Using the bar model to ease the transition from transforming arithmetic-numerical to algebraic equations: theoretical considerations and possible obstacles

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We discuss an approach to transforming and solving algebraic equations via the so-called bar model, based on the strategy of transposing. After developing a learning environment, we conducted design experiments to get insights into how students work with it. First, this paper aims to present the core idea of our learning environment. Second, we highlight the following difficulties that students face when working with the bar model: (1) the model itself, with its translation processes between graphical and symbolic representations (such as numbers, variables or operation signs), turned out to be a considerable learning content, (2) the transition from arithmetic-numerical contexts to general algebraic equations in the bar model seems to bear distinct conceptual obstacles which may even lead to misconceptions based on over-generalizations resulting from the bar model. We point to theoretical insights and implications for enhancing our learning environment.

Keywords: Bar model, equations, algebra, design research.

Introduction

In the transition from arithmetic to algebra, students face a variety of difficulties (e.g., Warren, 2003). Especially for low-achieving students, it is important to fill the concepts and procedures introduced in arithmetic-numerical contexts with meaning for transferring them into the field of algebra. One example of such an idea that originates from numerical considerations is the concept of equivalence with its strong connections to the procedure of equivalence transformations regarding solving equations. This process, however, is known to bear several problems for learners. Kieran (2006) gives a short overview of students' errors, like ignoring the minus sign or reduction errors. To work against these procedural errors, a deep understanding of the idea of equivalence could be helpful. The idea of equivalence represents a central part of solving equations in school algebra and can be characterized by three different perspectives (Prediger & Roos, in press): (1) equations are equivalent if they have the same set of solutions, (2) equations are equivalent if there exists an equivalence transformation that transforms one into the other[1], and (3) equations are equivalent if they describe the same situation. While the first and second characterizations stay close to the formal mathematical definition, the third can be used for giving meaning to equivalence transformations in a rather intuitive and visual way. Thus, especially for low-achieving students, this could be a promising approach to build up conceptual understanding for the mathematical procedure of solving equations.

Our focus is on developing a learning environment, i.e. teaching material, to ease the transition from transforming arithmetic-numerical to algebraic equations. With the help of the so-called bar model,

[1] An equivalence transformation can be seen as an application of a bijective function on both sides of the equal sign.

we want to emphasize the meaning of equivalent equations in order to develop the strategy of transposing for solving equations. Firstly, we summarize the theoretical background for creating our learning environment. Secondly, we report results of interview studies used to evaluate the learning environment to derive theoretical and practical conclusions.

Theoretical Background

Regarding the process of solving algebraic equations, Selter et al. (2012) differentiate between two formal strategies: *performing the same operation on both sides* ($B + C = A \Leftrightarrow B + C - C = A - C$) and *transposing* (put an expression on the other side of the equal sign by applying the respective inverse operation: $B + C = A \Leftrightarrow B = A - C$; compare Figure 1) (see also Kieran, 1992). While solving equations by performing the same operation on both sides should depict one main objective (Malle, 1993), especially at the beginning of the learning process, the strategy of transposing is considered more intuitive for students (Mason et al., 2005). Selter et al. (2012) particularly emphasize the close relationship of the idea of transposing to former arithmetical experiences. Therefore, the strategy of transposing can be linked back on the one hand to the concept of equivalent equations as equations describing the same situation and, on the other hand, to experiences that were made in arithmetical contexts. We explain both shortly.

Equivalent equations as describing the same situation – a graphical model

According to Malle (1993), object relationships represented in drawings (such as relationships between line segments) can be used to give variables, expressions, and formulae a meaningful interpretation. Such graphical models can also be used later on for making sense of transposing when solving equations (ibid.; see Figure 1).

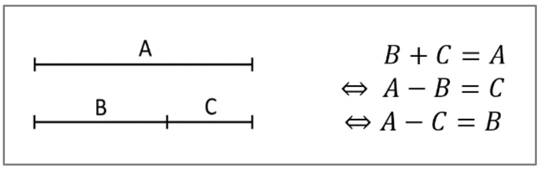


Figure 1: Bar model for representing transposing; based on Malle (1993, p. 220)

The Singapore bar model (e.g., Fong Ng & Lee, 2009) works similar to this approach. This model is used widely among others in Singapore's primary schools (e.g., Kaur, 2019) and discussed as a method for supporting students to get good results in problem solving activities for instance in international assessment studies like TIMSS (e.g., Beckmann, 2004).

