A state-of-the-art review on uncertainty analysis of rotor systems

Chao Fu, Jean-Jacques Sinou, Weidong Zhu, Kuan Lu, Yongfeng Yang

Mechanical Systems and Signal Processing 183 (2023) 109619

Contents lists available at ScienceDirect

Mechanical Systems and Signal Processing

journal homepage: www.elsevier.com/locate/ymssp

A state-of-the-art review on uncertainty analysis of rotor systems

Chao Fu [a], Jean-Jacques Sinou [b], Weidong Zhu [c], Kuan Lu [a], Yongfeng Yang [a,*]

[a] Institute of Vibration Engineering, Northwestern Polytechnical University, Xi'an 710072, China
[b] Laboratoire de Tribologie et Dynamique des Systèmes, UMR CNRS 5513, École Centrale de Lyon, 69134 Écully, France
[c] Department of Mechanical Engineering, University of Maryland, Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250, USA

ARTICLE INFO

Communicated by Luigi Garibaldi

Keywords:
Rotor system
Uncertainty
Faults
Stability
Reliability and optimization
Vibration control

ABSTRACT

Uncertainty handling and analysis of rotor systems have been active research areas during the last two decades. This paper provides a state-of-the-art review on the research progress of uncertainty treatment in rotor systems. Firstly, sources of uncertainties are identified and then classified according to their nature. Secondly, the most popular uncertainty analysis methods are summarized based on different types of uncertainties, including their basic principles, strengths and limitations. Thirdly, the dynamic characteristics of rotor systems under various uncertainties are described in detail, i.e., the natural characteristics and unbalanced responses, faulted rotor dynamics, stability problems and parameter identifications. Then, the reliability analysis, optimizations and vibration control for uncertain rotor systems are reviewed. Finally, comments on the latest research progress are made and outlooks on future research directions are highlighted. It is expected that this review will provide useful guidelines for designers and researchers towards the more efficient handling of uncertainty and robust dynamic analysis for rotor systems as well as insightful prospects, showing where extra efforts are needed.

1. Introduction

A rotating system is one of the most fundamental motion elements in mechanical systems and has wide applications in areas such as energy, aeronautics and astronautics. Many of these applications including aero-engines, gas turbines and electric generators serve a critical role in the civil and military sectors. One of the challenges for engineers and researchers working in rotating machinery is to provide a deep understanding of various dynamics phenomena encountered during machine operations and to model and predict the vibration responses of complicated rotor systems [1–4]. Thus, the robust dynamic characteristics analysis, design optimization, and effective maintenance of rotor systems are always hot research areas in the engineering community to ensure the safe and smooth operations of such rotating systems. Developments in engineering and science make extra demands on the performance and functionality of rotor systems, the robust design with high reliability, efficient optimization to satisfy different requirements, and precise dynamics assessments and control have become increasingly important. This requires not only taking into account the complexity of rotors through detailed mathematical modeling but also promoting an innovative vision and more sophisticated approaches to progress towards more reliable predictions of the dynamic behaviors of rotors. Thus the pressing needs originated from engineering draw the attention of researchers to the uncertainty quantification (UQ) [5] in rotating systems, which forms a consensus that UQ plays a prominent role in the modern rotordynamics and predictive maintenance. Advances in numerical tools and enormous improvements

* Corresponding author.
E-mail addresses: fuchao@nwpu.edu.cn (C. Fu), yyf@nwpu.edu.cn (Y. Yang).

https://doi.org/10.1016/j.ymssp.2022.109619
Received 5 October 2021; Received in revised form 3 April 2022; Accepted 9 June 2022
Available online 5 August 2022

Nomenclature

UQ	uncertainty quantification
QoI	quantity of interest
AMB	active magnetic bearing
MCS	Monte Carlo simulation
PCE	polynomial chaos expansion
RMT	random matrix theory
PDF	probability density function
CDF	cumulative distribution function
LHS	Latin hyper-cube sampling
PM	perturbation method
K-L	Karhunen-Loève
EOM	equation of motion
LST	least square technique
ROM	reduced order model
MET	maximum entropy theory
PD	probability distribution
MCMC	Markov Chain Monte Carlo
FST	fuzzy set theory
FEM	finite element method
MF	membership function
IA	interval algorithm
TIM	Taylor interval method
TIF	Taylor inclusion function
CIF	Chebyshev inclusion function
FRF	frequency response function
HBM	harmonic balance method
AFT	alternating frequency/time technique
ANM	asymptotic numerical method
PAI	polar angle interpolation
RBDO	reliability-based design optimization
RDO	robust design optimization
PVC	passive vibration control
AVC	active vibration control
LMI	linear matrix inequalities
PID	proportional-integral-derivative
LQR	linear quadratic regulator
RMSE	root mean square error
LOO	leave-one-out
QQ-plot	Quantile-quantile plot

in modern computing resources have made the missions possible. In fact, a large number of works [6–11] emerged on the UQ of rotordynamics and the number is rising rapidly with time.

The uncertain factors in practical large-scale rotor systems can be diversified regarding their generation process. Indeed, there are many ways uncertainties can be introduced into rotor systems and the dynamics are susceptible to these ubiquitous uncertainties. Manufacture errors will cause the geometries of essential components of rotor systems to vary from time to time although controlled by tolerance. The material properties can also have dispersions during this process which bring about anisotropy or defects. The external excitations, thermal effects during service, wear of critical components and material degradations are difficult to quantify or even sometimes unpredictable. In addition, model simplifications, measurement and simulation errors contribute to the complexity. Those variabilities have deep influences on the natural characteristics and dynamics of rotor systems, causing performance deviations from the initial design, vibration deteriorations and even unexpected dynamic behaviors. For example, the vibrations of well-balanced rotors on balancing machines can easily exceed the allowed amplitudes when assembled on a real aeroengine, which can degrade the performance of the machine during operations and cause critical safety issues.

When different uncertainties are considered, traditional and deterministic methods in rotor dynamic analysis will be insufficient to properly handle the aforementioned uncertainties associated with rotor systems [12–14]. The dynamic responses of an uncertain mechanical system can show moderate nonlinearity with respect to the uncertainty input although the system itself is linear. Therefore, certain techniques dedicated to the UQ of rotordynamics must be incorporated. From a computational point of view, UQ aims to quantitatively estimate various kinds of uncertainties and their predicted effects on the dynamics of rotor systems. However,

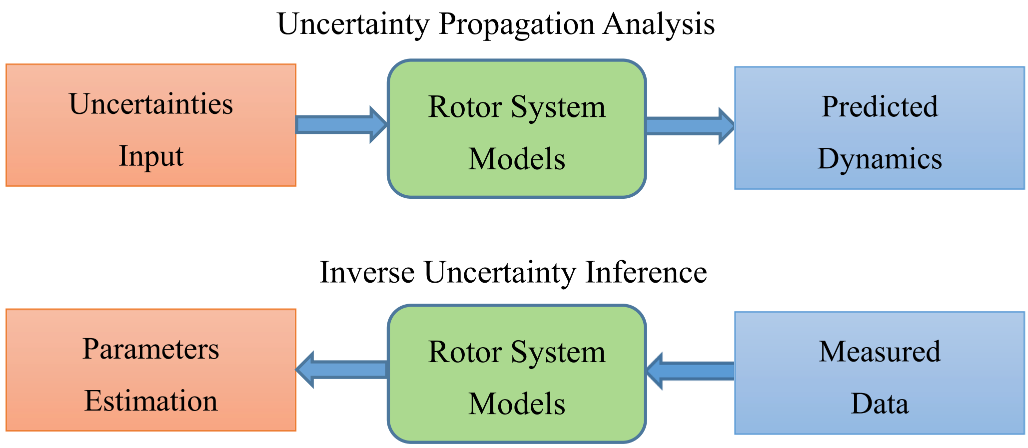

Fig. 1. Uncertainty propagation and inverse inference in rotor systems.

appropriate descriptions or mathematical modeling of uncertainties should be done in the first place. This preliminary stage, which is sometimes neglected, corresponds to one of the key phases essential for optimal use of the UQ tools and the proposal of adequate responses to the real problems faced by engineers and their need to robustly predict the vibratory behaviors of complex rotor systems. Generally, a specific taxonomy of various uncertainties is used to characterize their properties and sources although an appropriate selection is somewhat tricky due to the quality of information available and the discrepancy in perspectives across the disciplines, such as the structural reliability and robust control theory [15–17]. Nevertheless, the domains of the uncertainties have fundamental impacts on the applicability of different UQ methods for uncertainty analysis in rotordynamics. Researchers developed many models to suit different cases, such as random, fuzzy and interval variables. When the uncertainties are classified, suitable UQ methods need to be chosen according to their respective prerequisites, working mechanism, numerical performance and whether they are intrusive or non-intrusive. In different uncertainty domains, various UQ approaches [18–20], as well as solutions to improve efficiency and accuracy, have been proposed [21–23]. Not all procedures are available in specific situations and principles exist for selection [24].

To date, researchers have carried out many meaningful investigations in various aspects of rotor dynamics within the uncertain context and sensitivity analysis, such as calculating the Sobol index [25]. Typically, two types of uncertainty analysis frames can be found in the literature according to their computation purposes as shown in Fig. 1, i.e., forward uncertainty analysis or propagation [26] and backward uncertainty analysis or inverse inference [27]. The first category is mainly concerned with the uncertainty propagation from uncertainties in the system input to the dynamic output [28–31]. The input contains all kinds of sources and is represented by the rotor model parameters or loads, such as stiffness, damping, geometry and mass unbalance. Through the UQ performed based on the rotor physical model, the effects of different uncertainties on the dynamics are predicted. The output could be any of the dynamic characteristics such as the unbalanced response and it can be termed as the quantity of interest (QoI). The backward inference is data-driven and often involves estimating and reducing uncertainty in the input from measured data [32]. Of course, one can carry out forward uncertainty analysis based on reverse inference. The applications of the backward analysis typically concern parameter identifications and model calibration/updating. Currently, as evidenced by the literature, considerably more efforts have been put into the forward uncertainty analysis [33–35]. It is natural since the backward analysis is generally a tougher task. In essence, they are connected and rely on each other for reliable computations.

Although a large number of studies have been reported in rotordynamics with consideration of various uncertainties, a comprehensive state-of-the-art review in this field is unavailable. Thus, this work attempts to give a systematic overview of uncertainty sources identified in the literature and their classifications, popular analysis approaches used and their strengths and limitations as well as the reported rotor dynamic characteristics under uncertainty. The scope of this paper mainly covers uncertainty propagations but also summarizes essential efforts in the inverse analysis. For interested readers, the recent progress in reliability analysis of rotor systems and vibration control under uncertainty is provided as well. The discussions on the research trends and gaps and future directions that still need great efforts are summarized. It should be mentioned that this paper only includes works with explicit use of uncertainty. The scope of rotor systems in this review is mostly concerned with the classical shaft-disk-bearing rotor systems. Rotating elements in helicopters, pure motors, wind turbines blades and drones or unmanned aerial vehicles are not included because they largely belong to other disciplines.

The rest of this paper is organized as follows. Section 2 identifies the main sources and classifications of various uncertainties associated with rotor systems. The popular uncertainty analysis procedures and justifications are described in Section 3 according to the domain of uncertainty. Detailed dynamic characteristics of rotor systems under uncertainty are reviewed and discussed in Section 4. In Sections 5 and 6, the progress of rotor reliability assessment and vibration control with uncertainty is summarized. Section 7 identifies the main gaps and limitations, and the outlooks for future research directions are proposed. In the last section, conclusions are drawn.

2. Sources of uncertainty and classifications

In this section, the main sources of uncertainties in rotor systems studied in the literature are summarized and analyzed first. Then, the classifications and mathematical signatures for various uncertainties are explained, which is a prerequisite step to allow the modeling of rotor systems under uncertainty and choosing the appropriate uncertainty analysis methods.

2.1. Sources of uncertainty

An engineering rotor system is complicated in nature and almost every part of its life cycle can introduce uncertainties from manufacturing to its operations and further maintenance. Uncertainty makes the rotor simulation model not being able to represent real situations. In the literature, researchers presented inconsistent categorizations of rotor system uncertainties. This review attempts to identify and summarize those investigated uncertainties according to their types and relationships to the rotor dynamic system. From a perspective of general physical rotor systems, uncertainties can arise in the following main forms: (1) model parameters uncertainty, (2) boundary conditions uncertainty, (3) external loads uncertainty, (4) faults uncertainty, (5) observations uncertainty and (6) model uncertainty. It should be noted that the above categorization is not strictly rigorous or exhaustively according to the taxonomies in uncertain structural dynamics and overlay a little with each other although a unified approach cannot be found as well in the latter. However, the provided characterization method can help the rotordynamics community to identify and manage various uncertainties more efficiently based on the nature of the discipline and research interests. In the rest of this subsection, we explain elaborately the four types of uncertainties with their causes.

(1) Model parameters uncertainty. This type of uncertainty is often intuitively considered in the first place by many researchers in the field of structural dynamics, which can be explained by an easy and well-accepted understanding of their origins in real engineering practice. The physical model parameters of rotor systems exhibit uncertainty in different ways. First of all, the manufacture of rotors is controlled by tolerance and sometimes contains errors or defects. The materials used can distribute unevenly in components such as shafts and discs, namely, the material dispersions. Thus, the geometries and material properties of the system will show certain degrees of uncertainty. What also affects them is the working environment, e.g., temperature and medium. These are the so-called environmental variations and manufacturing tolerances [36,37]. Other reasons for parameter alterations are assembling errors, components wear and material degradation. The assembling state is mutually agreed to have significant influences on the dynamics of rotors and engineers can produce different assembles, which involve alignment, tightening of bolted joints and so on [38]. During long-time services, the materials of critical components will develop progressive asymmetrical wear and degradation due to scratches and friction during abnormal operations like rubbing, which is uncertain and very difficult to quantitatively describe at a specific time. It also contributes to the imbalance distributions of rotor systems as inherent randomness of a natural process. Although mass imbalance is often regarded as a typical fault, it cannot be completely eliminated and thus we treat it as a general model parameter that corresponds to the most common internal excitation in a rotor system. The uncertainties of this type finally take the form of variabilities in the model parameters of rotor systems, including shaft lengths and diameters [39], Young's modulus (shaft stiffness) [40], density and damping ratio [41], disc diameters and mass [42], residual imbalance distribution [43] and its magnitude and phase [44]. Some researchers also consider the uncertain axial stiffness [45] and cumulative pitch deviations [46]. Another sub-type is uncertainty in the external loads on rotor systems. Apart from the internal load of imbalance excitations, rotor systems are also subject to various external loads. They have direct effects on the dynamic responses of rotors and cannot escape from uncertainty as well. Specifically, these uncertainties are considered in external excitations on rotor systems including white noise disturbance [47], base excitations [48] and seismic excitations [49–51]. It should be noted that generally several origins for model parameter uncertainties are taken into account by combining several aspects that come from both geometries [52] and material properties of the rotor system. The main drawbacks of multi-considerations are the potential complexity in the uncertainty propagation analysis, and the difficulty of providing a precise understanding of the observed phenomena and the roles of individual uncertain parameters.

(2) Boundary conditions uncertainty. It refers specifically to the indeterministic variations in rotor supporting structures. This is also a major source of uncertainties and has aroused much attention of researchers worldwide. As explained in [41], the supporting stiffness of a rotor system consists of three main parts, i.e., the support structure, the squeeze film damper and the bearing. Nevertheless, they are affected significantly by the loads, assemble states and temperatures. The connecting structures can be varied and provide unpredictable stiffness to the rotor system. Most importantly, nearly all commonly used bearings in rotor systems, e.g., journal/slider bearings, rolling element bearings and active magnetic bearings (AMB) are exposed to uncertainties and they have become the special focuses of many researchers. Journal bearings operate with lubricant fluid forces to support the rotating shaft, which is significantly dependent on the loads, rotating speed and lubricant viscosity [53–55]. The latter has a strong relationship with lubricant temperature and the temperature can be difficult to control. What's more, a motor is unlikely to provide strictly constant drive torques on the rotor and subsequently the angular speed is unsteady. Recent researches also suggest that the micro-abrasive effects and surface roughness in hydrodynamic journal bearings also affect their performances [56–58]. As a result, fluid-induced forces acting on the shaft are uncertain or we can interpret as the bearing or seal dynamic coefficients, i.e., stiffness and damping, are uncertain quantities [59]. It is critical for the rotor system to understand bearing reaction forces since they are major factors causing vibration instabilities such as the well-known oil whirl and oil whip [60]. Similar fluid nature uncertainty is also found in labyrinth seals [61]. Combined with assembling and manufacturing errors, the asymmetric coefficient and relative phase of two bearings in a rotor system are studied as uncertainties [62]. Rolling bearings generally have multiple substructures and are susceptive to defects caused by manufacturing and wear. Ball-inner race contact uncertainties are considered when assessing the rotor's positioning precision [63]. The AMBs are very high-precision components and significant influences by uncertainty should be avoided.

Table 1
Sources of uncertainty in rotor systems.

Type	Sub-types	Parameters	References
Model parameters	Geometries	Shaft length and diameter, disc diameter and thickness	[36,39,46,91–93]
	Material properties	Young's modulus	[39,40,92,94]
		Density, damping, disc mass and inertial moments	[39,41,91,95]
	Imbalance	Amplitude and phase	[40,42–44,96–98]
Boundary conditions	Joints and couplings	Stiffness and damping	[36,41,45,92,94,99–102]
	Journal bearings	Lubricant viscosity and clearance	[43,60,62,67,103–109]
	Rolling bearings	Contact stiffness	[63,110]
	AMBs	Permeability and current	[64,111,112]
External loads	Noise excitations	Random process parameters	[47,69–72,11365–68]
	Axial loads	Amplitude and phase	
Typical faults	Asymmetry	Asymmetry depth	[73,114]
	Bow	Amplitude and phase	[73]
	Crack	Crack depth	[74,75,115]
	Misalignment	Lateral and angular magnitude	[73,84,116,117]
	Rubbing	Contact stiffness, initial clearance and friction coefficient	[65,77–79,118]
Observation	Measurement	Noise and errors	[86–89,119]
Model uncertainty	–	–	[39,90,120–122]

Wang et al. [64] established a two-degree-of- freedom AMB model with uncertain parameters induced by the environment and materials, such as magnetic permeability.

(3) External loads uncertainty. This category mainly corresponds to the variabilities in any external loads on rotor systems, such as the unsteady aerodynamic forces, axial loads [65–68] and external noise disturbances [47,69–72].

(4) Faults uncertainty. In this type, the uncertainties are introduced by typical faults in rotor systems. Even though this category can be considered to have correlations with model parameter uncertainty, we make this type a separate source to guide the reader's attention to their desired topics because faulted rotor dynamics are always the special focus of many engineers and researchers. Generally, the dynamics of rotor systems with typical faults are more difficult to understand and can show more complex vibration behaviors [1], which often are nonlinear, with consequently the appearance of new amplitude peaks. Currently, the investigated faults involving uncertainties are shaft asymmetry, misalignment, cracks, rubbing, initial bow and coupled faults. Faults related parameters can exhibit uncertainties. On the one hand, early faults are hard to detect, and their development is sometimes unpredictable. On the other hand, modeling or defining accurately the intermittent faults or non-surface defects is not an easy task even if a fault is diagnosed. Therefore, it is practically beneficial to model them within the uncertainty context. Shaft asymmetry typically depicts the irregular cross sections of the shafts due to manufacture or crack. The asymmetry depth is considered uncertain by Didier et al. [73]. Crack faults are often induced by defects, and wear and their forms are affected by the weight of the system. The bending stiffness of the cracked shaft is decreased compared with an intact one. Multiple researchers investigated the uncertain factors of crack faults such as crack depth [74–76]. As engines get more compact, the clearance between the stator and rotor is becoming smaller. Other faults and imbalances will increase the vibration amplitudes of rotors operating at high rotation speeds. Thus, rotor and casing contact is prone to occur. However, this process is rather complicated and involves several structures and disciplines, such as mechanics, friction and thermal effects. To predict the inherent variabilities in the rubbing process, works have been done to take the initial clearance, contact stiffness and friction coefficient uncertainties into consideration [77–79]. The bow fault represents the initial deformation of rotors caused by thermal effects or self-weight. The uncertainty in bow amplitude and phase was studied in Didier's work [73]. Uncertain misalignments in rotor systems are normally found in two cases, i.e., parallel and angular misalignments [80–82]. They generate reaction forces and moments in the coupling and deteriorate the vibrations of the system. Uncertainty in this fault is related to assembling and evolution of status during operation. The amplitudes, i.e., offset of centerlines in parallel misalignment and angle in angular misalignment, are variables to be studied [73,83–85].

(5) Observations uncertainty. It covers the errors and noise in measurement, data recording, storing and computation rounding or errors. Data or signal is deemed always accurate in traditional rotor dynamics analysis and its uncertainty has not been fully assessed yet. The data scarcity or information inadequacy is enriching observation uncertainty too. This type of uncertainty often interacts with others and influences the rotor system simultaneously, causing it difficult to accurately track and separate. From the reported works, some efforts are dedicated to the measurement uncertainty analysis in rotordynamics [86–89].

(6) Model uncertainty. Modeling error is another important type of uncertainty, which currently has not attracted sufficient attention in rotordynamics. It represents the discrepancy between the model output and any experimental observations caused by simplifications and imperfections of the constructed model, but this type of uncertainty is not included in the parameters of the established computational model (those uncertainties represented in the model are referred to as parametric uncertainties in this context). Note that although the computational model can be refined using model updating techniques, it is improved from certain angles and it is never identical to the real system. The nonparametric probabilistic method [5] is found to be capable of dealing with model and/or parametric uncertainties in computational models of rotor systems [39,90].

A brief summarization of the investigated sources and specific forms of uncertainties in rotor systems found in the literature is illustrated in Table 1 with representative references. It is worth mentioning that rotor systems can operate under multiple and multifold uncertainties described in the table. The number of uncertain parameters may increase significantly when dealing with

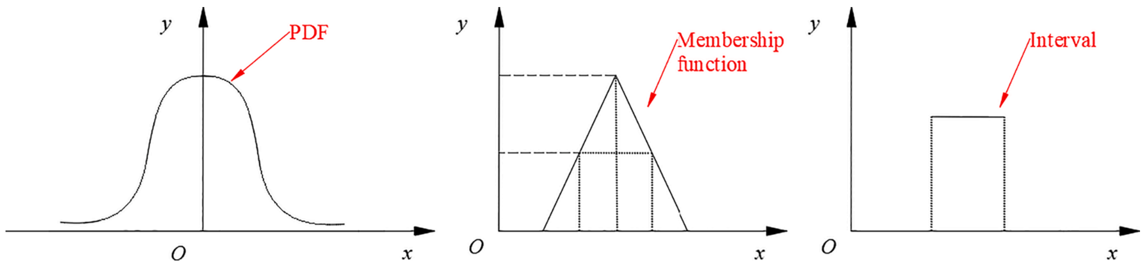

Fig. 2. The most popular uncertainty models used in uncertain rotordynamics.

realistic models and leads to major difficulties in uncertainty analysis. Therefore, this may require the development of advanced numerical techniques to be able to take all these uncertainties into account simultaneously with an acceptable computational time. We must point out that it would be impossible in the current context to provide a full list of all uncertainties in various rotor systems in engineering due to different structural configurations and environments. However, anyone designing and analyzing rotor dynamics with the explicit use of uncertainty analysis may consider the above-mentioned uncertainty sources.

2.2. Classifications

While tracking the sources provides the physical point of view of different uncertainties, classification of them is more concerning the mathematical aspect since a specific taxonomy will make the modeling of the rotor dynamic systems with various uncertainties and actual computations possible. The most recognized taxonomy proposed in risk assessment classifies various uncertainties into two types: aleatory and epistemic uncertainties [123–125]. Aleatory uncertainty is used to describe the inherent variability or randomness in a physical system and natural process. It is irreducible and intrinsic, which means human will not be able to suppress it from learning more knowledge or collecting more information and data. Therefore, it is also called objective uncertainty. The most common examples of this type of uncertainty in rotor systems are the manufacturing tolerance and error, environmental variations and the related model parameters. Instrumental or measurement errors are sometimes also included in this type. Epistemic uncertainty usually arises from incomplete knowledge of the physical system and human factors. Due to the lack of knowledge, simplifications and assumptions are made in any phase of the modeling. That is typical epistemic uncertainty. It can be reduced or even eliminated through gathering more information and gaining more knowledge of the rotor system under study. Thus, it is also named subjective or cognitive uncertainty. Typical examples are the boundary conditions and physical laws that remain misunderstood.

Generally, aleatory uncertainty is often associated with a probabilistic description based on the fact that enough statistical information is gathered. Quantifying epistemic uncertainty is more difficult since less prior information is available. The epistemic uncertainty can be processed by the non-probabilistic theories or probabilistic theories with PDFs assigned. It is important to understand the major difference in the treatment of these two types of uncertainties from the engineer's point of view. The epistemic uncertainties are related to the lack of information (i.e., evaluated points in the predefined design space). More model evaluations through more evaluated points in the design space of interest can significantly reduce epistemic uncertainties, the associated disadvantage is of course the increased cost. On the contrary, as far as aleatory uncertainties are concerned, adding more points would only lead to a better knowledge of the chosen probabilistic description and therefore a more reliable accuracy of output results via the uncertainty analysis process. Of course, one of the limitations associated with both aleatory and epistemic uncertainties is the compromise between reasonable calculation times and confidence in the accuracy and reliability of the output results. In other words, the engineer's expertise and the most accurate prior knowledge of the identification and quantification of various uncertainties are important assets for the construction of an efficient mathematical model to propagate uncertainties and deduce the associated vibrational response of a rotor system. This naturally puts the interest in the inverse uncertainty inference process back at the focus of the discussion.

Researchers tried to use different mathematical models to make the best of available information and reflect the lacking extent, such as evidence, fuzzy or imperfect and interval variables. In another sense, it aims to benefit most from the limited data. In uncertain rotor systems, the random or stochastic modeling of uncertainties is the most common practice [40,126–128] and relevant theoretical frameworks are more sophisticated. Then, the accurate stochastic modeling of uncertainties is a topic of primary importance [129]. Indeed, the construction of the prior stochastic models and the underlying parameters must be chosen to best fit the input data and to try to reproduce their impacts on the vibrational response, via the uncertainty propagation process, as faithfully as possible. Fuzzy variables are the second choice as these models need less prior information [53,100,106,130]. Interval modeling of uncertainties emerged only recently but is gaining momentum due to its relief of requirements in other models [41,42,95,131]. The most popular uncertainty models used in rotordynamics, i.e., the probabilistic, fuzzy and interval variables, are illustrated in Fig. 2.

Different modeling of uncertainties is important and will lead to different choices of the UQ approaches in further analyses [132]. One should decide the uncertainty model according to the specific problem encountered and how much prior information is available. Even if it is a common routine that an uncertain factor is described as aleatory, i.e., random variable, the researcher can choose to use the non-probabilistic models to avoid human assumptions when a reliable probability distribution model is not obtained. This is not rare because large-scale rotor systems are often equipped in heavy-duty rotating machineries, such as gas turbines and aero engines. It

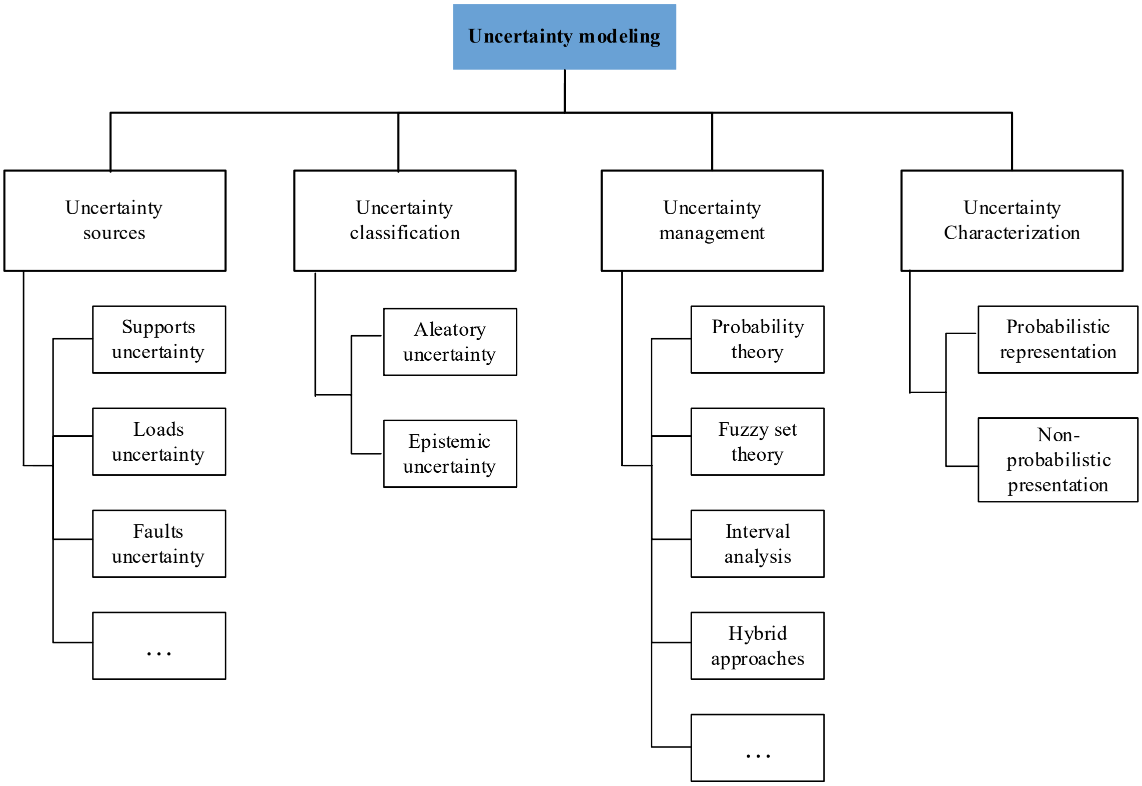

Fig. 3. General tree map of uncertainty modeling.

is obviously not easy to obtain sufficient statistical data because of the limited samples and high expenses for long-time tests. Occasionally, hybrid uncertainty modeling can be a better choice [13,36,118,133–136]. For example, the probability-interval hybrid uncertain model [137] is widely employed for mechanical systems subject to both probabilistic and non-probabilistic uncertain parameters.

