Rendezvous of Nonholonomic Robots via Output-Feedback Control under Time-varying Delays

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Abstract—We address the problem of making nonholonomic vehicles, with second-order dynamics and interconnected over a bidirectional network, converge to a formation centered at a non-prespecified point on the plane with a non-prespecified common orientation. We assume that only the Cartesian position of the center of mass of each vehicle and its orientation are available for measurement, but not the velocities. In addition, we assume that the interconnections are affected by time-varying delays. Our control method consists in designing a set of second-order systems that are interconnected with the robots’ dynamics through virtual springs and transmit their own coordinates to achieve consensus. This and the virtual elastic couplings with the vehicles make the latter achieve consensus too. To the best of our knowledge, output feedback consensus control of underactuated nonholonomic vehicles has been little studied, all the less in the presence of delays.

Index Terms—Rendezvous, consensus, autonomous vehicles, persistency of excitation, output feedback, differential-drive mobile robots.

I. INTRODUCTION

For first and second-order integrators the leaderless consensus problem, which consists in the state variables of all agents converging to a common value, is well-studied and solved under many different scenarios [1]. The solution to this problem is more complex if one considers the agents’ dynamics [2], [3], network constraints, such as communication delays [4], unavailability of velocity measurements [5], or nonholonomic constraints that restrict the system’s motion [6]. For autonomous vehicles, which, in contrast to mathematical models, do occupy a physical space, the leaderless consensus problem consists in making all robots converge to a rendezvous point while forming a pattern with an unknown center. This is done by specifying an offset position from the unknown center for each robot. It may be required that either only the positions [6], [7] or only the orientations [8] achieve a common equilibrium point, or that both positions and orientations converge to a common value [9].

Rendezvous control is useful, e.g., if a group of robots must converge to postures that form a desired geometric pattern in order to subsequently maneuver as a whole [10]. This is a typical two-stage formation problem. In the first, a rendezvous algorithm is required for the stabilization of the agents [11]–[13] and in the second a formation-tracking controller is employed [14]. From a systems viewpoint, rendezvous control of nonholonomic vehicles is inherently a set-point stabilization problem. In that regard, it presents the same technical difficulties as the stabilization of a single robot. In particular, that nonholonomic systems are not stabilizable via time-invariant smooth feedback [15], but either via discontinuous time-invariant control [12] or time-varying smooth feedback [16], [17].

In this paper we solve the rendezvous problem for force-controlled nonholonomic systems unequipped of velocity sensors and interconnected over a network with bidirectional interconnections affected by time-varying delays. Delays are common in network control systems [18]. Not relying on velocity measurements is desirable since these measurements are often contaminated by noise and velocity sensors may be unreliable [19]. Measurement delays and lack of velocity measurements have been addressed in the literature on consensus of nonholonomic systems, but not simultaneously. For instance, [19]–[21] address control problems without velocity measurements, and delays are considered in [20], but they all concern leader-follower formation tracking control; output-feedback consensus is addressed in [22] and [23], but delays are not considered. In [24] a controller achieving consensus formation in the presence of smooth time-varying delays is proposed, but the controller uses state feedback.

The design of the controller that we propose in this paper is inspired by the previous works [5] and [24]. The control approach consists in designing a group of independent second-order systems to achieve output consensus among themselves and, then, steer the plants to output consensus through a virtual mechanical coupling. The underlying idea is reminiscent of that used in [5], for flexible-joint manipulators, but in contrast to the latter, we do not use a high-gain nonlinear observer. As in [24], our controller relies on persistency of excitation to overcome the difficulties imposed by the nonholonomicity on set-point stabilization, but it is not a certainty-equivalence modification of the controller in that reference.

The remainder of this paper is organized as follows. In the next section we describe the nonholonomic second-order dynamic model and lay our main assumptions. In Sections III and IV we present some preliminary, but original, results on state- and output-feedback consensus control that are useful to explain and put our main results in perspective. The latter are presented in Section V. In Section VI we provide some realistic simulations using the Gazebo-ROS environment and we provide concluding remarks in Section VII.

II. MODEL AND PROBLEM FORMULATION

We consider a group of $N$ autonomous nonholonomic second-order vehicles modeled by the equations:

\[
\begin{align*}
\dot{x}_i &= \omega_i, \\
\dot{\omega}_i &= u_{wi}, \\
\dot{z}_i &= \varphi(\theta_i) v_i, \\
\dot{v}_i &= u_{vi}, \quad i \in \{1, 2 \ldots N\}
\end{align*}
\]

where

\[
\varphi(\theta_i) := \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \end{bmatrix}^T,
\]

\[
z_i = [x_i, y_i]^T \in \mathbb{R}^2
\]
denotes the Cartesian coordinates of the $i$th
vehicle on the plane, \( \theta_i \in \mathbb{R} \) denotes its orientation, and
\[
u_{\psi i} := \frac{1}{mT} (\tau_{1 i} + \tau_{2 i}), \quad \omega_{\psi i} := \frac{2R}{T_c} (\tau_{1 i} - \tau_{2 i})
\]
are the control inputs, which are defined in function of the wheel torques \( \tau_{1 i} \) and \( \tau_{2 i} \), the robot inertia \( I \), the mass \( m \), the wheel radius \( r \), and the wheel axle length \( R \).

Remark 1: For the purpose of analysis, the angles are defined as real variables, but this may entail undesired unwinding. In practice, \( \theta_i \in (-\pi, \pi] \). This is considered in Section VI.

