Did you know you can draw a huge number of infinite heights? The students' realization tree of the heights of a triangle

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In this paper we report on the analyses of the mathematical discourse of 7th-grade students about the solution of a task on the recognition of the heights of a triangle. Our aim is to describe through the commognitive lens the realizations that appear in the mathematical discourse of the classroom and to observe possible interactions between them. For this purpose, we construct and compare the expected and the actual students' realization trees, showing the richness of the realizations addressed by the participants in the mathematical discourse.

Keywords: Realization tree, commognition, heights, triangle.

Introduction

A wide literature in mathematics education has documented students' difficulties in drawing and recognizing the heights of a triangle and has highlighted that the most common difficulties are related to obtuse triangles and right triangles (Hershkowitz, 1989; Mariotti, 1995). Among the several reasons for explaining these difficulties, there is the influence of specific representations of the height, that are presented at school, and which become so popular and widespread that they are considered the concept itself by many students (Hershkowitz, 1989). In particular, at least in the Italian context, at the end of primary school students' idea of "height of a polygon" is strongly related to stereotyped representations of it and this is often the result of a univocal definition of height (Sbaragli, 2017).

Based on this scenario, and on the assumption that the notion of height is involved in many important mathematical results that students will face during their learning experiences, this study investigates the effects of a didactical approach for introducing students to the notion of heights of a triangle based on the presentation of multiple characterizations and representations. The effects are described in terms of the features of the height of a triangle, with respect to one side, that have become part of the classroom discourse.

Theoretical framework and research questions

We adopt Sfard's commognitive lens (2008) according to which discourse and its development are analyzed on their own, as main objects of research rather than as means to explore other constructs. We share the underlying assumption of this theory that thinking is not ontologically different from communicating and learning is a way of changing one's discourse. Commognition describes mathematical discourse as involving continuous transitions between *signifiers* that are "words or symbols that function as nouns in utterances" (Sfard, 2008, p. 154) and their *realizations* that are "perceptually accessible object[s] that may be operated upon" for producing narratives about the signifiers (Sfard, 2008, p. 154). These terms are used to emphasize that nothing is "there" to be represented, since mathematical objects are discursive objects. Moreover, some signifiers are more commonly used than others, but there is no one signifier that represents the object. This hierarchical role of realizations can be visually represented through *realization trees* (Sfard, 2008). In this study, we are interested in the classroom discourse on the heights of a triangle. The construction of the students' realization tree allows us to gain insights into its growth and richness as resulted from a didactical sequence, thus, giving information on the formation of the mathematical object "height of a triangle" in students' discourse. Usually when students deal with a new signifier, they use different realizations separately; the conscious reference to the same signifier by using many different realizations is a learning achievement which is the result of a *saming* process (Sfard, 2008).

One of the main elements characterizing the mathematical discourse are *narratives*, i.e. statements (written or spoken) formulated as descriptions of objects, relationships between objects or processes, which are subject to possible approval or rejection. The shared narratives of a classroom are those that are considered true; they may be different from those accepted by the scientific community. Moreover, mathematical discourse involves a consistent use of *routines*, i.e. specific repetitive patterns (Lavie et al., 2019) and of *visual mediators*, i.e. objects of symbolic, iconic, gestural nature to which we refer (Nachlieli & Tabach, 2012). For example, in geometry, the drawing of a figure can be a visual mediator in both paper-and-pencil and dynamic geometry environments.

In light of this theoretical lens, the research questions guiding our investigation are: (a) What realizations of the signifier *height of a triangle* can be identified in students' discourse? (b) Are there interactions between different realizations? If so, when do they occur?

Methodology

The data we present are collected in a 7th-grade class, during a video recorded lesson on the recognition of the heights of a triangle. The students were introduced to the mathematical object *height* through different possible realizations. In particular, two main realizations were proposed by the teacher, who was participating in a larger project aimed at designing and experimenting inclusive mathematical activities at middle school. We describe these two realizations as follows:

T1) *Segment perpendicular to a side, or to its extension, passing through the vertex not contained in that side (opposite).* It was discussed with students that this realization can be realized in two ways: as a segment drawn from the vertex (*T1-a*) or from the side (*T1-b*). The teacher also promoted the construction of heights using both physical (ruler and set square) and digital (*GeoGebra*) artifacts.