The above discussion illustrates the uncertainty modeling in general rotor systems, which involves the identification of uncertainty sources, classification of the uncertainty identified, management or handling of the uncertainty and lastly characterization using the probabilistic (parametric/non-parametric) and non-probabilistic representations. A general tree map of this process is shown in Fig. 3.

3. Uncertainty quantification methods

In this section, the popular uncertainty analysis methods in rotor dynamics are summarized, and their strengths and limitations are illustrated. It must be pointed out that UQ approaches shown in this section are by no means exhaustive and there are plenty of them in the structural uncertain dynamics and risk assessment. However, the most common and prevailed methods are compared according to the current research progress in uncertain rotor systems. Based on the uncertainty types and their mathematical models, two main kinds of uncertainty analysis methods are available, i.e., probabilistic methods [138] and non-probabilistic methods [139]. The popular probabilistic approaches in UQ are the Monte Carlo simulation (MCS) [140–142], polynomial chaos expansion (PCE) [143,144], random matrix theory (RMT) [145,146], Bayesian inference [147] and Kriging surrogate [148,149]. All methods have applications in uncertain rotordynamics. The common non-probabilistic methods include the fuzzy theory [150], evidence theory [151], imprecise theory [152] and interval approaches [153,154]. Among them, applications of the fuzzy and interval methods are common in rotordynamics uncertainty analysis. It should be noted that these methods are different choices to quantify uncertainties with their pros and cons.

In what follows, the core principles of the most popular methods applied in rotordynamics, and a comparison will be demonstrated.

3.1. Probabilistic methods

Probabilistic uncertainty analysis methods are widely used in uncertain rotor systems. They rely on rigorous probability theories with the prerequisite that the PDFs (or joint PDFs) of uncertainties are known. Specifically, there are many probabilistic or stochastic formulations reported in the literature for the purpose of uncertainty analysis in rotor systems, such as the MCS, PCE and Kriging. In this subsection, the principles and main ideas are explained.

3.1.1. Montel Carlo simulation

The MCS is a simple but powerful tool for statistical analysis, which is very intuitive and straightforward. It is capable of dealing with all stochastic problems with known PDFs and cumulative distribution functions (CDFs), such as the Gaussian distribution, uniform distribution and Gamma distribution. As an external sampling-based method, the MCS is non-intrusive and can be applied to various probabilistic problems in different disciplines regardless of the complexity or nonlinearity. It yields the unbiased mean and variance of stochastic results and further users can plot the confidence intervals or percentiles. The major disadvantage of MCS is its low convergence rate, which means it requires a large number of parameter samplings to produce accurate and reliable outputs and incurs overwhelming computation costs, especially when the original problem is already expensive to evaluate. Generally, tens of thousands of samples are used in uncertain rotor dynamics and often the MCS results are used as references [94,155]. Researchers have proposed to use the Latin hyper-cube sampling (LHS) method to improve the sampling efficiency and then reduce the cost of the crude MCS [62,156,157]. The LHS divides the input uncertainty into small portions with the same probability according to the distributions of the uncertainty and then produces samples in each segment. As a result, the sample numbers on hypercubes are the same and the total number is decreased considerably, making the application of the MCS to costly models possible.

3.1.2. Perturbation method

The perturbation method (PM) has been applied in stochastic analysis for a long time. The basic idea is that for a perturbed vibration system, where the perturbation is small compared to the original values, the fluctuations can be expressed by low order terms of zero-mean random quantities based on the Taylor expansions. Generally, a first- or two-order expansion is sufficient to obtain satisfactory approximations in linear or nonlinear problems. According to this theory, an uncertain parameter can be expressed in its first-order perturbation form [91]

$$b_j = b_{j0}(1+\varepsilon_j), \quad j = 1,2,\ldots,m \tag{1}$$

where b_j represents the jth uncertain parameter, b_{j0} is its original value, ε_j denotes the perturbation term, which should be small, and m is the number of uncertainties. The eigenvalues or critical speeds of the rotor system are uncertain as well and they are modeled by the two-order expansion as

$$\omega = \omega_0 + \sum_{j=1}^{m} \widehat{\omega}_j \varepsilon_j + \sum_{j=1}^{m} \sum_{k=1}^{j} \widehat{\omega}_{jk} \varepsilon_j \varepsilon_k, \quad j = 1,2,\ldots,m \tag{2}$$

where ω is the uncertain critical speed and ω_0 is the initial value, $\widehat{\omega}_j$ and $\widehat{\omega}_{jk}$ are the first- and second-order perturbation terms. Similarly, other quantities or matrices associated with uncertainties can be re-written in this way. Then, submit those perturbation formulations into the deterministic system equation and find the coefficients of the small perturbation term ε with the same order, which should be equal for both sides of the equation. The above deduction should eliminate the random perturbation terms but keep their coefficients and allow one to obtain a set of recurrence equations. The further evaluation procedure follows a deterministic routine and post-processing. It must be noted that including higher-order perturbations will significantly increase the calculation burden and complexity. Expected accuracy is only guaranteed for trivial fluctuations of the initial values, i.e., considerably small variations of the uncertainties.

3.1.3. Polynomial chaos expansion

The PCE is an efficient and powerful tool for stochastic uncertainty analysis in general dynamic systems. The stochastic finite element method (SFEM), where the PCE is associated with the Galerkin projection, proposed by Ghanem and Spanos [143] has been successfully applied to various uncertain problems. In this method, the stochastic process or variable is expressed by orthogonal bases of the standard random variables with Gaussian properties. The polynomial chaos bases stem from the homogeneous chaos theory of Wiener. Different probability models have their corresponding optimal orthogonal polynomials, e.g., the Hermite polynomials are associated with normal distributions and the Legendre polynomials are the bases for uniform distributions. As previously stated, this illustrates once again the need to precisely identify the stochastic modeling of uncertainties and the importance of the inverse uncertainty inference process in order to achieve results in accordance with real observations. The process of inverse uncertainty inference can allow a definition of a distribution law in agreement with experimental observations, which will have an impact on the choice of the PCE mathematical basis and consequently on the output results via the uncertainty propagation analysis.

A random variable input s is taken as an example to briefly explain the theory. By using the Karhunen-Loève (K-L) expansion, it can be expanded as

$$s = \bar{s} + \sum_{j=1}^{\infty} s_j \xi_j \tag{3}$$

where $\bar{s}$ is the mean of the random variable input s, s_j represents the jth expansion term and $\{\xi_1, \ \xi_2, \ \xi_3, \ \cdots\}$ is a set of orthogonal standard random variables. Generally, only finite terms are retained in calculation and the expansion in Eq. (3) is truncated to P terms. Then the QoI, X, which normally is the dynamic response, can be expressed by

$$X = \sum_{j=0}^{P} X_j \phi_j(\boldsymbol{\xi}) \tag{4}$$

where X_j denotes the jth PC coefficient to be solved and $\phi_j(\boldsymbol{\xi})$ is the jth order generalized scalar Askey-Wiener polynomial chaos based on the multi-dimensional random variable vector $\boldsymbol{\xi} = \{\xi_1, \xi_2, \cdots, \xi_m\}$.

In Eq. (4), the number of homogenous chaos or PCE coefficients is

$$P = \frac{(m+p)!}{m!p!} - 1 \tag{5}$$

in which p is the PC order. The expressions of polynomial chaos $\phi_j(\boldsymbol{\xi})$ are associated with the one-dimensional or multi-dimensional orthogonal polynomial depending on the dimension of uncertainty and type of probability model.

To evaluate the PC coefficients X_j, Eqs. (3) and (4) can be submitted into the equations of motion (EOM) of the deterministic system and the intrusive Galerkin projection is then used [143]. However, the process is intrusive and may be less friendly to complicated problems than the non-intrusive implementation, where the rotor dynamic problems deterministic solver can be called as a whole. In other words, the non-intrusive process allows the calculation of chaos expansion coefficients for response metrics of interest, based on a set of simulation response evaluations. This process can be seen as a simulation black box. Conversely, an intrusive method does not require the sampling of uncertainties in the propagation process as well as the associated verification of convergence. However, the intrusive PCE requires a rewriting of the equations of the problem which can sometimes be complicated and needs also modifications to a numerical solver. Here, we illustrate the procedures of the non-intrusive scheme. Multiplying both sides of Eq. (4) by $\phi_j(\boldsymbol{\xi})$ and rearranging the equation yield

$$X_j = \frac{E[X(\boldsymbol{\xi}) \cdot \phi_j(\boldsymbol{\xi})]}{E[\phi_j^2(\boldsymbol{\xi})]} \tag{6}$$

where $E[\cdot]$ denotes the expectation operator. In calculating Eq. (6), the orthogonality of polynomials is used as follows:

$$\phi_0(\xi) \equiv 1, E[\phi_i(\xi)\phi_j(\xi)] = \delta_{ij} E[\phi_j^2(\xi)] \tag{7}$$

with δ_{ij} being the Kronecker function. The expectation $E[\phi_j^2(\boldsymbol{\xi})]$ in Eq. (6) can be exactly calculated in closed form while $E[X(\boldsymbol{\xi}) \cdot \phi_j(\boldsymbol{\xi})]$ is expressed as

$$E[X_j \cdot \phi_j(\boldsymbol{\xi})] = \int \rho(\boldsymbol{\xi}) X_j \phi_j(\boldsymbol{\xi}) d\boldsymbol{\xi} \tag{8}$$

where $\rho(\boldsymbol{\xi})$ represents the weight. Eq. (8) may be evaluated numerically by the Gaussian quadrature. The zeros of orthogonal polynomials are used as collocations for uncertain parameters. The sparse grid technique can be applied to improve the computation efficiency in multi-dimensional uncertain problems [21], which will reduce the total collocations. In the case that the number of collocations and the number of unknown PC coefficients disagree, the least square technique (LST) can be applied as

$$\boldsymbol{X} = \left([\mathbf{T}(\boldsymbol{\xi})]^T \mathbf{T}(\boldsymbol{\xi})\right)^{-1} \mathbf{T}(\boldsymbol{\xi})]^T \widetilde{\boldsymbol{X}} \tag{9}$$

where $\boldsymbol{X} = [X_0, X_1, \cdots, X_{P-1}]^T$ is the vector of PC coefficients, the transform matrix $\mathbf{T}(\boldsymbol{\xi})$ is configured by the chaos basis $\{\phi(\boldsymbol{\xi})\}$ and $\widetilde{\boldsymbol{X}}$ represents the model output matrix at selected collocation sets. Their expressions can be found in [158]. When the PC coefficients are determined, the statistics of the system can be calculated, such as the mean and variance:

$$\overline{X} = X_0 \tag{10}$$

$$\sigma^2(X) = \sum_{i=1}^{P} X_i^2 E[\phi_j^2(\boldsymbol{\xi})] \tag{11}$$

Normally, low PC orders will produce satisfactory estimations of the random dynamics of uncertain rotor systems. The applications of the PCE include linear and nonlinear problems [40,78,80]. However, the number of uncertainties considered simultaneously is generally small if no additional techniques are applied, such as sparse grids. In addition, it is sometimes necessary to combine the PCE technique with other approaches to deal with non-linear problems in the presence of uncertainties [75]. In the case of a non-intrusive process, this is not a particular problem, except for the computation time and the storage of the data, because it aims to use a non-linear solver that is generally already implemented, for each sample of the distribution law. On the other hand, if an intrusive process is used, this requires a thorough rewriting of different steps in the solution process.

3.1.4. Kriging modeling

The Kriging surrogate model or meta-model is a non-intrusive spatial statistics-based technique also known as the Gaussian process modeling that aims to establish an unbiased estimation function. It is an interpolation tool that uses a small number of data produced from original systems to predict unknown quantities. The mapping between input and output parameters is approximated by the surrogate function to avoid repeated calls to the original system model. It should be noted that the Kriging surrogate is a method of interpolation based on the Gaussian process, but it can also be used for epistemic uncertainties, which is superior to the response surface method when non-linear behaviors and/or when functions with discontinuities or strong variations must be approximated. The method has been used in UQ, optimizations and parameter identifications [12,36,108,159–161]. For a predefined input vector $\mathbf{x}$, the system output $y(\mathbf{x})$, i.e., QoI, can be metamodeled by

$$y(\mathbf{x}) = f(\mathbf{x})^T\boldsymbol{\beta} + z(\mathbf{x}) \tag{12}$$

where $\boldsymbol{\beta}$ is the regression coefficients, $f(\mathbf{x})$ designates a linear combination of basis functions and $z(\mathbf{x})$ represents the realization of a stochastic process with zero mean and the covariance as

$$Cov[z(\mathbf{x}^{(1)}),\ z(\mathbf{x}^{(2)})] = \sigma^2 R(\theta,\ \mathbf{x}^{(1)},\ \mathbf{x}^{(2)}) \tag{13}$$

in which $\mathbf{x}^1$ and $\mathbf{x}^2$ are two input vectors, σ^2 is the variance of the stochastic process $z(\mathbf{x})$, θ denotes the correlation parameter vector for fitting the model and $R(\theta,\ \mathbf{x}^{(1)},\ \mathbf{x}^{(2)})$ defines the spatial correlation, which is calculated by

$$R(\theta,\ \mathbf{x}^{(1)},\ \mathbf{x}^{(2)}) = \prod_{j=1}^{m} R_j(\theta_j,\ x_j^{(1)},\ x_j^{(2)}) \tag{14}$$

with m being the number of inputs. The choice of the correlation function is a crucial step in the creation of the kriging *meta*-model. It is necessary to choose according to the nature of the rotor problem under study. The commonly used Kriging correlation functions are the linear, exponential, Gaussian, and Matern autocorrelation functions.

To fully determine and calibrate the surrogate model, the LHS can be used to obtain the samples of input. Then the optimal regression coefficients are computed by the LST as

$$\boldsymbol{\beta} = (\mathbf{F}^T\mathbf{R}^{-1}\mathbf{F})^{-1}\mathbf{F}^T\mathbf{R}^{-1}\mathbf{Y} \tag{15}$$

where $\mathbf{F}$ is a matrix of functions values of $f(\mathbf{x})$ calculated at samples of inputs, $\mathbf{Y}$ is the original system output values at inputs samples and $\mathbf{R}$ designates the correlation matrix whose entries are obtained from Eq. (14). Generally, a separate test data set from the original system is used to validate and improve the surrogate model apart from the data used to build it, which is called training data. The goodness of the model can be evaluated by various metrics, such as the root mean square error (RMSE) which is a classical metric to provide the average distance between the predicted values from the kriging *meta*-models and the reference values in the dataset (for a predefined number of comparison points). Classically, a convergence study by increasing the number of points used for the construction of the Kriging *meta*-model must be carried out to judge the relevance of the latter. More sophisticated indicators [149] such as the leave-one-out (LOO) strategy or the classical normal quantile–quantile plot graph (normal QQ-plot) can also be used as diagnostics to validate the relevance of kriging *meta*-models. One of the main advantages of these indicators is that they give also how well the *meta*-model predicts the values at unknown locations without requiring the use of additional computations.

To be noted that a non-intrusive *meta*-modeling method Polynomial-Chaos-Kriging (PC-Kriging) derived from the combination of PCE and Kriging has been proposed in [36] to predict the critical speeds and the dynamic response for rotordynamics with uncertainties, in the case of a high number of uncertain parameters and with parameters of different nature, namely parametric and random.

3.1.5. Random matrix theory

The RMT is a non-intrusive nonparametric modeling technique that does not need the prior PDFs of uncertainty. Therefore, it is generally deemed capable of dealing with the model form uncertainty. A reduced-order model (ROM), which serves as the mean model, can be used to improve efficiency in complex structures. It employs the maximum entropy theory (MET) to estimate the PDFs of random matrices and then the sampling method is applied to obtain the random responses of uncertain systems. The method is firstly proposed in the uncertain structural dynamics [146] and later introduced into the uncertain rotordynamics, such as the works of Murthy et al. [90,121,122], Gan et al. [39,120,162] and Feng et al. [163].

To obtain the PDF of a symmetric and positive definite matrix $\mathbf{A}_{n\times n}$, which usually refers to the mass, stiffness and damping matrices in rotor systems, the following three constraints should be fulfilled:

$$\widetilde{d}\mathbf{A} = 2^{n(n-1)/4}\prod_{1\leqslant i\leqslant j\leqslant n} \mathrm{d}\mathbf{A}_{ij} \tag{16}$$

where $p_{\mathbf{A}}$ and $\overline{\mathbf{A}}$ are the PDF and the mean of matrix $\mathbf{A}$, ν follows the Gama law and the expression of $\widetilde{d}A$ reads

$$\widetilde{d}\mathbf{A} = 2^{n(n-1)/4}\prod_{1\leqslant i\leqslant j\leqslant n} \mathrm{d}\mathbf{A}_{ij} \tag{17}$$

Then, the PDF of $\mathbf{A}$ can be obtained by using the MET as [39,146]

$$p_{\mathbf{A}} = c_{\mathbf{A}}(det(\mathbf{A}))^{\lambda-1}\exp\left(-\frac{n-1+2\lambda}{2}tr(\mathbf{A}^{-1}\mathbf{A}^T)\right) \tag{18}$$

with $tr(\cdot)$ being the trace operator, λ the Lagrange multiplier and

$$c_{\mathbf{A}} = \frac{(2\pi)^{-n(n-1)/4}((n-1+2\lambda)/2)^{n(n-1+2\lambda)/2}}{\left(\prod_{l=1}^{n}\Gamma((n-l+2\lambda)/2)\right)(\det(\mathbf{A}))^{(n-1+2\lambda)/2}} \tag{19}$$

where $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-x}dt,\ x > 0$ is the Gama function. A dispersion control parameter can further be defined:

$$\delta_{\mathbf{A}} = \left[\frac{1}{n-1+2\lambda}\left(1+\frac{(tr(\mathbf{A}))^2}{tr(\mathbf{A})^2}\right)\right]^{1/2} \tag{20}$$

When the dispersion control parameter $\delta_{\mathbf{A}}$ tends to zero, λ tends to infinity and the random matrix approaches its mean in probability. The Cholesky factorization can be applied here to the random matrix:

$$\mathbf{A} = \mathbf{L}_{\mathbf{A}}^T\mathbf{L}_{\mathbf{A}} \tag{21}$$

in which $\mathbf{L}_{\mathbf{A}}$ is the upper triangular matrix. It is convenient for the matrices not satisfying symmetric and positive definite in rotor systems, such as the skew-symmetric gyroscopic matrix, refer to the work of Murthy et al. [122] for more information. Based on the above theory and the factorization in Eq. (21), one can further use probabilistic methods to carry out stochastic analysis using the built PDFs, such as the MCS.

The MRT usually treats linear uncertain problems and its capability in nonlinear cases should be exploited and tested further. It cannot include the correlations between matrices and the physical meaning is not clear in the response space [163]. It should be also noted that the additional uncertainty can be introduced in determining the dispersion control parameter, which measures the uncertainty level in nonparametric modeling and should be estimated and calibrated from measurement data.

3.1.6. Bayesian inference

The Bayesian inference method is applied in rotor dynamic systems mainly for inverse uncertainty analysis, including fault diagnosis [164–166], model updating [167,168] and parameter identification [169,170], such as bearing parameters. In the Bayes scheme, expert knowledge is used in the process of model calibrations. Unknown parameters or uncertainties are given an estimated priori probability distribution (PD) without observations. The priori probability models will be updated from measurement or calculations where the likelihood of observations based on model prediction acts as the criteria. Then, the posterior PD is obtained using the updated data. The basic theory involves the conditional probability calculation. For instance, if A and B are two stochastic events, the probability of A given B is defined by

$$P(A|B) = \frac{P(AB)}{P(B)},\ P(B) > 0 \tag{22}$$

where $P(AB)$ represents the joint probability for the simultaneous occurrence of both events A and B. Then the Bayesian theorem is given as

$$P(B|A) = \frac{P(B)P(A|B)}{P(A)} = \frac{P(B)P(A|B)}{\int P(B)P(A|B)dB} \tag{23}$$

in which $P(B)$ is the priori PD indicating the current model and $P(A|B)$ denotes the likelihood function observing A when B happened. $P(B|A)$ is the posterior PD which represents the updated models of B with observations of A. The likelihood function is given in [166]. The above deduction is quite simple. However, the computational burden and priori probabilities will grow excessively in cases where a large number of random events are involved. Generally, the analytic posterior PDs cannot be obtained, and the Markov Chain Monte Carlo (MCMC) could be employed to approximate the posterior PDs by extracting random samples in the parameter space. The Metropolis-Hastings method is one of the representatives.

3.2. Fuzzy set theory

Fuzzy set theory (FST) is proposed by Zadeh [171] to model the uncertainties with limited or sparse data, i.e., imprecise information. In rotordynamics, the FST has been applied for uncertainty analysis [100,101,104,135], control [172] and fault diagnosis [130,173]. In contrast to regular crisp sets, which have fixed boundaries, the fuzzy sets use a membership function (MF) to define to what extent a number belongs to the sets. The degree of membership of a fuzzy variable can vary continuously between 0 and 1 while in crisp sets it is either 0 or 1. In the latter, it degrades to interval models. The FST is a possibility theory rather than probabilistic even though it has been successfully extended to stochastic analysis [132]. The MFs establish the relationships between uncertain variables and fuzzy sets. Without loss of generality, let us take a triangular fuzzy set, as shown in Fig. 2(b), for an example (complex cases such as trapezoidal fuzzy sets or multi-dimensional sets can be more complicated, but the principles are similar). Its MF can be given by

$$\mu_f(x) = \begin{cases} 0, & x \leqslant a \\ \dfrac{x-a}{b-a}, & a \leqslant x \leqslant b \\ \dfrac{x-c}{b-c}, & b \leqslant x \leqslant c \\ 0, & x \geqslant c \end{cases}, \quad \mu_f(x) \in [0,\ 1] \tag{24}$$

where a, b and c are the minimum, most likely and maximum values of the set, respectively. If the fuzzy variable equals the most likely value, i.e., b, the MF value is 1 and those for other fuzzy variables can be proportionally interpolated.

When the fuzzy uncertainty model is defined, the following fuzzy dynamics analysis can be described by two steps [132]. The first step is based on the α-cut method to discrete the fuzzy input. The α-cut method uses given MF values (α levels) to extract the corresponding variable intervals with the same MF value. There is no common rule that how many MF values should be included but the ultimate purpose is robust estimations. The second step is finding the possible QoI intervals based on optimization strategies with the constraints being the intervals at different α levels. Thus, the efficiency of FST relies largely on the optimization method.

3.3. Interval methods

Interval methods are newly emerged uncertainty analysis methods in rotor systems. The uncertainties are described by a pair of numbers denoting the lower and upper bounds in which range the uncertain variables are expected to vary. Therefore, they are also called uncertain-but-bounded quantities, i.e., their strict distributions are unknown but the intervals are always easier to know. This description of uncertainty is the easiest and simplest. The practice of interval uncertainty analysis in dynamic systems results in the interval output of the QoIs. However, the direct application of interval algorithm (IA) may lead to enormous overestimations although the basic arithmetic operations rules are readily available. The excessive estimation of output bounds leads to results of less significance. Efforts have been made to control the overestimations by proposing new methods. The PM can also be used in interval form besides the theory described in Section 3.1. The demonstration of it, however, is skipped because its core idea is similar to the Taylor interval method (TIM), which will be included in this sub-section. The recently established non-intrusive CIM will also be illustrated. It should be noted that correlations between interval variables are not considered, otherwise the convex modeling should be employed [174–176]. The readers are referred to [177,178] for more information about correlations of uncertainties and convex models.

3.3.1. Taylor interval method

The TIM is similar to the PM in terms of requiring that the uncertainty levels remain small in order to be effective. Similarly, it is an intrusive implementation indicating that uncertainty is injected into the modeling process of rotor systems [83,179–181]. The first-order Taylor expansion can be employed to describe output bounds of QoIs directly and explicitly, say the vibration response q:

$$\begin{cases} q^T_{\min}(t) = \min\limits_{h\in\mathbf{H}} q(t,\mathbf{H}) = q(t,\mathbf{H}^c) - \sum\limits_{j=1}^{m}\left|\dfrac{\partial q(t,\mathbf{H})}{\partial h_j}\right| \\ q^T_{\max}(t) = \max\limits_{h\in\mathbf{H}} q(t,\mathbf{H}) = q(t,\mathbf{H}^c) + \sum\limits_{j=1}^{m}\left|\dfrac{\partial q(t,\mathbf{H})}{\partial h_j}\right| \end{cases} \tag{25}$$

where t is time, superscript c denotes the mid-points of uncertain interval input and $\mathbf{H}$ is a vector of interval parameters defined by

$$\begin{cases} \mathbf{H} = [h_1,\ h_2,\ \cdots,\ h_i,\ h_m]^T,\ i \in [1,\ m] \\ h_i = [h_i^L, h_i^U] = [h_i^c - h_i^R, h_i^c - h_i^R],\ h_i^R = \dfrac{h_i^U - h_i^L}{h_i^c} \end{cases} \tag{26}$$

in which superscripts L, R and U represent the lower bound, interval radius and upper bound, respectively. For interval inputs in the system, a perturbation is given as

$$\begin{cases} \mathbf{H} = \mathbf{H}^c + \Delta\mathbf{H},\ |\Delta\mathbf{H}| \leqslant \mathbf{H}^R \\ h_i = h_i^c + \Delta h_i,\ |\Delta h_i| \leqslant h_i^R \end{cases} \tag{27}$$

Then those expansions, including inputs and outputs are submitted to the original EOM for further manipulations. It could be difficult to extend to higher orders since obtaining the partial derivative in the expansion is computationally expensive or even impossible. It is also termed the Taylor inclusion functions (TIF) in [182] and it was found the results have greater overestimation compared to the Chebyshev inclusion function (CIF), which will be explained in the next sub-section.

3.3.2. Chebyshev interval method

The CIM is a non-intrusive framework based on the Chebyshev orthogonal polynomials. Wu et al. [182] proposed the CIF for nonlinear dynamic systems under interval uncertainties, where a vehicle suspension problem and a double pendulum problem were studied. Till now, it has been extended into multiple research areas like gear systems, multi-body systems and pipe systems [183], due to its excellent performance and convenience. The main theory will be briefly explained. For any interval variable $x \in [a, b]$, the n-degree Chebyshev polynomial is defined by

$$C_n(x) = \cos n\theta,\ \text{with}\ \theta = \arccos\frac{2x-(b+a)}{b-a} \tag{28}$$

where $\theta \in [0,\ \pi]$. This projection translates any physical interval inputs into the standard variables in [-1, 1]. In multi-dimensional cases, it evolves to

$$C_{n_1,n_2,\cdots,n_m}(x_1,\ x_2,\ \cdots,\ x_m) = \cos n_1\theta_1 \cos n_2\theta_2 \cdots \cos n_m\theta_m \tag{29}$$

Then, the input interval variables are deemed as standard intervals after the projection without causing ambiguity. The n-order Chebyshev approximation for multi-dimensional uncertainty problems with the QoI $\mathbf{y}(t, \mathbf{x})$ is given by

$$\mathbf{y}(t,\mathbf{x}) = \sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\cdots\sum_{i_m=0}^{n}\frac{1}{2^p}\mathbf{y}_{i_1,i_2,\cdots,i_m}C_{i_1,i_2,\cdots,i_m}(\mathbf{x}) \tag{30}$$

where p represents the total number of zeros that appear in the set of subscripts $i_1, i_2, \cdots, i_m$, $\mathbf{y}_{i_1,i_2,\cdots,i_m}$ denotes the Chebyshev coefficients to be determined and $\mathbf{x}$ is the input interval vector $\mathbf{x} = [x_1,\ x_2,\ \cdots,\ x_m]^T \in [-1,\ 1]^m$.

To evaluate the coefficients, the multi-dimensional Mehler integration is used as

$$\mathbf{y}_{i_1,i_2,\cdots,i_m} = \left(\frac{2}{\ell}\right)^m \sum_{j_1=0}^{\ell}\sum_{j_2=0}^{\ell}\cdots\sum_{j_m=0}^{\ell}\mathbf{y}(t,\ cos\theta_{j_1},\ cos\theta_{j_2},\ \cdots,\ cos\theta_{j_m})\cos i_1\theta_{j_1}\cos i_2\theta_{j_2}\cdots\cos i_m\theta_{j_m} \tag{31}$$

where ℓ designates the number of interpolations, which should not be less than $n+1$ for the sake of accuracy. The interpolations are actually the zeros of degree Chebyshev polynomials, which can also be used for stochastic collocations [184]:

$$\begin{cases} \theta_{j_1},\ \theta_{j_2},\ \cdots,\ \theta_{j_m} \in \{\widehat{\theta}\} \\ \widehat{\theta}_j = \dfrac{2j-1}{\ell}\dfrac{\pi}{2},\ j = 1,\ 2,\ \cdots,\ \ell \end{cases} \tag{32}$$

The $\mathbf{y}(t,\ cos\theta_{j_1},\ cos\theta_{j_2},\ \cdots,\ cos\theta_{j_m})$ is the model output of the deterministic system at the interpolation point $\mathbf{x} = [cos\theta_{j_1},\ cos\theta_{j_2},\ \cdots,\ cos\theta_{j_m}]^T$. Thus, it is evidently shown that the original solver will be only executed at interpolations as a black box. After computing all the coefficients, the QoI output bounds are calculated as

$$\mathbf{y}^I = [\mathbf{y}^L, \mathbf{y}^U] = \frac{1}{2^m}\mathbf{y}_{0,\cdots,0} + [-1,1]\sum_{\substack{0\leqslant i_1,\ldots,i_m\leqslant n \\ i_1+\cdots+i_m\geqslant 1}}\left(\frac{1}{2}\right)^p \left|y_{i_1,i_2,\cdots,i_m}\right| \tag{33}$$

The above inclusion operation may lead to overestimations caused by the interval arithmetic.

