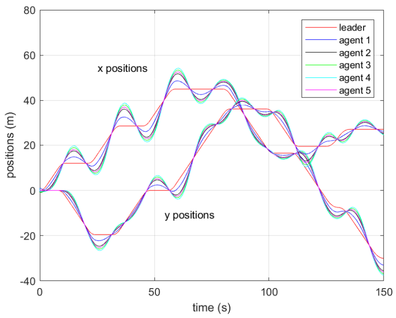
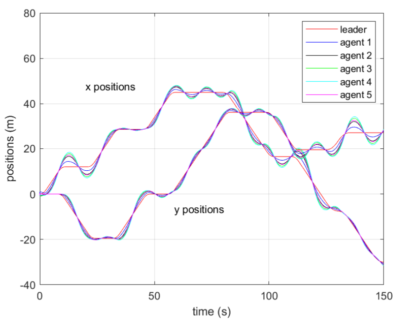
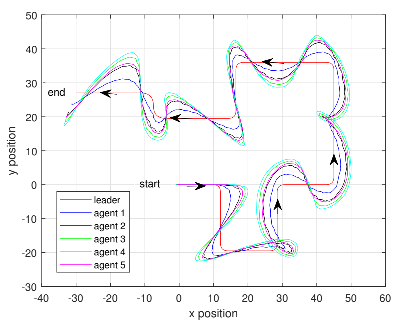


(a) Linear control law (9)

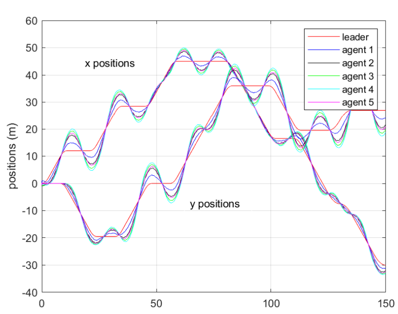


(b) Homogeneous control law (10)

Fig. 6: Positions of leader and agents for $h = 0.45$s



(a) Trajectories of leader and agents for $h = 0.6$s



(b) Positions of leader and agents for $h = 0.6$s

Fig. 7: Simulation results for homogeneous control law for $h = 0.6$s

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