It is required that the vehicles meet in formation around a non-predefined rendezvous point on the plane, denoted \( z_c := (x_c, y_c) \), and acquire a non-predefined common orientation, determined \( \theta_c \), modulo a given offset \( \delta_i = [\delta_{1i}, \delta_{2i}]^\top \), with \( i \leq N \), which determines the position of the \( i \)th vehicle relative to the unknown center of the formation. In other words, defining, \( \bar{z}_i := z_i - \delta_i \) (correspondingly, \( \bar{x}_i := x_i - \delta_{1i} \) and \( \bar{y}_i := y_i - \delta_{2i} \)) the control goal is to make
\[
\begin{align*}
\lim_{t \to \infty} v_i(t) &= 0, \quad \lim_{t \to \infty} \bar{z}_i(t) = z_c, \quad \forall i \leq N, \\
\lim_{t \to \infty} \omega_i(t) &= 0, \quad \lim_{t \to \infty} \theta_i(t) = \theta_c \quad \forall i \leq N.
\end{align*}
\]
This is a leaderless consensus problem. That is, neither the coordinates \((x_c, y_c)\) nor the angle \(\theta_c\) are determined a priori as a reference. They depend on the initial postures, on the systems’ nonlinear dynamics, and on network features. The rendezvous problem has been successfully solved under different conditions, but the originality of this paper resides in considering the realistic scenario determined by the following assumptions simultaneously.

First, owing to the fact that velocity measurements are often corrupted by noise and sensor defects we pose the following hypothesis.

Assumption 1: Only the coordinates \((z_i, \theta_i)\) are measured.

Second, we assume that a WiFi communication network is available over which the \( i \)th robot communicates with a set of neighbors, which we denote by \( N_i \). It is naturally assumed that once a communication is set between two robots \( i \) and \( j \in N_i \), the flow of information is bidirectional and is never lost. Whence the following.

Assumption 2: The network may be modeled using an interconnection graph that is undirected, static, and connected.

Remark 2: In graph theory, a graph is undirected if the nodes exchange information in both direction, it is static if the interconnection is constant, and it is connected if any node is reachable from any other node. The latter is a necessary condition to achieve consensus [1].

On the other hand, because the robots communicate through a WiFi network, the communication between the robots \( i \) and \( j \) is affected by non-constant time-delays. More precisely, we consider the following.

Assumption 3: The communication from the \( j \)th to the \( i \)th robot is subject to a variable time-delay denoted \( T_{ji}(t) \). It is assumed that the function \( t \mapsto T_{ji}(t) \) is bounded, has bounded time-derivatives, up to the second, and the upper-bound, denoted \( \overline{T}_{ji} \), is known.

Assumption 3, which is imposed only for technical reasons imposed by the method of formal analysis, carries certain conservatism in the supposition that the delays are differentiable. Indeed, time-delays over WiFi networks or the Internet may rather be of a non-smooth nature [9], [18]. In Section VI we provide realistic simulations in which Assumption 3 is violated. Yet, the formal solution to the rendezvous problem defined above under Assumptions 1, 2, and under discontinuous time-varying delays remains an open problem.

III. CONTROL ARCHITECTURE: STATE-FEEDBACK CASE

An essential feature of the model (1)–(2) is that it consists of two coupled second-order systems driven by independent control inputs. One system determines the linear motion and the other the angular one. Each of the latter being a second-order mechanical system, the control design starts by devising a consensus controller for (1) and (2) separately. To that end, we revisit a controller from the literature, but we provide an original analysis of robust stability that serves as design-basis for our dynamic output-feedback controller, presented in Section V.

A. Robust consensus control of second-order systems

The consensus problem for systems with dynamics
\[
\dot{\vartheta}_i = u_i \quad i \leq N, \quad u_i \in \mathbb{R}
\]
(that is steering \( \vartheta_i \rightarrow \vartheta_c, \dot{\vartheta}_i \rightarrow 0 \), and \( \dot{\vartheta}_i \rightarrow 0 \) with \( \vartheta_c \) constant and not imposed a priori) is now well understood in various settings. For instance, it is well known (see [1]) that if the systems modeled by (6) communicate over a network modeled by an undirected, static, and connected graph, the distributed control law, of proportional-derivative (PD) type,
\[
u_i = -d_i \dot{\vartheta}_i - p_i \sum_{j \in N_i} a_{ij} (\vartheta_i - \vartheta_j); \quad d_i, \ p_i > 0,
\]
where \( a_{ij} > 0 \) if \( j \in N_i \) and \( a_{ij} = 0 \) otherwise, solves the consensus problem. There are many reported ways to show this. For further development we provide here a simple and original proof based on Lyapunov’s direct method. Let \( \vartheta := [\vartheta_1 \cdots \vartheta_N]^\top \) and
\[
\ddot{\vartheta} := -\frac{1}{N} \vartheta + \varpi, \quad \varpi := [1 \cdots 1]^\top.
\]
Indeed, \( \dot{\vartheta} \) denotes a vector whose \( i \)th element corresponds to the difference between \( \vartheta_i \) and the average of all states, i.e., \( \vartheta_i := (1/N) \varpi_i \). In addition, under Assumption 2, \( \vartheta_i \) corresponds to the consensus equilibrium point. Now, to abbreviate the notation, we also define
\[
\Pi := I - \frac{1}{N} \varpi \varpi^\top.
\]
Note that \( \Pi = \Pi^\top \) and \( \|\Pi\| \leq 1 \), where \( \|\Pi\| \) corresponds to the induced norm of \( \Pi \), and \( \vartheta = \Pi \vartheta \).