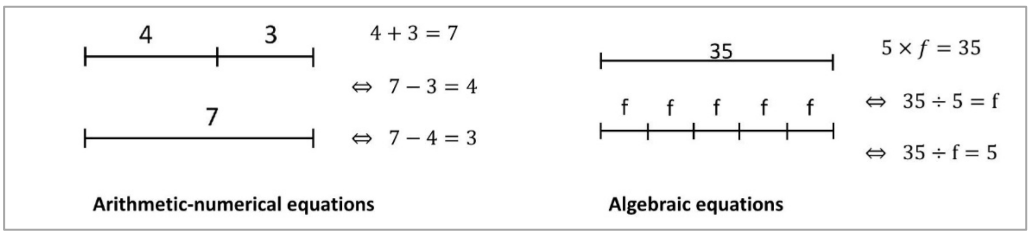


Figure 2: The bar model as a visual representation of equivalent equations

One bar model serves to describe three equations (see Figure 2) and that is why these equations are called equivalent. Following a semiotic point of view, the bar model can be considered a representational system. As with every representational system, working with the bar model presupposes specific knowledge being implicit (see Kempen and Biehler (2020) for a more detailed description) to work with. For example, two numbers are added by placing corresponding line segments next to each other, subtraction by crossing out or erasing. Multiplication is traced back to counting units in this model and is thus done by bundling line segments with equal length (see the five fs in Figure 2); division can be done by laying out a line segment with smaller segments of equal length. The equality of two values, expressions, etc., results from the phenomenon that two resulting line segments are equal. It becomes clear that one needs both an understanding of the mathematical operations and how they are represented or performed in the context of the bar model. To communicate such incidents, a distinct language is required that refers to the operations performed in the bar model and to the model itself. This language is closely related to the meaning-related language needed to describe basic operations (see Table 1).

Linking back the idea of equivalent equations to arithmetical contexts

For the strategy of *transposing* the relationship between the operations – the inverse operation of addition is subtraction etc. – is one key component that must be transmitted from arithmetic to algebra. To accomplish this transmission, it seems helpful to focus on the meanings of the operations involved as well as the related language that comes along with these meanings (see Table 1). The use of such meaning-related language seems also helpful when working with the bar model (see above).

Table 1: meanings and meaning-related language (see also Prediger & Roos, in press)

Meanings of the basic operations	Examples for meaning-related language
• Addition as putting together • Subtraction as taking away or determining the difference	• I have 2 and I **put it together** with 3, then I obtain **in total** 5. • I have 5, and I **take away** 3, so 2 remains. • I have 3, so I **need 2 more to reach** 5.
• Multiplication as counting in units • Division as sharing (partitive model) • Division as measuring (quotative model)	• I count in groups: 3 **groups / sets / units of** 2 are 6. • I **share** 6 **among** 3 people, so everybody gets 2. • 2 fits 3 times into the 6.

The learning environment

Based on the considerations above, we developed a learning environment to prepare the transition from arithmetic to algebra for transforming equations using the bar model (Prediger & Roos, in press). The steps in the learning environment are shown in Figure 3, although the paper focusses on step II and V.

Different representa-tions for equations **(I)**	The bar model in numerical contexts **(II)**	Equivalent equations in numerical contexts **(III)**	The bar model in algebraic contexts **(IV)**	Equivalent equations in algebraic contexts **(V)**

Figure 3: Steps in the learning environment towards transforming algebraic equations

In *step II (The bar model in numerical contexts)*, students get to know the bar model and start working with it. The key component here is the understanding of the bar model itself and how to perform and understand basic (arithmetic) operations in it (see Table 1, Figure 1). Learners need to connect the bar model as a graphical representation of equations with equations represented symbolically, and formulate verbally corresponding relationships. In *step V (Equivalent equations in algebraic contexts),* the objective is to detach the ideas of transforming equations from the use of the bar model so that students can transform equations also in rule-based procedures. Here, abstraction processes and the idea of reverse operation play a decisive role.

Research Question

Our focus is on developing a learning environment to ease the transition from transforming arithmetic-numerical to algebraic equations by using the bar model. The design research project (Gravemeijer & Cobb, 2006) builds and enhances the design based on empirical insights into students' learning processes. In this paper, we focus on the following research question: *Which obstacles become apparent when transforming and solving algebraic equations with the aid of the bar model?*

Methodology

Within the applied design research approach (Gravemeijer & Cobb, 2006), the learning environment created was sequenced into five steps (Figure 3). The target group of our learning environment consists of low-performing students who need a second chance to develop an understanding of the mathematical process of solving equations. The first two design experiment cycles addressed three low-achieving tenth graders in remediating mathematics classes, aiming to pass their Grade ten exam in a prevocational setting (Cycle 1) and three eighth graders of a German comprehensive school (Cycle 2). Data was collected while the students worked on equations in zoom-sessions in addition to their 'normal' math classes during the pandemic in January – May 2021. In total, 960 minutes of video data were collected and partially transcribed.