T2) *Height of the strip in which the triangle is inscribed.* This realization addresses the height of a triangle in terms of the height of the strip into which the polygon can be inscribed. A strip is determined by two parallel lines: one line contains a side of the triangle and the other the vertex that does not belong to the chosen side.

These two realizations are chosen because *T1* is very familiar among students at the end of primary school and it is largely proposed in school textbooks at different levels; on the contrary, *T2* was intentionally introduced for the first time by the teacher because it lays the foundation for looking at the length of the height as the class of congruent segments and for overcoming the common difficulty consisting in placing the height outside an obtuse triangle.

In this study we focus on a 1-hour lesson, conducted by the second author of the paper with the regular mathematics teacher. Students are given a task (Figure 1), involving an obtuse triangle.

Observe the figure.
Are all the four dotted segments heights of the triangle?
Explain why.

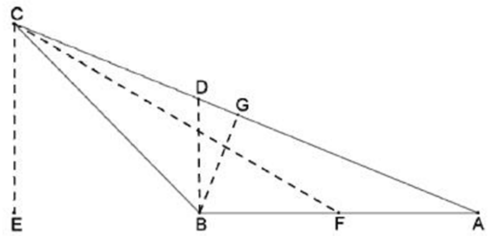

Figure 1: The heights recognition task

The task is designed for triggering the students' discussion on the four dotted segments. In order to answer the question, students have to argue why each of them can or cannot be a height of the given triangle. The explanations will inform us on the realizations that the class has developed until then.

The *a priori* realization tree

Building on Weingarden and colleagues' (2019) elaborations, the theoretical realization tree (Figure 2) is *a priori* constructed according to all the possible realizations of the height that students, who were exposed to specific geometrical activities on the height of a triangle, might refer to.

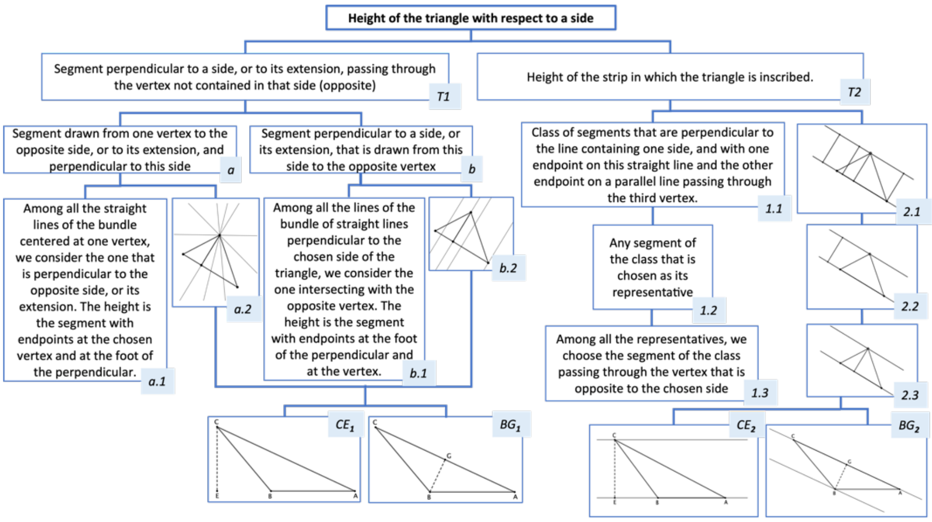

Figure 2: The a priori realization tree of the signifier *height of the triangle with respect to a side*

The tree has the root in the signifier "height of the triangle with respect to a side". From this root, two main branches with T1-realizations (on the left) and T2-realizations (on the right) develop; each branch contains both verbal and visual realizations. At the first level under the root, there are the main realizations that are explicitly used during the geometrical activities.

Focusing on the left side of the tree, the node at the first level splits into two branches (*a* and *b*). At this level the realizations (*a.1, a.2, b.1, b.2*) maintain some traces of the construction rituals. *T1-a* realizations characterize the height by starting from all the straight lines passing through a vertex and choosing the line that perpendicularly intersects the straight line upon which the side of the triangle lies; the second extreme of the height is reached in this way. Within the tree, we reported these realizations by referring to the bundle of straight lines centered in a vertex of the triangle. In these realizations the starting point is the chosen vertex. *T1-b* realizations characterize the height by starting from all the straight lines perpendicular to the side of the triangle and choosing the line that passes through the non-consecutive vertex. Within the tree, we reported these realizations by referring to the bundle of straight lines perpendicular to the chosen side of the triangle. In these realizations the starting point is the chosen side.