Next, we introduce the Laplacian matrix, \( L := [\ell_{ij}] \in \mathbb{R}^{N \times N} \), where
\[
\ell_{ij} = \left\{ \begin{array}{ll}
\sum_{k \in N_j} a_{ik} & i = j \\
-a_{ij} & i \neq j.
\end{array} \right.
\]
By construction, \( L \varpi_N = 0 \) and, after Assumption 2, \( L \) is symmetric, it has a unique zero-eigenvalue, and all of its other eigenvalues are strictly positive. Thus, \( \text{rank}(L) = N - 1 \). Also, the last term on the right-hand side of Equation (7) satisfies
\[
\col \left[ \sum_{j \in N_i} a_{ij} (\vartheta_i - \vartheta_j) \right] = L \ddot{\vartheta},
\]
where \( \col(\cdot) \) denotes a column vector of \( N \) elements \((\cdot)\). Indeed, by the definition of the Laplacian, we have
\[
\col \left[ \sum_{j \in N_i} a_{ij} (\vartheta_i - \vartheta_j) \right] = L \left[ \vartheta - \frac{1}{N} \varpi \varpi^\top \right] = L \varpi_N \varpi^\top.
\]
However, \( L \varpi_N = 0 \), so the right hand side of the equation above equals to \( L \varpi \), which corresponds to \( L \ddot{\vartheta} \), by definition. These identities are useful to write the closed-loop system (6)–(7) in the multi-variable form
\[
\ddot{\vartheta} = -D \ddot{\vartheta} - PL \vartheta,
\]
where \( P := \text{diag}(p_i) \) and \( D := \text{diag}(d_i) \), and to see that the Lyapunov function
\[
V(\vartheta, \ddot{\vartheta}) := \frac{1}{2} \left[ \ddot{\vartheta}^\top L \ddot{\vartheta} + \ddot{\vartheta}^\top P^{-1} \ddot{\vartheta} \right]
\]
is positive definite, even if $L$ is rank deficient. Indeed, the term
\[ \tilde{\alpha} \triangledown L \tilde{\alpha} \geq \lambda_2(L) \left| \tilde{\theta} \right|, \]
where $\lambda_2(L) > 0$ corresponds to the second eigenvalue of $L$ (that is, the smallest positive eigenvalue), not for any $\tilde{\alpha} \in \mathbb{R}^N$, but for $\tilde{\alpha}$ as defined in (8). Now, evaluating the total derivative of $V_1$ along the trajectories of (11) and using $L \tilde{\theta} = L \tilde{\theta}$ (again, this holds because $L \tilde{V}_1 = 0$) we see that
\[ V_1(\tilde{\theta}, \tilde{\theta}) = \tilde{\alpha} \triangledown P^{-1}D \tilde{\theta}. \]
(13)
Global asymptotic stability of the consensus manifold $\{ (\tilde{\theta}, \tilde{\theta}) = (0, 0) \}$ may be ascertained from (13) by invoking Barbashin-Krasovskiǐ’s theorem (also, but wrongly, known as LaSalle’s theorem). As a matter of fact, since the system is linear time-invariant, it is also globally exponentially stable and robust to external perturbations.

To see this more clearly, using $V_1$ it is possible to construct a simple strict Lyapunov function. This is useful to assess the robustness of system (6) in closed loop with the consensus control law defined in (7) in terms of input-to-state stability. Let

\[ V_2(\tilde{\theta}, \tilde{\theta}) := V_1(\tilde{\theta}, \tilde{\theta}) + \varepsilon \tilde{\alpha} \triangledown P^{-1} \tilde{\theta}, \quad \varepsilon \in (0, 1). \]
(14)
In view of the properties of $V_1$ it is clear that $V_2$ also is positive definite and radially unbounded, but only for all $\tilde{\theta}$ as defined in (8) and for sufficiently small values of $\varepsilon \in (0, 1)$. The total derivative of $V_2$ along the closed-loop trajectories yields
\[ V_2(\dot{\tilde{\theta}}, \dot{\tilde{\theta}}) = \dot{V}_1 + \varepsilon \dot{\tilde{\theta}} \triangledown \Pi \tilde{\theta} \tilde{\theta} \dot{\theta} - \tilde{\dot{\alpha}} \tilde{\alpha} \dot{\theta} \dot{\theta} + \dot{\tilde{\theta}} \tilde{\theta} \dot{\theta} \dot{\theta}, \]
(15)
which, in view of (13) and the fact that $\| \Pi \| \leq 1$, implies that
\[ \dot{V}_2(\tilde{\theta}, \tilde{\theta}) \leq -c_1 \frac{d_m}{p_M} \| \tilde{\theta} \| ^2 - \varepsilon c_2 \| \tilde{\alpha} \| ^2 + \dot{\tilde{\alpha}} \tilde{\alpha}, \]
(17)
with $\alpha := [\alpha_1 \cdots \alpha_N]^T$. It follows that the map $\tilde{\alpha} \rightarrow \dot{\tilde{\theta}}$ is state-stiately passive [25] and, also, the closed-loop system is input-to-state stable with respect to the input $\alpha$. •

From the previous analysis, we conclude that for the angular-motion subsystem (1) the controller
\[ u_{\omega i} = -d_{\omega i} \omega_i - p_{\omega i} \sum_{j \in \mathcal{N}_i} a_{ij} (\theta_i - \theta_j) + \alpha_i, \]
(18)
where $d_{\omega i}$ and $p_{\omega i} > 0$, ensures global asymptotic stability of the consensus manifold $\{ \omega_i = 0 \land \theta_i = \theta_j \}$ if $\alpha_i \equiv 0$ and the closed-loop system is input-to-state stable with respect to $\alpha_i$—cf. [24].