From Eqs. (31) and (32), we can notice that the quadrature points will increase in tensor form leading to the curse of dimension. In fact, there are several variants of this method. In [42], the dimension-wise Chebyshev method was employed to treat interval uncertainties individually under the assumption that they are independent. However, it involves derivative calculation. The LST is introduced to calculate the Chebyshev coefficients in a regression way similar to the response surface method in [158], where the collocations are largely reduced to improve the efficiency and the numerical Gaussian quadrature is avoided. The improved interval method is called the Chebyshev collocation method. It was further proved that the accuracy of the method mainly depends on the collocations rather than the basis functions [185]. Eq. (33) can be further processed by the MCS or an optimization method, such as the particle swarm optimization, to tighten the final interval of the output result to reduce overestimations [186]. Inspired by the above Chebyshev method, a polynomial surrogate model was constructed for the interval analysis of cracked rotor systems subject to interval uncertainties [74].

3.4. Summarization and comparison

Probabilistic or stochastic methods are developed based on mature statistical theories and their mathematical deductions are rigorous. Plenty of statistical information about the output can be obtained, such as any orders of the statistical moments, especially the ensemble mean and standard variance. One can predict a confidence interval using the stochastic responses and even the probability of any response values. That is very beneficial to engineering for a confident design and dynamic analysis. The prerequisite of implementing these methods is that the PDFs or joint PDFs of uncertain variables are already known or there is sufficient data to derive such a reliable distribution model. If the requirement is not met and hypotheses are used, the validity of results is questionable. Subjectively choosing a standard probability law model or introducing unreasonable assumptions may produce misinterpreted results and even incorrect conclusions. Looking more closely, the MCS is simple and reliable, but it requires a huge amount of tedious computation. The intrusive PM is best suited for small variations and cannot be used for large range uncertainties. The PCE may produce the spurious

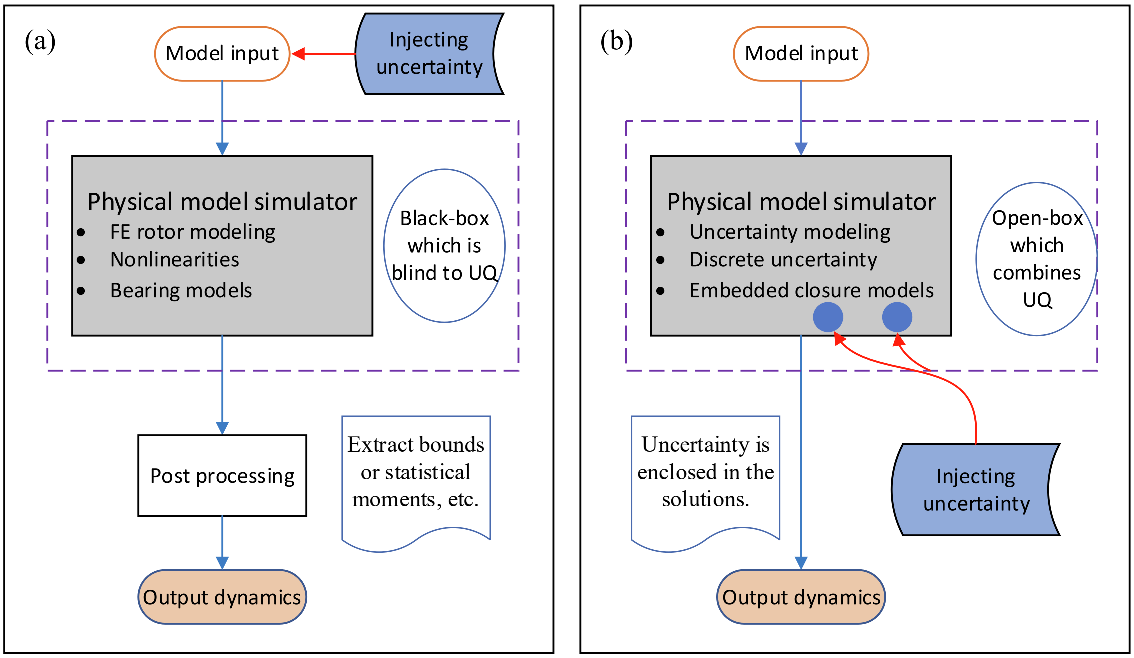

Fig. 4. Schematic demonstration of the general process of (a) non-intrusive, black-box and (b) intrusive, open-box.

peaks of dynamic systems near resonance and improvements can be found regarding the accuracy and efficiency [94,187]. In Kriging modeling, the order and correlation functions play a vital role in the accuracy of surrogate models. The non-intrusive PC-Kriging method derived from a combination of the PCE and Kriging allows modeling rotor system with aleatory and epistemic uncertainties simultaneously. In this case, classical indicators can be used to validate the relevance of such advanced Kriging-based surrogate modeling or increase the efficiency of surrogate models [36]. The capability of the RMT in dealing with uncertain nonlinear rotor systems remains exploited. The Bayesian inference suffers from obtaining the priori probability distributions for complicated rotor problems.

The fuzzy approaches, which are built on the basis of imprecise data, are applicable for the cases with partial true and partial false variables. The limited data is taken into consideration compared with interval methods, although it is not enough to derive the PDFs. It also leads to mathematical imprecisions [188]. When the uncertain input of rotor systems requires complex fuzzy expressions, finding the appropriate membership functions can be very difficult or it is somewhat empirical. Misleading results will be produced if they are not applied correctly.

Interval approaches can be easily applied to general uncertain dynamic systems since only the lower and upper bounds of uncertain inputs are needed, which brings much convenience in some cases. The major disadvantage of interval methods is that data within the interval input, if any, are not made use of. Essentially, the outputs are interval-based quantities similar to the inputs. However, the variability pattern of the system output interval ranges is lost completely [188]. For example, there may be a few scattered data observations for the uncertain parameters of rotor systems, which is highly possible, but only the dispersion bounds are used in interval uncertainty analysis. In addition, the interval methods usually give conservative dynamics estimations compared with other types of methods and they suffer from overestimations. The most notable situation is direct applications to complicated calculations, where the overestimations cumulate and finally give large-range interval predictions that obviously make no sense. The TIM is similar to the PM to some degree in that it cannot be used for large-range uncertainties. What's more, it is difficult to obtain high orders of the derivatives required in the TIM in order to be accurate in complex and nonlinear uncertain cases. The CIM and its variational versions may subject to the curse of dimension. Efforts have been made to improve its ability to deal with high-dimensional and nonlinear uncertain problems [189].

Apart from the above points, the way the UQ methods work, i.e., intrusive or non-intrusive, is of great importance in more complicated problems. The intrusive methods integrate the uncertainties in the modeling process of the rotor systems without any generality, which means an extensive reprogramming of both the existing codes and solvers for rotor dynamic systems and the UQ methods. Its advantage is that the solutions can be obtained accurately and efficiently. The PM, spectral methods (the K-L expansion) and TIM are typically intrusive. The non-intrusive approaches, however, treat the deterministic dynamic problem of rotor systems as a black box by surrogate models based on the theories such as orthogonal approximation. As such, the original solvers can be well preserved without much modification. These methods have excellent generality and will easily adapt to different systems and problems. The uncertainty in the final output will be extracted via the post-processing of the model outputs at parameter samples or collocation points. The MCS, Kriging and CIM or their variants are popular non-intrusive approaches. The PCE and Bayesian inference can be both intrusively and non-intrusively implemented. The non-intrusive methods can be implemented in connection with the

Table 2
Summarization of popular uncertainty analysis methods for rotor systems.

Type	Method	Strength	Limitations	References
Probabilistic	MCS	Intuitive Non-intrusive All PDFs supported	Low convergence rate	[47,62,82,89,99,117,157,194–198]
	PM	Can solve nonlinear problems	Small levels of uncertainty Difficulty in high order terms	[77,91,199–201]
	PCE	Fast Non-intrusive and intrusive	Cannot model correlation Spurious oscillations near vibration resonances	[40,44,45,78,80,109,116,156,202,203]
	Kriging	Fast Non-intrusive	Attention should be paid to correlation functions	[12,36,108,159–161,204]
	RMT	Nonparametric Prior PDFs not mandatory	Difficult to select dispersion control parameters	[39,90,121,122]
	Bayes	Backward uncertainty inference	Difficult to define the priori PD in complicated problems	[164–167,169]
Fuzzy	FST	Require limited information only Can handle partial true and partial false	Possibility and necessity measures required Difficult to define membership functions	[54,100,101,104,130,173,205,206]
Interval	IA	Straightforward	Large overestimation	[41,207]
	TIM	Can solve nonlinear problems	Small levels of uncertainty Difficulty in high order terms	[83,179,180]
	CIM	Fast Non-intrusive Suitable for large level uncertainty	High burden for a large number of uncertainties Possible large errors near vibration resonances	[38,42,75,97,131]

commercial software, unlike the intrusive process which generally remains handmade code and therefore is commonly used by experts. From a perspective of industrial versatility, the non-intrusive techniques are generally recommended for large-scale rotor systems, which involve complex modeling and solution process. One of the major drawbacks is that the non-intrusive approaches can be very costly in terms of computation time and data storage since a number of calculations are needed to converge and consequently obtain a correct representation of the outputs (for example to find the PCE coefficients which best match a set of response values obtained from a design of computer experiments in agreement with a PDF predefined). For the intrusive approaches, the required computational time is by nature very low compared to a non-intrusive process given that only one calculation is necessary. However, the required rewriting of the equations of the problem can sometimes be complicated and the modifications to a numerical solver are painful. Even if this does not represent a major theoretical problem in terms of mathematical rewriting [48,64,75], it can be quite complicated to set up when many uncertainties are considered.

For a better illustration, the differences between intrusive and non-intrusive UQ approaches in rotor dynamic analysis are depicted in Fig. 4.

Further, a primary concern in the UQ of the dynamics of rotor systems is the computational accuracy and efficiency. As pointed out previously, the various UQ techniques are divided into intrusive methods and non-intrusive methods. The intrusive methods are generally very efficient since they do not require repeated runs of the physic models while the non-intrusive methods often induce many run times of the deterministic model, i.e., they are referred to as the sampling-based methods. Naturally, one expects the UQ methods to give accurate estimations while they remain highly efficient. This involves the two basic sides in general uncertainty analysis. On the one hand, the calculated results must be accurate enough or the errors are in an acceptable range otherwise the efforts made will not make sense. On the other hand, the computational cost should be low. Obviously, the computation time required by the UQ methods depends on the specific machine used, the degree of complexity of the deterministic problem (such as the number of DOFs and nonlinearity) and the number of uncertainties included. However, we will discuss the required cost by comparing the needed total run times of the deterministic model, which excludes the performance difference of the machines. Then, the problem arises in twofold: the complex physical models and the curse of dimensionality. For general large-scale rotor models established, which is mandatory for industrial applications, the burden of a deterministic simulation can be overwhelmingly heavy, especially when nonlinearities are included. Moreover, the sampling-based UQ methods will normally require a tensorial number of uncertain parameter samples if many uncertainties are present, leading to the cost rising on an exponential basis. Then, the realization of uncertainty analysis will be nearly impossible if an enormous number of simulation runs are needed even though the modern computing resources are more advanced than before. However, the contradiction between accuracy and efficiency cannot be solved once for all because the issue is deeply rooted and the two sides are closely linked to each other. A compromise or trade-off is generally the way out by proposing dedicated techniques, such as the sparse girds and regressions [23], intending to maintain sufficient accuracy and reduce the simulation burden. Generally, the traditional MCS is a crude sampling method and its convergence is low. Thousands or tens of thousands of simulations are required to secure reliable dynamics output of the uncertain system. Its results are generally deemed to be accurate enough and therefore the MCS is frequently used to validate other probabilistic methods, such as the PCE. However, in some cases, MCS is not applicable due to the high cost of computational models of rotor systems. It is then necessary to implement robust estimators to validate the convergence of the solution obtained. Efforts can be found in the literature to improve the efficiency of the traditional MCS by proposing variational methods [190]. Similar to the traditional MCS, the scanning method, which evenly generates samples for interval uncertain parameters, is often employed as a reference to other interval methods, such as the CIM. The numerical performance

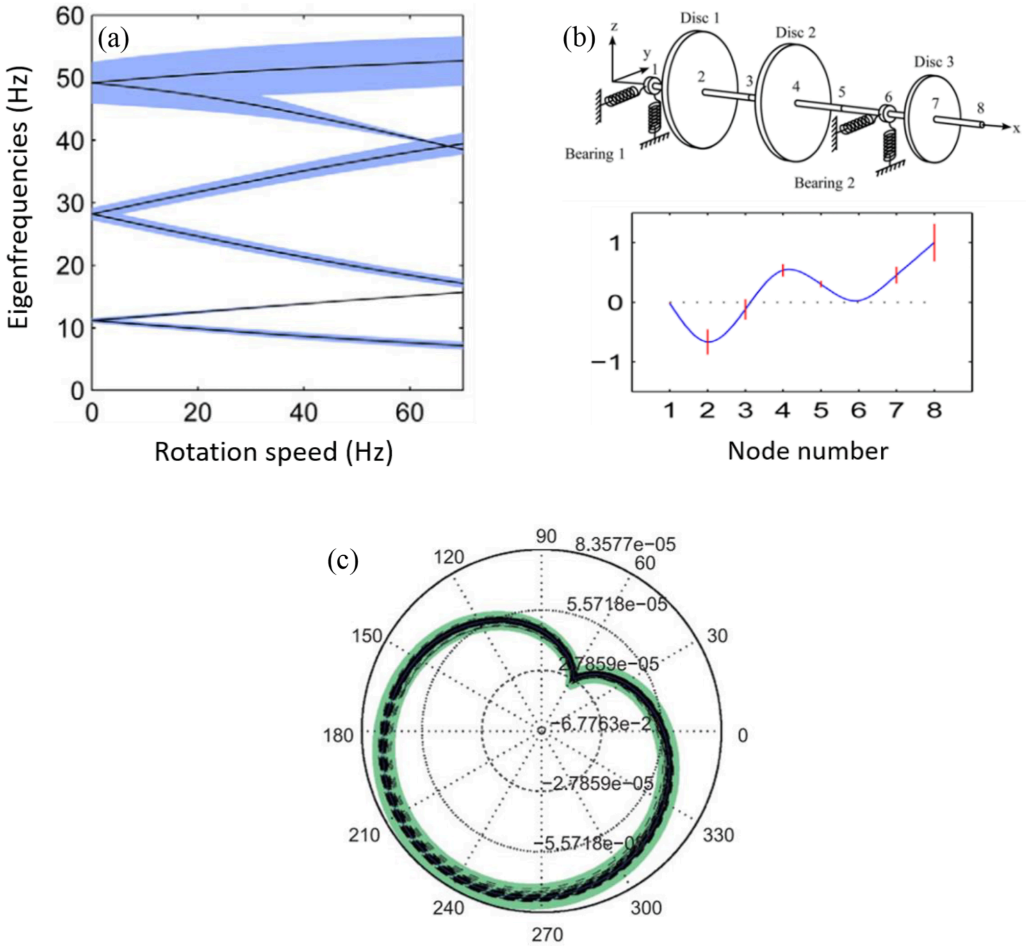

Fig. 5. Natural characteristics and shaft orbits of uncertain rotor systems: (a) the Campbell diagram [18], (b) the variability of the third forward mode shape (vertical lines indicate the standard variation) [18] and (c) orbits of an uncertain rotor with multiple faults [73].

of other UQ methods described previously is generally better than the MCS and scanning method, but one should judge individually based on the dimensions of both the physical model and uncertainties in the uncertain rotor dynamic problem under study.

It is also worth mentioning that physical parameters in rotor systems always stay strictly positive. Although the results are not likely to be influenced from a mathematical point of view, it is not rigorous in physics-based computations. Hence, the unreasonable parameter samples, if present, produced in the sampling-based methods like the MCS should be disregarded [13,40]. The advantages and disadvantages of the popular UQ methods described above are summarized in Table 2 and representative references of their applications to uncertain rotor dynamics analysis are provided as well. There are also other UQ approaches in the other disciplines but not yet applied to rotordynamics, such as the evidence theory or Dempster-Shafer theory [191], imprecise probability [192] and interval probability [193].

Based on the above discussions of different UQ methods as well as their pros and cons, we give some remarks regarding the choice of the approaches when one attempts to carry out the uncertainty analysis of rotor systems. The most important supports for choosing a UQ method are the details available and the computational budget [24]. The type of data available will affect the most appropriate method one should choose. Moreover, the differences between intrusive and non-intrusive implementations have significant meanings for convenience in complex engineering scenarios. On the other side, the sampling of uncertainties has to be considered according to the computational burden that one can bear. To sum up, the choice of the methods largely depends on the specific situations and the preference of users.

4. Dynamic characteristics with uncertainty

In this section, the detailed research progress of dynamic characteristics of uncertain rotor systems is summarized. The classification is made by the main content and the characteristics discussed in the works, which covers the general natural characteristics and dynamic response, investigations that include typical faults of rotor systems, stability problems and inverse parameter identification.

To enable a more effective and insightful discussion of the uncertain rotordynamics, the most classic EOMs of a generalized rotor-

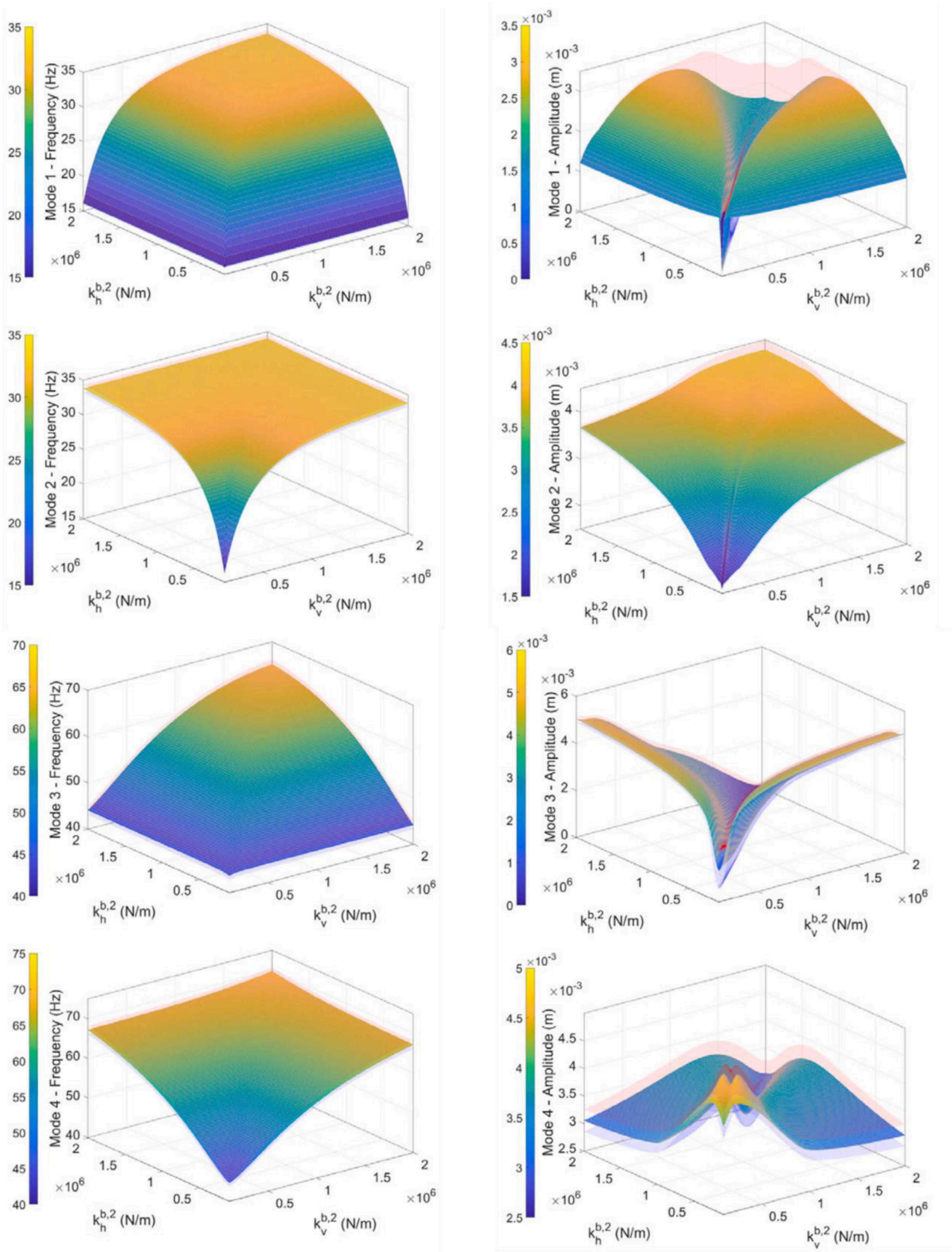

Fig. 6. The evolutions of the average and average ± standard deviation of the critical speeds and the associated maximum amplitudes for the first four modes of a rotor-bearing system with random parameters [36].

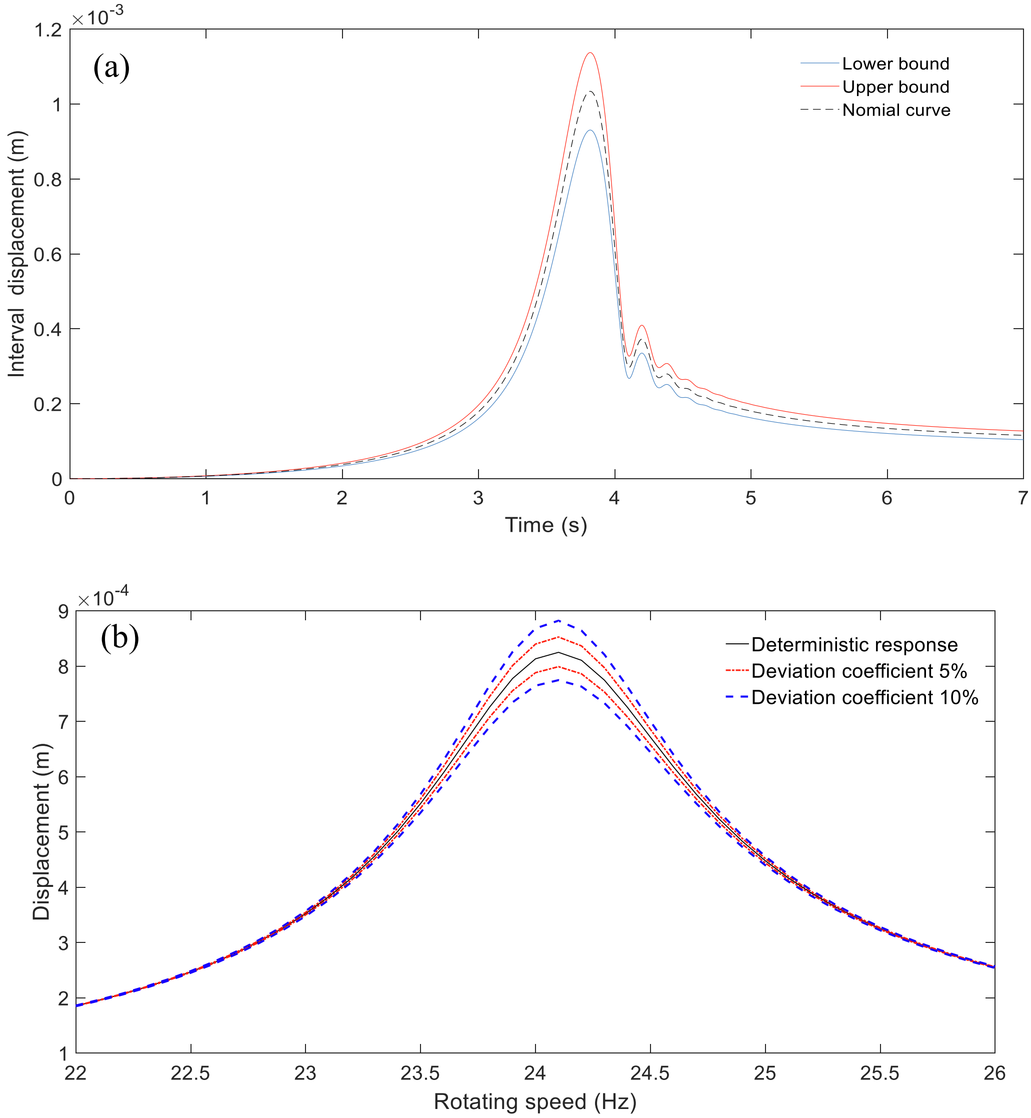

Fig. 7. Transient and steady-state responses of an uncertain overhung rotor system: (a) the accelerating transient responses with the interval unbalance [42] and (b) the steady-state responses with the interval damping [13].

bearing system based on the FEM are provided as follows:

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C} + \omega\mathbf{G})\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{W} + \mathbf{Q}_l + \mathbf{Q}_{nl} \tag{34}$$

where $\mathbf{x}$ is the vector of displacements, ω defines the rotational speed of the rotor, $\mathbf{M}$ and $\mathbf{G}$ are the global mass and gyroscopic matrices including mass and gyroscopic matrices of the shaft and rigid discs, $\mathbf{C}$ and $\mathbf{K}$ are the external damping and stiffness matrices of the shaft and bearing support. To be noted that $\mathbf{W}$ defines the vector of gravity forces, $\mathbf{Q}_l$ represents the linear forces on the rotor system (normally the imbalance force for the complete rotor system) and $\mathbf{Q}_{nl}$ denotes the nonlinear forces, which could be introduced by the presence of typical faults or bearing supports. If the rotor system is linear, $\mathbf{Q}_{nl}$ will be a vector of zeros. A brief description of the FE modeling, as well as the expressions of the matrices, are provided in Appendix A.

4.1. Natural characteristics and unbalanced responses

The natural characteristics in this paper specifically refer to the critical speeds and mode shapes of rotor systems, which are the fundamental points to be taken care of in designing and dynamic analysis. Further, the most concerned is the vibration responses of the systems under imbalance excitations and they are analyzed from different perspectives in the time domain and frequency domain as well as their sensitivities [208]. In practice, the vibrations of a rotor system should not exceed the allowed maximum amplitude to ensure safe operations of the whole equipment. The uncertainties stemming from various sources make the situation more complicated [92,194,209,210]. Based on Eq. (34) and the formulations given in Appendix A, we can gain insights into the impacted matrices and vectors in the EOMs of the rotor systems when different uncertainties are taken into consideration. For example, the bending stiffness of a shaft depends on the moments of inertia of the shaft cross sections I and Young's modulus E. Therefore, any uncertainties in the geometries of the shafts will affect the stiffness matrix $\mathbf{K}$ of the EOMs and the uncertain density will lead to variations in the mass matrix $\mathbf{M}$. Similarly, the uncertainties in the geometric and material properties of the discs can cause dispersions of the mass and stiffness matrices, and additionally the gyroscopic matrix $\mathbf{G}$, the linear imbalance excitation forces $\mathbf{Q}_l$ and the gravitation force $\mathbf{W}$. The variability of the external loads, if any, is reflected by changes in $\mathbf{Q}_l$ as well, such as the noise disturbances. Boundary conditions uncertainty is typically introduced by supports such as various bearings. It induces the uncertain stiffness and damping matrices ($\mathbf{K}$, $\mathbf{C}$), and sometimes the uncertain nonlinear forces $\mathbf{Q}_{nl}$, such as the nonlinear ball bearing forces and nonlinear oil film forces in journal bearing or simply a cubic stiffness term. The EOMs of rotor systems with typical faults will have additional variabilities depending on different faults and there are interactions between different matrices and vectors due to nonlinearities, which will be discussed later. To be noted that if the Rayleigh damping model is used, i.e., the proportional damping with respect to the mass and stiffness matrices, all the uncertainties affecting the mass and stiffness matrices will have an impact on the damping matrix.

To clarify key issues in the stochastic rotor systems, many works have been carried out based on the applications of the PCE [45,55,92,94,156,202]. Sarrouy et al. [18] presented the stochastic Campbell diagram and the corresponding mode shape variability based on the PCE with 10 % dispersions in the density of a disc, as shown in Fig. 5(a) and Fig. 5(b). Didier et al. [40] formulated the spectral implementation of PCE for an in-depth investigation of the frequency responses and further extended it to an uncertain rotor subject to support excitation [48]. Didier et al. [73] also showed the shaft orbits of an uncertain rotor system with multiple faults, see Fig. 5(c). Gan et al. [39] showed the unbalance responses of a Jeffcott rotor using the max, min and mean values based on the nonparametric modeling. The effects of disc position, i.e., the offset, and the gyroscopic force were discussed. It was found that uncertainties in the unbalanced force and the bounded noise excitation cause the widened variation ranges of critical speeds and the vibration amplitude. A comprehensive study was carried out by Sinou et al. [12] to exploit the use of Kriging surrogate modeling to predict the forward and backward critical speeds of a flexible rotor with probabilistic uncertainties. It was proved that the proposed method has high efficiency and accuracy. Further, an advanced Kriging surrogate model based on the PCE was proposed to establish a hybrid *meta*-model for the natural characteristics and responses sensitivity analysis of a rotor with many random and epistemic uncertainties [36], the evolutions of critical speeds and the associated maximum vibration amplitudes for the first four modes are illustrated in Fig. 6. To be noted that a sensitivity analysis based on the Sobol index can be directly performed from the Kriging *meta*-models without any additional cost. Liu et al. [91,200] used the Riccati perturbation transfer matrix method to study the critical speeds of a rotor system with random mass density and section diameter. Comparison between the first-order and two-order perturbations shows that the latter has slightly better accuracy, but the complexity is greatly increased. It was also demonstrated that the PM has significant errors for large uncertainty levels.