Focusing on the right side of the tree, starting from a realization with many heights (*1.1*, *2.1*), following the branch we can reach a realization where only one height passing through a vertex is considered (*1.3*, *2.3*). The coexistence of these different realizations in a student's tree presumes that a *saming* process was accomplished, as for the expert mathematicians.

The leaves at the lower level make explicit the realizations of the heights with respect to the given task. Although they are visualized only as visual realizations, since the given drawing provides the student with a common visual mediator, we use them also for verbal realizations such as "*CE*" or "*BG*" which make sense only coupled with the given drawing.

How the analyses were conducted

Data were cyclically analyzed, passing through several rounds of analyses. The preliminary step is the transcription of the classroom discussion by reporting all the components of students' discourse, i.e. verbal utterances, drawings, gestures. Working on this transcription, the first round is aimed at identifying the instances of realizations – labeled using a progressive number in square brackets – within the students' discourse. For example, an utterance as "*if you put the ruler on BA, you have also the height CE*" is coded as a verbal realization of the height with respect to the side *AB*.

The second round focuses on these instances of realization. The analysis aims at identifying the type of realization that students refer to. In particular, the coding is conducted according to the closeness of students' discourse with one of the descriptions of the height as a T1 or T2 realization. For example, a statement such as "*it is a height because it goes to the opposite vertex*" is coded as a verbal T1-realization of the segment traced from the side to the vertex, and the reference to the movement towards the opposite vertex allows us to label it as a T1-b realization. The closeness is also established according to the inferences of the researcher who knows the endorsed narratives developed by the class during the previous activities. For example, the expression "Paolo's method" is coded as a T2-realization because it was developed by the students for shortly referring to the ritual of drawing the heights inside the strip. This round allows us to qualitatively describe the characteristics of each realization and therefore provides useful information for answering the first research question.

These two rounds of analyses are fundamental for gathering the essential information on the realizations to be exploited for the construction of the actual realization tree. For the subsequent round, we start from an empty structure of the a priori realization tree; we read again the labeled transcription for filling the leaves with the corresponding realizations. For example, since the first instance of realization emerges in Alessio's discourse and it is coded as T1-b, we write "*[1]Alessio*" in the leaf named *b* (Figure 3). The unexpected realizations are included as new leaves with the label *NEW*. After this round of analyses we obtain a new tool for observing at a glance how the realizations were spread during the discussion. Moreover, this round allows us to observe which realizations of the height have become part of the classroom discourse and which not yet. The progressive numbers coupled with the name of the students allow us to keep track of the temporal development of the realizations during the lesson and of the students who participated in the mathematical discourse. The actual realization tree provides us with a tool for observing the possible interactions among different realizations, and therefore for answering the second research question.

Preliminary findings: the realization tree as a tool for analyses

The actual realization tree (Figure 3) highlights that there are some empty leaves for the expected realizations that are not addressed by students and new leaves for unexpected realizations that emerged within the classroom discourse.