Remark 4: The previous computations hold with obvious changes in the notation for the angular-motion dynamics $\dot{\theta}_i = u_{\omega i}$, which is equivalent to (1). This is used farther below.

B. On consensus in the linear motion

After the developments in Section III-A and with the purpose of designing two independent controllers for the angular and linear motion, it appears appealing to use the following control law for the subsystem (2). Let
\[ u_{vi} = -d_{vi} v_i - p_{vi} \varphi(\theta_i) \sum_{j \in \mathcal{N}_i} a_{ij} (\bar{z}_i - \bar{z}_j) \]
(19)
and let us replace the state variable $\theta_i$ with an arbitrary trajectory $\theta_i(t)$ which, for the time being we assume to be bounded and to have a bounded derivative $\omega_i(t)$, for all $t \geq 0$ and all $i \leq N$ (this technical assumption is relaxed later). Thus, the closed-loop linear-motion dynamics, formed by Eqs. (2) and (19), may be regarded as a time-varying subsystem, decoupled from the angular motion dynamics—cf. [25, p. 657], [26]. That is,
\[ \dot{\bar{z}}_i = -d_{vi} v_i - p_{vi} \varphi(\theta_i(t)) \sum_{j \in \mathcal{N}_i} a_{ij} (\bar{z}_i - \bar{z}_j). \]
(20a)
\[ \dot{\bar{v}}_i = -d_{vi} v_i - p_{vi} \varphi(\theta_i(t)) \sum_{j \in \mathcal{N}_i} a_{ij} (\bar{z}_i - \bar{z}_j). \]
(20b)
Next, akin to $V_1$ in (12), we define the Lyapunov function
\[ V_3(v, \bar{z}) := \frac{1}{2} \sum_{i = 0}^N \left[ \frac{1}{p_{vi}} v_i^2 + \frac{1}{2} \sum_{j \in \mathcal{N}_i} a_{ij} |\bar{z}_i - \bar{z}_j|^2 \right], \]
(21)
where $v := [v_1 \cdots v_N]^T$ and $\bar{z} := [\bar{z}_1 \cdots \bar{z}_N]^T$—cf. (10). This function is positive definite and radially unbounded in the velocities $v_i$ and the consensus errors. The total derivative of $V_3$ along the closed-loop trajectories of (20) yields
\[ \dot{V}_3(v, \bar{z}) = -v^T D_v P^{-1} v, \]
(22)
where $P_v := \mathrm{diag}[p_{vi}]$ and $D_v := \mathrm{diag}[d_{vi}]$.

Now, the system in (20) being non-autonomous, Barbashin-Krasovskiǐ’s theorem does not apply, but we may use Barbilat’s Lemma [25] to conclude (after integrating on both sides of (22)) that $v_i \rightarrow 0$ and $\bar{v}_i \rightarrow 0$. In turn, from (20b), we see that
\[ \lim_{t \rightarrow \infty} \varphi(\theta_i(t)) \sum_{j \in \mathcal{N}_i} a_{ij} (\bar{z}_i(t) - \bar{z}_j(t)) = 0. \]
This expression, however, does not imply that the consensus objective is reached. Indeed, note that the set of equilibria of the system in (20) corresponds to points belonging to the set
\[ U := \left\{ v_i = 0 \land \varphi(\theta_i) \sum_{j \in \mathcal{N}_i} a_{ij} (\bar{z}_i - \bar{z}_j) = 0 \right\}, \]
which admits points such that $\bar{z}_i \neq \bar{z}_j \in \mathbb{R}^2$ because rank $\varphi(\theta) = 1$.

This means that if orientation consensus is reached and, for instance, $\theta_i(t) \rightarrow 0$ then $\bar{z}_i \rightarrow 0$, but $\bar{v}_i \not\rightarrow \bar{v}_c$—see Eq. (3).

Remark 5: This shows that the consensus problem for nonholonomic systems cannot be treated as that for ordinary second-order systems—cf. [27].

To ensure consensus it is necessary that the set of equilibria correspond to the set $U \cap U^\perp$, where
\[ U^\perp := \left\{ v_i = 0 \land \varphi(\theta_i) \sum_{j \in \mathcal{N}_i} a_{ij} (\bar{z}_i - \bar{z}_j) \neq 0 \right\}, \]
where $\varphi(\theta_i) = [-\sin(\theta_i) \cos(\theta_i)]^T$ is the annihilator of $\varphi(\theta_i)$ hence, $\varphi(\theta_i) \perp \varphi(\theta_i)^T \varphi(\theta_i) = 0$.

