Multiplication as a matter of Grundvorstellungen, strategies and representations

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This paper is about the initial part of our research concerning the question of how to introduce multiplication in mathematics classes and how to support children's understanding of it. For that purpose, we developed a theoretical approach first which makes a distinction between basic ideas (*Grundvorstellungen*), strategies and representations of multiplication. On this basis, we are now able to investigate more closely how different aspects of the multiplication teaching fit together and which difficulties might arise in their interplay. In this paper, an example from a German mathematics textbook illustrates our theoretical reflections and distinctions.

Keywords: Multiplication, Grundvorstellungen, strategies, representations, textbooks.

Introduction

It is multiplication that, for primary school children, opens the door to larger whole numbers. In many countries, the action of repeated addition is commonly used to introduce multiplication. This initial approach often shapes the idea of multiplication and, thus, tends to become the dominant frame for interpreting multiplicative situations – for students as well as for primary teachers (Askew, 2018). However, relying exclusively on repeated addition proves to be critical for sustainable understanding of multiplication (Bakos & Sinclair, 2019; Askew, 2018). In relation to these findings, we plan a larger empirical study to have a closer look on the question of how multiplication is actually introduced in everyday classroom communication. But for now – and in this paper –, we focus on clarifying our theoretical perspective and, thus, differentiate between basic ideas in the sense of *Grundvorstellungen* (GVs), strategies and representations of multiplication. This way, we have become able to investigate more closely how different aspects of teaching multiplication fit together and which difficulties might potentially arise in their interplay.

Thus, in this paper, we present part of our theoretical framework and approach the following questions: 1) What are basic ideas of the mathematical concept of multiplication? 2) What are appropriate strategies for obtaining correct solutions to multiplication problems? 3) Which representations can help to teach and learn about those basic ideas and strategies?

We will present the example of a German textbook in order to illustrate the use of our theoretical reflections: What suggestions of handling the complex interaction of basic ideas, strategies and representations can be found in the textbooks?

On Grundvorstellungen

In order to be able to support children's understanding of multiplication, we have to ask what multiplication is all about: What does it actually mean to multiply? For our study, we follow vom

Hofe & Blum (2016, p. 226) and use the German term *Grundvorstellung* (GV; pural 'Grundvorstellungen'). Grundvorstellungen (GVs) characterize mathematical concepts and their applications to real-life situations. On a primarily normative level, they are descriptions of the relationship "between mathematical structures, individual-psychological processes, and subject-related contexts, or, in short: the relationships between mathematics, the individual, and reality." (ibid, p. 213). With regard to elementary school, the real contexts are, above all, everyday contexts of action. For the case of multiplication, then, the question is which everyday actions have an inherent multiplicative structure.

According to the GVs concept, three aspects can be distinguished: The *first* aspect is the "*constitution of meaning*" of a mathematical concept (ibid, p. 230). In our context, this means that children get to understand multiplication by linking the mathematical procedure to real-life contexts, situations, and actions. The *second* aspect is the "*generation of a corresponding mental representation*" (ibid, p. 230). In our context, this means that children approach multiplication by constructing mental representations which include exactly those aspects of the real-life context that are relevant from a mathematical perspective. The *third* aspect is the "*ability to apply*" the mathematical concept to real-life situations by recognizing a corresponding structure (ibid, p. 230). In our context, this means that children become able to apply the mathematical concept of multiplication to real-life contexts by recognizing multiplicative structures in the complexity of real-life situations.

We chose the GV concept for our research on multiplication for at least two reasons: *First*, the concept highlights that teachers (and mathematics educators) have to make decisions on a normative level: From an expert's point of view, which everyday actions have an inherent multiplicative structure and can be, for this reason, a suitable starting point for individual constitution of meaning? Thus, the GV concept focuses very clearly on the connection to real-life situations. *Second*, the GV concept allows us to differentiate between this normative level on the one hand and an empirical level on the other hand. Vom Hofe & Blum (ibid, p. 232) speak about a normative and a descriptive way of using the GV concept. When we use the concept in a descriptive way, we try to get as much information as possible about the mental representations that individual students *actuall*y have developed. Those mental representations might correspond to the intended ones more or less, but they are crucial when it comes to the actual processes of teaching and learning.

In the following, we will first use the GV concept in a normative way. Thus, we present two essential GVs of multiplication that can be found in the literature. Both of them ground on the activity of building units or – a bit shorter – of *unitizing* (Götze & Baiker, 2020; Lamon, 1994). This initial selection might need to be complemented and adapted in our ongoing research work.

Repeated addition

In Germany as in many other countries, repeated addition is the most common approach to introduce the basic idea of multiplication. Tasks then focus on *sequential situations* in which someone performs a specific *action* several times. An example from the German textbook "Zahlenbuch 2": A boy carries 3 books from a box behind him to a table in front and he does it exactly 7 times (figure 1, left). The boy says: "Always 3 books on a pile.". Two aspects are characteristic of the GV repeated addition: Someone bundles units and performs, one after another, a specific action on each of those units.

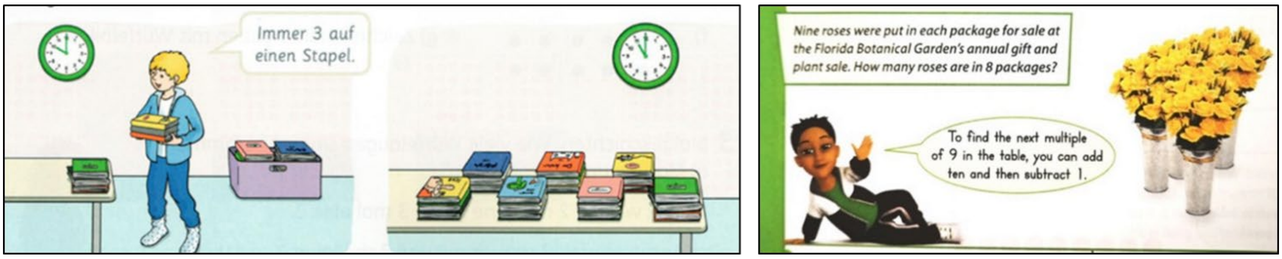

Figure 1: Dynamic and static situations

The story about carrying books from a box to a table can be told mathematically in two different ways. It can be described as a process of repeated addition (3+3+3+3+3+3+3) or, in order to shorten the story, as a process of multiplication ($7 \cdot 3$). The different meanings of multiplicand and multiplier fit perfectly well to the real-life situation (3 books on each pile, carrying 7 piles one after another) (Götze & Baiker, 2020). Thus, multiplication is introduced as a 'shortcut' of a repeated addition. However, is multiplication really nothing more than a certain form of adding?

In other tasks, the chronological sequence is not highlighted that much. For example, nine roses were put in each package for sale at the "Florida Botanical Garden's annual gift and plant sale" (figure 1, right, from the US-American textbook "enVision Mathematics Grade 3"). This is a rather static situation; all activities are already completed. The only question that brings us to repeated addition is the question of the total number: "How many roses are in 8 packages?" In order to find an answer to that question, the boy suggests to add (and subtract) repeatedly: "To find the next multiple of 9 in the table, you can add ten and subtract 1." Thus, repeated addition is mainly presented as a *strategy* to obtain a correct solution.

In fact, there are mathematics educators that come up with doubts about the GV of repeated addition and its potential for understanding multiplication. For example, Nunes & Bryant (2009, p. 9) summarize: "Finally, it is assumed that, in spite of the procedural links between addition and multiplication, these two forms of reasoning are distinct enough to be considered as separate conceptual domains." Similarly, Bakos and Sinclair (2019) state, together with Akew (2018), that the exclusive reference to repeated addition implies "limiting access to opportunities through which functionally thinking can emerge" (Akew 2018, p.1). Since several years, this position is supported from different sides. First, there are empirical studies that report a correlation between the use of addition strategies and the underachievement on multiplication problems (Baroody, 1999; Park & Nunes, 2001). Second, studies focus on successful forms of teaching multiplication which do not introduce multiplication as a 'shortcut' of addition, but as a mathematical operation in its own right (Park & Nunes, 2001). Third, some researchers stress that the GV of repeated addition does not allow to recognize the *functional* relation between multiplicand and multiplier (Askew, 2018). This last reference leads us directly to the second GV.

One-to-many correspondence

The concept of one-to-many correspondence refers to the activity of comparing quantities in a certain way (Vergnaud, 1983). A first way of comparing quantities is to compare them additively. For example, Sara has 7 playing cards more than Jonathan has. There are two sets of cards and we can determine the difference between these two sets by adding: If Jonathan collects 7 additional playing cards, he will have as many cards as Sara. Thus, additive reasoning stems from the (mental) action of joining and placing sets in one-to-one correspondence. A second way of comparing quantities is to compare them multiplicatively (Sinclair & Bakos, 2019). For example, “Amy’s Mum is making 2 pots of tomato soup. She wants to put 3 tomatoes in each pot of soup. How many tomatoes does she need?” In this case, there are more than two quantities and these quantities are not compared additively. Instead, the action is rather putting two variables in one-to-many correspondence (Nunes & Bryant, 2009, p. 11). Two aspects are characteristic of the GV of one-to-many correspondence: Someone performs an action that keeps the ratio between two variables (tomatoes, pots) constant and that leads to bundled units – at least in the end.

It is noteworthy that the concrete action can actually be performed in one way or another. For example, Amy’s Mum can put one tomato in each pot until there are 3 tomatoes in both pots *or* she can always put 3 tomatoes at once in a pot. Both actions lead to the same result, to a constant ratio between tomatoes and pots. Thus, in the GV of one-to-many correspondence, the focus is put on relations between quantities. In other words, the basic idea of one-to-many correspondences puts particular emphasis on the relation between the multiplicand and the multiplier. Accordingly, this GV stresses the asymmetry of multiplication as well as repeated addition does.

The example of cooking tomato soup (with surprisingly few tomatoes) is taken from a study conducted by Park & Nunes (2001, p. 768). In this intervention study, the researchers compare two treatment conditions: teaching of multiplication through repeated addition and teaching through one-to-many correspondence. Both groups made significant progress from pre- to posttest. But, at posttest, the group taught by one-to-many correspondence performed significantly better than the repeated addition group in multiplicative problems even after controlling for level of performance at pretest (Park & Nunes, 2001, p. 770). On this basis, the researchers come to the conclusion that teaching of multiplication should not be grounded in repeated addition, but in one-to-many correspondence (ibid, p. 772). Further research results strengthen this position, namely those about children’s informal knowledge about multiplication. Several studies report that many children already start school with a remarkable understanding of one-to-many correspondence and that this informal knowledge seems to be quite resistant (Nunes & Bryant, 2009, p. 12, 21). Moreover, many children, who have not been taught about multiplication yet, quite successfully use correspondence strategies in order to solve multiplicative reasoning problems (Kouba, 1989; Carpenter et al., 1993).

On strategies

Independent of the focussed GV, it is another important aspect of teaching multiplication to provide strategies for children which allow them to obtain correct solutions to multiplication problems in a flexible and efficient way (Nunes & Bryant, 2009). However, strategies taught in mathematics classes seem to be different in different countries. To start with, we draw on the German perspective and refer to core strategies (Götze & Baiker, 2020).

Knowing by heart

From our perspective, knowing by heart is not actually a *calculating* strategy. Thus, you will not have to calculate anymore if you know the solution to a multiplication problem by heart. Still, it can serve as a very helpful “tool for solution” – for example as part of addition strategies as we will see in the next paragraph (Rathgeb-Schnierer & Green, 2013, p. 354). Knowing by heart is not necessarily linked to (any) Grundvorstellung of multiplication.

Repeated addition

Repeated addition was introduced as a GV above. Although some researchers argue that it is not really a separate idea of multiplication, it is understood as an appropriate strategy to solve multiplication problems anyway (Götze & Baiker, 2020).

One possibility to realize repeated addition is to add every single unit: $3+3+3+3+3+3+3 = 21$. Alternatively, you can start from a result that you know by heart and add or subtract the 'missing' units: $7 \cdot 3 = 15+3+3 = 21$. However, the concrete calculation process may look like, the "procedural links between addition and multiplication" become obvious (Nunes & Bryant, 2009, p. 9).

Changing order

Changing the order of the factors is a helpful strategy when solving multiplication tasks. This strategy is based on the mathematical structure of commutativity. Nevertheless, it is quite difficult to link this strategy to real-life contexts and, thereby, to the intended GVs. It is much easier, but takes longer to carry 7 times 3 books from a box to a table than it would be to carry 3 times 7 books. If Amy's Mum put 2 instead of 3 tomatoes in each pot and took 3 instead of 2 pots, the soup might still taste the same, but there would be more of dishwashing to do.

On representations

As we can see so far, the interplay of GVs and strategies might be rather difficult in detail. In this regard, it is particularly relevant that all of them require representations in order to be accessible to children in mathematics classes (Kuhnke, 2013). For this very reason, representations are the third part of our theoretical perspective.

Real-life or didactical

Kuhnke (2013, p. 42) differentiates between real-life and didactical representations. Real-life representations take up real-life contexts that children probably already know, whereas didactical representations are specially made for teaching purposes and, therefore, are strongly adapted to the intended mathematical structure. With a view to GVs, this distinction is important because real-life representations are much more helpful for linking the mathematical concept of multiplication to typical application situations that children might know from their everyday lives outside school.

Real-life: Picture sequences

Picture sequences are real-life representations. They are well-known and widespread representations of multiplication and usually consist of two or more pictures telling a story of repeated actions. Thus, this representation is closely connected to the GV and the strategy of repeated addition. As we see in the story of the boy and the books above, the story-line itself is reduced to a minimum. Thus, links to every experiences are supposed to be realized and, at the same time, processes of abstracting and seeing the mathematical aspect within that story are meant to be enabled.

Real-life or didactical: Unstructured and structured quantities

Pictures of structured or unstructured quantities might be either realized in real-life or in didactical representations. This way of representing multiplication is based on the discrimination of multiplicand and multiplier. In the case of unstructured items, the action of unitizing is highlighted. The one-to many correspondences may be visualized as well as the concept of repeated addition. Strategies supported by this way of representing are unitizing and repeated addition.

Mainly didactical: Rectangular arrangements

In Germany, rectangular arrangements as a particular form of structured quantities are quite common. They might be either real-life or didactical representations and support a close link between geometry and arithmetic. Rectangular arrangements especially enable the visualization of the commutative structure of multiplication. Such structured arrangements can represent both GVs: The focus can be on repeated addition or on one-to-many correspondences. Strategies supported by this way of representing are unitizing and counting units, repeated addition and changing order.

5. First insights: Textbooks

How is multiplication introduced in textbooks? Do representations align with certain GVs? What strategies are introduced and supported? In the following, we present an example from our textbook analyses in order to illustrate the use of our theoretical distinction between GVs, strategies, and representations as an analytic framework. Thus, we ask for 1) representations in order of their appearance and analyze on this basis 2) which GVs are addressed and 3) which strategies are supported.

In the German textbook „eins zwei drei Mathematik 2“ (one two three mathematics 2), the introduction of multiplication is to be found on pages 72-73.

Context: A common classroom.

Representations: Rectangular arrangements embedded in the classroom situation, didactical rectangular arrangement, pre-structured representations of quantities

Tasks: Talk about the picture, find multiplicative structures in your own classroom, talk about quantities and about amounts of units, write multiplication tasks according to the representations given, draw representations

Addressed GVs: repeated addition

Potentially supported strategies: knowing by heart, changing order, repeated addition

In this introduction, the focus is exclusively on the GV of repeated addition, although the given everyday situation of a classroom would support a much wider spectrum. Such pictures basically offer the opportunity to include unstructured quantities which require the process of unitizing and support the concept of one-to-many correspondences. Besides, the tasks reduce the potentially wide range of supported concepts. Accordingly, this textbook conceptualizes multiplication solely in the context of repeated addition. In particular, multiplication is reduced to a certain way of writing and speaking. It is understood as a ‘shortcut’ for addition.

The next pages 74-75 focus on rectangular arrangements.

Context: Children working in a (math) class room
Representations: rectangular arrangements (one embedded in a story line showing the sequence of progression, one row after the other is uncovered)
Tasks: write the addition and the multiplication problem, show the multiplication problem on the hundred board, draw the multiplication problem and write the matching addition problem.
Addressed GVs: repeated addition
Potentially supported strategies: knowing by heart, repeated addition

On these pages, the focus is put on rectangular arrangements. These didactical representations offer the opportunity to refer to the commutative structure of multiplication and to introduce the changing of order as an appropriate strategy to solve multiplication tasks. Interesting enough, this potential is not used. Instead, children should 'translate' between one type of representation (pictures of rectangular arrangements) and another (symbolic representations, multiplication and addition problems). Accordingly, there is a focus on the structure accomplished by a differentiation in rows and columns. Again, repeated addition is the only addressed GV and the only addressed strategy as well. The misleading idea of multiplication being a different way of writing additions is strengthened.

6. Discussion

There are at least two GVs of multiplication: repeated addition and one-to-many correspondences. Both rely on the basic mathematical activity of unitizing (Lamon, 1994). However, repeated addition is the dominant approach in many countries including Germany, Italy, Taiwan, the US and Canada. First (German) textbook analyses confirm that this way of introducing multiplication provides a good basis to repeated addition as a *strategy*. Nevertheless, we found *representations* that might be used for addressing a much wider range of multiplicative situations. We can think of comparing quantities multiplicatively, of stressing the functional relation between multiplicand and multiplier and, in this way, of focusing on one-to-many correspondences. But, as first analyses indicate, these potentials concerning the *Grundvorstellungen* don't seem to be exhausted in the textbooks. Instead, we found the introductions of multiplication being mainly restricted to the GV and the strategy of repeated addition. This is a surprising result, especially as many researchers agree on the GV of one-to-many correspondence as very promising for supporting children's understanding of multiplication and their performance on multiplication problems.

On the basis of these first results, we regard the theoretical discrimination of GVs, strategies and representations as helpful for our work on the question of how to introduce multiplication in a meaningful and consistent way. Thus, we resume that these elements do not always complement each other in a useful way, but can actually be in conflict.

How do we plan to proceed in our larger project? On the one hand, we want to find out how teachers are supported in their teaching of multiplication in different national contexts in order to prove, deepen and complete our findings from the textbook analyses. On the other hand, we are in process to do research in mathematical classrooms to get insight into the ways introduction of multiplication is empirically realized in everyday classrooms. We have identified remarkable challenges from a theoretical perspective. Thus, the question arises how teachers actually face these challenges of introducing multiplication. At the moment, we observe (German) mathematics classes of grade 2 in order to reconstruct empirically how teachers actually work with the offers of their textbooks and meet the challenge of introducing multiplication in class discussions. This way, we hope to contribute to the scientific discussion about competing approaches to multiplication.

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