During the sessions, the design experiment leader watched the student work with the material. Whenever the student's work stayed unclear, she asked the students to explain their approach, thoughts, and solutions. Also, when students needed additional help or had questions, she explained the tasks in more detail.

In our analysis, we watched the videos and selected places where difficulties with the bar model or the transformation of equations occurred. For these places, we took a closer look into the corresponding transcripts. Two of the typical conceptual challenges appear when working in step II and step V of the learning environment. They will now be discussed based on the cases of Vivien and Anno. The problems discussed below can be considered prototypical for our sample in terms of their characteristics.

Tentative Results

Case of Vivien

The case of Vivien was already presented in Prediger & Roos (in press). She is an 18-year old girl in grade 10 participating in a remediating mathematics class. In this episode, the design experiment leader (DEL) talks with Vivien about the task in Figure 4 located in step II of the learning environment.

Three different running programs and their drawings

- John goes running every Saturday to train for a half marathon. He runs 21 km every time.
- Eva runs three times a week 7 km and
- Max runs every day 3 km from his place to his grandparents.

Assign Eva, Max and John's runs to the matching drawings.

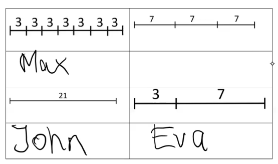
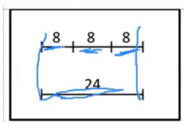


Figure 4: Vivien's solution to the running program task

1 Vivien: And Eva beneath, the one next to it [*refers to the lower right bar*]. Where there is written 3, then the middle line and then 7.
2 DEL: Here?
3 Vivien: Yes.
4 DEL: And can you explain how you came up with that or why?
5 Vivien: Because it says, “Eva runs three times a week 7 km”. And then I would say, the 3 stands for “three times a week” and the 7 for “7 km”.

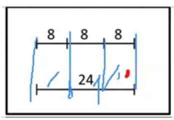
Vivien focused on the numbers while missing to connect the bar model with the intended mathematical operation. This problem is also mirrored in her explanation: Rather than grasping the additive structure of 3 and 7 in this part of the bar model (e.g., with meaning-related language of addition like “putting together”, see Table 1), she only articulates the numbers, not joint lengths: “Where there is written 3, then the middle line and then 7” (Line 1). Vivien shows difficulties with the distinction of additive and multiplicative structures in connection with the bar model. This is also reflected in her rather simple use of language: “And then” is the only connective between 3 and 7 that she uses, which does not allow her to distinguish an additive structure from a multiplicative structure. Moreover, Vivien fails in realizing the idea of multiplication displayed in the bar model on the upper right. Here, multiplication is shown as counting units. It becomes evident that a learner has to combine two facets of knowledge: First, the conceptual understanding of multiplication as counting units is needed for mathematizing the text on Eva’s run. Second, the representation of this multiplication (as several units consisting of 7 km each) in the context of the bar model has to be realized (three line segments of length seven are meant to represent "$3 \cdot 7$"). In this sense, the conceptual understanding of multiplication serves as a prerequisite for choosing the adequate bar model. However, based on this conceptual understanding, the corresponding representation in the bar model (the juxtaposition of three line segments [addition] of equal length [leading to multiplication]) has to be understood, too, as a matter of implicit knowledge.

Case of Anno: Anno is a 14-year old German 8th grader with average achievement in mathematics classes at the comprehensive school. In the beginning of his learning process, he displayed a good understanding of the bar model by explaining the meanings of the underlying operations. When talking about the bar model, he finds the correct corresponding symbolic equations and gives a correct explanation (see Figure 5).



$3 \times 8 = 24$

Anno: And these are 3 lines of 8 each, so to speak 3 pieces of wood that have been put together or have the same length as the 24 [*indicates his statements with blue*][...].



$24 : 8 = 3$

Anno: Now if I have a lot of such small things of 8 each here and would put them into the 24 [*draws in new markings*] […] one simply sees here if one would extend all the lines, it would result in 3 times, therefore 3 compartments.

Figure 5: Anno explaining corresponding equations in the bar model

Anno refers directly to the significance of the operations when he speaks of pieces that have been "put together" or of the "three compartments" that have been placed in the 24. Thus, although the bar model is new to him, he seems to have a good intuitive understanding of the operations and the corresponding connections between the bar model and the symbolic representation. In the following episode, the design experiment leader and Anno discuss the task shown in Figure 6 (step V, see Figure 3). This task was designed to initiate a detachment from the bar model with which the students had worked before. Detaching is necessary because the multiplication with the rate r can hardly be visualized as counting in units in the bar model (and scaling up and down is not known by Anno). Therefore, from the tasks, which were solved before with the bar model, the idea of the reverse operation is to be transferred.