Roughly speaking, the controller must “pull” out non-equivalently-equal-to-zero trajectories that may remain “trapped” in $U$ and away from $U^\perp$. To that end, we endow the angular-motion controller with a term that incorporates an external function of time (smooth and bounded) and acts as a perturbation to the angular-motion closed-loop dynamics. This “perturbation” is designed to persist as long as
\[ \varphi(\theta_i) \sum_{j \in \mathcal{N}_i} a_{ij} (\bar{z}_i - \bar{z}_j) \neq 0. \]
More precisely, let $\psi_i$, $\bar{\psi}_i$, and $\bar{\psi}_i$ be bounded (belong to $L_\infty$) let $\psi_i$ be persistently exciting, i.e., let there exist $T$ and $\mu > 0$ such that
\[ \int_{t-T}^{t} \psi_i(s)^2 ds \geq \mu, \forall t \geq 0. \]
(23)
Then, for the control law in (18), we define
\[ \alpha_i(t, \theta_i, \bar{z}_i) := k_{\alpha i} \psi_i(t) \varphi(\theta_i)^\perp (\bar{z}_i - \bar{z}_j), \quad k_{\alpha i} > 0. \]
Thus, while $\alpha_k$ injects excitation into the system, which ensures that the position consensus errors converge, it acts as a bounded (hence harmless) perturbation on the angular-motion dynamics —cf. (18). Indeed, the state-feedback controller defined by (18), (19), and (24) ensures full consensus, in position and orientation, for the closed-loop system, even in the presence of delays; this is shown in [24].

IV. Control architecture: output-feedback case
As in the case where state feedback is available, the output-feedback control design relies on the dichotomy of the system’s dynamics (1b)–(2). Let us consider, first, the angular-motion dynamics, (1). Note that, expressed as $\dot{\theta}_i = u_{\theta_i}$, this system corresponds to an elementary Newtonian force-balance equation with unitary inertia. The problem at hand still is to synchronize the angular positions $\theta_i$ for $N$ such systems, but since $\omega_i$ is not available, we cannot use the control law in (18) —with $\alpha_k \equiv 0$—. Yet, it appears reasonable to conjecture that the objective $\theta_i \to \theta_j$ for all $i, j \leq N$ may be achieved by coupling the subsystems $\dot{\theta}_i = u_{\theta_i}$, via torsional springs, to virtual second-order oscillators for which the states are available and are synchronized by design —see Fig. 1 for an illustration.

![Fig. 1: Schematic representation of coupled mass-spring-damper systems: angular motion. It is the controller state variable, $\vartheta_{wi}$ that is transmitted to neighboring robots and, correspondingly, $\vartheta_{wji}$ is received from neighbors in the set $N_i$.](image)

More precisely, consider the dynamic system
\[
\dot{\vartheta}_{wi} + d_{wi} \ddot{\vartheta}_{wi} + p_{wi} \sum_{j \in N_i} a_{ij} (\vartheta_{wi} - \vartheta_{wj}) = \nu_{wi}
\]
(25)

where $\nu_{wi}$ is an external input to be defined, the state $\vartheta_{wi} \in \mathbb{R}$, and $d_{wi}, p_{wi} > 0$.

As we showed in Section III-A, for (25) consensus is achieved, that is, there exists a real constant $\omega_i$, such that $\vartheta_{wi} \to \vartheta_{wi}, \ddot{\vartheta}_{wi} \to 0$, for all $i \leq N$, provided that $d_{wi}, p_{wi} > 0$, and $\nu_{wi} = 0$. On the other hand, the system in (25) defines a passive map $\nu_{wi} \mapsto \ddot{\vartheta}_{wi}$. Furthermore, the system (1b) also defines a passive map, $u_{\theta_i} \mapsto \vartheta_{wi}$. Hence, it appears natural to hinge the systems (25) and (1) by setting
\[
\nu_{wi} := -u_{\theta_i}, \quad u_{\theta_i} := -k_{\omega_i}(\theta_i - \vartheta_{wi}), \quad k_{\omega_i} > 0.
\]
(26)

That is, the coupling $-k_{\omega_i}(\theta_i - \vartheta_{wi})$ may be interpreted as the force exerted by a torsional spring that hinges the (angular) positions of the two subsystems —again, see Fig. 1. Therefore, consensus among the angular positions $\theta_i$ is achieved indirectly by imposing consensus on the dynamic controllers’ variables $\vartheta_{wi}$. Consensus among the plants’ variables $\theta_i$ is achieved in view of the virtual mechanical coupling in (26). As a matter of fact, the control law in (26) is inspired by how joint flexibility in robot manipulators is modeled —cf. [28], [29] and the fact that consensus in the link positions may be achieved by applying a consensus control law on the actuator dynamics —cf. [5]. Then, we have the following original statement on output-feedback consensus control of second-order systems $\theta_i = u_i$.

**Proposition 1 (Output feedback orientation consensus):** Consider a group of differential-drive robots, each with dynamic model (1), in closed loop with the dynamic controller defined by (25), (26) and under Assumptions 1 and 2. Then, for any initial conditions $(\theta_{i0}, \omega_{i0}, \vartheta_{wi0}, \dot{\vartheta}_{wi0}) \in \mathbb{R}^4$ there exist constants $\theta_c$ and $\vartheta_c \in \mathbb{R}$ such that, for all $i$ and $j \leq N$,
\[
\lim_{t \to \infty} \theta_i(t) = \lim_{t \to \infty} \theta_j(t) = \theta_c, \quad \lim_{t \to \infty} \omega_i(t) = 0,
\]
\[
\lim_{t \to \infty} \vartheta_{wi}(t) = \lim_{t \to \infty} \vartheta_{wj}(t) = \vartheta_c, \quad \lim_{t \to \infty} \dot{\vartheta}_{wi}(t) = 0.
\]