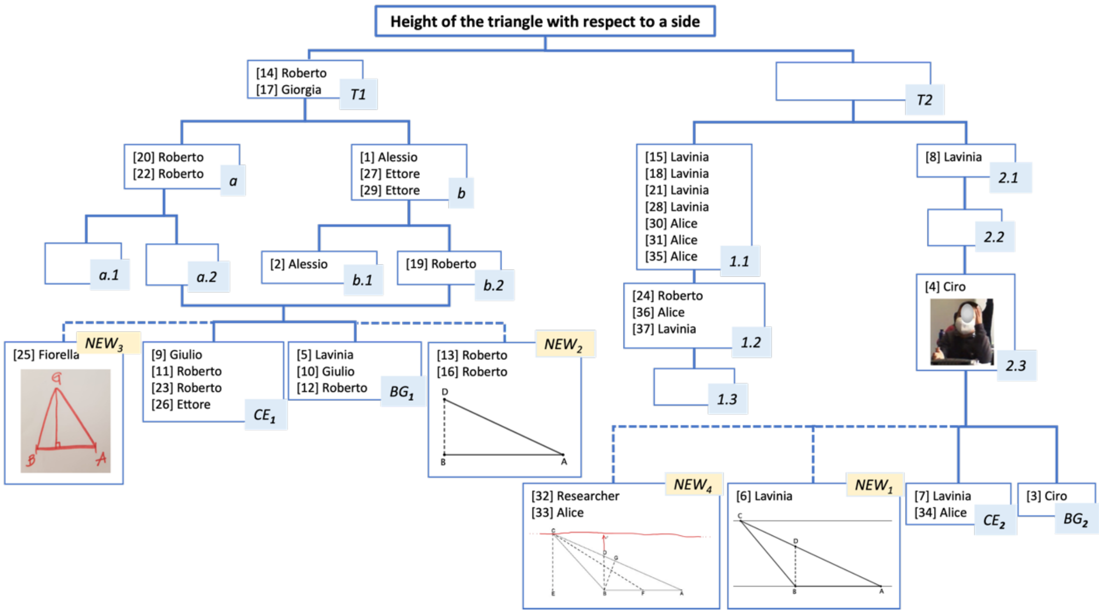

Figure 3: The classroom realization tree of the signifier *height of the triangle with respect to a side*

Comparing the a priori and the actual realization trees

As regards the expected realizations that are not addressed by students, we notice the bundle of straight lines centered in a vertex of the triangle. Although students refer to realizations of height that are drawn from the vertex towards the side of the triangle ([20], [22]), in their discourse we do not find elements that suggest the choice of the perpendicular line among the many passing through the vertex. This aspect can be justified by frequent references to the construction ritual through ruler and set square. Among the T1-realizations, this ritual shifts much of the students' focus toward realizations that implicitly make use of the bundle of parallel lines ([2], [19]); in this way, the perpendicularity is embedded into the use of the set square. Among the T2-realizations, the empty leaf in Figure 3 shows that there are no explicit references to the strip in which the triangle is inscribed. Rather, the students' discourse seems to be linked to the construction ritual, that is Paolo's method, which involves a line parallel to the side in focus.

As regards the unexpected realizations, there are leaves (see *NEW* in Figure 3) which report on the discourse on segments different from *CE* as realizations of the height for the side *AB*, and realizations which refer to triangles different from *ABC*. In the former case, students' discourse is focused on the segment *BD*, after that Lavinia triggers the attention on it ([6]) as a possible T2-realization of the height. Then she adds further details ([15]):

Lavinia: If you put the ruler on the segment AB, okay, are you with me? And then you take the set square. You start drawing some heights and BD is one of those.

Roberto accepts, to some extent, Lavinia's idea of considering *BD* as a height, but he addresses a different triangle ([16]):

Roberto: I think you're looking at a hidden triangle ABD and you think BD is a height.

Claims like this lead us to include another leaf in the actual tree (see New_2 Figure 3). This is a T1-realization since Roberto's discourse does not contain any reference to the strip.

The last new T2-realization is introduced by Alice who proposes a refined version of Lavinia's realization of the height. Alice starts describing how to draw a height for *AB* which passes through point *B* ([31]); the researcher supports her description by adding new visual elements on the visual mediator, which is shared with the whole classroom. Comparing the new realization with the segment *BD*, Alice explains why *BD* is not a suitable realization of height for *AB* ([33]):

Alice: Therefore, BD cannot be a height, because it has to reach at least... the intersection.

Building on Anita and Lavinia's discourse, students consider the possibility of having more than one height for *AB* and continue arguing on these T2-realizations.

Another unexpected realization is the one proposed by Fiorella. She considers another triangle, *ABG,* and shares her drawing with the classmates (see New_3 Figure 3). In particular, Fiorella uses this new visual mediator to justify the possibility for *BD* to be accepted as a realization of height, and in line with Roberto's discourse she stresses the need to change the reference triangle. However, she disagrees with Roberto concerning the choice of such a triangle.

Moving from T1 to T2 realizations

By looking at the progressive numbering of the labels in the realization tree (Figure 3) it is possible to observe the distribution of T1 and T2 realizations within the classroom discourse. Generally speaking, each student seems to be tied to a certain type of realization. Indeed, during the entire lesson some students, as Alessio, Roberto, and Giulio, focus on T1-realizations while other students, as Lavinia and Ciro, focus on T2-realizations. Although most of the students remain anchored in a certain type of realization and they do not spontaneously move back and forth between different types of realizations, we can notice that when Roberto moves from the T1 branch to the T2 ([23], [24]) some other students start reporting on T2-realizations in their discourse. Therefore, the global classroom discourse goes through a shift from T1 to T2 realizations. This shift in discourse develops around the segment *BD*. Lavinia claims that *BD* is a height, addressing T2-realizations ([15]). She refers to the use of the set square, describing the construction ritual for drawing "several heights" (T2-realization). She does not speak about moving the set square in order to trace the only segment that intercepts the vertex of the triangle. On the other hand, other students argue that *BD* is not a height, bringing arguments based on T1-realizations. In this respect, the exchange between Lavinia and Roberto is very interesting:

Lavinia: So BA is the segment. Did you know you can draw a huge number of infinite heights?
Roberto: But if they don't go to the opposite vertex...
Lavinia: But that doesn't matter! They don't have to go!

The turning point occurs when Roberto changes perspective and switches from the left to the right side of the tree, that is, when a T2-realization of height enters his discourse ([24]). This shift is shown in the tree: between [23] and [24] Roberto changes side. The corresponding discourse is:

Roberto: Do you know how to make a height? You have to put a ruler and a set square, from the opposite vertex you have to draw a line to the opposite side or to its extension, that's what the EC side is for.

Roberto: If you really want to make a height here [he points at a position on AB], at least you have to use Paolo's method.

The reference to "Paolo's method" seems to be key in shifting the attention of the class towards T2-realizations. Indeed, after Roberto's intervention there are many references to T2-realizations: [28], [30], [31], [32], [33], [34], [35], [36], [37].

At this point Roberto tries to convince Lavinia that in order for the segment *BD* to be a height it lacks an important feature: the point *D* does not intercept the strip.

Roberto: Oh thanks [*sarcastic tone*], but then DB is not a height if it doesn't reach there.

The first realizations of height in his discourse involve a segment with endpoints at one vertex and at the opposite side of the triangle, so his interventions are all placed in the left branch of the realization tree. Then Roberto seems to include in his tree, through a *saming* process, also segments not passing through any vertex of the triangle, and this happens when he starts participating in Lavinia's discourse.

The transition from T1 to T2 realizations seems to play a fundamental role in the classroom discourse. By remaining anchored to only T1-realizations of height, students fail to convince Lavinia that *BD* is not acceptable. When the focus of Roberto's discourse shifts to T2-realizations also the other students become participants in Lavinia's discourse and begin to use the same words and visual mediators. Finally, the classroom discourse finds a synthesis in the realization constructed by the researcher under the students' guidance (see New_4 Figure 3).

Conclusion

The analyses of the classroom discourse allow us to conclude that students spontaneously refer to both T1 and T2 realizations of height. By looking at the realization tree (Figure 3) it is possible to grasp the richness of the classroom discourse, involving different realizations of height: students refer to most of the expected realizations and they also introduce new ones. Each student remains essentially tied to one type of realization; therefore, we do not observe in their discourse interactions between T1 and T2 realizations. However, we can notice how the interaction is promoted by the attempt to match two discourses on two different realizations. This is evident in the discussion between Roberto and Lavinia, who refer to a T1 and a T2-realization of the height, respectively, while they are searching for a shared narrative about *BD*. The interaction becomes necessary for them to participate in the same discourse and Roberto plays a key role in moving from a T1 towards a T2-realization. In this way, he succeeds in finding a common ground with Lavinia and also in involving the whole classroom in the discourse. In particular, the "use of the set square" and "Paolo's method" are shared narratives referring to rituals that have been culturally constructed by the classroom and they are linked to both T1 and T2-realizations.

Comparing the two realization trees, we can claim that students include in their discourse most of the expected realizations. From the educational point of view, this is relevant since the richness of the realization tree seems to be a marker of the effectiveness of an unconventional – but in line with the solid findings of the research in mathematics education – introduction to the height of a polygon which explicitly works on different realizations. From the theoretical point of view, the study presents a first attempt of using realization trees as tools for *a priori* and *a posteriori* analysis of the classroom discourse in the domain of geometry, where the visual mediator (i.e. the drawing) has a specific and prominent role (Mariotti, 1995). In this perspective, the classroom realization tree needs further elaboration to become a more effective tool in terms of reporting on the students' discourse behind the construction of the tree and the role of the teacher's intervention in supporting such a discourse.

Although this is an exploratory study and the findings are quite local, we hope that considerations emerging from this paper can provide educators and researchers with food for thought, in particular concerning the didactical value of promoting a widespread use of many different realizations for the same signifier as a means for triggering the students' participation into mathematical discourse.

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