Flatness of interconnected linear systems and applications to electrical systems

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Abstract: In this paper, we completely describe flatness of the simplest class of interconnected systems: we consider an interconnection of (controllable) single-input linear subsystems in dimension two with a star topology and a linear interconnection dynamics (with no inputs acting directly on the interconnection variable). First, we observe that even if each subsystem is flat, flatness of the global interconnected system is not necessarily preserved. Then, we give necessary and sufficient verifiable conditions for flatness of the interconnected system. When the interconnected system is flat, we analyze how its flat output depends on the interconnection variable and how it can be expressed in function of the flat outputs of each subsystem. Finally, we show how our results can be applied to electrical power systems.

Keywords: Interconnected systems, star topology, flatness, electrical power systems.

1. INTRODUCTION

Systems consisting of the interconnection of a certain number of subsystems are ubiquitous in current and emerging technologies. From power distribution and energy grids, fleets of connected autonomous vehicles or drones, natural processes in biology and genetics, to online social networks, they are all characterized by their complexity. When considering interconnected systems as a whole, the interconnection plays an essential role in the control applications and call upon the development of novel adapted tools for control design. An important question is: when suitable properties are ensured for each subsystem, does the interconnected system share the same properties? In other words, are these properties preserved by the interconnection? A property that is very useful in applications (for instance, for trajectory tracking, constructive controllability or trajectory generation) is that of flatness (see, e.g., Fliess et al. [1995, 1999], Lévine [2009] and references therein).

The fundamental property of flat systems is that all their solutions may be parameterized by m functions and their time-derivatives, m being the number of controls. More precisely, the control system $\Xi : \dot{x} = F(x, u)$, where x is the state defined on a open subset X of $\mathbb{R}^n$ and u is the control taking values in an open subset U of $\mathbb{R}^m$, is *flat* if we can find m functions, $\varphi_i(x, u, \ldots, u^{(l)})$, for some $l \geq 0$, such that

$$x = \gamma(\varphi, \ldots, \varphi^{(r-1)}) \text{ and } u = \delta(\varphi, \ldots, \varphi^{(r)}), \qquad (1)$$

for a certain integer r, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ is called a *flat output*. Therefore all state and control variables can be determined from the flat outputs without integration and all trajectories of the system can be completely parameterized. Although flatness has received a lot of attention because of its important applications, the problem of giving necessary and sufficient verifiable conditions in order to check whether a system is flat remains widely open. Of course, there exist many results proving flatness for particular systems but less results characterizing flatness for classes of systems, and even less for classes of interconnected systems. For instance, to the best of our knowledge, characterizing flatness for interconnected systems taking into account both the (controllable) several subsystems and the interconnection, in general, has never been addressed in the literature. Most of the works on networked and interconnected systems that we are aware of (see, e.g., Schenk and Lunze [2018], Schenk et al. [2021] where flatness-based controllers ensure trajectory tracking, or Kaczor et al. [2020] where flatness and machine-learning methods are used to reduce oscillations in elastically coupled objects) consider flatness of each subsystem independently (i.e., flatness of decoupled agents). In Zolfaghari et al. [2020], a control method based on flatness of a global system of n bidirectional power converters connected in parallel is proposed in order to provide desirable output characteristics and robustness against unmodeled dynamics and unknown inputs. Nevertheless, also in this case the interconnection (common voltage bus) among the power converters (subsystems) is not described as a state variable, and therefore not included in the flatness investigation.

In this framework, our aim is to completely describe flatness of the simplest class of interconnected systems: we consider an interconnection of (controllable) single-input linear subsystems in dimension two with a star topology and a linear interconnection dynamics (with no control inputs acting directly on the interconnection variable). It is well known that for linear systems, flatness and controllability are equivalent. It follows that each subsystem is flat and one of the most natural questions is whether the interconnected system is also flat. We show that even in the simplest possible case flatness is not always preserved. Our goal is, on one hand, to give necessary and sufficient verifiable conditions to check flatness of the interconnected system and, on the other hand, when the interconnected system is flat, to understand how its flat output depends on the interconnection variable and how it can be expressed in function of the flat outputs of each subsystem.

From a theoretical point of view, solving in this paper the problem in the simplest case is interesting for few reasons; firstly, it yields a complete analysis of flatness for a well defined class of interconnected systems, secondly, it shows what kind of difficulties one must face when trying to characterize flatness for more general interconnected systems, thirdly, it allows to observe new phenomena (like non preservation of flatness or non-uniqueness of flat outputs when the interconnected system is flat). Even if the systems considered for investigating flatness of interconnected systems seem simple, they are relevant for several application. For example, the considered class of systems falls into the ones describing some types of power converters, as the (bidirectional or monodirectional) buck converters that are considered in electrical microgrids, see, e.g., Iovine et al. [2022], Sira-Ramirez and Silva-Ortigoza [2006]. Driven by this application, in the sequel we propose a general analysis but we focus on linear single-input linear subsystems of dimension two and an interconnection with star topology described by linear dynamics. Finally, we produce an example that shows how flatness for the whole set of subsystems composing the global system can be a powerful tool to select desired trajectories while allowing for a selection of desired characteristics of the control inputs.

The paper is organized as follows. In Section 2, we recall the definition of flatness and formalize the problem studied in the paper. In Section 3, we give our main result and illustrate it by an application to electrical power systems in Section 4. We present the proof of our main result in Section 5.