**Proof:** Consider the function
\[
W_3(\dot{\vartheta}_i, \vartheta_i, \omega_i, \theta_i) := W_1(\vartheta_i, \dot{\vartheta}_i) + W_2(\theta_i, \dot{\omega}_i, \omega_i),
\]
where $\vartheta_i := [\vartheta_{w1} \cdots \vartheta_{wN}]^T$.
\[
W_1(\vartheta_i, \dot{\vartheta}_i) := \frac{1}{2} \sum_{i \leq N} \left[ \frac{\dot{\vartheta}_i^2}{p_{wi}} + \frac{1}{2} \sum_{j \in N_i} a_{ij} (\vartheta_{wi} - \vartheta_{wj})^2 \right];
\]
(27)
\[
W_2(\theta_i, \dot{\omega}_i, \omega_i) := \frac{1}{2} \sum_{i \leq N} \left[ \frac{\omega_i^2}{p_{wi}} + k_{\omega_i}(\theta_i - \vartheta_{wi})^2 \right].
\]
(28)

The function $W_2$ corresponds to the total energy of the mass-spring (closed-loop) system $\dot{\theta}_i = -k_{\omega_i}(\theta_i - \vartheta_{wi})$; the first term is the kinetic energy and the second the potential energy “stored” in the torsional spring of stiffness $k_{\omega_i}$. Akin to $W_3$ in (21) and $V_1$ in (12), the function $W_3$ is positive definite and radially unbounded in the consensus errors and the velocities. The total derivative of $W_3$ along the closed-loop trajectories yields
\[
\dot{W}_3(\dot{\vartheta}_i, \vartheta_i, \omega_i, \theta_i) = -\sum_{i \leq N} \frac{d_{wi}}{p_{wi}} \ddot{\vartheta}_{wi}^2.
\]
(29)

Then, the system being autonomous, we may invoke Barbashin-Krasovskii’s theorem. First, we see that $\dot{W}_3 = 0$ if and only if $\ddot{\vartheta}_{wi} = 0$. This implies that $\vartheta_{wi} = 0$ and $\dot{\vartheta}_{wi} = \text{const}$ for all $i \leq N$. From (25) and $\nu_{wi} := k_{\omega_i}(\theta_i - \vartheta_{wi})$ we conclude that $\dot{\theta}_i = \text{const}$, i.e., $\omega_i = \dot{\omega}_i = 0$. In turn, from $\dot{\vartheta}_{wi} = -k_{\omega_i}(\theta_i - \vartheta_{wi}) = -\nu_{wi} = 0$ and (25) we obtain
\[
\sum_{j \in N_i} a_{ij} (\vartheta_{wi} - \vartheta_{wj}) = 0 \quad \text{and} \quad \theta_i = \vartheta_{wi} \forall i, j \leq N.
\]

After Assumption 2, it follows that the only solution to these equations is $\theta_i = \vartheta_{wi} = \theta_c$ for all $i, j \leq N$.

![Fig. 2: Schematic representation of coupled mass-spring-damper systems: linear motion. The controller state variable $\vartheta_{vi}$ is sent to neighboring robots and, correspondingly, $\vartheta_{vji}$ is received from neighbors in $N_i$.](image)
Then, the dynamical system (30) is coupled to the double (nonholonomic) integrator (2). In contrast to the case of the angular motion, however, for the linear motion the control input \( u_{vi} \) must incorporate the change of coordinates defined by \( \varphi \). Therefore, we define

\[
u_{vi} := -\varphi(t_i)^T k_{vi} (\tilde{z}_i - \vartheta_{vi}) \quad k_{vi} > 0 \tag{31}\]

—cf. Eq. (26).

Thus, the controller (25)–(26) achieves consensus for the angular-motion dynamics (1) via output feedback while the controller (30)–(31) steers the linear-motion dynamics (2) in the presence of measurement delays, is given by output feedback while the controller (30)–(31) steers the linear-motion dynamics (2), in the presence of measurement delays, is given by

\[
u_{vi} := \sum_{j \in N_i} a_{ij} (\vartheta_{vi} - \vartheta_{vj} (t - T_{ji}(t))) \tag{33}\]

On the other hand, for the angular motion dynamics, we introduce

\[
u_{\omega i} := -k_{\omega i} (\vartheta_i - \vartheta_{\omega i}) + \alpha_i (t, \vartheta_i, e_{vi}), \tag{34a}\]

\[
\dot{\vartheta}_{\omega i} = -d_{\omega i} \vartheta_{\omega i} - k_{\omega i} (\vartheta_{\omega i} - \vartheta_i) - p_{\omega i} e_{v_i}, \tag{34b}\]

where

\[e_{vi} := \sum_{j \in N_i} a_{ij} (\vartheta_{vi} - \vartheta_{vj} (t - T_{ji}(t))) \tag{35}\]

All constant parameters are defined as above. In addition, in order to be used with an output-feedback controller, the function \( \alpha_i \) is redefined —cf. Eq. (24)— as

\[
\alpha_i (t, \vartheta_i, e_{vi}) := k_{\alpha i} \psi_i (t) \varphi(t_i)^T (\vartheta_{vi} - \tilde{z}_i), \tag{36}\]

where \( k_{\alpha i} > 0, \psi_i \) is twice differentiable, bounded, with bounded second derivatives and \( \varphi(t_i) \) is persistently exciting —cf. Section III-B. That is, \( \alpha_i \) in (34a) fulfills the same role as explained above. Then, our main statement is the following.