You know an example of such a formula from percentage calculation:

$\underbrace{amount}_{a} = \underbrace{rate}_{r} \times \underbrace{base}_{b}$

i) What are the equivalent equations here?

The same is now possible with every formula. So I finally don't have to learn so many by heart…

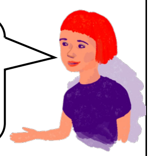


Figure 6: Anno's task for detaching from the bar model due to other meanings of multiplication

After Anno finds the correct equivalent equations ($a = r \times b; r = a \div b; b = a \div r$) the interviewer asks how he came up with his solutions:

Anno: Yes, because it was always in the beginning [*referring to former tasks in the learning environment*] larger [value] was always calculated by the two smaller ones. If you now assume that somehow these are the two smaller ones, like 3 times 2 or something, the larger value is calculated by these two [*marks the rate and the base*]. This [*referring to a*] you can then divide by the two, by the rate and the base, I suppose.

Also in the following, Anno continues explaining his strategy of transposing equations by using ideas of smaller or bigger numbers. For him, the number or variable that stands alone opposite the multiplication on one side of the equation has to be the largest. If one wants to obtain an equivalent equation, this only makes sense if one divides the larger number by a smaller number. He shows here – similar to Vivien – a non-sufficient focus on the underlying operations with respect to their structures. Although Anno described the meaning of multiplication and division as inverse operations in earlier tasks, he can only superficially exploit this idea when developing a strategy without the bar model. He uses a method that is based on the magnitude of numbers ("if one now assumes […] these are the two smaller […] the larger value is calculated by these two"). This strategy might be considered an over-generalization resulting from the bar model: The one number alone on one side of the equation is always bigger than the two numbers on the other side. In fact, multiplication in the bar model is based on the aspect that at least one of the factors is a natural number. Albeit Anno's strategy can be helpful when working with equations with natural numbers, the strategy fails when multiplying with factors smaller than 1.

Discussion and Conclusion

The insights we gained in our design experiments are the following:

Regarding the understanding of the bar model, some students have considerable difficulties connecting representations (iconic-symbolic-verbal) relying on conceptual understanding of basic operations. For such students, it is hard to use the model to make sense of equivalence transformations of equations. Even the step before, representing one side of the equal sign in the bar model already depicts challenges. These students focus on the given numbers in the model instead of on the underlying mathematical structures. This phenomenon has also been reported in the context of multiplication for grade 5 students (Prediger, 2019). However, our sample consists of students in grade 10. Other students who have acquired an adequate understanding of the meaning of equivalent equations within the bar model are not necessarily able to develop an appropriate strategy (like transposing) when prompted to detach it from the bar model. Especially regarding algebraic equations with a multiplicative structure, the concept of inverse operations should be used, not an idea concerning the magnitude of numbers that are not transmissible to decimal numbers below 1.

Based on these findings regarding students' obstacles when using the bar model for solving equations, we want to highlight the following theoretical implications. First, when working with the bar model, a profound understanding of basic mathematical operations has to be considered essential. This understanding is not only necessary for performing respective operations when solving equations in arithmetical or algebraic contexts; learners need to have an appropriate conceptual understanding to understand which operations are illustrated in the bar model or perform operations in the bar model themselves. Although Koleza (2015) found that third graders already understand multiplication after short instruction with the bar model, our preliminary results show that this does not have to be the case even for students in grade 8. Moreover, the bar model must first be seen as a learning object in its own right before it can aid learning. In this sense, the bar model is neither self-evident nor self-explanatory, as learners need specific knowledge to work with it. Besides, the student's attention must be directed from a focus on the numbers to a focus on the underlying operations (see also Prediger (2019)) and its representation in the bar model. In the case of Anno, the extensive work with the bar model led to an over-generalization of a respective strategy for working with equations. Furthermore, this overgeneralization might be considered a considerable misconception. Accordingly, the idea of inverse operation has to be highlighted in the bar model to focus rather on the operations than on the 'length' in the bar model. Following these theoretical insights, we draw the following related practical implications: (1) Especially for low achieving students, the repetition of the meanings of the basic operations seems necessary. Accordingly, we plan to extend our learning environment with a new part at the beginning to ensure respective prerequisites. (2) The conceptual meanings of the basic operations need to be more closely related to the bar model so that the bar model is brought in more explicitly as an independent object of learning (step II, Figure 3). (3) Tasks must be added that focus on the application of the idea of inverse operations; this idea should be followed throughout the whole learning environment. In addition, tasks have to be incorporated to help students detach from the bar model and thus work against overgeneralizations and misconceptions.

Based on the results of our analysis, we are planning the next cycle of our research project for the end of 2021.

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