Rotor dynamic balancing is an important step before it is assembled into an engine, which ensures the vibration amplitudes fall within the allowed range. Datz et al. [44] quantified the effects of the stochastic uncertainty in the balancing weights on the influence coefficients and further the response amplitudes. A heavy-duty high-speed gas turbine rotor from industry was investigated experimentally. Recently, Zhao et al. [175] analyzed the performance of a transient balancing scheme with uncertainty propagation based on a Bently rotor test rig.

The PDFs of the critical speeds were obtained in [101] for the fuzzy random rotor model. Qiu and Rao [100] applied a fuzzy approach to analyze the dynamics of a rotor system with nonlinear restoring forces. The bifurcation, orbits and Poincare map were presented and compared under different fuzzy stiffness. The intrusive interval analysis was applied to a finite element (FE) rotor system in [41] where the geometric properties, support stiffness and imbalance excitation were modeled as interval inputs. The mode shapes and steady-state responses were compared with the results from MCS. Further, the CIM was used to quantify the dispersions and a correction method was proposed to improve the accuracy of response bounds near resonance [97]. Fu et al. [13,42,131] investigated the transient, steady-state vibration responses of an uncertain rotor with an overhung disk under interval or hybrid uncertainties using the non-intrusive UQ methods, as shown in Fig. 7. A few works also studied the uncertain dynamics of dual-rotor-bearing systems [96,161,211]. Capiez-Lernout et al. [212] defined the blade manufacturing tolerances for the mistuned industrial bladed disks and Capiez-Lernout et al. [213] further analyzed the mistuning effects and relevant uncertain dynamics based on the nonlinear ROMs using the nonparametric probabilistic approach. Picou et al. [214] presented a robust analysis of the influences of the geometric nonlinearities in rotating bladed disks with the intentional and unintentional mistuning. Xie et al. [215] proposed a monitoring method of the blade damage by using the frequency domain statistics extracted from the random vibration of the shaft. The effects of friction parameter uncertainties in blades system with underplatform dampers were investigated by Yuan et al. [216] and the nonlinear dynamical response under uncertainty was analyzed.

The dynamics of rotor systems under the fuzzy or random uncertainties associated with different kinds of bearings are the most heated research topics [53–55,60,103,104,108,111,117,195,217–223], among which the journal bearings are major focuses of many researchers. For uncertainties in journal bearings, three main points were generally investigated, i.e., the rough surfaces, radial clearance and lubricant viscosity (caused by lubricant temperature variations). Those uncertainties will be reflected in the dynamic coefficients of journal bearings and further the vibration behaviors of the rotor-bearing systems [224]. Envelopes of the uncertain transient responses of a flexible rotor supported by journal bearings were discussed by Cavalini et al. [104] based on fuzzy analysis. Kriging metamodel was used to represent the thermohydrodynamic models of three kinds of bearings in a Francis hydropower unit [108], i.e., a cylindrical journal bearing, a tilting-pad journal bearing and a tilting-pad thrust bearing.

The research on nonlinear rotordynamics under uncertainty is more difficult than the linear studies since there are interactions

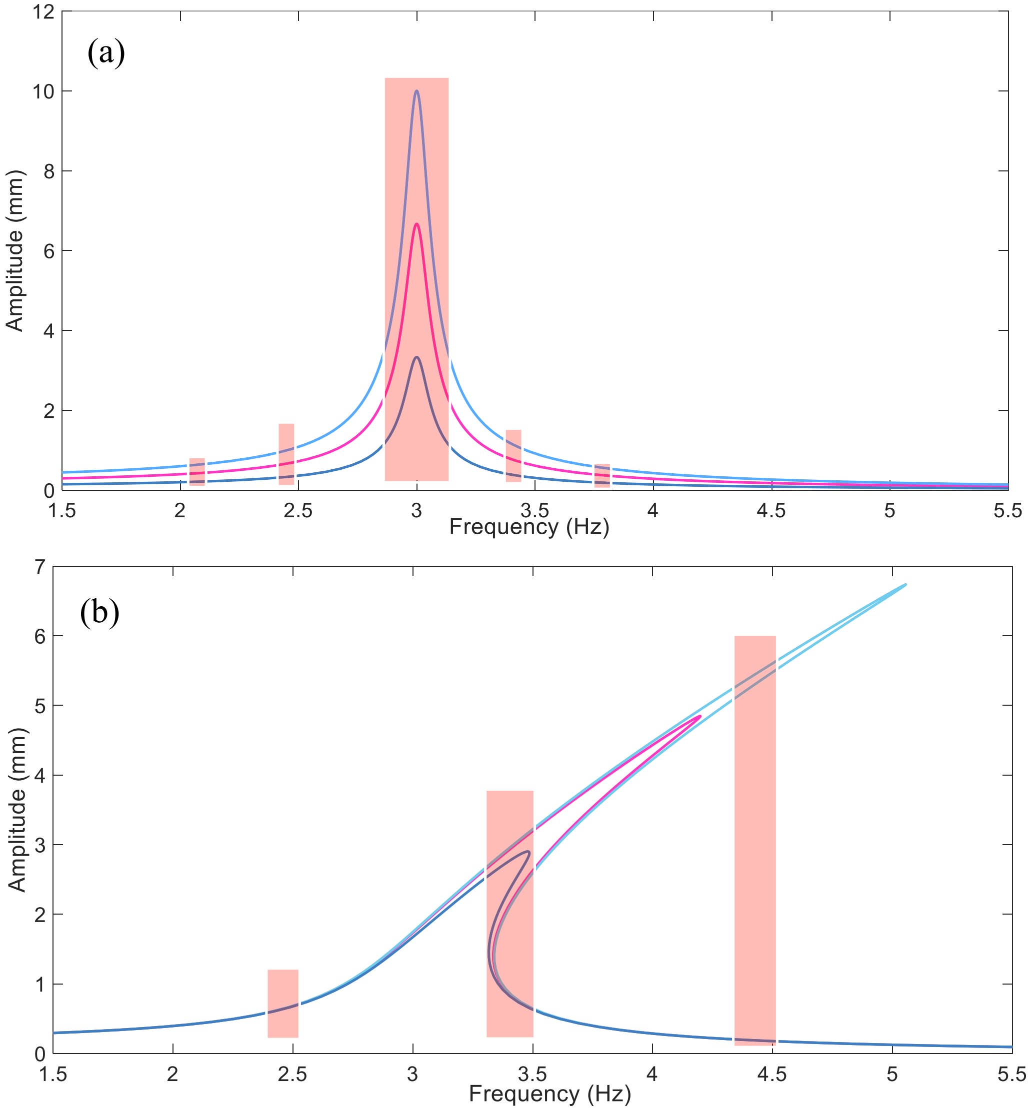

Fig. 8. Fixed frequency views: (a) linear FRFs and (b) nonlinear FRFs.

between nonlinearities and uncertainties. Some papers investigated the nonlinear dynamics of the rotor-AMBs systems [64,225,226] and cracked rotor systems [47,75,76,115,227,228]. Nonlinear analysis techniques, such as the alternating frequency/time technique (HB-AFT), orthogonal polynomial approximation and linearization, are used to deal with nonlinearities. The detailed introduction of the above contributions will be presented in the later dedicated sections. Here, we discuss a special and critical issue posed in nonlinear uncertain dynamics. Nonlinearity in uncertain rotor systems will bring great challenges to frequency response function (FRF) computations [155]. In FRFs of linear uncertain rotor systems, there is always one single corresponding amplitude for a frequency. However, the FRFs for nonlinear rotors have turning points which means one frequency may correspond to multiple amplitudes, including stable and unstable solutions. Thus, the original quantification method by using the frequency as a control parameter in linear FRFs will fail in nonlinear cases. As shown in Fig. 8, each curve can be a sampled FRF from an uncertain and nonlinear dynamic system. In Fig. 8(a), the nonlinear terms are neglected and the system degrades to linear cases. It is noticed that the UQ method will easily find sampled values from every FRF at fixed frequencies. In Fig. 8(b), the situation is much more complicated. There may be two or three solutions for one single frequency. Moreover, the path-following techniques produce different frequency steps for different sample FRFs. This can be concluded as sample solutions obtained from different FRFs are not on the ‘same condition’ and are further incomparable from the perspective of UQ methods. Sinou et al. [155,229] treated the frequency as an uncertain input and expanded on the PC basis apart from other random parameters. Then, the envelopes of the stochastic responses were obtained for nonlinear systems with cubic stiffness and also contact problems based on the HB-AFT. An intrusive PCE was implemented in the HBM process to build the nonlinear response from a limited number of runs of the computational model. Sarrouy et al. [230,231] proposed to use the phase of vibration as a new control parameter in linear stochastic systems to eliminate multimodality and discontinuity in FRFs. This

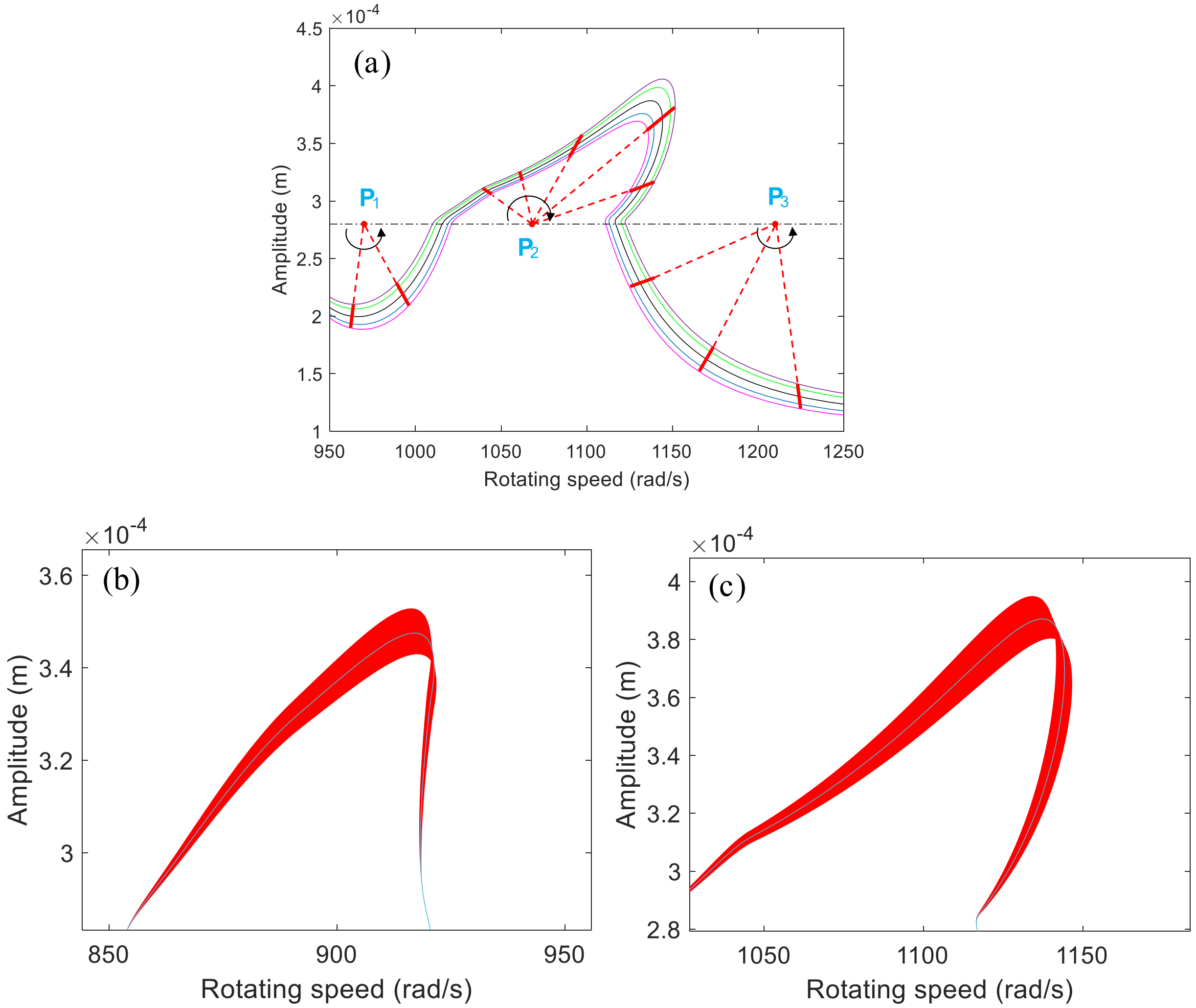

Fig. 9. Polar angle interpolation for nonlinear FRF uncertainty prediction [234]: (a) the polar sampling view, (b) and (c) accurate predictions of the multi-solution response bounds.

viewpoint may also be inspiring for solving the multi-solution issue in nonlinear systems with uncertainties although till now no relevant research has been reported to clarify how to transfer back to the frequency measure from the phase measure. Peradotto et al. [232] applied the asymptotic numerical method (ANM) to stochastic FRFs calculation of a rotor system with cubic stiffness and verified the results by MCS. Panunzio et al. [233] further generalized the method to deal with the Gibbs phenomenon and introduced an arc-length ratio, which ensures resonance points of FRFs or the turning points are reached simultaneously. Fu et al. [234] proposed a polar angle interpolation (PAI) method as a non-intrusive post-processing technique to tune the different nonlinear FRF samples of a rub-impact dual-rotor system, as shown in Fig. 9. The fixed frequency view in the traditional sense is replaced by the polar angle view, which has only one amplitude for a given polar angle (the amplitude samples are regenerated via interpolation with a fixed polar step length). However, it must be pointed out that the PAI method may be clumsy or even fail in nonlinear FRFs with extreme curvature or there are more than three amplitudes for a fixed frequency.

From the reported works on rotordynamics with uncertainty, several general points can be summarized. An essential feature found is that the deviations of QoIs depend on the uncertainty level of inputs. Normally, larger variations of inputs on rotor systems lead to more significant dispersion of the output QoIs although different forms may be presented according to the type of uncertainty used, such as variance or standard variance in stochastic analysis and bounds in interval cases. The second point that should be pointed out is that the effects of uncertainty can be different for rotor systems regarding different model configurations and physical values. In fact, sensitivities of uncertainties play vital roles, and they naturally vary from case to case. Finally, it is noticed that influence patterns depend on whether the inherent characteristics of the rotor system under study will be affected by the input uncertainties. For instance, damping in a linear rotor system mainly controls the vibration amplitudes and its effects are mainly found in resonance regions. Thus, the uncertainty of damping in linear rotor systems causes the deviations of vibration responses only near the critical speeds and no obvious changes can be found in critical speeds themselves. Imbalance uncertainty affects vibration amplitudes only as well, but it has

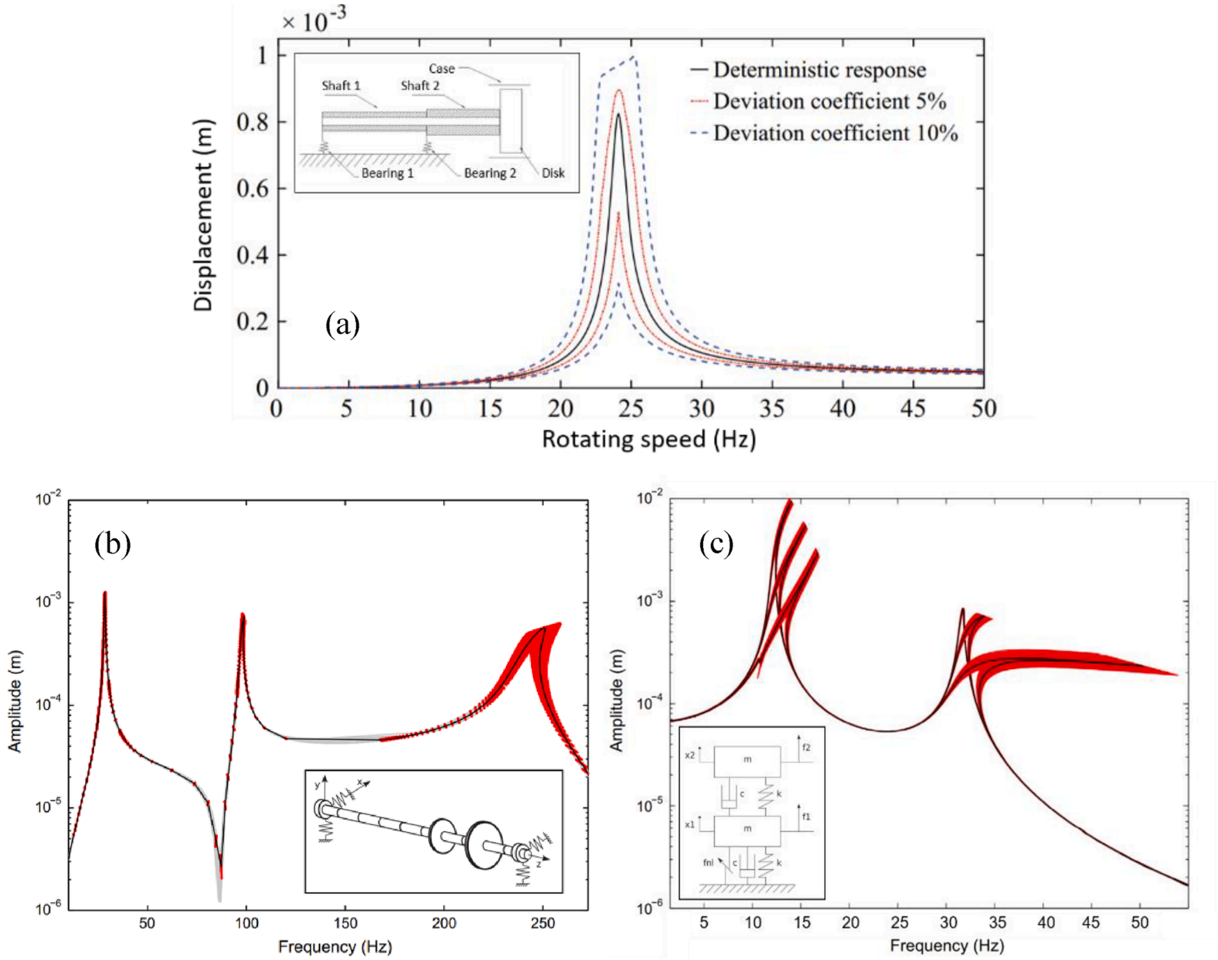

Fig. 10. The FRFs of uncertain dynamical systems: (a) linear rotor system, (b) nonlinear rotor system and (c) nonlinear two-degrees-of-freedom system. (Reproduced from [42,155,229])

universal influences at all rotation speeds. For uncertainties that alter the natural characteristics of rotor systems, the results are totally different. As evidenced by the literature, the lower bound in interval problem or lower envelope in stochastic problem has a single sharp peak while the upper bound or upper envelope has an expanded resonance band [93]. The bandwidth depends on the level of input uncertainty. In the band region, amplitudes are high compared to deterministic values. However, the amplitude at the very critical speed has not deviated very much from the deterministic resonance peak but is almost identical. The new resonance peaks in upper bounds locate at new rotation speeds. Whether the vibration peak is shifted to the left or right depends on the uncertainty considered. This phenomenon was found both in linear and nonlinear uncertain rotor systems as well as general mass-spring vibrators [155,229], as shown in Fig. 10. For anti-resonance peaks, the pattern is reversed.

4.2. Dynamics of faulted rotors

Faults are common in rotating machinery for a variety of reasons and they significantly affect the dynamics of rotor systems. If one fault occurs and proper maintenance is not carried out in time, other secondary faults will successively appear. For example, misalignment or bow of the shaft can lead to high vibration amplitudes which may induce contact of the rotor and stator, i.e., rub-impact. Faults reduce the lifetime of rotating machinery and can even cause catastrophic accidents. Thus, fault mechanism and diagnosis are important research areas in rotordynamics. The uncertainty associated with common faults has attracted the attention of researchers in recent years. As summarized in Section 2.1, uncertainty analysis was extended to rotor systems with several typical faults, such as bow, misalignment, asymmetry, crack and rub-impact faults. It is worth clarifying once again that the mass imbalance of a rotor system is not classified as a fault but as a model parameter to distinguish from Section 4.1. in this paper although the rotor imbalance is admitted as one of the most common faults in rotor systems.

4.2.1. Bow fault

The bow corresponds to an initial deformation of the rotor which can be caused by thermal effects or gravity effects on off-line machines. The fault adds a linear force to the EOMs [73]:

$$\mathbf{f}^{bow} = \mathbf{f}_c^{bow}\cos(\omega t + \varphi^{bow}) + \mathbf{f}_s^{bow}\sin(\omega t + \varphi^{bow}) \tag{35}$$

where the constant φ^{bow}, $\mathbf{f}_c^{bow}$ and $\mathbf{f}_s^{bow}$ are the initial phase of the bow, the amplitudes for the cosine and sine terms, respectively. Thus, we can notice that the uncertainty in a bow fault will have influences on the linear force vector $\mathbf{Q}_l$.

Didier et al. [73] studied the stochastic nonlinear responses of a rotor system subjected to uncertainties on both the amplitude and the phase of the bow. A high variability on the first and third harmonic components due to uncertainties in bow characteristics was observed whereas no sensitivity on the second and fourth harmonic components was detected. These results can be explained by the fact that the forces that model the bow are linear and the rotor model studied has coupling terms between odd harmonics.

4.2.2. Misalignment fault

Misalignment fault in rotor systems often refers to the parallel and angular mismatch of shafts connected by couplings due to manufacturing or assembling errors. The effects of the parallel misalignment are represented by additional forces at the coupling node [73]:

$$f_{m1} = Nk_{m1}\delta_{m1}[\sin(\omega t), 1 - \cos(\omega t), 0, 0]^T \tag{36}$$

where N, k_{m1} and δ_{m1} denote the number of bolts in the coupling, the stiffness of bolts and the amplitude of the parallel misalignment, respectively. From Eq. (36), it is evident that the uncertainty in the parallel misalignment adds to the variability of the linear forces $\mathbf{Q}_l$.

An angular misalignment will bring two terms to the EOMs of the rotor system, i.e., a parametric stiffness and a reaction force [73]:

$$\begin{cases} \mathbf{K}_{m2} = \frac{1}{2}(3k_{m1} + k_{m2})r^2 \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} + \frac{1}{2}k_{m2}r^2 \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \cos(2\omega t) & \sin(2\omega t) \\ & & \sin(2\omega t) & -\cos(2\omega t) \end{bmatrix} \\ \mathbf{f}_{m2} = -\frac{1}{2}\delta_{m2}r^2[0,\ 0,\ 3k_{m1} + k_{m2} + k_{m2}\cos(2\omega t),\ k_{m2}\sin(2\omega t)]^{\mathrm{T}} \end{cases} \tag{37}$$

where r, k_{m2} and δ_{m2} are the radius of bolts, the additional stiffness and the amplitude of the angular misalignment, respectively. As shown in Eq. (37), the variability related to the angular misalignment induces changes in the system stiffness matrix and the linear force vector. There will be couplings between the first a few orders of frequency components in the dynamic responses due to the non-synchronous stiffness and forces.

Asymmetry of shaft, misalignment and bow coupled faults were investigated by Didier et al. [73] to study the stochastic nonlinear dynamic responses of a rotor system. Uncertainties in the mass imbalance, Young's modulus, depth of asymmetry, amplitude of bow, relative displacement of the parallel misalignment and angular misalignment were considered. Both the envelopes of the stochastic FRFs and shaft orbits were presented. Misalignment fault in rotating systems is mainly induced by assembling errors. The lateral offset between the centerlines of two parallel shafts is termed parallel misalignment and the angle between two consecutive shafts is called angular misalignment. Li et al. [116] modeled angular misalignment in a rotor system with cubic support stiffness. Considering uncertainties in damping, stiffness coefficient and angular misalignment fault, the authors illustrated the mean response time history, orbit and PDFs of amplitude, in which P-bifurcation was observed. The initial angular misalignment uncertainty of rolling element bearings stemmed from mounting was studied in [82]. The PDFs and CDFs for rating life and fatigue life of the bearings and spindle were discussed in detail by two cases. It was found that the rating life of the spindle system is greatly reduced by the uncertain misalignment in rear bearings. Recently, interval analysis was introduced to misalignment fault studies [83]. Fu et al. [84] used a Legendre interval collocation method to quantify the effects of misalignment uncertainty on the dispersions of $n\times$ harmonic components. The results show that the uncertainty in parallel misalignment propagates into the $1\times$ and $3\times$ components while uncertainty in angular misalignment mainly affects the $2\times$ and $4\times$ components. Misalignment in the uncertain journal bearings and the stochastic lubrication were investigated as well [235].

4.2.3. Asymmetry and crack fault

Asymmetry is a common fault in rotating machinery that is classically due to the geometry of shafts, asymmetric coupling, or the presence of cracks. For example, a breathing transverse crack can be modeled by recalculating the moments of inertia of the damaged shaft element and the stiffness loss due to the crack. Then the final form of the stiffness matrix of the cracked rotor system is expressed as [75,227]

$$\begin{cases} \widetilde{\mathbf{K}} = \mathbf{K} - g(t)\mathbf{K}^{crack},\ g(t) = \dfrac{1 - \cos(\omega t)}{2} \\ \mathbf{K}^{crack} = diag(\mathbf{0},\ \cdots,\ \mathbf{0},\ \mathbf{K}_j^{loss},\ \mathbf{0},\ \cdots,\ \mathbf{0}) \end{cases} \tag{38}$$

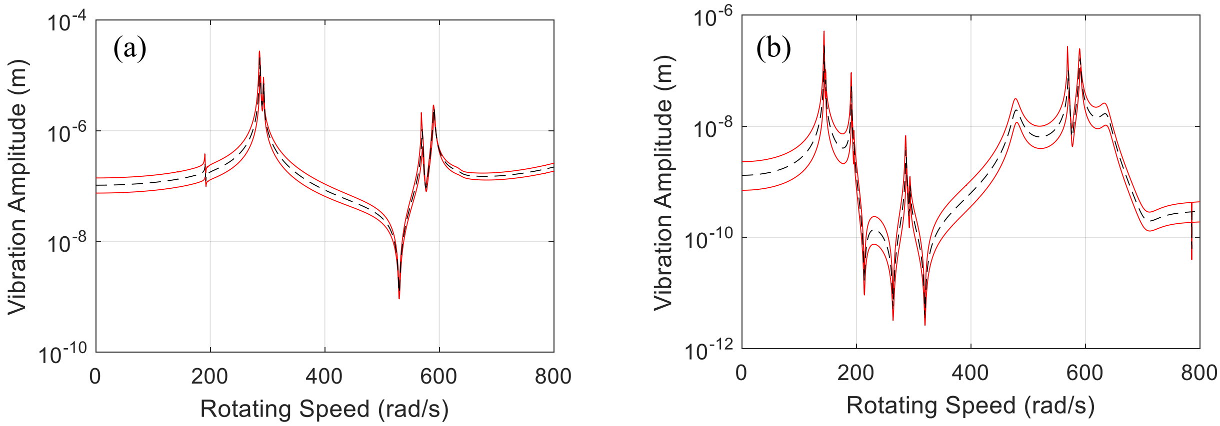

Fig. 11. Harmonics analysis of a cracked rotor system with the interval crack depth [75]: (a) 2× component and (b) 4× component.

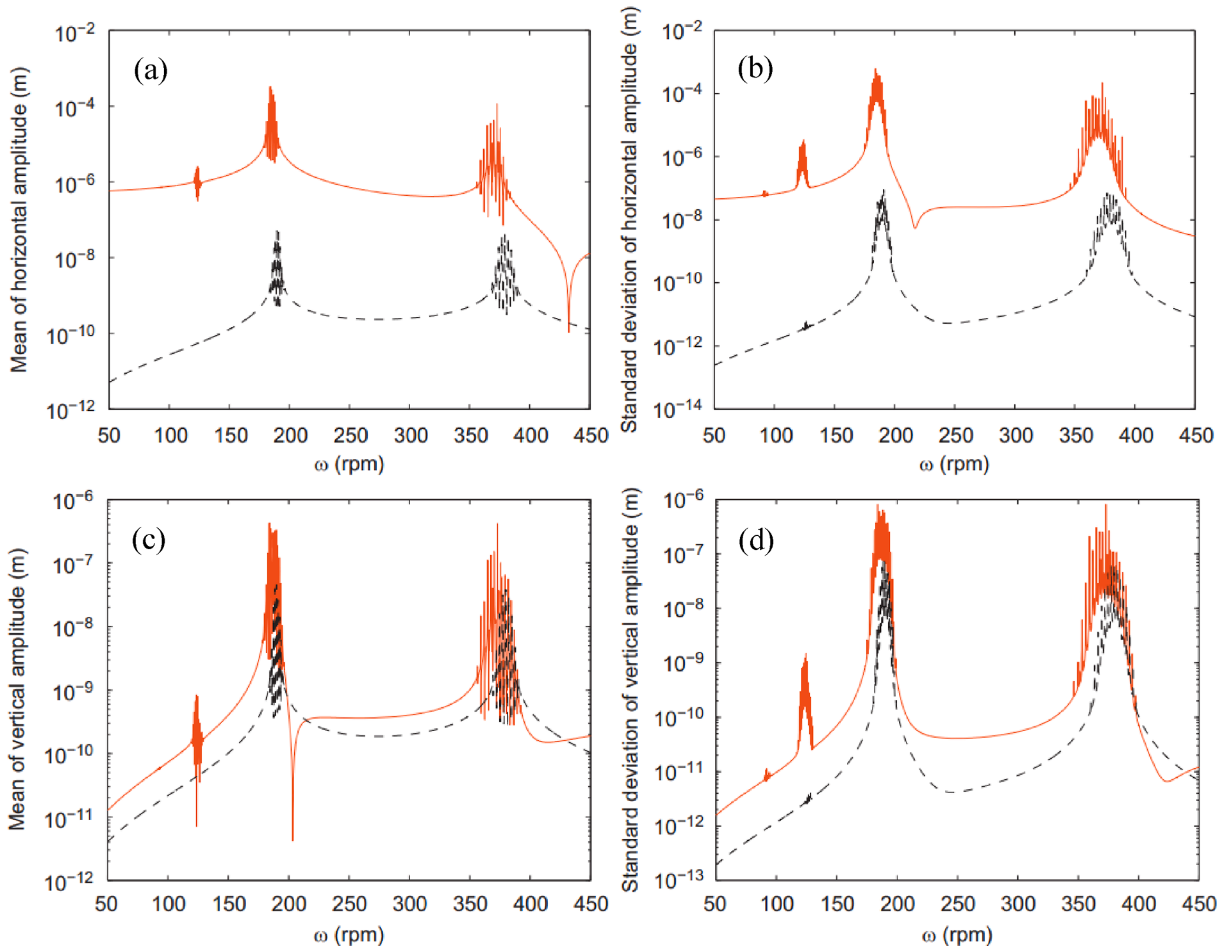

Fig. 12. The 2× displacements for a cracked rotor with stochastic uncertainties in Young's modulus and excitations (solid lines for the non-dimensional crack depth 1 and dashed lines for 0.5) [76]: (a) mean of horizontal response, (b) standard variance for horizontal response, (c) mean of vertical response and (d) standard variance for vertical response.

where $g(t)$, $\mathbf{K}_j^{loss}$ and $\mathbf{K}^{crack}$ are the breathing function to simulate the time-variant behaviors of the crack geometry (a cosine function is commonly used), the stiffness reduction of the damaged shaft element (j indicates the cracked element number), and the augmented

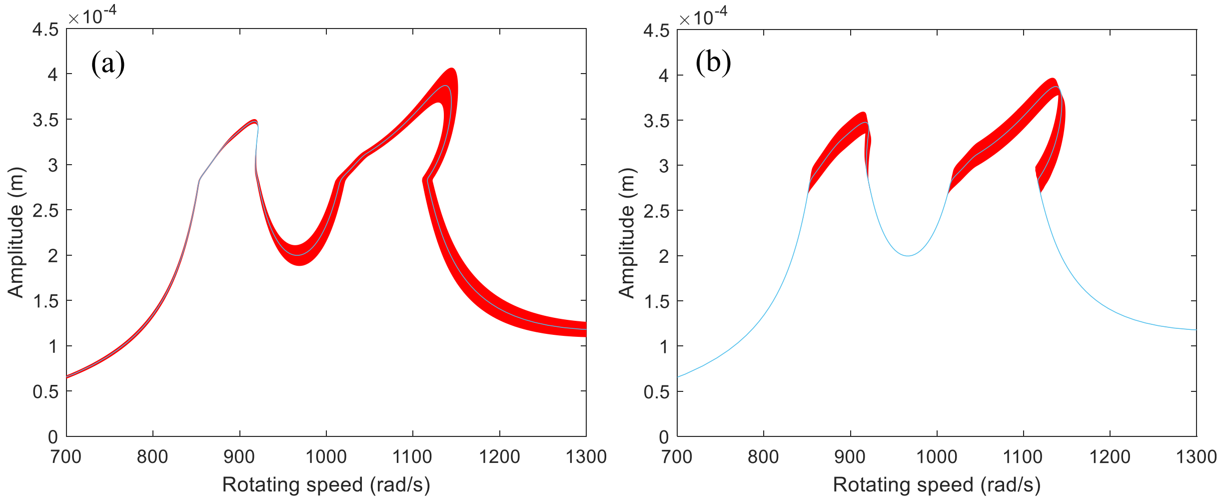

Fig. 13. The amplitude-frequency responses of an uncertain rub-impact dual-rotor system (Red-uncertain response ranges; blue solid line-deterministic) [234]: (a) interval unbalance excitations and (b) interval rotor/stator contact stiffness.

$\mathbf{K}_j^{loss}$ in global form, respectively. ω corresponds to the rotational speed of the rotor system. The parametric stiffness matrix $\widetilde{\mathbf{K}}$ will interact with the displacement vector $\mathbf{x}$, introducing the couplings of the frequency components in the vibration responses as evidenced in the HBM deduction process [75]. Thus, complicated behaviors may be observed when uncertainties exist in the cracked system.

The quantification of the effects of uncertain geometrical properties, such as the depth of the asymmetry for a rotor shaft, on the variability of the nonlinear response of a rotor system, was investigated by Didier et al. [73]. Results illustrated that uncertainty in the asymmetric section properties induces evident variations of the static, $2\times$, $3\times$ and $4\times$ harmonic components while the variation of the first harmonic component remains very small. Consequently, variations on the global nonlinear response were not drastically affected because the first harmonic component represents the predominant vibration of the global nonlinear response of the rotor system. Sinou and Jacquelin [94] studied the nonlinear response of a rotor system with uncertain stiffness and asymmetric coupling that involves time-dependent terms.

A crack in the shaft is a frequently seen fault in rotor systems because of wear or defects. It breaches the structural integrity of rotor systems and lots of efforts have been put to detect crack faults [157]. The depth of a breathing crack and other model parameters are considered as interval variables to reveal their impacts on the frequency components in the nonlinear responses of a rotor system based on the interval descriptions [75], as demonstrated in Fig. 11. Sinou et al. [76] investigated the stochastic nonlinear responses of a rotor system with a transverse breathing crack when the stiffness of the shaft and the excitation forces were uncertain. The harmonic responses, illustrated in Fig. 12, are found to have spurious peaks near resonances and sub-resonances due to the nonlinearities and the high sensitivity of the system to uncertainties. The spurious peaks in the responses of uncertain mechanical systems are deemed as inaccurate estimations. Considering a crack in the middle span of a rotor (this causes high sensitivities of the responses to uncertainties), Fu et al. [228] revealed the roles of the Chebyshev surrogate model parameters in the prediction accuracy and efficiency. It is found that the Chebyshev order is the key factor in reducing those false peaks. The role of the PC order and the improvement strategy for reducing the spurious peaks are also discussed in [94,233]. Breathing cracks can evolve to open cracks. Subsequently, the dynamics of an uncertain hollow-shaft rotor system with an open crack were studied [236] using the CIM. Leng et al. [47] studied the bifurcation and chaos response of a cracked rotor subject to random disturbance. The above-mentioned investigations suggest that it is practically insufficient to use linear approach-based condition monitoring techniques, i.e., monitoring frequency shifts and variations in modal parameters, for detecting damages with the presence of uncertainties, which lead to variations of the monitored features as well. The recommended damage indicators are the appearance of the $n\times$ harmonic components at sub-critical speeds, i.e., $1/n$ of the critical speeds. Moreover, interactions between the imbalance and breathing cracks lead to more significant deviations in the horizontal direction than in the vertical direction [76]. In other works, the Kriging surrogate model was used to identify crack damage in a rotor system based on the super-harmonic nonlinear characteristics [204,227]. Chen and Guo [237] considered measurement uncertainty and proposed the interval Gauss function as kernels of an interval crack fault diagnosis method. An optimization method was then applied to improve the classification accuracy which was verified via an example.

4.2.4. Rub-impact fault

Rub-impact fault represents the contact between the rotor and stator as the clearance between them is reached when the rotor experiences severe vibrations. In practice, the rub-impact fault is usually caused by other faults that significantly increase the vibration amplitudes of the rotor or the rotor and casing are not concentric. During their contact, the rotor continues to operate which generates the tangential friction force and normal impact force on the disc node, which can be written in the fixed coordinates frame as [234]

$$\begin{cases} F_x = H(x_r,\ y_r)K_{rub}(1 - \dfrac{\delta_{rub}}{\sqrt{x_r^2+y_r^2}})(x_r - \mu y_r) \\ F_y = H(x_r,\ y_r)K_{rub}(1 - \dfrac{\delta_{rub}}{\sqrt{x_r^2+y_r^2}})(\mu x_r + y_r) \end{cases},\ H(x_r,\ y_r) = \begin{cases} 0,\ if\ \sqrt{x_r^2+y_r^2} - \delta_{rub} < 0 \\ 1,\ if\ \sqrt{x_r^2+y_r^2} - \delta_{rub} \geqslant 0 \end{cases} \tag{39}$$

where $H(x_r,\ y_r)$ and (x_r, y_r) are the Heaviside function and the displacements of the disc center. Symbols δ_{rub}, μ and K_{rub} represent the radial clearance, friction coefficient and contact stiffness between the rotor and stator, respectively. The forces expressed in Eq. (39) are non-smooth and piecewise nonlinear, causing difficulties in the UQ of a rub-impact rotor.

Most researchers focused on the dynamic analysis and fault diagnosis of rubbing rotors in a deterministic sense. Researchers gradually paid attention to uncertainty propagation in rotor systems with rubbing events [77]. Yang et al. [201] presented the stochastic bifurcation and chaos of a rub-impact rotor system considering random stiffness and random excitation. Results showed that the nonlinear response can be promoted when the rotational speed is near the 1/2 first-order critical speed and suppressed over it. Guo et al. [238] modeled uncertainty in a Jeffcott rotor/stator with synchronous full annular rub as weak random disturbances. The diffusion and persistence under different conditions are studied, especially the mechanism of noise-induced escape from synchronous full annular rub towards dry friction backward whirl with large vibration amplitudes. Mixed aleatory and epistemic uncertainties were considered in [134] and the likelihood method was applied to obtain the nonlinear responses of a Jeffcott rotor. Ma et al. [78] studied a rubbing rotor based on the gPCE formulation in conjunction with the HB-AFT. Uncertainties in the rub-impact fault-related parameters were investigated to reveal their effects on nonlinear steady-state responses of the rotor model. The above investigations used the analytical Jeffcott rotor model as a research object to highlight the propagation mechanism of various uncertainties. However, this simple model may limit the practical applications. Fu et al. [79] extended the interval analysis to an overhung rotor system subject to rub-impact in which the clearance between rotor and stator, contact stiffness and friction coefficient were used as uncertain variables. However, it lacks exploitation of deeper nonlinear characteristics since time history responses can exhibit limited information. Further, Fu et al. [234] studied the deviation ranges of the nonlinear FRFs of a dual-rotor rub-impact system subject to interval uncertain parameters and found interesting variation patterns when the model parameters (such as the disk imbalance) and fault-related parameters (such as the rotor/stator contact stiffness) are uncertain, as presented in Fig. 13.

Several studies exploited dynamics under other faults within an uncertain context. Garoli et al. [166] proposed to combine the gPCE and Bayesian inference to identify the bearing wear fault parameters. Experimental measurements are used in the comparison of the stochastic and deterministic approaches. A fuzzy fault tree analysis was carried out by Li et al. [173] where the uncontained event of an engine rotor was on the top of a fault list covering over twenty faults in total. Castro et al. [222] applied a deep learning method to diagnose the ovalization fault of hydrodynamic journal bearings for model-based condition monitoring. The results proved that the deep learning method used is a powerful tool to predict ovalization faults in bearings. Tian et al. [239] investigated the effects of the uncertain defect size of the inter-shaft bearing in a dual-rotor system. The local defect size variability on the ball bearing inner raceway caused the main characteristics of the system to vary tremendously. An entropy method was used to diagnose the rolling bearing fault in a rotor-bearing system [240].

From a pragmatic point of view, the uncertainty treatment in faulted rotor systems attempts to gain a deeper understanding of the vibration behaviors of the faulted systems under uncertainty, and further promote the more robust detection or monitoring of these faults based on the uncertain responses. It is evident from the above-reported literature that many issues remain to be exploited on the topic of uncertainty analysis for faulted rotor systems and the robust diagnosis of typical faults. The vibrations of rotor systems with faults are already complicated and the quantitative uncertainty treatment is extremely challenging. Currently, simple rotor models are often used, and experimental works are insufficient.

4.3. Stability analysis

Instability issues in rotating systems are aroused by multiple sources. A rotor operating in unstable regions is very dangerous because the vibrations and noise produced will aggravate significantly [65]. From the reported literature on stability analysis with uncertainty, three main aspects are concerned, i.e., axial loads, oil film bearings and AMBs. Other investigated sources include uncertainty in support motion [241] and parametric excitations (not axial forces) which were modeled by a combination of harmonic terms and stationary stochastic processes [242]. Szolc and Konowrocki [81] modeled the overhung rotor system in a heavy blower applied in the mining industry and the stability and sensitivity were discussed in detail with stochastic imperfections included. In [243], normal variables were used to model the relative phase of two bearings to represent the uncertainty of fit position and installation. A 3D solid element rotor model was considered and quasimodes analysis for determining stable regions was carried out based on Hill's determinant method. The rotor-AMBs systems with uncertainty often involve vibration control and thus will be summarized in Section 6.

In rotor-bearing systems, axial forces at two ends of the shaft induced by external actions can cause the system to experience parametric instabilities. Lateral deflections of a slightly bent shaft with creep will increase gradually with time due to the axial loads, which are generally compressive. The shaft can collapse if a certain threshold is reached. Young et al. [66] studied the lateral vibrations of a rotor-ball bearing system with axial forces represented by the sum of a static force and a random process with a zero mean, which were analyzed by the stochastic average method. The first- and second-moment stability criteria were determined proving that the first-moment response is always stable. Bai and Zhang [67] obtained the Lyapunov exponent of a stochastically excited rotor system

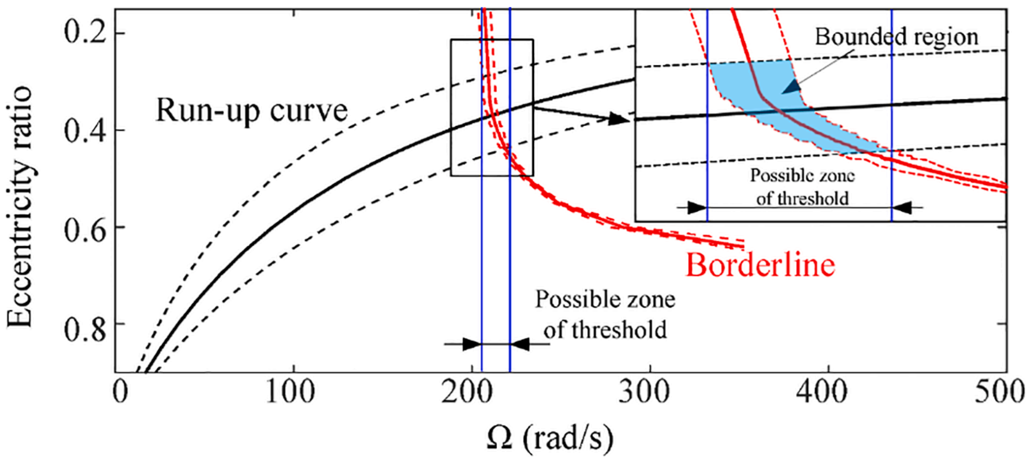

Fig. 14. Stochastic stability thresholds of a rotor with misaligned journal bearings [55].

and presented the almost sure asymptotic stability, where a dissipated quasi-integrable-Hamiltonian system with a stationary ergodic process was used to represent the rotor. Results showed that translation motions of the rotor system are generally stable, but angular motions can significantly affect the almost sure asymptotic stability. Pavlović et al. [244] revealed the influence of rotatory inertia on stochastic stability and concluded that it cannot be neglected in short shafts where transverse shear should be considered. Further, the cases of viscoelastic rotating shafts were investigated [127,245,246] showing viscoelasticity increases the almost sure stability.

Oil-film bearings are prone to cause instabilities in many rotor systems. However, faithful and accurate models required to predict the reliable dynamic behaviors of rotor-bearing systems are difficult to establish. In fact, the dynamic coefficients are changing with external factors such as load, temperature and rotation speed. Therefore, it is appropriate to introduce uncertainty representations for robust dynamic analysis. Sun et al. [109] investigated the stability of an uncertain rotor supported on misaligned journal bearings and demonstrated that the stability threshold increases with misaligned angles. As shown in Fig. 14, the stochastic stability thresholds can take any intersections of the 95 % confidence intervals of the run-up curves and the borderlines. Moustafa and El-Awady [105,198] realized that single-valued quantities are not adequate for oil-film bearing dynamic coefficients and they introduced interval analysis for stability studies of rotor-bearing systems, where the bearing clearance, oil film stiffness and damping coefficients, as well as the shaft flexural rigidity, were interval uncertainties. Garoli and de Castro [80] investigated the period-doubling (oil whirl) and quasi-periodic motions (oil whip) induced by the fluid-induced instabilities in a rotor-bearing system with uncertainties in bearings, namely, the uncertain lubricant viscosity and radial clearance. Further, Visnadi and de Castro [195] pointed out that uncertainty in the radial clearance has more influence on the stability than that in lubricant viscosity. Rough surfaces in hydrodynamic journal bearings were modeled by transverse and longitudinal long narrow ridges and valleys in [217]. The stability characteristics of a rigid rotor supported on the imperfect hydrodynamic journal bearings were then investigated by the so-called SFEM. It was shown that transverse roughness significantly increases the rotor's stability while isotropic roughness slightly decreases it and longitudinal roughness reduces it.

It is found that existing works on stability analysis of rotor systems with uncertainty mainly used analytical rotor models. Moreover, instabilities induced by other events of rotor systems under uncertainty have not been studied, such as dry friction.

4.4. Parameter identification

Parameter identification is a typical inverse inference process aiming to identify unknown model parameters in a rotor system. Subsequently, uncertainty analysis in this area essentially fits into backward inference such as Bayesian inference. Estimation of unknown parameters under uncertainty such as measurement error is meaningful for robust system identification [247], accurate modeling and damage assessment [248]. Although model updating of rotor systems is closely related to parameter estimation and calibration, it is a dedicated topic in general dynamic systems and has its own principles [167,168,249]. This paper, which concentrates on uncertainty treatment, will not cover much of that area.

The most common research objectives in the literature concerning parameter identification with uncertainty in rotor systems are to identify imbalance [170], bearing properties [89,160,199,250,251] and damages [166,252]. Mao et al. [253] proposed a micro-genetic algorithm with the advance and retreat method to identify the eccentricity of the spindle-tool system subject to random inputs and outputs. Identification results for random deviations up to 5 % showed that error in the identified eccentricity for both motor and wheel is no greater than 0.47 %. Further, Mao et al. [180] used mixed interval and random descriptions for the uncertainty in unbalance identification process based on perturbation of transient responses of the rotor. In [254], the authors proposed to identify unbalance using an uncertainty-based combination of signal processing techniques, including Fourier transform and quasi-harmonic fitting of signal denoised with Hilbert-Huang transform, Hilbert vibration decomposition, and wavelet packet decomposition. The simulations and experimental results showed the validity of the data fusion, which reduces measurement uncertainty between 10 and

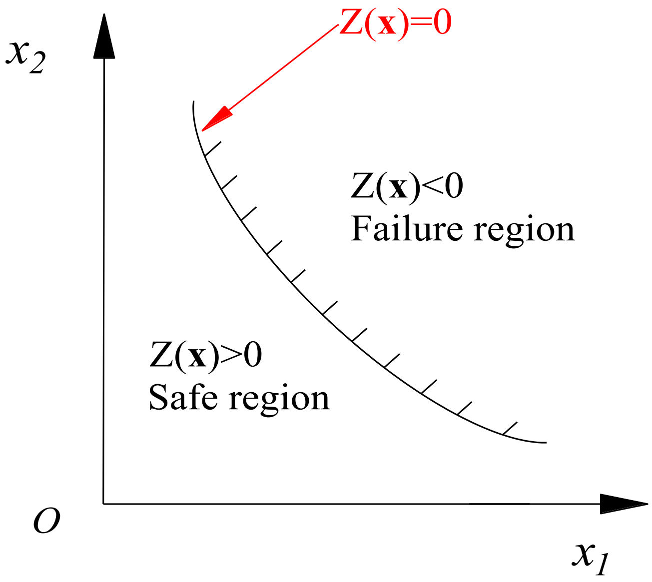

Fig. 15. Safe region, limit state function and failure region.

40 %.

Tiwari and Vyas [255] carried out the stiffness parameter estimations of rolling element bearings based on random responses of the nonlinear rotor-bearing system, in which the random uncertainty primarily comes from progressive deterioration of the bearing surfaces and subsurfaces. It can also identify the magnitude and angular location of mass imbalance and damping ratio in the system. Experimental validation proved the effectiveness of the proposed method. Tyminski et al. [169] employed Bayesian inference to estimate dynamic coefficients of hydrodynamic journal bearings. The MCMC method was applied to evaluate the influences of journal bearing's uncertainties on the overall behavior of these components. A rotor-journal bearing platform was developed to obtain prior distributions of random parameters. Then, various case studies were performed that confirmed the efficiency of the Bayesian method on rotor-bearing system parameters identification. Garoli et al. [256] proposed the combined implementation of the Bayesian inference and gPCE for identification of the AMB parameters and subsequently a sensitivity analysis was accomplished. Lu et al. [204] established a Kriging surrogate model based on the super-harmonic features, i.e., the 2× responses, to identify crack damages in rotor systems. Results suggest that a few samples will be sufficient for the surrogate model and it is robust to noise. It was also pointed out that more crack features are needed to improve the reliability of the surrogate method for real industrial applications. Han et al. [257] proposed a method based on an improved Kriging surrogate model with multi-point addition criterion and evolutionary algorithm for structural parameters identification in rotor systems, such as the bearing parameters and clearance in rub-impacts. Numerical verification with 10 % Gaussian noise and experimental validation of rotor-bearing system experiencing rub-impact were conducted, justifying the causes of the discrepancies between the experimental and FEM results were inevitable uncertainties in material properties and boundary conditions.

5. Reliability assessment and optimization

This section summarizes the reliability analysis of rotor systems in uncertain scenarios and optimizations of design and dynamic analysis, which is important for risk assessment [258]. These two aspects can be investigated jointly to obtain optimal solutions in a reliability sense.

5.1. Reliability assessment

Reliability generally represents the probability that a system performs its expected function without any failure within a certain period of time (before any failure occurs) [259]. It has profound significance in analyzing the reliability of rotor systems in the phases of design and operation [120,260–262] and further guides the fatigue life assessment [263,264]. Li et al. [265] reviewed the recent advances in reliability analysis of aero-engine rotor systems.

Reliability assessment usually consists of two major steps: The first is to determine failure modes defined by the limit state functions and the second is to calculate the probabilities of failure modes or reliability indices. A fundamental effort to calculate reliability is the following integral of the PDFs or joint PDFs of uncertainties over the entire safe region:

$$R = Pr(Z > 0) = \int_{Z>0} f(X)dX \tag{40}$$

where X is the random responses, $f(X)$ designates the joint PDF and Z represents the state function. As illustrated in Fig. 15, the safe region, limit state function and failure region can be defined by values of the state functions as

$$\begin{cases} Z = g(X,\ \Omega) \leqslant 0,\ \text{safe state} \\ Z = g(X,\ \Omega) > 0,\ \text{failure state} \end{cases} \tag{41}$$

where Ω denotes the rotation speed. The expression of the state function g depends on the failure modes of specific problems. Once it is defined, the reliability index can be calculated as

$$\beta = \frac{\mu_g}{\sigma_g} \tag{42}$$

with μ_g and σ_g being the mean and standard variance of g.

Reliability criteria and calculation of the integral in Eq. (40) can be difficult for nonlinear rotor systems. When interval uncertainty is taken into consideration, then it evolves to interval-probability reliability analysis, where the limit sate functions form a belt defined by upper and lower bounds. Then, the robust optimization normally follows to fully determine reliability bounds. Zhang et al. [266] developed a statistical fourth-moment method for the reliability analysis of a rubbing Jeffcott rotor where the reliability integral was obtained by the Edgeworth series technique. In the rubbing rotor problem, the state function was defined as whether rubbing occurs:

$$Z = g(r,\ \delta) = \delta - r \tag{43}$$

where δ is the clearance between the rotor and stator and $\boldsymbol{r}$ denotes the radial deflection of the rotor. Further, the reliability sensitivity was obtained by derivatives [267]. Yang et al. [118] used a likelihood-based approach for reliability analysis of the nonlinear rub rotor/stator system with mixed aleatory and epistemic uncertainties. Bayesian techniques were employed to reduce the cost of computation. Effects of rotation speed, friction coefficient and clearance on the system reliability were discussed. The proposed method enables the incorporation of different kinds of data, such as sparse point data, probabilistic distribution and intervals.

Stress-strength interference in nonlinear rotor systems was investigated by Zhu et al. [268] using commercial software and the reliability model was established. The state function was defined by the material strength and dynamic stress of the rotor system. The sensitivity of reliability to the mean and variance of basic random variables was then calculated. It was revealed that the increase of the mean of shaft radius and material strength tends to make the rotor reliable while others tend to make it fail and an increase of variances of all parameters tends to make the system invalid. Bai et al. [269] applied probabilistic and nonprobabilistic hybrid reliability analysis to the blisk of aero-engine using the dynamic substructural extremum response surface decoupling method, where the tuned and mistuned blisks were considered. The authors presented the maximum failure probabilities of various dynamic characteristics of the considered blisk, including the natural frequency, modal displacement, modal stress, modal strain energy and vibration responses. Zhang et al. [63] examined the dynamic reliability of the rotor's positioning precision induced by varying compliance vibration of ball bearings. The contact uncertainty coefficient was modeled as the Bernoulli distribution to calculate the instantaneous contact probability. They found that increases in the mean and variance of elastic models, clearance and ball diameter decrease the dynamic reliability of the system. Recently, Ma et al. [270] studied the reliability of a rubbing rotor system based on the Kriging surrogate modeling strategy and pointed out that the gap between the rotor and stator plays a vital role in the reliability of the system.

5.2. Optimization under uncertainty

Optimization techniques help designers to find the best solutions in design space in terms of various parameters such as the geometric dimensions and bearing locations [271,272]. Global optimal solutions are obtained with additional efforts to avoid being trapped in local optimums for non-convex or nonlinear problems [114]. Traditional optimizations generally exclude uncertainties in rotor systems. Optimizations of problems with various uncertainties taken into consideration were firstly proposed by Taguchi et al. [273], which is called robust optimization. Robust optimizations of rotor systems involving uncertainty mainly fall into two categories: reliability-based design optimization (RBDO) and robust design optimization (RDO). To be more specific, RBDO imposes some probability constraints in the process of optimization while RDO finds optimal designs that are insensitive to the variabilities of uncertainties. The fundamental theory of RBDO can be expressed as

$$\begin{aligned} &\textit{minimize}\ F(x) \\ &\textit{subject to}\ P_f = P[G_j(X) \leqslant 0] \leqslant p_j \end{aligned} \tag{44}$$

where $F(X)$ represents the objective function for optimization, P_f is the probability of the current design satisfying constraints imposed, $G(X)$ denotes the limit state function derived from the design requirement specified constraints and p_j is the target failure probability for the jth design variable. A value of $G(X)$ less than zeros implies failures of the system. A typical formulation of RDO is given in [159] as

$$\begin{aligned} &\textit{find}\ \boldsymbol{d},\ \boldsymbol{\mu}_X \\ &\textit{minimizing}\ \widetilde{f} = \frac{1-\alpha}{\widehat{u}} \mathrm{E}\left[f(\boldsymbol{d},\ \boldsymbol{X},\ \boldsymbol{P}) + \frac{\alpha}{\widehat{\sigma}} \sigma[f(\boldsymbol{d},\ \boldsymbol{X},\ \boldsymbol{P})]\right] \end{aligned} \tag{45}$$

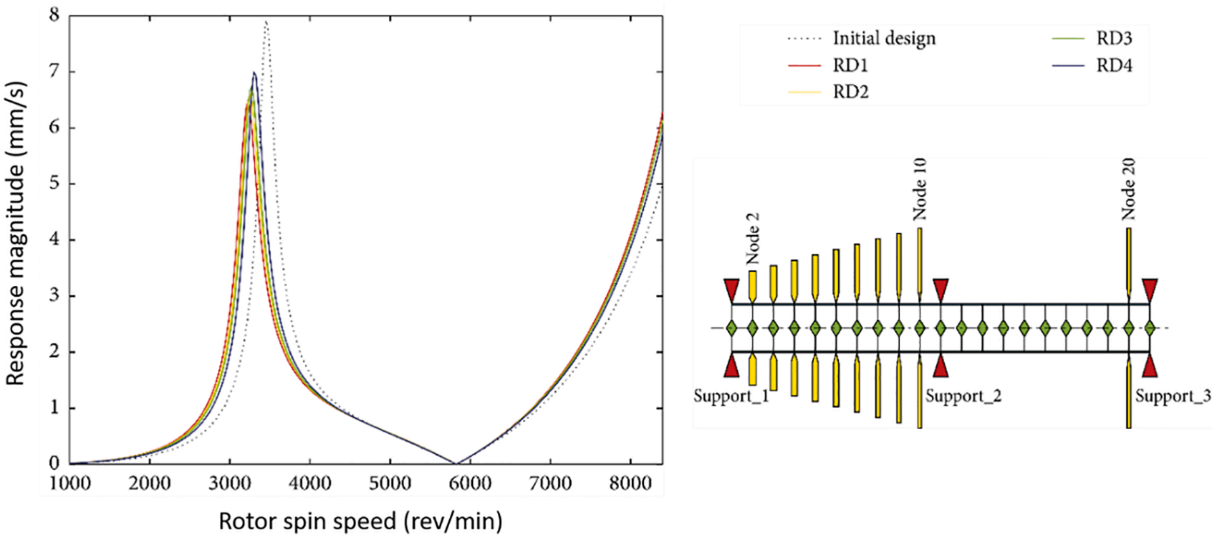

Fig. 16. Robust optimizations of a gas turbine low-pressure rotor system [102].

and

$$subject\ to \begin{cases} \mathrm{E}[g(\boldsymbol{d},\ \boldsymbol{X},\ \boldsymbol{P}) - \beta\sigma[g(\boldsymbol{d},\ \boldsymbol{X},\ \boldsymbol{P})]] \geqslant 0 \\ \sigma[c(\boldsymbol{d},\ \boldsymbol{X},\ \boldsymbol{P})] \leqslant \sigma^U \\ \boldsymbol{d}^L \leqslant \boldsymbol{d} \leqslant \boldsymbol{d}^U \\ \boldsymbol{\mu}_X^L \leqslant \boldsymbol{\mu}_X \leqslant \boldsymbol{\mu}_X^U \end{cases} \tag{46}$$

where $\boldsymbol{d}$ denotes the deterministic design variables, $\boldsymbol{X}$ represents random design variables, $\boldsymbol{P}$ is random parameters, $\boldsymbol{\mu}_X$ is the mean of $\boldsymbol{X}$, $\widetilde{f}$ denotes the objective function, g is the constraints function, c is constraints on variance, superscripts L and U represent lower and upper values, β is the feasibility index, notations with a cap mean normalization parameters and α is the weight factor ranging from 0 to 1. It can also be adapted to optimize just the mean or variance by setting the value of α to 0 or 1.

These two types of optimizations with or without probability constraints have seen a range of applications in uncertain rotor systems. In designing a rotor system, it is expected that working rotation speeds stay away from critical speeds of the system to avoid resonance vibrations. In the probabilistic domain, Xu et al. [102] proposed an intelligent RDO method in combination with different techniques including the PCE, radial basis function (RBF) neural network and evolutionary algorithms for a gas turbine rotor system with uncertain support stiffness. Then, the robust solutions were obtained by the multi-object optimization algorithm NSGA-II. It can be seen from Fig. 16 that the optimized design yields the lower mean and standard variations of the responses and the margin of the critical speed is larger than the initial design. Ritto et al. [107] considered uncertainties in many parameters of a rotor model, such as bearing stiffness and diameter, as the design parameters to ensure working speeds are as far away as possible from natural frequencies, where the Campbell diagram was adopted to derive the objective function and penalty functions were introduced. A multi-stage centrifugal compressor with stochastic uncertainty in the residual mass unbalances and stiffness of journal bearings was investigated by Stocki et al. [159]. The study aimed to minimize the weight of the compressor while maintaining robustness to uncertainties, reducing the risks of dangerous stress concentrations, and rubbing events. Dynamic balancing is an important and powerful tool to reduce vibrations induced by mass imbalance. Li et al. [274] proposed a robust balancing scheme by formulating a convex optimization problem to handle stochastic uncertainties in the influence coefficient as well as in vibration responses. The second-order cone programming was then used to solve the optimization problems. Application of the method to a 1150 MW nuclear turbine-generator system showed that the proposed robust balancing method outperforms other methods, such as the min–max balancing. The excellent performance of robust balancing solved by convex optimization problems was experimentally validated by Huang [275]. Mass flow rate and pressure ratio in stall margin were modeled as stochastic uncertain parameters and the RBDO based on the Pareto-optimal analysis was carried out for the NASA stage 37 axial compressor [276]. After optimization, the probability of safe operation (no stall) increases to 95 % with improved efficiency and weight. Jia et al. [277] used the six-sigma method to optimize vibration responses of a rotor considering the uncertain distribution of mass imbalance, i.e., uncertain amplitude and phase. Experiments showed that the standard deviations of vibration amplitudes at disks and accelerations at bearings decreased between 14 % and 46 %.

For design parameters that are subject to uncertainty but cannot be modeled as stochastic variables, non-probabilistic optimization can be a viable alternative. Shiau et al. [278] applied interval descriptions of the inner radius of shafts and proposed an interval genetic algorithm (IGA) to minimize shaft weight and/or transmitted force. Also based on the IGA, Hong et al. [279] optimized a rotor system based on many design parameters to ensure there is an adequate margin between working speeds and critical speeds. Feng et al. [280] focused on a drive-shaft system with both random and interval hybrid uncertain parameters to optimize the axial friction force on the

Table 3
Representative works on optimization of uncertain rotor systems.

Type	Objectives	Uncertainties	Method	Reference
Probabilistic	Minimize mean and variance of response	Support stiffness uncertainty	An intelligent robust design approach	Xu et al. [102]
Probabilistic	Adequate margins between working speeds and critical speeds	Stiffness of bearing, diameter and mass	A global optimization algorithm based on a restart procedure and local searches	Ritto et al. [107], Lopez et al. [272]
Probabilistic	Minimize weight	Residual unbalances and journal-bearing stiffness	LHS-based reliability optimization	Stocki et al. [159]
Probabilistic	Reduce the largest residual vibration amplitude	Influence coefficient matrix and initial vibration data	Convex optimization	Li et al. [274], Huang [275]
Probabilistic	Maximize efficiency and minimize weight	Mass flow rate and pressure ratio in stall margin	Reliability-based design optimization	Hong et al. [276]
Probabilistic	Minimize vibration amplitude	Magnitude and phase of unbalance	Six-sigma method	Jia et al. [277]
Non-probabilistic	Minimize shaft weight and/or transmitted force	Inner radius of shaft	Interval genetic algorithm	Shiau et al. [278]
Non-probabilistic	Adequate margins between working speeds and critical speeds, bearing load	Many model parameters	Interval genetic algorithm	Hong et al. [279]
Hybrid	Lower bound of reliability above 99.9 %	Axial friction force	Reliability-based optimization	Feng et al. [280]

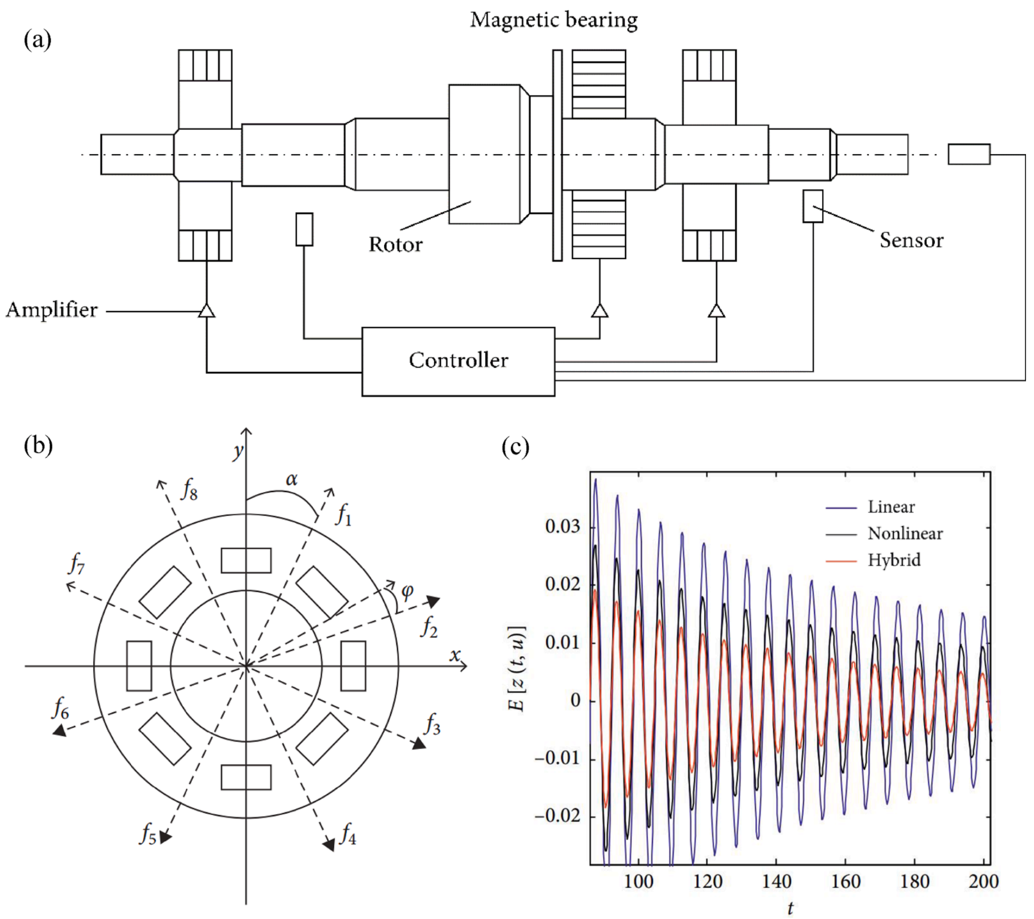

Fig. 17. Control of an uncertain rotor-ABM system [64]: (a) rotor system, (b) magnetic forces and (c) time history of the rotor controlled by different methods.

system. The deterministic system model was verified by experiments. Due to hybrid uncertainty input, the lower bound of reliability of the system was considered as the objective function which is restricted to be no less than 99.9 %.

A summarization of works done in robust optimization of rotor systems under uncertainty is given in Table 3 with representative references. The types of uncertainty used, optimization objectives, uncertainties included, and key methods applied are also illustrated. From Table 3, it can be seen that most of the studies were carried out using stochastic representations and a few applied interval models. Yet, no fuzzy-type reliability analysis of rotor systems is found. Moreover, hybrid reliability assessment in this field concerning multiple types of uncertainties has just emerged and many issues remain to be solved.

6. Vibration control with uncertainty

Vibration control of rotor systems seeks to suppress the vibration amplitudes and improve stability to ensure reliable operations [281]. It can be divided into passive vibration control (PVC), such as the squeeze film dampers and nonlinear energy sinks, and active vibration control (AVC), such as the AMBs, depending on whether the control strategy needs external energy. If uncertainty is involved, it is further classified as robust control [282,283]. The AVC with uncertainty is frequently investigated while the PVC under uncertainty can be fitted into the analysis of the uncertain dynamics.

In rotor systems equipped with AMBs, rotating shafts are levitated by magnetic forces, and this avoids mechanical contact between bearings and journals, reducing wear of components. Control strategies can be conveniently applied to AMBs working as a mechatronic system. Parameter changes or uncertainties in AMBs can bring dispersions, which then require the robust control schemes for better control performances. Carvalho et al. [172] proposed a robust neuro-fuzzy controller which achieves an optimal balance between vibration attenuation performance and robustness. Better performances in terms of energy efficiency and vibration attenuation than the proportional-integral-derivative (PID) controller were observed via the experiments of a flexible rotor-AMBs test rig. Lauridsen et al. [112] designed three controllers, i.e., a H_∞ controller, a μ controller and a linear parameter varying controller for AMBs subject to uncertain and changing dynamic seal forces. Notable performance improvement was observed in the robust control compared with traditional controls. Liu et al. [284] used the lumped system uncertainties to model the external disturbances and parameter variations in a rotor-AMBs system. Then, feedback linearization was applied for transforming mismatched uncertainties into virtual inputs. The PID with the second-order differential was proposed to guarantee the closed-loop stability and avoid the power amplifier saturation. Simulation and experiment results proved its efficacy. Wang et al. [64] investigated the uncertain magnetic permeability in an AMB and found that this random uncertainty affects the Hopf bifurcation of the rotor-AMB system shown in Fig. 17(a). Fig. 17(b) shows the magnetic forces generated by the AMB and they are directly linked to the permeability. Then, a hybrid feedback control method was proposed to control the Hopf bifurcation behaviors based on a linear control scheme and a nonlinear stochastic strategy. Results showed that the hybrid method has the best performances, as evidenced by Fig. 17(c). Considering leakages in the magnetic circuit and the non-flat shape of the magnetic circuit at the gap in electromagnets, Kato et al. [225] revealed the effects of parametric uncertainties in the electromagnetic force model on the equilibrium positions and their stability. In addition, the stability of the AMB was improved by choosing the proper parameter value based on feedback linearization. Li et al. [285] developed an accurate nominal model for a high-speed rotor test rig with AMBs, and a robust controller synthesis was proposed with uncertainty representations. The advanced control methods based on H_∞ and μ synthesis successfully validated the rotor model established. Ren et al. [286] defined an interval type-2 model-based fuzzy logic controller for a rotor-AMB system to achieve fast and stable levitation by compensating the uncertainty through the well-designed membership functions. Experiments show the advantages of the proposed controller over the conventional linear quadratic regulator (LQR), H_∞ and type-1 fuzzy logic controllers.

Nonlinear terms, unbalance, parameter variations, and uncertain terms were considered by Inoue et al. [287] for vibration control and unbalance estimation of a nonlinear rotor system supported on a single-row deep groove ball bearing. The disturbance observer was designed to suppress the vibration amplitudes all over the rotation speeds with mass unbalances being identified. Therefore, the proposed method serves as a dynamic balancing strategy as well. Chen et al. [288] formulated the robust-stable and quadratic finite-horizon optimal active vibration controller with low trajectory sensitivity. Linear matrix inequalities (LMI) were used to represent the robust stabilizability condition. It was found that the method can effectively suppress the vibration of the flexible rotor system and also avoid the possibilities of both spillover induced and time-varying parameters perturbations induced instability. Koroishi et al. [17] incorporated electromagnetic actuators to consider parametric uncertainties and the LMI was adopted to determine the gains of the electromagnetic actuator in the LQR control. It was found the method is robust for parameters with an uncertainty level -20% to $+20$ %. The LMI was also employed in a rotor system subject to seismic excitations [49], in which the performances of the H_2, H_∞ and mixed H_2/H_∞ controllers were compared. Results obtained indicate that the mixed H_2/H_∞ control delivers the best suppressions in both the frequency and time domain given that the performance index is chosen properly. Riemann et al. [289] found that the influences of gyroscopic terms in rotor systems are important, which require speed-dependent control. They considered uncertainties in rotation speeds and proposed to use the mixed μ synthesis techniques. Uncertainties in flexible mode frequencies and their modal damping values of the plant model, uncertainties in the current and position stiffness values of the AMBs and uncertain rotation speed were studied in [290]. They showed that the deployment of an add-on controller for gyroscopic effect cancellation is feasible.

7. Discussions and outlooks

Due to discipline differences, the development and applications of uncertainty analysis in rotordynamics lag behind those of structural dynamics. Yet, the uncertainty handling has aroused increasing attention in the research community of rotordynamics in recent years to carry out robust design, dynamic analysis and fault diagnosis. So far, many successful attempts have been made to apply different kinds of UQ methods concerning various uncertainties. The reported dynamics cover many aspects of rotordynamics, including the natural characteristics, transient/steady-state responses, faulted system vibrations and diagnosis, reliability analysis, optimizations and control. Although significant progress has been made, several gaps or directions need more effort. Giving an exhaustive list of topics of interest for future developments is not possible. However, some interesting outlooks for future research are as follows.

(1) Identify uncertainty sources and their variation coefficients based on measured data. In the literature, the stochastic variable, stochastic process, fuzzy number and interval variable are introduced in the UQ of rotor systems. The obtained analysis results can serve the purpose of sensitivity and influence pattern investigation of uncertainties and guide the design and dynamic analysis. In fact, many of these uncertainty models in the literature are pre-defined and chosen subjectively by the researchers. Moreover, the uncertainty level is chosen as percentages of the deterministic values and it generally is defined as 1 %, 5 % and 10 %, etc. This process of identifying uncertainty and its uncertainty level has no standard protocol and varies between researchers. A basic goal of the UQ is finding out the main uncertainties and evaluating their influences, and then reducing the uncertainties or minimizing the potential risks. The gap here is that it lacks an effective method to derive major uncertainty sources and levels from measurements and observations from real rotordynamic machines. Bridging this gap will allow much more easy engineering implementations and make uncertainty analysis more case-sensitive. Indeed, this preliminary choice directly impacts the final results given by the uncertainty propagation analysis and users should use discretion in choosing the tools to build an effective uncertainty model. Examples using different theories to determine uncertainties from experimental data can be found in [291–294]. As previously stated, the first key point in a robust design process for rotor systems in the presence of uncertainties is inverse uncertainty inference.
(2) Develop more effective and powerful UQ methods. The various uncertainty analysis methods currently applied in rotordynamics have their strengths, but also limitations. The primary concerns are of course the accuracy and efficiency or an excellent trade-off between them. If the number of uncertainties is large, say tens or hundreds, the required computation cost of both probabilistic and non-probabilistic methods will be prohibitive even though there are some improvements based on certain techniques [295]. The application and enhancement of non-probabilistic approaches should be further exploited for practical situations when data does not support known probability law and avoid introducing too many human assumptions. In addition, recently emerged machine learning methods [296,297] may possibly contribute to the state-of-the-art. However, the development of such methods must go hand in hand with a mastered vision of the most important physical phenomena involved in rotordynamics. In other words, such developments should not lead to black boxes only but should support the solid development of efficient and reliable design tools, which are adapted to real needs and problems encountered in rotordynamics.
(3) More efforts should be devoted to the nonlinear dynamics of uncertain rotor systems. It is a major challenge to most of the research practice and UQ methods when the rotor system under study experiences strong nonlinearity, which is very common due to the nonlinear support structures, typical faults and the nonlinearities in many kinds of bearings. The deeper nonlinear behaviors and mechanisms need to be revealed more thoroughly and also how system nonlinearity interacts with different uncertainties. Indeed, even if a few works have already achieved considerable progress on this subject, some issues have not been settled for good and these advanced methods are not yet transferable to real industry because of the scientific expertise and mathematical models associated with such approaches. Moreover, the decent solution of nonlinear systems with multiple solutions under uncertainties, and the characterization and evolution of the associated nonlinear signatures, are still open problems.
(4) Experimental rotordynamics research with uncertainty. As can be evidenced from the publications, most of the contributions were published without experimental validations. This is natural since specific uncertainty is extremely difficult to accurately simulate and control, especially for the relatively complex rotor systems. Numerical simulations on uncertainty analyses are important for the understanding of the phenomena associated with rotor systems under uncertainty. However, the variabilities of the natural characteristics, critical dynamic responses and any other important findings should be also experimentally verified and it is mandatory to demonstrate the various techniques proposed are reliable. Some works present good results that are validated experimentally, for example, the analyses of measurement uncertainty [88], the parameter identification of faults or bearings [166,256], characterizing the influence of uncertainty in rotor balancing [44,175], control problems [225] and validations of nonlinear FRFs [298]. However, the number of experimental studies on this topic is limited and it is far from sufficient because standard strategies and mature experiment methods are still missing. How to design effective auxiliary devices and controllably carry out dedicated experimental investigations are future directions of great significance. This point naturally goes in close connection with the notion of inverse uncertainty inference.
(5) Many issues need to be solved within the industrial context. Existing researches show very few applications to large-scale industrial rotating systems. Most of the publications were completed on analytical rotor models or laboratory system scales. It will certainly make meaningful progress if the uncertainty analysis is applied in practical scenarios with efforts to promote the most advanced theory. For these cases, the ROMs can alleviate the computation burden to some extent [299,300]. It also remains unclear which UQ methods will benefit most from the joint use of the reduced-order modeling.

8. Concluding remarks

Given the significance of uncertainty analysis for rotor systems, this study aims to deliver a comprehensive review on the latest progress in this area. Sources of various uncertainty and their classifications are illustrated in detail. The popular probabilistic, fuzzy and interval types of uncertainty analysis methods are described and their merits and limitations in applications are discussed. Many aspects of existing publications on uncertain dynamic characteristics of rotor systems are summarized as well as reliability, optimization and vibration control issues. Discussions on the progress made point out that the main directions that need more effort in this area are building uncertainty models and their variation levels based on measurements, analyzing the uncertain dynamics of nonlinear rotor systems, developing more efficient and powerful uncertainty analysis methods that can treat a large number of uncertain variables, devoting to the experimental validations and engineering scenario applications. We hope this review can provide useful information for designers on the uncertainties in rotor systems that need to identify, the methods to choose the correct models for them based on available data and how to select the most suitable analysis methods according to different research purposes.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This research was sponsored by Shanghai Sailing Program (Grant No. 22YF1452000), the Natural Science Foundation of Shaanxi Province (Grant No. 2022JQ-011), the Fundamental Research Funds for the Central Universities (Grant No. G2021KY0601) and the National Natural Science Foundation of China (Grant No. 11972295).

We appreciate the valuable comments from the reviewers and the fruitful discussions with Dr. Xiaoxiao Liu at Xi'an Jiaotong University.

Appendix A

The main objective of this appendix is to give one of the most classical linear formulations for general rotor systems composed of shaft, disks and bearings. More complex formulations for dual rotors, the addition of faults (misalignment, bow, crack, rub-impact, etc) or specific bearing elements exist. The interested reader is referred to the references and discussions in this review for more details.

In the following we consider a two-node Timoshenko beam element of length l corresponding to the nodal displacement $\mathbf{q} = [v_1 \quad w_1 \quad \theta_1 \quad \psi_1 \quad v_2 \quad w_2 \quad \theta_2 \quad \psi_2]^T$, as shown in Fig. A1.

For a symmetrical solid shaft, the expression of the element mass matrix $\mathbf{M}^e = \mathbf{M}_1^e + \mathbf{M}_2^e$ (the sum of the translational and rotary mass matrices), the stiffness matrix $\mathbf{K}^e$, the skew-symmetric gyroscopic matrix $\mathbf{G}^e$ and the damping matrix $\mathbf{C}^e$ can be written by

$$\mathbf{M}_1^e = \frac{\rho S l}{420}\begin{bmatrix} 156 & 0 & 0 & -22l & 54 & 0 & 0 & 13l \\ & 156 & 22l & 0 & 0 & 54 & -13l & 0 \\ & & 4l^2 & 0 & 0 & 13l & -3l^2 & 0 \\ & & & 4l^2 & -13l & 0 & 0 & -3l^2 \\ & & & & 156 & 0 & 0 & 22l \\ & & & & & 156 & -22l & 0 \\ & & Sym. & & & & -4l^2 & 0 \\ & & & & & & & -4l^2 \end{bmatrix}$$

$$\mathbf{M}_2^e = \frac{\rho I}{30l}\begin{bmatrix} 36 & 0 & 0 & -3l & -36 & 0 & 0 & -3l \\ & 36 & 3l & 0 & 0 & -36 & 3l & 0 \\ & & 4l^2 & 0 & 0 & -3l & -l^2 & 0 \\ & & & 4l^2 & 3l & 0 & 0 & -l^2 \\ & & & & 36 & 0 & 0 & 3l \\ & & & & & 36 & -3l & 0 \\ & & Sym. & & & & 4l^2 & 0 \\ & & & & & & & 4l^2 \end{bmatrix}$$

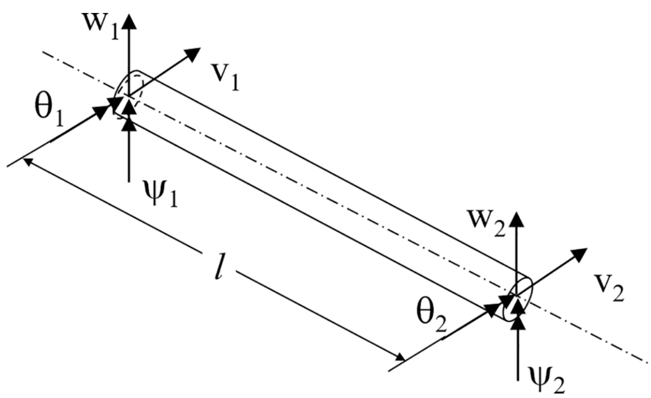

Fig. A1. Degrees of freedom for a Timoshenko beam element.

$$\mathbf{K}^e = \frac{EI}{(1+\beta)l^3}\begin{bmatrix} 12 & 0 & 0 & 6l & -12 & 0 & 0 & 6l \\ & 12 & -6l & 0 & 0 & -12 & -6l & 0 \\ & & (4+\beta)l^2 & 0 & 0 & 6l & (2-\beta)l^2 & 0 \\ & & & (4+\beta)l^2 & -6l & 0 & 0 & (2-\beta)l^2 \\ & & & & 12 & 0 & 0 & -6l \\ & & & & & 12 & 6l & 0 \\ & & Sym. & & & & (4+\beta)l^2 & 0 \\ & & & & & & & (4+\beta)l^2 \end{bmatrix}$$

$$\mathbf{G}^e = \frac{\rho I}{15l}\begin{bmatrix} 0 & -36 & 3l & 0 & 0 & 36 & 3l & 0 \\ & 0 & 0 & 3l & -36 & 0 & 0 & 3l \\ & & 0 & -4l^2 & 3l & 0 & 0 & l^2 \\ & & & 0 & 0 & 3l & -l^2 & 0 \\ & & & & 0 & -36 & -3l & 0 \\ & & & & & 0 & 0 & -3l \\ & & skew-sym. & & & & 0 & -4l^2 \\ & & & & & & & 0 \end{bmatrix}$$

and $\mathbf{C}^e = \alpha\mathbf{K}^e$. E and ρ are Young's modulus of elasticity and density of the rotor shaft. I is the second moment of area about any axis perpendicular to the rotor axis. The shear coefficient is given by $G = \frac{E}{2(1+\nu)}$ with the shear modulus $\beta = \frac{12EI}{GSl^2}$. ν is Poisson's ratio, S is the area of the cross section and α is a proportional factor to stiffness.

For the element damping matrix, one of the most classical forms is to consider the proportional damping $\mathbf{C}^e = \beta\mathbf{K}^e$ (or $\mathbf{C}^e = \beta\mathbf{K}^e + \alpha\mathbf{M}^e$) can be used, where the constant β is the proportional factor to stiffness (where the constant β and α are the proportional factors to stiffness and mass, respectively).

By considering that the rotor axis is along the x-direction and the disc is symmetric, the element mass and gyroscopic matrices $\mathbf{M}_d$ and $\mathbf{G}_d$ for one disc corresponding to its degrees of freedom $[u_d \quad v_d \quad \theta_d \quad \psi_d]^T$ are given by

$$\mathbf{M}_d = \begin{bmatrix} m_d & 0 & 0 & 0 \\ 0 & m_d & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & 0 & 0 & I_d \end{bmatrix} \quad \text{and} \quad \mathbf{G}_d = \omega\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_p \\ 0 & 0 & I_p & 0 \end{bmatrix}$$

where m_d is the mass of the disc and (I_p, I_d) are the polar moment of inertia about the rotor axis and the diametral moment of inertia about any axis perpendicular to the rotor axis, respectively.

When considering the modeling of the bearing support, a particularly useful type is to consider the linear stiffness and damping. However, many formulations exist depending on the specific bearing components of the rotor applications studied, see [1–4] for more details.

The above formulation illustrates the main formulation based on the FEM and the detailed expressions for different elements. Assembling of these matrices leads to the global EOMs of a general rotor-bearing dynamic system, as given in Eq. (34). The specific forms of additional matrix and vectors due to faults are breifly discussed in their respective sections and readers are referred to relevant references cited for more detailed modeling of these faults.

References

[1] M.I. Friswell, J.E. Penny, S.D. Garvey, A.W. Lees, Dynamics of Rotating Machines, Cambridge University Press, New York, 2010.
[2] Y. Ishida, T. Yamamoto, Linear and Nonlinear Rotordynamics: A Modern Treatment with Applications, John Wiley & Sons, New York, 2013.
[3] M. Lalanne, G. Ferraris, Rotordynamics Prediction in Engineering, John Wiley & Sons, New York, 1998.
[4] D. Childs, Turbomachinery Rotordynamics: Phenomena, Modeling, and Analysis, John Wiley & Sons, New York, 1993.
[5] C. Soize, Uncertainty Quantification: An Accelerated Course with Advanced Applications in Computational Engineering, Springer International Publishing, Berlin, 2017.
[6] M.F. Dimentberg, B. Ryzhik, L. Sperling, Random vibrations of a damped rotating shaft, J. Sound Vib. 279 (2005) 275–284.
[7] R. Fatehi Mohammad, A. Ghanbarzadeh, S. Moradi, A. Hajnayeb, Global sensitivity analysis improvement of rotor-bearing system based on the Genetic Based Latine Hypercube Sampling (GBLHS) method, Struct. Eng. Mech. 68 (2018) 549–561.
[8] C. Fu, X. Ren, Y. Yang, W. Deng, Application and comparative analysis of orthogonal polynomials in uncertain rotor dynamic response calculation, Journal of Aerospace, Power 33 (2018) 2228–2234.
[9] P.C. Barbosa, F.A. Lara-Molina, I.B. da Silva, A.A. Cavalini, V. Steffen, Uncertain and sensitivity analyses of a composite shaft, Meccanica 55 (2020) 35–48.
[10] M.F. Dimentberg, A. Naess, Nonlinear vibrations of a rotating shaft with broadband random variations of internal damping, Nonlinear Dyn. 51 (2008) 199–205.
[11] L. Schuchard, M. Voigt, R. Mailach, P. Dumstorff, A. de Lazzer, H. Almstedt, Improved rotor design with combined 3D-2D probabilistic approach, in: ASME Turbo Expo 2021: Turbomachinery Technical Conference and Exposition, Virtual, Online. June 7-11, 2021, pp. V09BT27A002.
[12] J.-J. Sinou, L. Nechak, S. Besset, Kriging metamodeling in rotordynamics: Application for predicting critical speeds and vibrations of a flexible rotor, Complexity 2018 (2018) 1264619.
[13] C. Fu, Y. Xu, Y. Yang, K. Lu, F. Gu, A. Ball, Response analysis of an accelerating unbalanced rotating system with both random and interval variables, J. Sound Vib. 466 (2020), 115047.

[14] S. Ding, Y. He, X. Zheng, A probabilistic approach for three-dimensional variation analysis in aero-engine rotors assembly, Int. J. Aeronaut. Space Sci. 22 (2021) 1092–1105.
[15] J. Qiu, J. Zhang, Y. Ma, Resonance reliability sensitivity analysis for torsional vibration of gear-rotor systems with random parameters. ASME 2017 International Mechanical Engineering Congress and Exposition, Tampa, Florida, USA, 2017. V014T014A001.
[16] B. Zhao, L. Xie, H. Li, S. Zhang, B. Wang, C. Li, Reliability analysis of aero-engine compressor rotor system considering cruise characteristics, IEEE Trans. Reliab. 69 (2019) 245–259.
[17] E.H. Koroishi, F.A. Lara-Molina, A.S. Borges, V. Steffen, Robust control in rotating machinery using linear matrix inequalities, J. Vib. Control 22 (2015) 3767–3778.
[18] E. Sarrouy, O. Dessombz, J.J. Sinou, Stochastic analysis of the eigenvalue problem for mechanical systems using polynomial chaos expansion-Application to a finite element rotor, J. Vib. Acoust. 134 (2012), 051009.
[19] K. Sepahvand, K. Nabih, S. Marburg, Collocation-based stochastic modeling of uncertain geometric mistuning in bladed rotor, Procedia IUTAM 13 (2015) 53–62.
[20] Y. Shi, J. Zhou, X. Lai, Y. Xu, W. Guo, B. Liu, Stability and sensitivity analysis of the bending-torsional coupled vibration with the arcuate whirl of hydro-turbine generator unit, Mech. Syst. Sig. Process. 149 (2021), 107306.
[21] G. Blatman, B. Sudret, An adaptive algorithm to build up sparse polynomial chaos expansions for stochastic finite element analysis, Probab. Eng. Mech. 25 (2010) 183–197.
[22] T. Wei, W. Gou, C. Fu, Y. Yang, Transient analysis of speed-varying rotor with uncertainty based on interval approaches, Discr. Dyn. Nat. Soc. 2018 (2018) 5904724.
[23] F. Nobile, R. Tempone, C.G. Webster, A sparse grid stochastic collocation method for partial differential equations with random input data, SIAM J. Numer. Anal. 46 (2008) 2309–2345.
[24] E. Acar, G. Bayrak, Y. Jung, I. Lee, P. Ramu, S.S. Ravichandran, Modeling, analysis, and optimization under uncertainties: A review, Struct. Multidiscip. Optim. 64 (2021) 2909–2945.
[25] E.O. Smith, J.H.S. de Baar, A.J. Neely, A Sobol' sequence parametric analysis of rotor thermal bow in gas turbines, in: Proceedings of the ASME Turbo Expo 2018: Turbomachinery Technical Conference and Exposition, Oslo, Norway, 2018, pp. V07AT33A010.
[26] R.C. Smith, Uncertainty Quantification: Theory, Implementation, and Applications, SIAM, 2013.
[27] R.C. Aster, B. Borchers, C.H. Thurber, Parameter Estimation and Inverse Problems, Elsevier, 2018.
[28] J. Hong, J. Wang, M. Chen, Y. Ma, Dynamic response analysis of rotor system with uncertain parameters via interval analysis method, in: Turbo Expo: Power for Land, Sea, and Air, American Society of Mechanical Engineers, 2012, pp. 593-600.
[29] G. Wang, Y. Ma, T. Li, J. Li, J. Hong, Modelling of misaligned rotor system in aero-engines and interval method investigation, in: ASME 2013 International Mechanical Engineering Congress and Exposition, November 15-21, San Diego, California, USA, 2013, pp. V04AT04A041.
[30] C. Fu, Y. Xu, Y. Yang, F. Gu, A. Ball, The uncertain vibrations of a rotor operating with angular acceleration based on Taylor expansion, in: Advances in Asset Management and Condition Monitoring, Springer International Publishing, Cham, 2020, pp. 1105–1113.
[31] X. He, G. Li, P. Xing, L. Lu, H. Gao, H. Zhang, G. Wang, Experimental and numerical investigation on dynamic characteristics of ship propulsion shafting under uncertainty based on displacement response, Ocean Eng. 237 (2021), 109637.
[32] G. Casella, R.L. Berger, Statistical Inference, Pacific Grove, Duxbury, CA, 2002.
[33] C. Fu, X. Ren, Y. Yang, Non-probabilistic analysis of a double-disk rotor system with uncertain parameters, J. Vibroeng. 20 (2018) 1311–1321.
[34] X. Guo, H. Ma, X. Zhang, Z. Ye, Q. Fu, Z. Liu, Q. Han, Uncertain frequency responses of clamp-pipeline systems using an interval-based method, IEEE Access 8 (2020) 29370–29384.
[35] K. Lu, Statistical moment analysis of multi-degree of freedom dynamic system based on polynomial dimensional decomposition method, Nonlinear Dyn. 93 (2018) 2003–2018.
[36] E. Denimal, J.-J. Sinou, Advanced kriging-based surrogate modelling and sensitivity analysis for rotordynamics with uncertainties, Eur. J. Mech. A. Solids 90 (2021), 104331.
[37] B. Xu, C. Zang, G. Zhang, Robust tolerance design for rotor dynamics based on possibilistic concepts, Arch. Appl. Mech. 92 (2022) 755–770.
[38] Y. Ma, Y. Wang, C. Wang, J. Hong, Nonlinear interval analysis of rotor response with joints under uncertainties, Chin. J. Aeronaut. 33 (2020) 205–218.
[39] C. Gan, Y. Wang, S. Yang, Y. Cao, Nonparametric modeling and vibration analysis of uncertain Jeffcott rotor with disc offset, Int. J. Mech. Sci. 78 (2014) 126–134.
[40] J. Didier, B. Faverjon, J.-J. Sinou, Analysing the dynamic response of a rotor system under uncertain parameters by polynomial chaos expansion, J. Vib. Control 18 (2011) 712–732.
[41] Y. Ma, Z. Liang, M. Chen, J. Hong, Interval analysis of rotor dynamic response with uncertain parameters, J. Sound Vib. 332 (2013) 3869–3880.
[42] C. Fu, X. Ren, Y. Yang, W. Qin, Dynamic response analysis of an overhung rotor with interval uncertainties, Nonlinear Dyn. 89 (2017) 2115–2124.
[43] R. Lasota, R. Stocki, P. Tauzowski, T. Szolc, Polynomial chaos expansion method in estimating probability distribution of rotor-shaft dynamic responses, Bull. Polish Acad. Sci. Techn. Sci. 63 (2015) 413–422.
[44] J. Datz, M. Karimi, S. Marburg, Effect of uncertainty in the balancing weights on the vibration response of a high-speed rotor, J. Vib. Acoust. 143 (2021), 061002.
[45] Y. Li, Z. Luo, J. Li, X. Hou, Analysis of bolted joint rotor system with uncertain axial stiffness, J. Northeastern Univ. 40 (2019) 700–705.
[46] B. Yuan, G. Liu, L. Liu, Quasi-static characteristics and vibration responses analysis of helical geared rotor system with random cumulative pitch deviations, Appl. Sci. 10 (2020) 4403.
[47] X. Leng, G. Meng, T. Zhang, T. Fang, Bifurcation and chaos response of a cracked rotor with random disturbance, J. Sound Vib. 299 (2007) 621–632.
[48] J. Didier, J.-J. Sinou, B. Faverjon, Multi-dimensional harmonic balance with uncertainties applied to rotor dynamics, J. Vib. Acoust. 134 (2012), 061003.
[49] Y. Chen, C. Zhu, Active vibration control based on linear matrix inequality for rotor system under seismic excitation, J. Sound Vib. 314 (2008) 53–69.
[50] K.B. Kim, J.N. Yang, Y.K. Lin, Stochastic response of flexible rotor-bearing system to seismic excitations, Probab. Eng. Mech. 1 (1986) 122–130.
[51] Z. Wang, X. Xiong, Q. Meng, X. Yang, Q. Xi, Random response of rotor system on the filtered gaussian white noise earthquake ground acceleration, Open Mech. Eng. J. 8 (2014) 214–218.
[52] J. Wang, X. Zheng, Review of geometric uncertainty quantification in gas turbines, J. Eng. Gas Turbines Power 142 (2020), 070801.
[53] A.A. Cavalini, A.G.S. Dourado, F.A. Lara-Molina, V. Steffen, Uncertainty analysis of a tilting-pad journal bearing using fuzzy logic techniques, J. Vib. Acoust. 138 (2016), 061016.
[54] F.A. Lara-Molina, A.A. Cavalini Jr., V. Steffen Jr., N.V. Cabrera, Tilting-pad journal bearing subjected to fuzzy type-2 uncertain parameters, J. Vib. Acoust. 141 (2019), 061008.
[55] X. Sun, K. Sepahvand, Z. Liu, S. Marburg, Rotor dynamics analysis under uncertainty in lubricant film. Proceedings of the 23rd International Congress on Acoustics, September 9-13, Aachen, Germany, 2019.
[56] J. Ma, C. Fu, H. Zhang, F. Chu, Z. Shi, F. Gu, A.D. Ball, Modelling non-Gaussian surfaces and misalignment for condition monitoring of journal bearings, Measurement 174 (2021), 108983.
[57] J. Ma, H. Zhang, S. Lou, F. Chu, Z. Shi, F. Gu, A.D. Ball, Analytical and experimental investigation of vibration characteristics induced by tribofilm-asperity interactions in hydrodynamic journal bearings, Mech. Syst. Sig. Process. 150 (2021), 107227.
[58] J. Ma, H. Zhang, Z. Shi, F. Chu, F. Gu, A.D. Ball, Modelling Acoustic Emissions induced by dynamic fluid-asperity shearing in hydrodynamic lubrication regime, Tribol. Int. 153 (2021), 106590.
[59] Y. Tarasevich, E. Savchenko, Probabilistic method for evaluation of dynamic characteristics of rotor in annular seals, Procedia Eng. 39 (2012) 238–243.
[60] G.Y. Garoli, H.F.d. Castro, Analysis of a rotor-bearing nonlinear system model considering fluid-induced instability and uncertainties in bearings, J. Sound Vib. 448 (2019) 108–129.

[61] X. Xiong, Y. Zhou, Y. Wang, Influences of randomly uncertain factors on dynamic coefficients of an interlocking labyrinth seal-rotor system, Appl. Sci. 10 (2022) 39.
[62] Y. Zuo, J. Wang, W. Ma, Quasimodes instability analysis of uncertain asymmetric rotor system based on 3D solid element model, J. Sound Vib. 390 (2017) 192–204.
[63] Y. Zhang, Y. Liu, Modeling of the rotor-bearing system and dynamic reliability analysis of rotor's positioning precision, Proc. Instit. Mech. Eng. Part O: J. Risk Reliab. 235 (2020) 491–508.
[64] J. Wang, S. Ma, P. Hao, H. Yuan, Hopf bifurcation and control of magnetic bearing system with uncertain parameter, Complexity 2019 (2019) 1641953.
[65] P. Pietkiewicz, S. Banaszek, G. Żywica, The stability loss of the rotor-slide bearings system under random load variations, J. Vibroeng. 19 (2017) 4921–4935.
[66] T.H. Young, T.N. Shiau, Z.H. Kuo, Dynamic stability of rotor-bearing systems subjected to random axial forces, J. Sound Vib. 305 (2007) 467–480.
[67] C. Bai, H. Zhang, Almost sure asymptotic stability of rotor systems subjected to stochastical axial loads, Mech. Mach. Theory 58 (2012) 192–201.
[68] H. Zhang, C. Bai, Y. Mao, Stochastic finite element modeling and response analysis of rotor systems with random properties under random loads, J. Mech. Sci. Technol. 29 (2015) 3083–3090.
[69] O. Gaidai, M. Dimentberg, A. Naess, Rotating shaft's non-linear response statistics under biaxial random excitation, by path integration, Int. J. Mech. Sci. 142–143 (2018) 121–126.
[70] O. Gaidai, P. Dou, A. Naess, M. Dimentberg, Y. Cheng, R. Ye, Nonlinear 6D response statistics of a rotating shaft subjected to colored noise by path integration on GPU, Int. J. Non Linear Mech. 111 (2019) 142–148.
[71] O. Gaidai, A. Naess, M. Dimentberg, Response statistics of rotating shaft with non-linear elastic restoring forces by path integration, J. Sound Vib. 400 (2017) 113–121.
[72] M.F. Dimentberg, A. Naess, L. Sperling, Response of a rotating shaft to uniaxial random excitation, J. Appl. Mech. 79 (2012), 044503.
[73] J. Didier, J.-J. Sinou, B. Faverjon, Study of the non-linear dynamic response of a rotor system with faults and uncertainties, J. Sound Vib. 331 (2012) 671–703.
[74] C. Fu, X. Ren, Y. Yang, K. Lu, W. Qin, Steady-state response analysis of cracked rotors with uncertain-but-bounded parameters using a polynomial surrogate method, Commun. Nonlinear Sci. Numer. Simul. 68 (2019) 240–256.
[75] C. Fu, X. Ren, Y. Yang, K. Lu, Y. Wang, Nonlinear response analysis of a rotor system with a transverse breathing crack under interval uncertainties, Int. J. Non Linear Mech. 105 (2018) 77–87.
[76] J.-J. Sinou, B. Faverjon, The vibration signature of chordal cracks in a rotor system including uncertainties, J. Sound Vib. 331 (2012) 138–154.
[77] Y. Zhang, B. Wen, Q. Liu, Uncertain responses of rotor-stator systems with rubbing, JSME International Journal Series C 46 (2003) 150–154.
[78] Z. Zhang, X. Ma, H. Hua, X. Liang, Nonlinear stochastic dynamics of a rub-impact rotor system with probabilistic uncertainties, Nonlinear Dyn. 102 (2020) 2229–2246.
[79] C. Fu, D. Zhen, Y. Yang, F. Gu, A. Ball, Effects of bounded uncertainties on the dynamic characteristics of an overhung rotor system with rubbing fault, Energies 12 (2019) 4365.
[80] G.Y. Garoli, H.F. de Castro, Generalized polynomial chaos expansion applied to uncertainties quantification in rotating machinery fault analysis, J. Braz. Soc. Mech. Sci. Eng. 42 (2020) 610.
[81] T. Szolc, R. Konowrocki, Research on stability and sensitivity of the rotating machines with overhung rotors to lateral vibrations, Bull. Polish Acad. Sci. Techn. Sci. 69 (2021) e137987.
[82] Y. Zhang, M. Zhang, L. Xie, K. Zhang, The effect of the uncertain initial angular misalignment on fatigue life of spindle-bearing system, Forsch. Ingenieurwes. 85 (2021) 39–56.
[83] C. Wang, Y. Ma, D. Zhang, J. Hong, Interval analysis on aero-engine rotor system with misalignment, in: Turbo Expo: Power for Land, Sea, and Air, American Society of Mechanical Engineers, 2015, pp. V07AT30A002.
[84] C. Fu, Y. Yang, K. Lu, F. Gu, Nonlinear vibration analysis of a rotor system with parallel and angular misalignments under uncertainty via a Legendre collocation approach, Int. J. Mech. Mater. Des. 16 (2020) 557–568.
[85] Z. Li, J. Jiang, Z. Tian, Stochastic dynamics of a nonlinear misaligned rotor system subject to random fluid-induced forces, J. Comput. Nonlinear Dyn. 12 (2016), 011004.
[86] A. Shrivastava, A.R. Mohanty, K.M. Pekpe, J.P. Cassar, Estimation of strain in a rotating shaft using deterministic–stochastic approach, in: Data-Driven Modeling for Sustainable Engineering, Springer International Publishing, Cham, 2020, pp. 375–386.
[87] A. Shrivastava, A.R. Mohanty, Detection of unbalance in a rotor-bearing system using a deterministic-stochastic approach, in: 24th International Congress on Sound and Vibration, July 23-27, London, United Kingdom, 2017.
[88] S. Lahriri, I. Ferreira Santos, Experimental quantification of contact forces with impact, friction and uncertainty analysis, Tribol. Int. 66 (2013) 93–104.
[89] L. Medina, R. Ruiz, S. Dı́ az, A simple approach to determine uncertainty bounds on bearing rotordynamic coefficients identification, in: Turbo Expo: Power for Land, Sea, and Air, 2008, pp. 1279-1287.
[90] R. Murthy, J.C. Tomei, X.Q. Wang, M.P. Mignolet, A. El-Shafei, Nonparametric stochastic modeling of structural uncertainty in rotordynamics: Unbalance and balancing aspects, J. Eng. Gas Turbines Power 136 (2014), 062506.
[91] B. Liu, Eigenvalue problems of rotor system with uncertain parameters, J. Mech. Sci. Technol. 26 (2012) 1–10.
[92] S. Zhou, X. Wu, H. Li, Y. Sun, Critical speed analysis of flexible rotor system with stochastic uncertain parameters, J. Vib. Eng. Technol. 5 (2017) 319–328.
[93] C. Fu, X. Ren, Y. Yang, Vibration analysis of rotors under uncertainty based on Legendre series, J. Vib. Eng. Technol. 7 (2019) 43–51.
[94] J.-J. Sinou, E. Jacquelin, Influence of Polynomial Chaos expansion order on an uncertain asymmetric rotor system response, Mech. Syst. Sig. Process. 50–51 (2015) 718–731.
[95] Z. Zheng, Y. Xie, D. Zhang, Numerical investigation on the gravity response of a two-pole generator rotor system with interval uncertainties, Appl. Sci. 9 (2019) 3036.
[96] C. Fu, G. Feng, J. Ma, K. Lu, Y. Yang, F. Gu, Predicting the dynamic response of dual-rotor system subject to interval parametric uncertainties based on the non-intrusive metamodel, Mathematics 8 (2020) 736.
[97] Y. Ma, Y. Wang, C. Wang, J. Hong, Interval analysis of rotor dynamic response based on Chebyshev polynomials, Chin. J. Aeronaut. 33 (2020) 2342–2356.
[98] T.S. Sankar, S.A. Ramu, R. Ganesan, Stochastic finite element analysis for high speed rotors, J. Vib. Acoust. 115 (1993) 59–64.
[99] R. Timbó, T.G. Ritto, Impact of damper seal coefficients uncertainties in rotor dynamics, J. Braz. Soc. Mech. Sci. Eng. 41 (2019) 165.
[100] Y. Qiu, S.S. Rao, A fuzzy approach for the analysis of unbalanced nonlinear rotor systems, J. Sound Vib. 284 (2005) 299–323.
[101] Q. Xu, Q. Zhao, H. Yao, B. Wen, Application of fuzzy random finite element method on rotor dynamics, J. Vibroeng. 16 (2014) 1854–1863.
[102] B. Xu, C. Zang, G. Zhang, Intelligent approach to robust design optimization of a rotor system due to its support stiffness uncertainty, Shock Vib. 2020 (2020) 2564679.
[103] E.H. Koroishi, A.A. Cavalini Jr, A.M. de Lima, V. Steffen Jr, Stochastic modeling of flexible rotors, J. Braz. Soc. Mech. Sci. Eng. 34 (2012) 574–583.
[104] A.A. Cavalini, F.A. Lara-Molina, T.d.P. Sales, E.H. Koroishi, V. Steffen, Uncertainty analysis of a flexible rotor supported by fluid film bearings, Latin Am. J. Solids Struct. 12 (2015) 1487–1504.
[105] K.A.F. Moustafa, Stability of journal bearing-rotor systems with interval-bearing parameters, J. Vib. Control 5 (1999) 941–953.
[106] A. Cavalini, A. Silva, F. Lara-Molina, V. Steffen, Dynamic analysis of a flexible rotor supported by hydrodynamic bearings with uncertain parameters, Meccanica 52 (2017) 2931–2943.
[107] T.G. Ritto, R.H. Lopez, R. Sampaio, J.E. Souza de Cursi, Robust optimization of a flexible rotor-bearing system using the Campbell diagram, Eng. Optim. 43 (2011) 77–96.
[108] J. Silva Barbosa, L. Campanine Sicchieri, A. Dourado, A. Cavalini, V. Steffen, Kriging approach dedicated to represent hydrodynamic bearings, J. Eng. Gas Turbines Power 143 (2021), 061016.
[109] X. Sun, K.K. Sepahvand, S. Marburg, Stability analysis of rotor-bearing systems under the influence of misalignment and parameter uncertainty, Appl. Sci. 11 (2021) 7918.

[110] R. Tiwari, N.S. Vyas, Estimation of non-linear stiffness parameters of rolling element bearings from random response of rotor-bearing systems, J. Sound Vib. 187 (1995) 229–239.
[111] G. Garoli, R. Pilotto, R. Nordmann, H. de Castro, Validation of the stochastic response of a rotor with uncertainties in the AMBs, in: 12th International Conference on Vibrations in Rotating Machinery, CRC Press, 2020, pp. 205–214.
[112] J. Lauridsen, I. Santos, Design of robust AMB controllers for rotors subjected to varying and uncertain seal forces, Mechanical, Eng. J. 4 (2017) 16–00618.
[113] E. Hashish, T.S. Sankar, Finite element and modal analyses of rotor-bearing systems under stochastic loading conditions, J. Vibr. Acoust. Stress Reliab. Design 106 (1984) 80–89.
[114] H. Liao, Global resonance optimization analysis of nonlinear mechanical systems: Application to the uncertainty quantification problems in rotor dynamics, Commun. Nonlinear Sci. Numer. Simul. 19 (2014) 3323–3345.
[115] Y. Yang, Q. Wu, Y. Wang, W. Qin, K. Lu, Dynamic characteristics of cracked uncertain hollow-shaft, Mech. Syst. Sig. Process. 124 (2019) 36–48.
[116] Z. Li, J. Jiang, Z. Tian, Non-linear vibration of an angular-misaligned rotor system with uncertain parameters, J. Vib. Control 22 (2014) 129–144.
[117] Z. Yang, Y. Zhang, K. Zhang, S. Li, Wear analysis of angular contact ball bearing in multiple-bearing spindle system subjected to uncertain initial angular misalignment, J. Tribol. 143 (2021), 091703.
[118] L. Yang, K. He, Y. Guo, Reliability analysis of a nonlinear rotor/stator contact system in the presence of aleatory and epistemic uncertainty, J. Mech. Sci. Technol. 32 (2018) 4089–4101.
[119] J. Muelaner, Z. Wang, P. Keogh, J. Brownell, D. Fisher, Uncertainty of measurement for large product verification: Evaluation of large aero gas turbine engine datums, Meas. Sci. Technol. 27 (2016), 115003.
[120] C. Gan, Y. Wang, S. Yang, Nonparametric modeling on random uncertainty and reliability analysis of a dual-span rotor, J. Zhejiang Univ.-Sci. A 19 (2018) 189–202.
[121] R. Murthy, M.P. Mignolet, A. El-Shafei, Nonparametric stochastic modeling of uncertainty in rotordynamics—Part I: Formulation, J. Eng. Gas Turbines Power 132 (2010), 092501.
[122] R. Murthy, M.P. Mignolet, A. El-Shafei, Nonparametric stochastic modeling of uncertainty in rotordynamics—Part II: Applications, J. Eng. Gas Turbines Power 132 (2010), 092502.
[123] F.O. Hoffman, J.S. Hammonds, Propagation of uncertainty in risk assessments: The need to distinguish between uncertainty due to lack of knowledge and uncertainty due to variability, Risk Anal. 14 (1994) 707–712.
[124] W.L. Oberkampf, J.C. Helton, C.A. Joslyn, S.F. Wojtkiewicz, S. Ferson, Challenge problems: Uncertainty in system response given uncertain parameters, Reliab. Eng. Syst. Saf. 85 (2004) 11–19.
[125] A.D. Kiureghian, O. Ditlevsen, Aleatory or epistemic? Does it matter? Struct. Saf. 31 (2009) 105–112.
[126] A.M. Ferreira, H.F. de Castro, Stochastic modeling of rotordynamics for electrical fuel pump application, in: K.L. Cavalca, H.I. Weber (Eds.), Proceedings of the 10th International Conference on Rotor Dynamics – IFToMM, Springer International Publishing, Cham, 2019, pp. 431–442.
[127] R. Pavlović, P. Kozić, S. Mitić, I. Pavlović, Stochastic stability of a rotating shaft, Arch. Appl. Mech. 79 (2009) 1163–1171.
[128] L.P. Volpi, D.M. Lobo, T.G. Ritto, A stochastic analysis of the coupled lateral–torsional drill string vibration, Nonlinear Dyn. 103 (2021) 49–62.
[129] C. Soize, Stochastic modeling of uncertainties in computational structural dynamics-Recent theoretical advances, J. Sound Vib. 332 (2013) 2379–2395.
[130] X. Zuo, L. Wang, Y. Yue, A new similarity measure of generalized trapezoidal fuzzy numbers and its application on rotor fault diagnosis, Mathe. Probl. Eng. 2013 (2013), 824706.
[131] C. Fu, X. Ren, Y. Yang, Y. Xia, W. Deng, An interval precise integration method for transient unbalance response analysis of rotor system with uncertainty, Mech. Syst. Sig. Process. 107 (2018) 137–148.
[132] A. Dourado, A. Cavalini, V. Steffen, Uncertainty quantification techniques applied to rotating systems: A comparative study, J. Vib. Control 24 (2017) 3010–3025.
[133] B. Cai, W. Shangguan, H. Lü, T. Bo, Hybrid uncertainties-based analysis and optimization design of powertrain mounting systems, Science China: Technological Sciences 63 (2020) 838–850.
[134] L. Yang, J. Zhang, Y. Guo, Uncertainty representation and quantification for a nonlinear rotor/stator system with mixed uncertainties, J. Vibroeng. 18 (2016) 4836–4851.
[135] E. Jacquelin, M.I. Friswell, S. Adhikari, O. Dessombz, J.-J. Sinou, Polynomial chaos expansion with random and fuzzy variables, Mech. Syst. Sig. Process. 75 (2016) 41–56.
[136] C. Fu, K. Lu, Y.D. Xu, Y. Yang, F.S. Gu, Y. Chen, Dynamic analysis of geared transmission system for wind turbines with mixed aleatory and epistemic uncertainties, Appl. Math. Mech. 43 (2022) 275–294.
[137] C. Jiang, J. Zheng, X. Han, Probability-interval hybrid uncertainty analysis for structures with both aleatory and epistemic uncertainties: A review, Struct. Multidiscip. Optim. 57 (2018) 2485–2502.
[138] G. Stefanou, The stochastic finite element method: Past, present and future, Comput. Methods Appl. Mech. Eng. 198 (2009) 1031–1051.
[139] M. Faes, D. Moens, Recent trends in the modeling and quantification of non-probabilistic uncertainty, Arch. Comput. Methods Eng. 27 (2020) 633–671.
[140] G. Fishman, Monte Carlo: Concepts, Algorithms, and Applications, Springer Science & Business Media, Berlin, Germany, 2013.
[141] J. Hammersley, Monte Carlo Methods, Springer Science & Business Media, Berlin, Germany, 2013.
[142] C.P. Robert, G. Casella, Monte Carlo Statistical Methods, Springer, New York, 1999.
[143] R.G. Ghanem, P.D. Spanos, Stochastic Finite Elements: A Spectral Approach, Courier Corporation, New York, 2003.
[144] D. Xiu, Numerical Methods for Stochastic Computations: A Spectral Method Approach, Princeton University Press, Princeton, 2010.
[145] M.L. Mehta, Random Matrices, Elsevier, San Diego, 2004.
[146] C. Soize, A nonparametric model of random uncertainties for reduced matrix models in structural dynamics, Probab. Eng. Mech. 15 (2000) 277–294.
[147] G.E. Box, G.C. Tiao, Bayesian Inference in Statistical Analysis, John Wiley & Sons, Amsterdam, 2011.
[148] S.N. Lophaven, H.B. Nielsen, J. Søndergaard, Dace,, A Matlab Kriging Toolbox, Citeseer (2002).
[149] O. Roustant, D. Ginsbourger, Y. Deville, DiceKriging, DiceOptim: Two R packages for the analysis of computer experiments by kriging-based metamodeling and optimization, J. Stat. Softw. 51 (2012) 1–55.
[150] M. Hanss, Applied Fuzzy Arithmetic, Springer-Verlag, Berlin Heidelberg, 2005.
[151] G. Shafer, A Mathematical Theory of Evidence, Princeton University Press, 1976.
[152] T. Augustin, F.P. Coolen, G. De Cooman, M.C. Troffaes, Introduction to Imprecise Probabilities, John Wiley & Sons, New York, 2014.
[153] G. Alefeld, J. Herzberger, Introduction to Interval Computation, Academic Press, New York, 2012.
[154] R.E. Moore, R.B. Kearfott, M.J. Cloud, Introduction to Interval Analysis, SIAM, Philadelphia, PA, 2009.
[155] J.-J. Sinou, J. Didier, B. Faverjon, Stochastic non-linear response of a flexible rotor with local non-linearities, Int. J. Non Linear Mech. 74 (2015) 92–99.
[156] R. Stocki, R. Lasota, P. Tauzowski, T. Szolc, Scatter assessment of rotating system vibrations due to uncertain residual unbalances and bearing properties, Comput.Assisted Methods Eng. 19 (2017) 95–120.
[157] T. Szolc, P. Tauzowski, J. Knabel, R. Stocki, Nonlinear and parametric coupled vibrations of the rotor-shaft system as fault identification symptom using stochastic methods, Nonlinear Dyn. 57 (2009) 533–557.
[158] J. Wu, Z. Luo, N. Zhang, Y. Zhang, A new uncertain analysis method and its application in vehicle dynamics, Mech. Syst. Sig. Process. 50–51 (2015) 659–675.
[159] R. Stocki, T. Szolc, P. Tauzowski, J. Knabel, Robust design optimization of the vibrating rotor-shaft system subjected to selected dynamic constraints, Mech. Syst. Sig. Process. 29 (2012) 34–44.
[160] F. Han, X. Guo, H. Gao, Bearing parameter identification of rotor-bearing system based on Kriging surrogate model and evolutionary algorithm, J. Sound Vib. 332 (2013) 2659–2671.
[161] X. Chen, H. Zhang, C. Zou, J. Zhai, Q. Han, Research on the prediction method of unbalance responses of dual-rotor system based on surrogate models, SN Applied Sciences 2 (2019) 12.

[162] W. Huang, C. Gan, Bifurcation analysis and vibration signal identification for a motorized spindle with random uncertainty, Int. J. Bifurcation Chaos 29 (2019) 1950001.
[163] W. Feng, B. Liu, H. Ding, H. Shen, Review of uncertain nonparametric dynamic modeling, Journal of Vibration and Shock 39 (2020) 1–9.
[164] J. Shao, M. Xu, R. Wang, Bayesian model-based fault diagnosis for the rotor, Aircraft Eng. Aerospace Technol. 81 (2009) 19–24.
[165] B. Xu, Intelligent fault inference for rotating flexible rotors using Bayesian belief network, Expert Syst. Appl. 39 (2012) 816–822.
[166] G.Y. Garoli, D.S. Alves, T.H. Machado, K.L. Cavalca, H.F. de Castro, Fault parameter identification in rotating system: Comparison between deterministic and stochastic approaches, Struct. Health Monitor. 20 (2021) 3182–3200.
[167] Z. Taherkhani, H. Ahmadian, Stochastic model updating of rotor support parameters using Bayesian approach, Mech. Syst. Sig. Process. 158 (2021), 107702.
[168] D. Giagopoulos, A. Arailopoulos, I. Zacharakis, E. Pipili, Finite element model developed and modal analysis of large scale steam turbine rotor: quantification of uncertainties and model updating. 2nd ECCOMAS Thematic Conference on Uncertainty Quantification in Computational Sciences and Engineering, Rhodes Island, 2017.
[169] N.C. Tyminski, F.W.S. Tuckmantel, K.L. Cavalca, H.F. de Castro, Bayesian inference applied to journal bearing parameter identification, J. Braz. Soc. Mech. Sci. Eng. 39 (2017) 2983–3004.
[170] G.Y. Garoli, N.C. Tyminski, H.F. de Castro, Stochastic collocation approach to bayesian inference applied to rotating system parameter identification, in: K. L. Cavalca, H.I. Weber (Eds.), Proceedings of the 10th International Conference on Rotor Dynamics-IFToMM, Springer International Publishing, Cham, 2019, pp. 401–415.
[171] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.
[172] F. Carmo Carvalho, M.V. Fernandes de Oliveira, F.A. Lara-Molina, A.A. Cavalini, V. Steffen, Fuzzy robust control applied to rotor supported by active magnetic bearing, J. Vib. Control 27 (2020) 912–923.
[173] Y. Li, H. Huang, S. Zhu, Y. Liu, N. Xiao, An application of fuzzy fault tree analysis to uncontained events of an areo-engine rotor, Internat. J. Turbo Jet Eng. 29 (2012) 309–315.
[174] J. Wang, Y. Yang, Q. Zheng, W. Deng, D. Zhang, C. Fu, Dynamic response of dual-disk rotor system with uncertainties based on Chebyshev convex method, Appl. Sci. 11 (2021) 9146.
[175] S. Zhao, X. Ren, Q. Zheng, K. Lu, C. Fu, Y. Yang, Transient dynamic balancing of the rotor system with uncertainty, Mech. Syst. Sig. Process. 171 (2022), 108894.
[176] Z. Jia, Y. Yang, Q. Zheng, W. Deng, Dynamic analysis of Jeffcott rotor under uncertainty based on Chebyshev convex method, Mech. Syst. Sig. Process. 167 (2022), 108603.
[177] C. Jiang, X. Han, G. Lu, J. Liu, Z. Zhang, Y. Bai, Engineering, Correlation analysis of non-probabilistic convex model and corresponding structural reliability technique, Comput. Methods Appl. Mech. Eng. 200 (2011) 2528–2546.
[178] Y. Ben-Haim, I. Elishakoff, Convex Models of Uncertainty in Applied Mechanics, Elsevier, 2013.
[179] Y. Hao, M. Chen, J. Hong, Y. Ma, Interval analysis method of rotordynamics based on Taylor expansion method, Journal of Aerospace, Power 29 (2014) 571–577.
[180] W. Mao, N. Zhang, J. Li, C. Fu, A proposed unbalance parameters identification method with mixed uncertainty, Arch. Appl. Mech. 91 (2021) 1879–1885.
[181] T. Wei, F. Li, G. Meng, W. Zuo, D. Yao, Bounds for uncertain structural problems with large-range interval parameters, Arch. Appl. Mech. 91 (2021) 1157–1177.
[182] J. Wu, Y. Zhang, L. Chen, Z. Luo, A Chebyshev interval method for nonlinear dynamic systems under uncertainty, Appl. Math. Model. 37 (2013) 4578–4591.
[183] X. Guo, Y. Cao, H. Ma, C. Xiao, B. Wen, Dynamic analysis of an L-shaped liquid-filled pipe with interval uncertainty, Int. J. Mech. Sci. 217 (2022), 107040.
[184] E. Jacquelin, S. Adhikari, M.I. Friswell, J.-J. Sinou, Role of roots of orthogonal polynomials in the dynamic response of stochastic systems, J. Eng. Mech. 142 (2016) 06016004.
[185] J. Wu, Dynamics uncertainty research based on interval arithmetic using Chebyshev polynomials, Huazhong University of Science and Technology, Wuhan, China, 2013.
[186] D. Yan, Y. Zheng, W. Liu, T. Chen, Q. Chen, Interval uncertainty analysis of vibration response of hydroelectric generating unit based on Chebyshev polynomial, Chaos, Solitons Fractals 155 (2022), 111712.
[187] E. Jacquelin, S. Adhikari, J.-J. Sinou, M.I. Friswell, Polynomial chaos expansion in structural dynamics: Accelerating the convergence of the first two statistical moment sequences, J. Sound Vib. 356 (2015) 144–154.
[188] S. Chowdhury, P. Champagne, P.J. McLellan, Uncertainty characterization approaches for risk assessment of DBPs in drinking water: A review, J. Environ. Manage. 90 (2009) 1680–1691.
[189] J. Wu, Z. Luo, N. Zhang, W. Gao, A new sequential sampling method for constructing the high-order polynomial surrogate models, Eng. Comput. 35 (2018) 529–564.
[190] R.Y. Rubinstein, D.P. Kroese, Simulation and the Monte Carlo Method, John Wiley & Sons, 2016.
[191] R.R. Yager, On the dempster-shafer framework and new combination rules, Inf. Sci. 41 (1987) 93–137.
[192] M. Oberguggenberger, J. King, B. Schmelzer, Classical and imprecise probability methods for sensitivity analysis in engineering: A case study, Int. J. Approx. Reason. 50 (2009) 680–693.
[193] K. Weichselberger, The theory of interval-probability as a unifying concept for uncertainty, Int. J. Approx. Reas. 24 (2000) 149–170.
[194] H. She, C. Li, G. Zhang, Q. Tang, Statistical investigation on the coupling mode characteristics of a blade-disk-shaft unit, Mech. Based Des. Struct. Mach. (2021), https://doi.org/10.1080/15397734.2021.1956328.
[195] L.B. Visnadi, H.F. de Castro, Influence of bearing clearance and oil temperature uncertainties on the stability threshold of cylindrical journal bearings, Mech. Mach. Theory 134 (2019) 57–73.
[196] S. Murugan, R. Ganguli, D. Harursampath, Aeroelastic response of composite helicopter rotor with random material properties, J. Aircraft 45 (2008) 306–322.
[197] V. Vishwakarma, A. Sinha, Forced response statistics of a bladed rotor with geometric mistuning, AIAA J. 53 (2015) 2776–2781.
[198] K. Moustafa, H. El-Awady, Interval rotor-bearing systems stability via Monte Carlo simulation, Int. J. Model. Simul. 28 (2008) 202–208.
[199] W. Mao, N. Zhang, D. Feng, J. Li, A proposed bearing load identification method to uncertain rotor systems, Shock Vib. 2021 (2021) 6615761.
[200] B. Liu, X. Yin, K. Jian, Y. Wu, Perturbation transfer matrix method for eigendata of one-dimensional structural system with parameter uncertainties, Appl. Math. Mech. 24 (2003) 801–807.
[201] Y. Yang, Y. Wang, Z. Gao, Nonlinear analysis of a rub-impact rotor with random stiffness under random excitation, Adv. Mech. Eng. 8 (2016) 1–12.
[202] E. Jacquelin, S. Adhikari, J.-J. Sinou, M.I. Friswell, Polynomial chaos expansion and steady-state response of a class of random dynamical systems, J. Eng. Mech. 141 (2015) 04014145.
[203] R. Kumar, S. Gupta, S.F. Ali, Random field modeling and analysis of rotor bladed disc sector using a data driven PCE based approach, in: ASME 2019 Gas Turbine India Conference. Volume 1: Compressors, Fans, and Pumps; Turbines; Heat Transfer; Structures and Dynamics, Chennai, Tamil Nadu, India, 2019, pp. V001T005A007.
[204] Z. Lu, Y. Lv, H. Ouyang, A super-harmonic feature based updating method for crack identification in rotors using a kriging surrogate model, Appl. Sci. 9 (2019) 2428.
[205] F.A. Lara-Molina, E.H. Koroishi, V. Steffen Jr., Uncertainty analysis of flexible rotors considering fuzzy parameters and fuzzy-random parameters, Latin Am. J. Solids Struct. 12 (2015) 1807–1823.
[206] V. Carvalho, B. Rende, A. Silva, A. Cavalini, V. Steffen, Robust balancing approach for rotating machines based on fuzzy logic, J. Vib. Acoust. 140 (2018), 051018.
[207] A.D. Dimarogonas, Interval analysis of vibrating systems, J. Sound Vib. 183 (1995) 739–749.
[208] F.A. Lara-Molina, A.A. Cavalini, E.H. Koroishi, V. Steffen, Sensitivity analysis of flexible rotor subjected to interval uncertainties, Latin American Journal of Solids and Structures 16 (2019) e188.

[209] M. Friswell, U. Prells, J. Penny, Determining uncertainty bounds for eigenvalues, Proc. ISMA (2004) 3055–3064.
[210] D.J. Ramos, A.R. Ferraz, G.B. Daniel, T.G. Ritto, Dynamic analysis of rotating systems considering uncertainties in the bearings' parameters, Internat. Conf. Rotor Dyn., Springer (2018) 460–474.
[211] C. Fu, K. Lu, Y. Yang, Z. Xie, A. Ming, Nonlinear vibrations of an uncertain dual-rotor rolling bearings system with coupling misalignment, J. Nonlinear Math. Phys. 29 (2022) 388–402.
[212] E. Capiez-Lernout, C. Soize, J.-P. Lombard, C. Dupont, E. Seinturier, Blade manufacturing tolerances definition for a mistuned industrial bladed disk, J. Eng. Gas Turbines Power 127 (2005) 621–628.
[213] E. Capiez-Lernout, C. Soize, M. Mbaye, Mistuning analysis and uncertainty quantification of an industrial bladed disk with geometrical nonlinearity, J. Sound Vib. 356 (2015) 124–143.
[214] A. Picou, E. Capiez-Lernout, C. Soize, M. Mbaye, Robust dynamic analysis of detuned-mistuned rotating bladed disks with geometric nonlinearities, Comput. Mech. 65 (2020) 711–730.
[215] J. Xie, J. Liu, J. Chen, Y. Zi, Blade damage monitoring method base on frequency domain statistical index of shaft's random vibration, Mech. Syst. Sig. Process. 165 (2022), 108351.
[216] J. Yuan, A. Fantetti, E. Denimal, S. Bhatnagar, L. Pesaresi, C. Schwingshackl, L. Salles, Propagation of friction parameter uncertainties in the nonlinear dynamic response of turbine blades with underplatform dampers, Mech. Syst. Sig. Process. 156 (2021), 107673.
[217] R. Turaga, A. Sekhar, B. Majumdar, Stability analysis of a rigid rotor supported on hydrodynamic journal bearings with rough surfaces using the stochastic finite element method, Proc. Institut. Mech. Eng., Part J: J. Eng. Tribolo. 212 (1998) 121–130.
[218] Z. Li, J. Jiang, Effects of rough bearing surface on the stochastic dynamics of a rotor-bearing system subject to random fluid-induced forces, in: P. Pennacchi (Ed.), Proceedings of the 9th IFToMM International Conference on Rotor Dynamics, Springer International Publishing, Cham, 2015, pp. 1157–1166.
[219] S.S. Rao, Y. Qiu, Analysis of rotor bearing systems with uncertainties: A fuzzy approach, in: ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, August 30-September 2, San Diego, California, USA, 2009, pp. 1247-1257.
[220] Y. Liu, Y. Zhang, Z. Wu, Stochastic dynamic analysis of the rotor–bearing system considering the randomness of the radial clearance, J. Braz. Soc. Mech. Sci. Eng. 41 (2019) 529.
[221] H.A.P. Da Silva, R. Nicoletti, Design of tilting-pad journal bearings considering bearing clearance uncertainty and reliability analysis, J. Tribol. 141 (2019), 011703.
[222] D.S. Alves, G.B. Daniel, H.F.d. Castro, T.H. Machado, K.L. Cavalca, O. Gecgel, J.P. Dias, S. Ekwaro-Osire, Uncertainty quantification in deep convolutional neural network diagnostics of journal bearings with ovalization fault, Mech. Mach. Theory 149 (2020), 103835.
[223] B. Geng, Y. Zuo, Z. Jiang, K. Feng, C. Wang, J. Wang, A double integral method for quantitative evaluation of influence on thin-walled casing response caused by bearing uncertainties, Chin. J. Aeronaut. 33 (2020) 2372–2381.
[224] C. Fu, W. Zhu, J. Ma, F. Gu, Static and dynamic characteristics of journal bearings under uncertainty: A non-probabilistic perspective, J. Eng. Gas Turbines Power 144 (2022), 071012.
[225] J. Kato, T. Inoue, K. Takagi, S. Yabui, Nonlinear analysis for influence of parametric uncertainty on the stability of rotor system with active magnetic bearing using feedback linearization, J. Comput. Nonlinear Dyn. 13 (2018), 071004.
[226] T.C. Yeh, S.L. Chen, C.H. Lin, Robust control of a magnetic rotor-bearing system with estimation of uncertainty bound, Internat. Conf. Syst. Sci. Eng. 2021 (2021) 397–402.
[227] J.-J. Sinou, E. Denimal, Reliable crack detection in a rotor system with uncertainties via advanced simulation models based on kriging and Polynomial Chaos Expansion, Eur. J. Mech. A. Solids 92 (2022), 104451.
[228] C. Fu, W. Zhu, Y. Yang, S. Zhao, K. Lu, Surrogate modeling for dynamic analysis of an uncertain notched rotor system and roles of Chebyshev parameters, J. Sound Vib. 524 (2022), 116755.
[229] J. Didier, J.-J. Sinou, B. Faverjon, Nonlinear vibrations of a mechanical system with non-regular nonlinearities and uncertainties, Commun. Nonlinear Sci. Numer. Simul. 18 (2013) 3250–3270.
[230] E. Sarrouy, Phase driven study for stochastic linear multi-dofs dynamic response, Mech. Syst. Sig. Process. 129 (2019) 717–740.
[231] E. Sarrouy, E. Pagnacco, E.S. de Cursi, A constant phase approach for the frequency response of stochastic linear oscillators, Mech. Industry 17 (2016) 206.
[232] E. Peradotto, A.M. Panunzio, L. Salles, C. Schwingshackl, Stochastic methods for nonlinear rotordynamics with uncertainties, in: Turbo Expo: Power for Land, Sea, and Air, American Society of Mechanical Engineers, 2015, pp. V07AT30A013.
[233] A.M. Panunzio, L. Salles, C.W. Schwingshackl, Uncertainty propagation for nonlinear vibrations: A non-intrusive approach, J. Sound Vib. 389 (2017) 309–325.
[234] C. Fu, W. Zhu, Z. Zheng, C. Sun, Y. Yang, K. Lu, Nonlinear responses of a dual-rotor system with rub-impact fault subject to interval uncertain parameters, Mech. Syst. Sig. Process. 170 (2022), 108827.
[235] J. Ma, C. Fu, W. Zhu, K. Lu, Y. Yang, Stochastic analysis of lubrication in misaligned journal bearings, J. Tribol. 144 (2022), 081802.
[236] C. Fu, Y. Xu, Y. Yang, K. Lu, F. Gu, A. Ball, Dynamics analysis of a hollow-shaft rotor system with an open crack under model uncertainties, Commun. Nonlinear Sci. Numer. Simul. 83 (2020), 105102.
[237] Y. Chen, C. Guo, An interval fault diagnosis method for rotor cracks, Comput. Electr. Eng. 87 (2020), 106752.
[238] K. Guo, J. Jiang, Z. Li, Diffusion and persistence of rotor/stator synchronous full annular rub response under weak random perturbations, J. Vib. Eng. Technol. 8 (2020) 599–611.
[239] B. Tian, Z. Yu, L. Xie, Y. Zhang, Dynamic analysis of the dual-rotor system considering the defect size uncertainty of the inter-shaft bearing, J. Mech. Sci. Technol. 36 (2022) 575–592.
[240] P. Kumar, S. Narayanan, S. Gupta, Dynamic parameters estimation and fault identification from random response of rolling element bearing in a rotor bearing system, in: ASME 2019 Gas Turbine India Conference. Volume 1: Compressors, Fans, and Pumps; Turbines; Heat Transfer; Structures and Dynamics, Chennai, Tamil Nadu, India. December 5-6, 2019, pp. V001T005A011.
[241] N. Driot, A. Berlioz, C.H. Lamarque, Stability and stationary response of a skew Jeffcott rotor with geometric uncertainty, J. Comput. Nonlinear Dyn. 4 (2009), 021003.
[242] N. Sri Namachchivaya, Mean square stability of a rotating shaft under combined harmonic and stochastic excitations, J. Sound Vib. 133 (1989) 323–336.
[243] Y. Zuo, J. Wang, A method for dynamic analysis of three-dimensional solid element rotors with uncertain parameters, J. Eng. Gas Turbines Power 139 (2017), 054501.
[244] R. Pavlović, P. Kozić, G. Janevski, Influence of rotatory inertia on stochastic stability of a viscoelastic rotating shaft, Theoret. Appl. Mech. 35 (2008) 363–379.
[245] R. Pavlović, P. Rajković, I. Pavlović, Dynamic stability of the viscoelastic rotating shaft subjected to random excitation, Int. J. Mech. Sci. 50 (2008) 359–364.
[246] J. Deng, Dynamic stability of a viscoelastic rotating shaft under parametric random excitation, Int. J. Non Linear Mech. 84 (2016) 56–67.
[247] Z.S. Zutavern, D.W. Childs, Identification of rotordynamic forces in a flexible rotor system using magnetic bearings, J. Eng. Gas Turbines Power 130 (2008), 022504.
[248] L. Xiang, Z. Wang, G. Tang, Empirical parameter study of uncertain rotor coupling system, Journal of Vibration, Measurement and Diagnosis 33 (2013) 450–455.
[249] Y. Kang, Z. Qiu, H. Zhang, Z. Shi, F. Gu, Model updating for rotor-discs system and its application in dynamic coefficients identification of journal bearings, Measurement 173 (2021), 108645.
[250] Y. Kim, B. Yang, A.C. Tan, Bearing parameter identification of rotor-bearing system using clustering-based hybrid evolutionary algorithm, Struct. Multidiscip. Optim. 33 (2007) 493–506.
[251] R.R. Mutra, J. Srinivas, R. Rządkowski, An optimal parameter identification approach in foil bearing supported high-speed turbocharger rotor system, Arch. Appl. Mech. 91 (2021) 1557–1575.
[252] D. Wang, C. Hua, D. Dong, B. He, Z. Lu, Crack parameters identification based on a kriging surrogate model for operating rotors, Shock Vib. 2018 (2018) 9274526.

[253] W. Mao, Q. Tang, D. Feng, Identification of eccentricity of a motorized spindle-tool system with random parameters, Mech. Sci. 12 (2021) 715–723.
[254] M. Tarabini, D. Scaccabarozzi, Uncertainty-based combination of signal processing techniques for the identification of rotor imbalance, Measurement 114 (2018) 409–416.
[255] R. Tiwari, N.S. Vyas, Parameter estimation in imbalanced non-linear rotor-bearing systems from random response, J. Sound Vib. 208 (1997) 1–14.
[256] G.Y. Garoli, R. Pilotto, R. Nordmann, H.F. de Castro, Identification of active magnetic bearing parameters in a rotor machine using Bayesian inference with generalized polynomial chaos expansion, J. Braz. Soc. Mech. Sci. Eng. 43 (2021) 552.
[257] F. Han, X. Guo, C. Mo, H. Gao, P. Hou, Parameter identification of nonlinear rotor-bearing system based on improved kriging surrogate model, J. Vib. Control 23 (2015) 794–807.
[258] J. Liu, S. Ding, G. Li, Influence of random variable dimension on the fast numerical integration method of aero engine rotor disk failure risk analysis. ASME 2020 International Mechanical Engineering Congress and Exposition, 2020. , pp. V014T014A039.
[259] L. Song, G. Bai, Multi-surrogate collaboration approach for creep-fatigue reliability assessment of turbine rotor, IEEE Access 8 (2020) 39861–39874.
[260] M.R. Motley, Y.L. Young, Influence of uncertainties on the response and reliability of self-adaptive composite rotors, Compos. Struct. 94 (2011) 114–120.
[261] C. Su, F. Guo, Q. Shi, Y. Zhang, Reliability sensitivity estimation of rotor system with oil whip and resonance, Adv. Mech. Eng. 9 (2017) 1–15.
[262] B. Zhao, L. Xie, Y. Zhang, J. Ren, X. Bai, B. Qin, An improved dynamic load-strength interference model for the reliability analysis of aero-engine rotor blade system, Proc. Instit. Mech. Eng., Part G: J. Aerospace Eng. 235 (2020) 1355–1373.
[263] R.C. McClung, M.P. Enright, H.R. Millwater, G.R. Leverant, S. Hudak, W. Johnson, B. Hillberry, A Software Framework For Probabilistic Fatigue Life Assessment Of Gas Turbine Engine Rotors, ASTM International West Conshohocken, PA, 2004.
[264] X. Guan, J. He, E.M. Rasselkorde, J. Zhang, W.A. Abbasi, S.K. Zhou, Probabilistic fatigue life prediction and structural reliability evaluation of turbine rotors integrating an automated ultrasonic inspection system, J. Nondestr. Eval. 33 (2014) 51–61.
[265] X. Li, L. Song, G. Bai, Recent advances in reliability analysis of aeroengine rotor system: A review, Internat. J. Struct. Integ. 13 (2022) 1–29.
[266] Y. Zhang, B. Wen, A.Y.T. Leung, Reliability analysis for rotor rubbing, J. Vib. Acoust. 124 (2001) 58–62.
[267] Y. Zhang, B. Wen, Q. Liu, Reliability sensitivity for rotor-stator systems with rubbing, J. Sound Vib. 259 (2003) 1095–1107.
[268] L. Zhu, Y. Wang, Z. Lin, H. Chen, G. Yao, W. Wang, Stress reliability sensitivity analysis for a flexibility rotor, in: Global Reliability and Prognostics and Health Management (PHM-Shanghai), IEEE, 2020, pp. 1–6.
[269] B. Bai, W. Zhang, B. Li, C. Li, G. Bai, Application of probabilistic and nonprobabilistic hybrid reliability analysis based on dynamic substructural extremum response surface decoupling method for a blisk of the aeroengine, Internat. J. Aerospace Eng. 2017 (2017) 5839620.
[270] X. Ma, Z. Zhang, H. Hua, Uncertainty quantization and reliability analysis for rotor/stator rub-impact using advanced Kriging surrogate model, J. Sound Vib. 525 (2022), 116800.
[271] B.S. Yang, S.P. Choi, Y.C. Kim, Vibration reduction optimum design of a steam-turbine rotor-bearing system using a hybrid genetic algorithm, Struct. Multidiscip. Optim. 30 (2005) 43–53.
[272] R.H. Lopez, T.G. Ritto, R. Sampaio, J.E.S. de Cursi, A new algorithm for the robust optimization of rotor-bearing systems, Eng. Optim. 46 (2014) 1123–1138.
[273] G. Taguchi, E.A. Elsayed, T.C. Hsiang, Quality Engineering in Production Systems, McGraw-Hill College, New York, 1989.
[274] G. Li, Z. Lin, P.E. Allaire, Robust optimal balancing of high-speed machinery using convex optimization, J. Vib. Acoust. 130 (2008), 031008.
[275] B. Huang, Optimum Balancing Of High Speed Uncertain Flexible Rotor Systems Using Convex Optimization, University of Virginia, 2007.
[276] S. Hong, S. Lee, S. Jun, D.-H. Lee, H. Kang, Y.-S. Kang, S.-S. Yang, Reliability-based design optimization of axial compressor using uncertainty model for stall margin, J. Mech. Sci. Technol. 25 (2011) 731–740.
[277] S. Jia, L. Zheng, Q. Mei, Flexible rotor optimization design with considering the uncertainty of unbalance distribution, Int. J. Simul. Multi. Des. Optim. 11 (2020) 19.
[278] T.N. Shiau, C.H. Kang, D.S. Liu, Interval optimization of rotor-bearing systems with dynamic behavior constraints using an interval genetic algorithm, Struct. Multidiscip. Optim. 36 (2008) 623–631.
[279] J. Hong, Z. Yang, Y. Ni, Y. Ma, Robust optimization of flexible rotor systems with uncertain parameters via interval analysis method, in: Turbo Expo: Power for Land, Sea, and Air, American Society of Mechanical Engineers, 2019, pp. V07AT32A004.
[280] H. Feng, W. Shangguan, S. Rakheja, Hybrid uncertainties-based analysis and optimization methods for axial friction force of drive-shaft systems, J. Sound Vib. 511 (2021), 116320.
[281] H. Sira-Ramirez, G. Silva-Navarro, F. Beltran-Carvajal, On the GPI balancing control of an uncertain Jeffcot rotor model, in: 2007 4th International Conference on Electrical and Electronics Engineering, 2007, pp. 306–309.
[282] C.R. Knospe, S.M. Tamer, S.J. Fedigan, Robustness of adaptive rotor vibration control to structured uncertainty, J. Dyn. Syst. Meas. Contr. 119 (1997) 243–250.
[283] S. Li, H. Jiang, B. Shen, Design of uncertain rotor-sliding-bearing system based on H∞ control theory, in: 3rd International Symposium on Systems and Control in Aeronautics and Astronautics, IEEE, 2010, pp. 880–884.
[284] C. Liu, G. Liu, J. Fang, Feedback linearization and extended state observer-based control for rotor-AMBs system with mismatched uncertainties, IEEE Trans. Ind. Electron. 64 (2016) 1313–1322.
[285] G. Li, Z. Lin, P.E. Allaire, J. Luo, Modeling of a high speed rotor test rig with active magnetic bearings, J. Vib. Acoust. 128 (2005) 269–281.
[286] G. Ren, Z. Chen, H. Zhang, Y. Wu, H. Meng, D. Wu, H. Ding, Design of interval type-2 fuzzy controllers for active magnetic bearing systems, IEEE/ASME Trans. Mechatron. 25 (2020) 2449–2459.
[287] T. Inoue, J. Liu, Y. Ishida, Y. Yoshimura, Vibration control and unbalance estimation of a nonlinear rotor system using disturbance observer, J. Vib. Acoust. 131 (2009), 031010.
[288] S.H. Chen, W.H. Ho, J.H. Chou, F. Lu, Design of robust-stable and quadratic finite-horizon optimal active vibration controllers with low trajectory sensitivity for the uncertain flexible rotor systems via the orthogonal-functions approach and the hybrid Taguchi-genetic algorithm, J. Vib. Control 18 (2011) 924–940.
[289] B. Riemann, M.A. Sehr, R.S. Schittenhelm, S. Rinderknecht, Robust control of flexible high-speed rotors via mixed uncertainties, IEEE 2013 (2013) 2343–2350.
[290] A. Sahinkaya, J.T. Sawicki, Robust control of active magnetic bearing systems with an add-on controller to cancel gyroscopic effects: Is it worth it? J. Vib. Control 27 (2020) 2754–2764.
[291] X. Wang, I. Elishakoff, Z. Qiu, Experimental data have to decide which of the nonprobabilistic uncertainty descriptions-convex modeling or interval analysis-to utilize, J. Appl. Mech. 75 (2008), 041018.
[292] P. Langer, K. Sepahvand, C. Guist, J. Bär, A. Peplow, S. Marburg, Matching experimental and three dimensional numerical models for structural vibration problems with uncertainties, J. Sound Vib. 417 (2018) 294–305.
[293] M. Friswell, J. Coote, M. Terrell, S. Adhikari, J. Fonseca, N. Lieven, Experimental data for uncertainty quantification, in: 23rd International Modal Analysis Conference, Orlando, Florida, USA, Citeseer, 2005.
[294] M.R. Fatehi, A. Ghanbarzadeh, S. Moradi, A. Hajnayeb, Determination of random matrices dispersion parameters for nonparametric modeling of stochastic dynamic systems with experimental verification, Int. J. Appl. Mech. 10 (2018) 1850101.
[295] K. Lu, Y. Yang, Y. Xia, C. Fu, Statistical moment analysis of nonlinear rotor system with multi uncertain variables, Mech. Syst. Sig. Process. 116 (2019) 1029–1041.
[296] K.P. Murphy, Machine Learning: A Probabilistic Perspective, MIT Press, Cambridge, MA, 2012.
[297] E. Capiez-Lernout, C. Soize, Nonlinear stochastic dynamics of detuned bladed-disks with uncertain mistuning and detuning optimization using a probabilistic machine learning tool, Int. J. Non Linear Mech. 143 (2022), 104023.
[298] T. Roncen, J.-J. Sinou, J.P. Lambelin, Non-linear vibrations of a beam with non-ideal boundary conditions and uncertainties-Modeling, numerical simulations and experiments, Mech. Syst. Sig. Process. 110 (2018) 165–179.
[299] B.K. Beachkofski, Probabilistic rotor life assessment using reduced order models, Shock Vib. 16 (2009) 581–591.
[300] K. Lu, Y. Jin, Y. Chen, Y. Yang, L. Hou, Z. Zhang, Z. Li, C. Fu, Review for order reduction based on proper orthogonal decomposition and outlooks of applications in mechanical systems, Mech. Syst. Sig. Process. 123 (2019) 264–297.