**Proposition 2** (Main result): Consider the system (1)–(3), under Assumptions 1–3, in closed-loop with (32)–(36). Then, for any initial conditions \( (\theta_{\alpha i}, \omega_{\alpha i}, \dot{\theta}_{\alpha i}, \dot{\omega}_{\alpha i}, \dot{\vartheta}_{\alpha i}, \vartheta_{\alpha i}, \vartheta_{\omega i}, \vartheta_{\omega i}) \in \mathbb{R}^{11} \), there exist constants \( \beta_i \in \mathbb{R} \) and \( z_i \in \mathbb{R}^2 \) such that (4) and (5) hold, if

\[
d_{\alpha i} > \frac{1}{2} p_{\alpha i} \sum_{j \in N_i} a_{ij} \beta_i + \frac{\sigma_{\alpha i}^2}{\beta_i}, \tag{37}\]

\[
d_{\omega i} > \frac{1}{2} p_{\omega i} \sum_{j \in N_i} a_{ij} \left[ \varepsilon_i + \frac{\sigma_{\omega i}^2}{\varepsilon_i} \right] \tag{38}\]

for some \( \beta_i, \varepsilon_i > 0 \), for all \( i \leq N \).

The conditions (37) and (38) impose bounds on the controller’s damping gains, depending on the bounds on the delays (and their derivatives). Note that these conditions are completely distributed; a different bound is required for each vehicle individually.

The arguments behind the statement of Proposition 2 rely on the observed dichotomy of the model (1)–(2) and the separate control designs for the linear- and angular-motion dynamics. Indeed, the resulting closed-loop equations have an underlying cascaded structure:

![Fig. 3: Schematic representation of the closed-loop system. Even though the systems are feedback interconnected, they may be regarded as in cascade [26], whence the feedback represented by a dashed arrow.](image)

VI. SIMULATION RESULTS

We used the simulator Gazebo-ROS and the Robot Operating System (ROS) interface to evaluate the performance of our controller in a scenario that reproduces as closely as possible that of a laboratory experimental benchmark. We employed the model of a PIONEER 3-DOF wheeled robot [31], available in Gazebo’s library. For simplicity, it is assumed that all the robots have the same inertial and geometrical parameters given by \( m = 5.64 \) kg, \( I = 3.115 \) kg m\(^2\), \( r = 0.09 \) m and \( R = 0.157 \) m. It must be underlined that for this robot the center of mass is not located on the axis joining the two wheels. Consequently, in this case, the Coriolis terms \( \bar{\mathbf{C}} \) and \( \bar{\mathbf{C}}_e \) appear on the left-hand side of Eqs. (1b) and (2b) respectively. Akin to an actual experimentation set-up, these constitute dynamic effects not considered in the model for which the controller is designed.

The six PIONEER 3D-X robots communicate over the undirected connected graph like the one illustrated in Fig. 4, below.

![Fig. 4: Communication topology: undirected connected graph](image)

Then, to emulate the time-varying delays \( T_{ji}(t) \), which are different for each pair of robots, we use randomly generated signals following a normal distribution with mean \( \mu = 0.3 \), variance \( \sigma^2 = 0.0003 \) and a sample time of 10 ms —see Fig. 5 for the illustration of one of such delays. Such time delay (non-smooth but piece-wise continuous) does not satisfy Assumption 3 since its time-derivative is bounded only almost everywhere (that is, except at the points of discontinuity). However, it is considered in the simulations since it is closer to what is encountered in a real-world set-up.
The initial postures of the robots are given in the 2nd-4th columns of Table I, below.

**TABLE I: Initial conditions and offsets**

<table>
<thead>
<tr>
<th>Index</th>
<th>$x_i$ [m]</th>
<th>$y_i$ [m]</th>
<th>$\delta_i$ [rad]</th>
<th>$\delta_{x_i}$ [m]</th>
<th>$\delta_{y_i}$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>7</td>
<td>1.57</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>13</td>
<td>0.0</td>
<td>1</td>
<td>2</td>
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<tr>
<td>3</td>
<td>2</td>
<td>9</td>
<td>-0.39</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>-2</td>
<td>6</td>
<td>0.39</td>
<td>-2</td>
<td>0</td>
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<tr>
<td>5</td>
<td>1</td>
<td>3</td>
<td>-0.39</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>4</td>
<td>-0.39</td>
<td>1</td>
<td>-2</td>
</tr>
</tbody>
</table>

The desired formation at rendezvous corresponds to a hexagon determined by desired offsets $\delta_i = (\delta_{x_i}, \delta_{y_i})$ with respect the unknown center of the formation. These constants are presented in the last two columns of Table I.

The control gains were set to $k_{x_i} = 1$, $k_{y_i} = 2$, $d_{x_i} = 3$, $p_{x_i} = 0.4$, $d_{y_i} = 2$, $p_{y_i} = 0.1$, for all $i \in [1, 6]$. These values correspond to magnitudes compatible with the emulated physics of the PIONEER 3D-X robots in Gazebo-ROS and are chosen so that the poles of the 2nd-order system $\dot{x} = -d_{x_i}x - p_{x_i}x$ have negative real parts and the system have an over-damped step-response. The functions $\alpha_i$ were taken as in (36) with $\alpha_{x_i} = 0.4$ and using the following multi-periodic function (any persistently-exciting function applies):

$$\psi_i(t) = 2.5 + \sin(2\pi t) + 0.3\cos(6\pi t) - 0.5\sin(8\pi t) - 0.1\cos(10\pi t) + \sin(\pi t) \quad \forall i \leq 6.$$  

(41)

The robots appear to achieve consensus, i.e., to meet at a non-predefined rendezvous point in hexagonal formation and with common non-predefined orientation —see Fig. 6. The center of the formation is located at $(-3.6, -4)$ and the consensus orientations settle at $\theta_c \approx -2.932$ rad. Note that the center of the formation and the common orientation do not correspond to the average of the vehicles’ initial conditions. Hence, the simulations illustrate that for networks of nonholonomic vehicles, the initial conditions do not determine the consensus point, as is the case of linear systems interconnected over static undirected connected graphs.