2. DEFINITIONS AND PROBLEM STATEMENT

The fundamental property of flat systems is that all their solutions may be parameterized by m functions and their time-derivatives, where m is the number of controls. Fix an integer $l \geq -1$ and denote $U^l = U \times \mathbb{R}^{ml}$ and $\bar{u}^l = (u, \dot{u}, \ldots, u^{(l)})$. For $l = -1$, the set U^{-1} is empty and $\bar{u}^{-1}$ in an empty sequence.

Definition 1. The system $\Xi : \dot{x} = F(x, u)$, $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, is *flat* at $(x^*, \bar{u}^{*l}) \in X \times U^l$, for $l \geqslant -1$, if there exist a neighborhood $\mathcal{O}^l$ of $(x^*, \bar{u}^{*l})$ and m smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(l)})$, $1 \leqslant i \leqslant m$, defined in $\mathcal{O}^l$, having the following property: there exist an integer r and smooth functions γ_i, $1 \leqslant i \leqslant n$, and δ_j, $1 \leqslant j \leqslant m$, such that

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(r-1)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(r)}) \quad (2)$$

for any C^{l+r}-control $u(t)$ and corresponding trajectory $x(t)$ that satisfy $(x(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and is called a *flat output*.

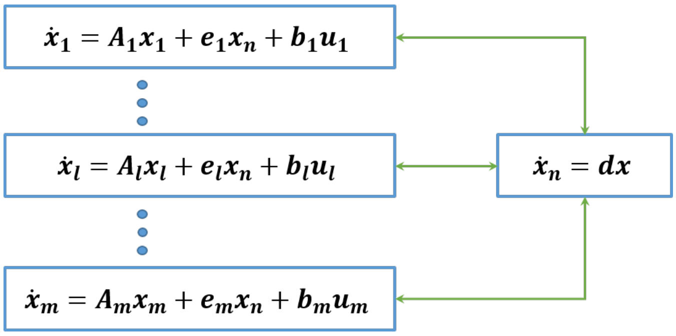

Fig. 1. Network of interconnected linear control systems with star topology.

The goal of this paper is to study flatness of the simplest class of interconnected systems. We consider m interconnected agents with a star topology, where each agent is a single-input two-dimensional control system and the interconnection has a linear dynamics (see Figure 1). The motivation of studying this particular class of interconnected systems comes from the fact that it covers a large class of electrical power systems and related controllers, see, for instance Iovine et al. [2022], Sira-Ramirez and Silva-Ortigoza [2006].

Without coupling, each subsystem (labeled by the lower-index ℓ) of Figure 1 is described by

$$\mathcal{S}_\ell : \dot{x}_\ell = A_\ell x_\ell + b_\ell u_\ell, \quad (3)$$

where $x_\ell = (x_\ell^1, x_\ell^2)^\top \in \mathbb{R}^2$ is the state, $u_\ell \in \mathbb{R}$ is the control, A_ℓ is a constant (2×2)-matrix and b_ℓ is a constant vector in $\mathbb{R}^2$.

The network is assumed to have a star topology, meaning that all subsystems are coupled through an interconnection variable common to all (with no control inputs acting on it) of them and with linear dynamics. In the following, we denote the interconnection variable by x_n, where $n = 2m + 1$ and stands for the state dimension of the network. More precisely, for $1 \leqslant \ell \leqslant m$, the dynamics of each interconnected subsystem is given by

$$\mathcal{S}_{\ell,c} : \dot{x}_\ell = A_\ell x_\ell + e_\ell x_n + b_\ell u_\ell, \quad (4)$$

where the index c indicates the coupling, and $e_\ell = (e_\ell^1, e_\ell^2)^\top \in \mathbb{R}^2$; while the evolution of the interconnection variable x_n is described by

$$\dot{x}_n = \sum_{\ell=1}^{m} (d_\ell^1 x_\ell^1 + d_\ell^2 x_\ell^2) + d_n x_n = dx, \quad (5)$$

where $d_\ell = (d_\ell^1, d_\ell^2) \neq (0, 0)$, meaning that all subsystems are involved in the interconnection, d denotes the constant row-vector $d = (d_1^1, d_1^2, \ldots, d_m^1, d_m^2, d_n)$ and $x = (x_1^1, x_1^2, \ldots, x_m^1, x_m^2, x_n)^\top$ corresponds to the state of the global system.

All coupled m subsystems (4) and the interconnection dynamics (5) form the overall plant

$$\mathcal{S} : \dot{x} = Ax + Bu, \quad (6)$$

with state $x = (x_1^1, x_1^2, \ldots, x_m^1, x_m^2, x_n)^\top \in \mathbb{R}^n$, where $n = 2m + 1$, control $u = (u_1, \ldots, u_m)^\top \in \mathbb{R}^m$, and (block) matrices

$$A = \begin{pmatrix} A_1 & \dots & 0 & e_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & A_m & e_m \\ d_1 & \dots & d_m & d_n \end{pmatrix}, \tag{7}$$

and

$$B = \begin{pmatrix} b_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & b_m \\ 0 & \dots & 0 \end{pmatrix}. \tag{8}$$

It is well known, see Fliess et al. [1995], that flatness of linear control systems is equivalent to controllability, as recalled in the following proposition:

Proposition 1. Consider the subsystem $\mathcal{S}_\ell : \dot{x}_\ell = A_\ell x_\ell + b_\ell u_\ell$. The following conditions hold:

(i) $\mathcal{S}_\ell$ is flat if and only if it is controllable, i.e., $\mathrm{rk}\,(b_\ell \; A_\ell b_\ell) = 2$. Moreover, any flat output is of the form $\varphi_\ell = C_\ell x_\ell$, where C_ℓ is any nonzero row-vector such that

$$C_\ell b_\ell = 0, \quad C_\ell A_\ell b_\ell \neq 0. \tag{9}$$

(ii) $\mathcal{S}_\ell$ can be transformed via a change of coordinates $\tilde{x}_\ell = T_\ell x$ and a feedback transformation $\tilde{u}_\ell = \alpha_\ell x_\ell + \beta_\ell u_\ell$, with T_ℓ an invertible constant matrix, α_ℓ and β_ℓ constant scalars such that $\beta_\ell \neq 0$, into the form:

$$\tilde{\mathcal{S}}_\ell : \begin{cases} \dot{\tilde{x}}^1_\ell = \tilde{x}^2_\ell \\ \dot{\tilde{x}}^2_\ell = \tilde{u}_\ell, \end{cases} \tag{10}$$

where $\varphi_\ell = \tilde{x}^1_\ell$.

Proposition 1 is an immediate consequence of Fliess et al. [1995] and shows that flatness of linear systems reduces to controllability. Moreover, contrary to nonlinear systems for which flatness is a local and generic property (see Fliess et al. [1999], Lévine [2009]), for linear systems, flatness is global: the function φ_ℓ defining the flat output (there is only one because $\mathcal{S}_\ell$ is a single-input system) is globally defined and the desired description (2) holds globally and is globally invertible. Finally, observe that the flat output φ_ℓ depends on the state only and that its existence (or equivalently, that of the nonzero row-vector C_ℓ satisfying (9)) is a direct consequence of the controllability of $\mathcal{S}_\ell$. All properties mentioned above are actually valid for any linear control system with an arbitrary number of inputs and arbitrary state dimension (and not only for single-input two-dimensional linear systems).

The transformation T_ℓ bringing $\mathcal{S}_\ell$ into form (10) is defined with the help of the flat output. Indeed, from the definition of the matrix C_ℓ and the controllability of $\mathcal{S}_\ell$, it follows that $\tilde{x}^1_\ell = C_\ell x_\ell$, $\tilde{x}^2_\ell = C_\ell A_\ell x_\ell$, is a global change of coordinates that, together with the invertible feedback transformation $\tilde{u}_\ell = C_\ell A^2_\ell x_\ell + (C_\ell A_\ell b_\ell) u_\ell$, transforms $\mathcal{S}_\ell$ into (10), which is clearly flat with $\varphi_\ell = \tilde{x}^1_\ell$ being a flat output. Form (10) is the single-input Brunovsky canonical form, see Brunovsky [1970], in dimension two, i.e., simply a double integrator.

From now on, we will assume that each (uncoupled) agent $\mathcal{S}_\ell$ is flat. In other words, we assume that each $\mathcal{S}_\ell$ is controllable (which is also natural from an application point of view, and not only for flatness), i.e., $\mathrm{rk}\,(b_\ell \; A_\ell b_\ell) = 2$, for $1 \leqslant \ell \leqslant m$. At this point, several natural questions concerning the interconnected system $\mathcal{S}$ arise:

(Q1) If each subsystem $\mathcal{S}_\ell$ (when no coupling is considered) is flat, then is the interconnected system $\mathcal{S}$ necessarily flat?

(Q2) If $\mathcal{S}$ is flat, then is the m-tuple $(\varphi_1, \dots, \varphi_m)$, where φ_ℓ is a flat output of $\mathcal{S}_\ell$ (i.e., it satisfies Proposition 1), a flat output of $\mathcal{S}$? In general, if $\mathcal{S}$ is flat, then which are the relations between the flat outputs of each subsystem $\mathcal{S}_\ell$ and that of the interconnected one $\mathcal{S}$?

(Q3) What is the role played by the interconnection variable for flatness of the interconnected system $\mathcal{S}$?

We answer these questions in the next section of the paper.

3. MAIN RESULTS

Before stating our main result, given by Theorem 1 below, let us introduce some notations that will be used in condition (C4). For $1 \leqslant \ell \leqslant m$, we denote by $\mathscr{C}_\ell$ the controllability matrix associated to the system $\mathcal{S}_\ell : \dot{x}_\ell = A_\ell x_\ell + b_\ell u_\ell$ (when no coupling is considered), i.e., $\mathscr{C}_\ell = (b_\ell \; A_\ell b_\ell)$, and by $\mathscr{C}_\ell^{-1}$ its inverse. For $1 \leqslant \ell \leqslant m$ and fixed $1 \leqslant j \leqslant m$, let

$$M_{\ell j} = \begin{cases} -[\mathscr{C}_\ell^{-1} e_\ell d_j b_j]_{21} \times Id, & \ell \neq j, \\ A_j - [\mathscr{C}_j^{-1}(A_j^2 + e_j d_j - d_n A_j) b_j]_{21} \times Id, & \ell = j, \end{cases} \tag{11}$$

where Id is the identity 2×2-matrix, $[\cdot]_{21}$ denotes the entry in the second row and first column of the corresponding vector, e_ℓ, d_j and d_n are associated to the interconnection variable x_n of the interconnected system $\mathcal{S}$ (see (4)-(6)). Define next the block diagonal matrix

$$M_j = \mathrm{diag}(M_{1j}, \dots, M_{mj}). \tag{12}$$

Finally, introduce the row, respectively, column, vectors

$$\bar{d} = (d_1, \dots, d_m) \in \mathbb{R}^{1\times 2m} \text{ and } \bar{b} = (b_1, \dots, b_m)^\top \in \mathbb{R}^{2m}. \tag{13}$$

Theorem 1. Consider the interconnected system $\mathcal{S} : \dot{x} = Ax + Bu$, given by (6). The following conditions are equivalent:

(C1) $\mathcal{S}$ is flat;

(C2) $\mathcal{S}$ is controllable;

(C3) $\mathrm{rk}\,(B \; AB \; A^2B) = 2m + 1 = n$;

(C4) There exists $1 \leqslant j \leqslant m$ such that $\bar{d} \times M_j \times \bar{b} \neq 0$, where M_j, $\bar{b}$ and $\bar{d}$ are associated to $\mathcal{S}$ and defined by (12) and (13);

(C5) $\mathcal{S}$ can be transformed via a change of coordinates $\tilde{x} = Tx$ and a feedback transformation $\tilde{u} = \alpha x + \beta u$, where α, β and T are constant matrices, with β and T invertible, into the form:

$$\tilde{\mathcal{S}} : \begin{cases} \dot{\tilde{x}}^1_\ell = \tilde{x}^2_\ell \\ \dot{\tilde{x}}^2_\ell = \tilde{u}_\ell, \; 1 \leqslant \ell \leqslant m, \\ \dot{\tilde{x}}_n = \displaystyle\sum_{\ell=1}^{m} (\tilde{d}^1_\ell \tilde{x}^1_\ell + \tilde{d}^2_\ell \tilde{x}^2_\ell) + \tilde{d}_n \tilde{x}_n = \tilde{d}\tilde{x}, \end{cases} \tag{14}$$

with $\tilde{x} = (\tilde{x}^1_1, \tilde{x}^2_1, \dots, \tilde{x}^1_m, \tilde{x}^2_m, \tilde{x}_n)^\top$, where $\tilde{x}_n = x_n$, and $\tilde{d} = (\tilde{d}^1_1, \tilde{d}^2_1, \dots, \tilde{d}^1_m, \tilde{d}^2_m, \tilde{d}_n)$ being such that there exists an integer $1 \leqslant j \leqslant m$ such that

$$\tilde{d}^1_j + \tilde{d}^2_j \tilde{d}_n \neq 0. \tag{15}$$

Moreover, for any fixed integer $1 \leqslant j \leqslant m$ satisfying condition (C4) (or, equivalently, (C5)), define the m-tuple of smooth functions $\varphi = (\varphi_1, \ldots, \varphi_m)$, where

- $\varphi_\ell(x_\ell) = C_\ell x_\ell$, for $1 \leqslant \ell \leqslant m$, $\ell \neq j$, is the flat output of the ℓ-th subsystem $\mathcal{S}_\ell : \dot{x}_\ell = A_\ell x_\ell + b_\ell u_\ell$ when no coupling is considered,

and

- $\varphi_j(x) = \bar{C}_j \bar{x}_j$, with $\bar{x}_j = \left(x_j^1, x_j^2, x_n\right)^\top \in \mathbb{R}^3$, is such that

$$\bar{C}_j \bar{b}_j = \bar{C}_j \bar{A}_j \bar{b}_j = 0 \text{ and } \bar{C}_j \bar{A}_j^2 \bar{b}_j \neq 0,$$
$$\text{with } \bar{A}_j = \begin{pmatrix} A_j & e_j \\ d_j & d_n \end{pmatrix} \text{ and } \bar{b}_j = \begin{pmatrix} b_j \\ 0 \end{pmatrix}. \quad (16)$$

Then $\varphi = (\varphi_1, \ldots, \varphi_m)$ is a flat output of the interconnected system $\mathcal{S}$.

Proof. See Section 5.

Theorem 1 shows that even for the simplest possible class of interconnected systems, flatness may not be preserved (so the answer to Question 1 is negative) and for the interconnected system to be flat we need additional conditions. Indeed, as expected, the (linear) interconnected system $\mathcal{S}$ is flat if and only if it is controllable. Controllability of interconnected systems is not a trivial problem, it depends on the type of interconnection and has been studied in many works (for some examples, see, Blackhall and Hill [2010], Kempker et al. [2012], Trumpf and Trentelman [2018] and the references therein).

From condition (C3), it follows immediately that for the class of systems considered here, the dimension of the controllable space is either $n-1 = 2m$ (the uncontrollable, equivalently non flat, case) or $n = 2m+1$ (the controllable, equivalently flat, case). Indeed, it can be easily shown (see the proof of Theorem 1) that for the interconnected system $\mathcal{S}$, we always have $\mathrm{rk}\,(B\ AB) = 2m$, so only one new direction can be lost because of the interconnection (which can destroy flatness).

Item (C4) gives a necessary and sufficient condition ensuring flatness (equivalently, controllability) of the interconnected system $\mathcal{S}$ in terms of matrices and vectors describing the uncoupled subsystems dynamics, the interconnection dynamics and how the interconnection variable acts on the subsystems. Its verification involves algebraic operations only, see (11)-(13). Notice however that in order to check (C4), we have to compute the inverses of all controllability matrices $\mathscr{C}_\ell$ for the uncoupled subsystems $\mathcal{S}_\ell$, $1 \leqslant \ell \leqslant m$, but since each subsystem is two-dimensional, this computation is, in general, immediate. In the $\tilde{x}$-coordinates of form (14), condition (C4) simply translates into the existence of an integer $1 \leqslant j \leqslant m$, associated to the $(\tilde{x}_j^1, \tilde{x}_j^2)$-dynamics (which is simply the original subsystem $\mathcal{S}_j$ transformed via an x_n-dependent change of coordinates and a suitable invertible feedback transformation, see the proof of Theorem 1), such that $\tilde{d}_j^1 + \tilde{d}_j^2 \tilde{d}_n \neq 0$. It is now clear that it is the dynamics of $\tilde{x}_n = x_n$ that determines whether the interconnected system $\tilde{\mathcal{S}}$ is flat or not (see Question 3).

The interconnected system $\mathcal{S}$ can always be brought into (14) and the change of coordinates leaves the interconnection variable unchanged (i.e., $\tilde{x}_n = x_n$). Notice, in particular, that in the new coordinates all vectors $\tilde{e}_\ell$ are zero (there is no direct influence of $\tilde{x}_n$ on the dynamics of $(\tilde{x}_\ell^1, \tilde{x}_\ell^2)$) and that the dynamics of the interconnected variable is still linear (since the change of coordinates is a linear one). Moreover, according to the proof of Theorem 1, in the $\tilde{x}$-coordinates, we always have $\tilde{x}_\ell^1 = \varphi_\ell(x_\ell) = C_\ell x_\ell$, where $\varphi_\ell(x_\ell)$ satisfies Proposition 1, that is, $\tilde{x}_\ell^1 = \varphi_\ell(x_\ell)$ is a flat output of the uncoupled subsystem $\mathcal{S}_\ell$. Recall that this was also the change of $\tilde{x}_\ell^1$-coordinate that had to be performed when transforming the uncoupled $\mathcal{S}_\ell$ into the double-integrator (10), see Proposition 1 and the comments following it. On the other hand, now the change of $\tilde{x}_\ell^2$-coordinate is of the form $\tilde{x}_\ell^2 = C_\ell(A_\ell x_\ell + e_\ell x_n)$ and, in general, it depends explicitly on the interconnection variable x_n.

The last statement of Theorem 1 describes the relations between the flat outputs of each subsystem $\mathcal{S}_\ell$, $1 \leqslant \ell \leqslant m$, and that of the interconnected system $\mathcal{S}$, thus answering Question 2. Consider the interconnected system represented in $\tilde{x}$-coordinates (and given by (14)). It is easy to see that $\varphi = (\tilde{x}_1^1, \ldots, \tilde{x}_m^1)$ (i.e., in the original x-coordinates, $\varphi = (\varphi_1, \ldots, \varphi_m)$, where each φ_ℓ is of the form $\varphi_\ell(x_\ell) = C_\ell x_\ell$ and is a flat output of the subsystem $\mathcal{S}_\ell$) can never be a flat output of $\tilde{\mathcal{S}}$. Indeed, with the help of φ, $\dot{\varphi}$ and $\ddot{\varphi}$, we can express all states $\tilde{x}_\ell^1$, $\tilde{x}_\ell^2$, and all controls $\tilde{u}_\ell$ but we will never be able to express $\tilde{x}_n$, thus contradicting flatness. Therefore, for the considered network (with a star topology, for which the connection variable has its own dynamics), the collection of flat outputs φ_ℓ of each subsystem can never be a flat output of the overall system. However the functions φ_ℓ determine all, except one, of its components and there are as many such flat outputs as integers j satisfying condition (C4) (or equivalently, (C5)). Indeed, for any j verifying the aforementioned conditions, any $(m-1)$-tuple of functions $\varphi_1, \ldots, \varphi_{j-1}, \varphi_{j+1}, \ldots, \varphi_m$, flat outputs of the uncoupled subsystems $\mathcal{S}_1, \ldots, \mathcal{S}_{j-1}, \mathcal{S}_{j+1}, \ldots, \mathcal{S}_m$, completed by a suitable function φ_j, given by (16), forms a flat output of the interconnected system $\mathcal{S}$. Observe that $\varphi_j = \bar{C}_j \bar{x}_j$, where $\bar{x}_j = \left(x_j^1, x_j^2, x_n\right)^\top \in \mathbb{R}^3$, necessarily involves the interconnection variable (if $\bar{C}_j$ is of the form $\bar{C}_j = (C\,0)$, then condition (16) implies that we have $Cb_j = CA_j b_j = 0$ and since $\mathrm{rk}\,(b_j\ A_j b_j) = 2$, the only possible solution is the trivial one), and it is actually a flat output of the three-dimensional system $\bar{\mathcal{S}}_j : \dot{\bar{x}}_j = \bar{A}_j \bar{x}_j + \bar{b}_j \bar{u}_j$, where $\bar{x}_j = \left(x_j^1, x_j^2, x_n\right)^\top$, and $\bar{A}_j$ and $\bar{b}_j$ are defined by (16).

Since flat outputs are not unique, a natural question is how to take advantage of their non uniqueness. When considering practical application, the answer to that question depends on the control objective that we want to achieve. For some examples, we send the reader to Kaminski et al. [2018] where changing the flat output is applied to the global motion planning of a non holonomic car in the presence of singularities, and to Do et al. [2022] where the question “among a set of flat outputs, which is better suited for control synthesis from the viewpoint of disturbance rejection performance?” is studied for a fixed-wing UAV model. Another situation when having several choices for the flat output may be useful is when measurements are needed and some outputs contain less measurement noise. In that case, it may be interesting to use more derivatives of a particular output component to decrease the number of derivatives of another (more sensitive) component.

4. ELECTRICAL CIRCUITS AND FLATNESS

The model we consider describes a Direct Current (DC) microgrid with star topology, which is standard in case of electrical distribution systems, and takes into account both currents and voltages of inductances and capacitors in the power converters (the subsystems of the global model), respectively, and the voltage in the DC bus (which plays the role of the interconnection variable). We focus on the situation where only buck converters are taken into account, and refer to the linear averaged models obtained when using Pulsed Width Modulation (PWM), see Mohan et al. [1995]. We use the standard Quasi-Stationary Line (QSL) approximation of the power lines, as in Iovine et al. [2022]. Then, we consider the following microgrid composed by m buck converters connected to the common DC bus:

$$\mathcal{S}:\begin{cases}\begin{cases}\dot{v}_\ell = \dfrac{1}{R_\ell C_\ell}(v_{DC}-v_\ell)-\dfrac{1}{C_\ell}i_\ell \\ \dot{i}_\ell = \dfrac{E_\ell}{L_\ell}u_\ell - \dfrac{R_{0\ell}}{L_\ell}i_\ell - \dfrac{1}{L_\ell}v_\ell\end{cases}, 1\le \ell\le m, \\ \dot{v}_{DC} = \dfrac{1}{C_{DC}}\displaystyle\sum_{l=1}^{m}\dfrac{1}{R_\ell}(v_\ell - v_{DC})\end{cases} \tag{17}$$

with $E_\ell > 0$ being the energies of the storage devices connected to the DC bus via the power converters, $v_\ell > 0$ the voltages and $i_\ell \in \mathbb{R}$ the currents of the power converters, and $v_{DC} > 0$ the voltage of the DC bus, i.e., the interconnection variable. The values of the constants R_ℓ, C_ℓ, L_ℓ, $R_{0\ell}$ and C_{DC} are positive, and introduced in Iovine et al. [2017]. The control inputs u_ℓ are the duty cycles of the power converters.

Flatness of two interconnected supercapacitors. In the sequel, we show that the system in (17) can be described by the class of systems in (6). Without loss of generality, we refer here to only two subsystems, i.e., two power converters, with states $x_\ell = (v_\ell, i_\ell)^\top$, $\ell = 1, 2$, interconnected via the DC bus voltage v_{DC}. Indeed, we consider $x = (\,v_1\;\; i_1\;\; v_2\;\; i_2\;\; v_{DC}\,)^\top$ and $u = (u_1\;\; u_2\,)^\top$, and therefore the A and B matrices in (7) and (8), respectively, can be defined as

$$A = \begin{pmatrix} -\frac{1}{R_1C_1} & -\frac{1}{C_1} & 0 & 0 & \frac{1}{R_1C_1} \\ -\frac{1}{L_1} & -\frac{R_{01}}{L_1} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{R_2C_2} & -\frac{1}{C_2} & \frac{1}{R_2C_2} \\ 0 & 0 & -\frac{1}{L_2} & -\frac{R_{02}}{L_2} & 0 \\ \frac{1}{R_1C_{DC}} & 0 & \frac{1}{R_2C_{DC}} & 0 & -\frac{1}{C_{DC}}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \end{pmatrix} \tag{18}$$

$$B = \begin{pmatrix} 0 & \frac{E_1}{L_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E_2}{L_2} & 0 \end{pmatrix}^\top. \tag{19}$$

The interconnected system described by matrices (18) and (19) is flat and both couples $\varphi = (v_1, v_{DC})$ and $\tilde{\varphi} = (v_{DC}, v_2)$ form a flat output. For $(\varphi_1, \varphi_2) = (v_1, v_{DC})$, the flat description (2) becomes:

$$v_1 = \varphi_1 \tag{20}$$
$$v_{DC} = \varphi_2 \tag{21}$$
$$i_1 = \frac{1}{R_1}(\varphi_2 - \varphi_1) - C_1\dot{\varphi}_1 \tag{22}$$
$$u_1 = \frac{1}{E_1}\left(\frac{L_1}{R_1}\dot{\varphi}_2 + \frac{R_{01}}{R_1}\varphi_2 - L_1C_1\ddot{\varphi}_1 - (\frac{L_1}{R_1} + R_{01}C_1)\dot{\varphi}_1\right) + \frac{1}{E_1}(1 - \frac{R_{01}}{R_1})\varphi_1 \tag{23}$$
$$v_2 = C_{DC}R_2\dot{\varphi}_2 + (1 + \frac{R_2}{R_1})\varphi_2 - \frac{R_2}{R_1}\varphi_1 \tag{24}$$
$$i_2 = -C_2C_{DC}R_2\ddot{\varphi}_2 - (C_2(1 + \frac{R_2}{R_1}) + C_{DC})\dot{\varphi}_2 - \frac{1}{R_1}\varphi_2 - C_2\frac{R_2}{R_1}\dot{\varphi}_1 + \frac{1}{R_1}\varphi_1 \tag{25}$$
$$\begin{aligned} u_2 = \frac{1}{E_2}\Big[&-\left(L_2C_2(1 + \frac{R_2}{R_1}) + L_2C_{DC} + R_{02}C_2C_{DC}R_2\right)\ddot{\varphi}_2 \\ &- \left(L_2\frac{1}{R_1} + R_{02}C_2(1 + \frac{R_2}{R_1}) + C_{DC}(R_{02} - R_2)\right)\dot{\varphi}_2 \\ &- L_2C_2C_{DC}R_2\varphi_2^{(3)} + (1 + \frac{R_2}{R_1} - \frac{R_{02}}{R_1})\varphi_2 \\ &- L_2C_2\frac{R_2}{R_1}\ddot{\varphi}_1 + (L_2\frac{1}{R_1} - R_{02}C_2\frac{R_2}{R_1})\dot{\varphi}_1 \\ &+ (\frac{R_{02}}{R_1} - \frac{R_2}{R_1})\varphi_1\Big], \end{aligned} \tag{26}$$

and trajectories for (v_1, v_{DC}) can be chosen arbitrarily. Several options are available. A first one would be to select independently the two flat outputs (i.e., how the two subsystems evolve over time), e.g.,

$$\varphi_1 = a_1e^{-\lambda_1 t} + a_2e^{-\lambda_2 t} + \bar{v}_{DC} \tag{27}$$
$$\varphi_2 = a_3e^{-\lambda_3 t} + a_4e^{-\lambda_4 t} + a_5e^{-\lambda_5 t} + \bar{v}_{DC}, \tag{28}$$

where $\bar{v}_{DC}$ is the desired equilibrium for v_{DC}. Another option, of interest for electrical systems as of current use, is to properly share the control effort among the subsystems. To this purpose, it is possible to consider φ_2 as in (28), and to use the constraint $u_1 = u_2$ to compute the needed φ_1.

5. PROOF OF THEOREM 1

Consider the interconnected system $\mathcal{S}$, given by (6), for which each $\mathcal{S}_\ell$ is assumed controllable. The equivalence of (C1) and (C2) follows from Fliess et al. [1995]. From (7)-(8), it is immediate that

$$\begin{aligned}\mathrm{rk}\,(B\;AB) &= \mathrm{rk}\begin{pmatrix} b_1 & A_1b_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_m & A_mb_m \\ 0 & d_1b_1 & \dots & 0 & d_mb_m \end{pmatrix} = \mathrm{rk}\begin{pmatrix} \mathscr{C}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathscr{C}_m \\ \xi_1 & \dots & \xi_m \end{pmatrix} \\ &= 2m, \end{aligned}$$

where $\mathscr{C}_\ell$ is the controllability matrix of $\mathcal{S}_\ell$ and $\xi_\ell = (0, d_\ell b_\ell)$. Hence, controllability of $\mathcal{S}$ is equivalent to $\mathrm{rk}\,(B\;AB\;A^2B) = 2m + 1 = n$, that is, the remaining direction is necessarily obtained with A^2B, showing the equivalence of (C2) and (C3). By a straightforward computation, it can be shown that

$$\mathrm{rk}\,(B\;AB\;A^2B) = \mathrm{rk}\left(\begin{array}{ccc|ccc} \mathscr{C}_1 & \dots & 0 & (A_1^2 + e_1d_1 - d_nA_1)b_1 & \dots & e_1d_mb_m \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \mathscr{C}_m & e_md_1b_1 & \dots & (A_m^2 + e_md_m - d_nA_m)b_m \\ \hline \xi_1 & \dots & \xi_m & d_1A_1b_1 & \dots & d_mA_mb_m \end{array}\right). \tag{29}$$

Multiplying the above matrix by the invertible block diagonal matrix $\mathrm{diag}(\mathscr{C}_1^{-1}, \dots, \mathscr{C}_m^{-1}, 1)$ does not change its rank. Thus, denonting $\mathscr{C}^{-1} = \mathrm{diag}(\mathscr{C}_1^{-1}, \dots, \mathscr{C}_m^{-1})$ and by M the $2m\times m$-matrix in the upper right corner of (29), we get

$$\mathrm{rk}\begin{pmatrix} Id & \mathscr{C}^{-1}M \\ \hline \xi & \zeta \end{pmatrix} = 2m+1,$$

where $\xi = (0, d_1b_1, \ldots, 0, d_mb_m)$ and $\zeta = (d_1A_1b_1, \ldots, d_mA_mb_m)$, which is equivalent to

$$\mathrm{rk}\begin{pmatrix} Id & \mathscr{C}^{-1}M \\ \hline 0 & \widetilde{\zeta} \end{pmatrix} = 2m+1,$$

where $\widetilde{\zeta}$ is the row-vector whose components are given by

$$\widetilde{\zeta}_\ell = d_\ell A_\ell b_\ell - \sum_{k=1}^{m} d_k b_k [\mathscr{C}^{-1}M]_{2k,\ell},\ 1 \leqslant \ell \leqslant m, \tag{30}$$

where $[\mathscr{C}^{-1}M]_{2k,\ell}$ denotes the entry in the $2k$-th row and ℓ-th column of $\mathscr{C}^{-1}M$. The above rank being $2m+1$ is equivalent to the existence of an integer $1 \leqslant j \leqslant m$ such that $\widetilde{\zeta}_j \neq 0$, which can be equivalently written in the compact form $\bar{d}\times M_j \times \bar{b} \neq 0$, where M_j, $\bar{b}$ and $\bar{d}$ are defined by (12) and (13) (see the paragraph before Theorem 1). This shows the equivalence of (C3) and (C4).

We next show the equivalence of (C2) and (C5). For $1 \leqslant \ell \leqslant m$, introduce the new coordinates $\tilde{x}^1_\ell = C_\ell x_\ell$ and $\tilde{x}^2_\ell = C_\ell(A_\ell x_\ell + e_\ell x_n)$, where C_ℓ is given by Proposition 1(i) (i.e., C_ℓ is such that $C_\ell b_\ell = 0$ and $C_\ell A_\ell b_\ell \neq 0$), and $\tilde{x}_n = x_n$ (i.e., the interconnection variable is kept unchanged), which is a valid change of coordinates for the interconnected system $\mathcal{S}$, and apply the invertible feedback transformation $\tilde{u}_\ell = C_\ell(A^2_\ell x_\ell + e_\ell dx_n) + (C_\ell A_\ell b_\ell)u_\ell$, $1 \leqslant \ell \leqslant m$. This brings $\mathcal{S}$ into form (14). Controllability is invariant with respect to the applied transformations, and a simple calculation shows that it is equivalent to the existence of an integer $1 \leqslant j \leqslant m$ such that $\tilde{d}^1_j + \tilde{d}^2_j\tilde{d}_n \neq 0$, where $\tilde{d} = (\tilde{d}^1_1, \tilde{d}^2_1, \ldots, \tilde{d}^1_m, \tilde{d}^2_m, \tilde{d}_n)$ describes the dynamics of the interconnection variable $\tilde{x}_n = x_n$ in the new $\tilde{x}$-coordinates.

To prove the last statement of Theorem 1, bring the system $\mathcal{S}$ into form (14), for which to simplify notation, we will drop the tildes. For any j satisfying (C5), i.e., such that $d^1_j + d^2_j d_n \neq 0$, define $\varphi_j = \bar{C}_j\bar{x}_j$, where $\bar{x}_j = (x^1_j, x^2_j, x_n)^\top$, and the nonzero row-vector $\bar{C}_j$ verifies

$$\bar{C}_j\bar{b}_j = \bar{C}_j\bar{A}_j\bar{b}_j = 0 \ \text{ and } \ \bar{C}_j\bar{A}^2_j\bar{b}_j \neq 0,$$
$$\text{with } \ \bar{A}_j = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ d^1_j & d^2_j & d_n \end{pmatrix} \text{ and } \ \bar{b}_j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{31}$$

Notice that such a row-vector $\bar{C}_j$ always exists due to condition $d^1_j + d^2_j d_n \neq 0$, which is equivalent to the controllability of the system $\bar{\mathcal{S}}_j : \dot{\bar{x}}_j = \bar{A}_j\bar{x}_j + \bar{b}_j\bar{u}_j$, with $\bar{A}_j$ and $\bar{b}_j$ given by (31), and that φ_j is actually a flat output of $\bar{\mathcal{S}}_j$. Take for instance, $\bar{C}_j = (-d^2_j, 0, 1)$, which verifies the above condition, and introduce the coordinates $\hat{x}^1_j = -d^2_j x^1_j + x_n$, $\hat{x}^2_j = \sum_{\ell\neq j}(d^1_\ell x^1_\ell + d^2_\ell x^2_\ell) + d^1_j x^1_j + d_n x_n$, and $\hat{x}^3_j = \sum_{\ell=1}^m \big(d_n d^1_\ell x^1_\ell + (d^1_\ell + d_n d^2_\ell)x^2_\ell\big) + (d_n)^2 x_n$, followed by a suitable static feedback transformation (invertible with respect to u_j), to bring the system $\mathcal{S}$ into the form:

$$\begin{aligned} \dot{x}^1_\ell &= x^2_\ell, & \dot{\hat{x}}^1_j &= \hat{x}^2_j, \\ \dot{x}^2_\ell &= u_\ell, & \dot{\hat{x}}^2_j &= \hat{x}^3_j + \sum_{\ell\neq j} d^2_\ell u_\ell \\ 1 \leqslant \ell &\leqslant m, \ell \neq j, & \dot{\hat{x}}^3_j &= \hat{u}_j, \end{aligned} \tag{32}$$

which is clearly flat with $\varphi = (x^1_\ell, \ldots, \hat{x}^1_j, \ldots, x^1_m)$ being a flat output. We deduce that the original interconnected system $\mathcal{S}$ is also flat at x^*, with the flat output $\varphi = (\varphi_1, \ldots, \varphi_m)$, where

- for $1 \leqslant \ell \leqslant m$, $\ell \neq j$, the function $\varphi_\ell = C_\ell x_\ell$ is a flat output of the ℓ-th linear subsystem $\mathcal{S}_\ell : \dot{x}_\ell = A_\ell x_\ell + b_\ell u_\ell$ (when no coupling is considered),
- for $\ell = j$, the function $\varphi_j = \bar{C}_j\bar{x}_j = \bar{C}_j\,(x_j\ x_n)^\top$ is a flat output of the linear system $\dot{\bar{x}}_j = \bar{A}_j\bar{x}_j + \bar{b}_j\bar{u}_j$, with $\bar{A}_j$ and $\bar{b}_j$ given by (16).

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