In addition, for the purpose of graphic illustration, following [3], we define the following synchronization errors, as the difference between each robot’s variables and the corresponding average:

$$e_{zi} := \bar{z}_i - \frac{1}{N} \sum_{j \in \mathcal{N}_i} z_j, \quad e_{\theta i} := \bar{\theta}_i - \frac{1}{N} \sum_{j \in \mathcal{N}_i} \theta_j,$$  

(42)

That is, the limits in (4) and (5) hold if the error trajectories $e_{zi}(t)$ and $e_{\theta i}(t)$ as defined above converge to zero, but the errors in (42) do not correspond to variables actually used by the controller nor measured for that matter. These errors are illustrated in Fig. 7 below.

From the top plot in Fig. 7 one can appreciate that the position synchronization errors $e_{zi}(t)$ tend to a steady state-error —a keen observer will notice that the hexagon in Fig. 6 is actually not quite so. The reason is that in the Gazebo-ROS simulation, after a transient, the amplitude of the input torques becomes considerably small in absolute value —see Fig. 8 below. Now, the presence of a steady-state error and the persistency-of-excitation effect in the controller maintain the input torques oscillating (periodically in this case due to the choice of $\psi_i(t)$ in (41)), but, physically, they result insufficient to overcome the robots’ inertia and friction forces that oppose their forward and angular motions. It seems fitting to say that in numerical simulations using Matlab, hence without considering the same physical phenomena, it may be appreciated that the synchronization errors tend to zero asymptotically and so do the control torques —see [30]. Also, we emphasize that in the Gazebo-ROS environment $\theta_i$ is defined in $(-\pi, \pi)$ to avoid unwinding, whence the apparent discontinuity appreciated in the bottom plot of Fig. 7.
The dynamic output-feedback controller for rendezvous of differential-drive robots that we propose has the neat physical interpretation of a second-order mechanical system itself and performs well even in the presence of discontinuous time-varying delays. Some readers may see a resemblance of our angular-motion controller with the input torques — Gazebo-ROS simulation.

Fig. 8: Input torques — Gazebo-ROS simulation.

VII. CONCLUSIONS

The dynamic output-feedback controller for rendezvous of differential-drive robots that we propose has the neat physical interpretation of a second-order mechanical system itself and performs well even in the presence of discontinuous time-varying delays. Some readers may see a resemblance of our angular-motion controller with the input torques — Gazebo-ROS simulation.

Fig. 8: Input torques — Gazebo-ROS simulation.
imply that \((L \otimes I_2) \varphi_i = 0\) which, in view of the properties of \(L\), implies the existence of \(\varrho_i \in \mathbb{R}^n\) such that \(\dot{\varrho}_i = 1_N \otimes \varrho_i\), or \(\varrho_i = \varrho_i \) for all \(i \leq N\). Hence, from the third equation in (40), we see that if \(\dot{\varrho}_i, \varrho_i\), and \(c_{\varrho_i} \to 0\), then \(\lim_{t \to \infty} \varrho_i(t) = \lim \varrho_i(t)\) and \(\lim \varrho_i(t) = \varrho_i \). The statement follows.

**Proof of Claim 3:** From the second equation in (40) it follows that, since \(\dot{\varrho}_i, \varrho_i\), and \(c_{\varrho_i} \to 0\),

\[
\varphi(\dot{\varrho}_i) (\varrho_i - \varrho_i) = 0, \quad \omega_i \varphi(\dot{\varrho}_i) (\varrho_i - \varrho_i) = 0.
\]

On the other hand, the solutions of the equation

\[
\varphi(\dot{\varrho}_i) (\varrho_i - \varrho_i) = 0
\]

are of the form \((\varrho_i - \varrho_i) = c_1 \varphi(\dot{\varrho}_i) \) with \(c_1 \in \mathbb{R}\) while the solutions of the equation

\[
\omega_i \varphi(\dot{\varrho}_i) (\varrho_i - \varrho_i) = 0
\]

are of the form \((\varrho_i - \varrho_i) = c_2 \varphi(\dot{\varrho}_i) \) with \(c_2 \in \mathbb{R}\). Therefore, (48) and (49) hold together if and only if \(c_2 = 0\), with \(c := c_1/c_2\). In turn, the latter may hold only if either \(c = 0\) or \(\omega_i = 0\). Now, if \(c = 0\) then \(\varrho_i \to 0\). Thus, (47) imply that either \(\varrho_i \to 0\), which is to be showed, or \(\omega_i \to 0\). In the latter case \(\lim_{t \to \infty} \omega_i(t) = \omega_i \) and since \(\dot{\varrho}_i \to 0\) we obtain, from Barbalat’s Lemma, that \(\varrho_i \to 0\). From a similar argument we conclude that \(\varrho_i \to 0\). Next, we show that \(\varrho_i, \dot{\varrho}_i, \dot{\varrho}_i \to 0\) and \(\psi_i \to 0\) — see (36) — to imply that \(\varrho_i \to 0\), so the proof ends.

To that aim, we recall that the total derivative of \(\mathcal{V}\) in (45a) along the trajectories of \(\Sigma_{\mathcal{V}}\) satisfies \(\dot{\mathcal{V}} \leq - \sum_{i,N} \lambda_{\mathcal{V}} \varrho_i \varrho_i - \frac{1}{P_c\alpha_i} \omega_i\)

The following are the references: