Non-autonomous $L^q(L^p)$ maximal regularity for complex systems under mixed regularity in space and time

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NON-AUTONOMOUS $\mathrm{L}^q(\mathrm{L}^p)$ MAXIMAL REGULARITY FOR COMPLEX SYSTEMS UNDER MIXED REGULARITY IN SPACE AND TIME

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ABSTRACT. We show non-autonomous $\mathrm{L}^q(\mathrm{L}^p)$ maximal regularity for families of complex second-order systems in divergence form under a mixed Hölder regularity condition in space and time. To be more precise, we let $p, q \in (1, \infty)$ and we consider coefficient functions in $\mathrm{C}_t^{\beta+\varepsilon}$ with values in $\mathrm{C}_x^{\alpha+\varepsilon}$ subject to the parabolic relation $2\beta + \alpha = 1$. To this end, we provide a weak (p, q)-solution theory with uniform constants and establish a priori higher spatial regularity. Furthermore, we show p-bounds for semigroups and square roots generated by complex elliptic systems under a minimal regularity assumption for the coefficients.

1. INTRODUCTION

In this article, we investigate the non-autonomous parabolic problem

$$\text{(CP)} \qquad \partial_t u(t, x) - \operatorname{div}_x A(t, x)\nabla_x u(t, x) = f(t, x), \qquad u(0) = 0.$$

The precise setting will be discussed in Section 1.1 below. Our interest lies in the $\mathrm{L}_t^q(\mathrm{L}_x^p)$ maximal regularity property, where $p, q \in (1, \infty)$. Roughly spoken, this means that, for a right-hand side f in $\mathrm{L}_t^q(\mathrm{L}_x^p)$, there is a unique solution u of the equation[1], and that one has that $\partial_t u(t, x)$ and $\operatorname{div}_x A(t, x)\nabla_x u(t, x)$ lie again in the space $\mathrm{L}_t^q(\mathrm{L}_x^p)$, with corresponding estimate against the data f.

Maximal regularity is a classical problem in mathematical analysis that has been studied for decades. One reason for this massive interest is that maximal regularity estimates allow the treatment of highly non-linear problems using powerful linearization techniques [4,36]. Meanwhile, the autonomous case, that is to say, when the family of elliptic operators does not depend on t, is well understood. In the Hilbertian case, this is due to de Simon [39], whereas the case of UMD Banach spaces is characterized by the seminal work of Weis [42]. The non-autonomous case is much harder; thus this question is still widely open till this day.

Let us take a step back and consider a generalization of Problem (CP). To this end, note that (CP) remains meaningful if we replace, for fixed t, the elliptic operator in divergence form $-\operatorname{div}_x A(t, x)\nabla_x$ by a sesquilinear form $a_t \colon V \times V \to \mathbb{C}$, where $V \subseteq H$ is the form domain and does not depend on t. Lions showed in [32] that this generalized problem

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[1] We will explain our notion of a solution thoroughly in Definition 2.4.

has maximal regularity for right-hand sides f in the space $\mathrm{L}^2_t(V^*)$, where V^* is the anti-dual space of V. We emphasize that this result only assumes that the forms a_t depend measurably on t. Motivated by this, the question of maximal regularity with right-hand sides in the space $\mathrm{L}^2_t(H)$ became known as Lions' non-autonomous maximal regularity problem. We will only review some contributions to this problem in the sequel. For a thorough background, we refer the reader to the survey [5].

By a counterexample of Fackler [21], it is known that, even for a family of forms that depends $\mathrm{C}^{1/2}$ on t, non-autonomous maximal regularity can fail. On the other hand, Acquistapace and Terreni [2] introduced already in the '80s a condition to show non-autonomous maximal regularity in Hölder spaces. Owing to progress in the theory of pseudo-differential operators, this condition was later rediscovered by Hieber and Monniaux [27] to show $\mathrm{L}^2_t(H)$ maximal regularity in a fairly abstract setting. Using a Hörmander criterion, this approach furthermore gives $\mathrm{L}^q_t(H)$ maximal regularity. Still relying on the Acquistapace–Terreni condition, Portal and Željko [35] used techniques from vector-valued harmonic analysis to replace the Hilbert space H by a UMD Banach space X. The condition of Acquistapace and Terreni is hard to grasp, but was verified under more accessible regularity conditions in several works: Using a $\mathrm{C}^{1/2+\varepsilon}$ dependence on t, Ouhabaz and Spina [34] showed maximal regularity in $\mathrm{L}^q_t(H)$ for a general family of forms, where as Fackler [20] treated $\mathrm{L}^q_t(\mathrm{L}^p_x)$ maximal regularity in the case of complex elliptic operators in divergence form. Concerning $\mathrm{L}^q_t(H)$ maximal regularity for a family of forms, the $\mathrm{C}^{1/2+\varepsilon}$ condition was further relaxed by Haak and Ouhabaz [25] to a Dini-condition. Also, Fackler [22] was able to show the $\mathrm{L}^q_t(\mathrm{L}^p_x)$ maximal regularity for operators in divergence form under the weaker time regularity $\mathrm{W}^{1/2+\varepsilon,p}_x$, when $p \geq 2$. At this place, we would also like to mention a result by Achache and Ouhabaz [1], which imposes a $\mathrm{W}^{1/2,2}_x$ regularity condition in conjunction with an additional very weak Dini-condition.

Besides the approaches based on the work of Acquistapace and Terreni, another successful strategy to attack Lions' maximal regularity problem emerged. As we have mentioned earlier, Lions showed maximal regularity for right-hand sides in $\mathrm{L}^2_t(V^*)$, which does not require any regularity in t except measurability. Dier and Zacher improved this result, to allow, on the one hand, an even larger space of right-hand sides, and, on the other hand, they deduced furthermore that the fractional derivative of order $1/2$ has optimal regularity, that is to say, the unique solution u of (CP) moreover satisfies $\partial_t^{\frac{1}{2}} u \in \mathrm{L}^2_t(V^*)$. This, combined with a commutator argument, leads to the regularity condition $\mathrm{W}^{1/2+\varepsilon,2}_t$ by Dier and Zacher [14], and a $\mathrm{BMO}^{1/2}$ condition by Auscher and Egert [7]. We emphasize that these results highly rely on the Hilbertian structure. For instance, they use Fourier techniques and the Lax–Milgram lemma. Therefore, $\mathrm{L}^q_t(\mathrm{L}^p_x)$ maximal regularity cannot be expected from this approach.

In all what we have considered so far, we supposed that form domains are constant, but the associated operators in L^2_x or L^p_x might have varying domains. If we assume that also the domains of the operators are constant, there are results requiring only very weak regularity in time. For instance, using perturbation techniques, Prüss and Schnaubelt [37] showed non-autonomous $\mathrm{L}^q_t(X)$ maximal regularity for families with continuous dependence on time. Under further structural assumptions, Gallarati and Veraar [23, 24] even showed maximal regularity under measurable dependence on t. We observe that, in the case of elliptic operators in divergence form, independence of the domains on t can be enforced by spatial regularity of the coefficients, for instance using Lipschitz regularity.

If we write β for the regularity in t, and α for the regularity in x, then the presented regularity requirements all lead to the relation $2\beta + \alpha = 1$. Dier and Zacher employed this relation in [14]. To be more precise, they showed $\mathrm{L}^2_t(\mathrm{L}^2_x)$ maximal regularity for problem (CP) with the regularity condition $\mathrm{W}^{\beta+\varepsilon,1/\beta}_t(\mathrm{W}^{\alpha+\varepsilon,d/\alpha}_x)$ for the real and scalar-valued coefficient function $A(t,x)$. The Hilbertian setting is indispensable for their approach, as well as the whole-space constellation in x.

The goal of this article is to pick up the mixed regularity relation $2\beta + \alpha = 1$ used by Dier and Zacher, but to show $\mathrm{L}^q_t(\mathrm{L}^p_x)$ maximal regularity in Theorem 1.1. Furthermore, we also treat complex systems, which seems to be completely new in the literature. This is based on improvements in the elliptic theory, notably Theorems 2.9 and 2.11. Finally, we show existence and uniqueness of so-called *weak (p,q)-solutions* for (CP), with explicit dependence of the implicit constants on the coefficients, in Theorem 3.1. This is our substitute for the weak solutions provided by Lions.

1.1. **Precise setting and main result.** Fix a finite time $T > 0$ and a dimension $d \geq 2$. We are going to consider non-autonomous parabolic problems in $(0,T) \times \mathbb{R}^d$. In all what follows, the symbols t and x are supposed to be quantified over $(0,T)$ and $\mathbb{R}^d$, and t^* is one such fixed time. Start with a bounded coefficient function

$$A\colon (0,T) \times \mathbb{R}^d \to \mathbb{C}^{dm\times dm} \qquad \text{satisfying} \qquad |A(t,x)| \leq \Lambda, \tag{1}$$

describing a non-autonomous complex system of size $m \geq 1$. We define for each fixed $t^* \in (0,T)$ an elliptic operator in divergence form in the following way: Consider the bounded sesquilinear form[2]

$$a_{t^*}\colon \mathrm{W}^{1,2}_x \times \mathrm{W}^{1,2}_x \to \mathbb{C}, \qquad a_{t^*}(u,v) = \int_{\mathbb{R}^d} A(t^*,x)\nabla_x u(x) \cdot \overline{\nabla_x v(x)}\,\mathrm{d}x.$$

To ensure its (uniform) ellipticity, we suppose that there exists $\lambda > 0$ such that

$$\sum_{k,\ell=1}^{m} \mathrm{Re}(A(t,x)^{k\ell}\xi^k \,|\, \xi^\ell) \geq \lambda|\xi|^2 \qquad (\xi \in \mathbb{C}^{dm}). \tag{2}$$

Note that we have implicitly identified $\mathbb{C}^{dm} \cong (\mathbb{C}^d)^m$. Using the form a_{t^*}, we define the operator

$$\mathcal{L}_{t^*}\colon \mathrm{W}^{1,2}_x \to \mathrm{W}^{-1,2}_x \quad \text{via} \quad \langle \mathcal{L}_{t^*}u, v\rangle_{\mathrm{W}^{-1,2}_x, \mathrm{W}^{1,2}_x} = a_{t^*}(u,v) \qquad (u,v \in \mathrm{W}^{1,2}_x).$$

Here, $\mathrm{W}^{-1,2}_x$ is the space of conjugate-linear functionals on $\mathrm{W}^{1,2}_x$.

The aim of this paper is to study the regularity theory of solutions to the non-autonomous parabolic problem associated with $\{\mathcal{L}_t\}_{0<t<T}$ given by

$$\partial_t u(t) + \mathcal{L}_t u(t) = f(t), \qquad u(0) = 0. \tag{P}$$

The solution concept for (P) will be made precise in Definition 2.4. The main result of this article is the following.

[2]Here, ∇_x denotes the gradient in the variable x. For the sake of readability, let us agree to omit the underlying sets $(0,T)$ and $\mathbb{R}^d$ in the notation of function space; instead, we will indicate the underlying set by the indices t and x. For instance, we will simply write $\mathrm{W}^{1,2}_x$ instead of $\mathrm{W}^{1,2}_x(\mathbb{R}^d)$ and so on.

Theorem 1.1. *Let $\alpha, \beta, \varepsilon > 0$ such that $2\beta + \alpha = 1$, and assume that the coefficient function A is in the class $\mathrm{C}_t^{\beta+\varepsilon}(\mathrm{C}_x^{\alpha+\varepsilon})$. Then, given $f \in \mathrm{L}_t^q(\mathrm{L}_x^p)$, there is a unique weak (p, q)-solution u of (P) such that $\mathcal{L}_t u(t) \in \mathrm{L}_t^q(\mathrm{L}_x^p)$ in conjunction with the estimate $\|\mathcal{L}_t u(t)\|_{\mathrm{L}_t^q(\mathrm{L}_x^p)} \lesssim \|f\|_{\mathrm{L}_t^q(\mathrm{L}_x^p)}$, that is to say, problem (P) admits maximal regularity.*

Agreement 1. Throughout this article, we consider the numbers Λ and λ from (1) and (2), as well as the numbers α, β, and ε from Theorem 1.1, as fixed. Moreover, we reserve the symbol M for the $\mathrm{C}_t^{\beta+\varepsilon}(\mathrm{C}_x^{\alpha+\varepsilon})$-norm of A. We refer to the numbers d and m as dimensions, and they are also considered fixed, likewise the integrability parameters p and q.

1.2. **Roadmap.** Our proof follows the classical approach due to Acquistapace and Terreni, but incorporates an *a priori* improvement of weak solutions in the spatial variable. In this roadmap, we intend to give the reader an overview of our strategy.

The starting point is a weak solution theory for the generalized problem (P'). This generalization permits us to use an approximation argument later on. Classically, this is due to Lions in the Hilbertian situation. Fackler used the result of Prüss and Schnaubelt [37] to have a (p, q)-version of Lions' result at hand. We cannot do this, as [37] does not yield implied constants that are uniform in the coefficients. However, we will need such a control for the *a priori* improvement of weak solutions in the spatial variable. We will come back to this at the very end of this roadmap. Hence, instead, we employ a framework of Dong and Kim [18] to treat complex systems in divergence form over spaces of the type $\mathrm{L}_t^q(\mathrm{W}_x^{-1,p})$. This will be done in Section 3, and consists of relating their notions with ours, as well as verifying an oscillation condition.

As is classical in the Acquistapace–Terreni approach, we derive a representation formula for weak (p, q)-solution in Section 4.1. The formula reads

$$u(t^*) = \int_0^{t^*} \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}(\mathcal{B}_{t^*} - \mathcal{B}_s)u(s)\,\mathrm{d}s + \int_0^{t^*} \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)} f(s)\,\mathrm{d}s.$$

Note that the operator $\mathcal{B}_{t^*} + \kappa$ replaces the operator $\mathcal{L}_{t^*}$ when passing from (P) to (P'). For maximal regularity, we have to estimate the term $(\mathcal{B}_{t^*} + \kappa)u(t^*)$. Formally, this leads to the operators

$$\begin{aligned} S_1(u)(t^*) &\mapsto \int_0^{t^*} (\mathcal{B}_{t^*} + \kappa)\mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}(\mathcal{B}_{t^*} - \mathcal{B}_s)u(s)\,\mathrm{d}s, \\ S_2(f)(t^*) &\mapsto (\mathcal{B}_{t^*} + \kappa)\int_0^{t^*} \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)} f(s)\,\mathrm{d}s. \end{aligned}$$

The commutation between $(\mathcal{B}_{t^*}+\kappa)$ and the integral in S_1 will be justified during the proof of our main result. Consequently, to establish maximal regularity, we have to bound the operators S_1 and S_2. This is the topic of Section 4. Observe, however, that the operator S_2 acts on the data f, but S_1 acts on the weak (p, q)-solution u. This has the following effect: For S_2, we plainly desire to show $\mathrm{L}_t^q(\mathrm{L}_x^p)$-bounds. These will follow from the theory of operator-valued pseudo-differential operators. However, for S_1, the target space is still $\mathrm{L}_t^q(\mathrm{L}_x^p)$, but higher regularity of weak solutions lets us vary the norm of the data space. To be more precise, in the classical approach as employed by Fackler [22], the data space is $\mathrm{L}_t^q(\mathrm{W}_x^{1,p})$. The fundamental gain in our approach is that we will replace that data space by the space $\mathrm{L}^q_t(\mathrm{W}^{1+\alpha,p}_x)$. This has the effect that less restrictive kernel bounds for S_1 compared to [22] suffice. We give more details on this in a moment.

Let us come back to the operator S_2. The classical approach is to rewrite this operator as a pseudo-differential operator. This will be presented in Section 4.4. To do so, we have to restrict to a class of more regular right-hand sides f. This is, however, not a restriction, since we can use a standard approximation argument for the equation. This will be explained in Step 1 in the proof of Theorem 1.1 in Section 6. We emphasize that this approximation argument does not rely, yet, on the explicit control of implicit constants for weak (p,q)-solutions. Eventually, [29] leads to boundedness of S_2 provided we can verify that $(\tau, s) \mapsto 2\pi i\tau(2\pi i\tau + (\mathcal{B}_s + \kappa))^{-1}$ in an *R-Yamazaki symbol*. The definition of an R-Yamazaki symbol and the verification of this condition are presented in Lemma 4.5. This uses two ingredients. First, that the coefficients are $\mathrm{C}^\varepsilon_t(\mathrm{L}^\infty_x)$. Second, that the operators $(\mathcal{B}_{t^*} + \kappa)$ are uniformly R-sectorial, that is to say, their resolvents are R-bounded on a common sector, and the implied constants are independent of time.

Uniform R-sectoriality is treated in Section 2.5. On the one hand, we have to carefully trace the constants in well-known results on R-boundedness (more precisely, the approach based on off-diagonal bounds from [31]). On the other hand, we combine the elliptic solvability theory of Dong and Kim (see Proposition 2.6) with recent advances for the Kato square root property [10] to eventually prove L^p_x-boundedness for the semigroup generated by $-(\mathcal{B}_{t^*} + \kappa)$ with uniform constants in Theorem 2.9. This result is complemented by further insights on elliptic operators with minimal spatial regularity in Section 2. In contrast to [22], we are able to also treat complex systems. This is because we do not rely on the Gaussian bounds from [9] anymore.

We come back to the operator S_1. As already mentioned, the plan is to show $\mathrm{L}^q_t(\mathrm{W}^{1+\alpha,p}_x) \to \mathrm{L}^q_t(\mathrm{L}^p_x)$ boundedness. This will turn out to be sufficient owing to the *a priori* estimate $\|u\|_{\mathrm{L}^q_t(\mathrm{W}^{1+\alpha,p}_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}$ for weak solutions – this is the higher spatial regularity that was already alluded before. The boundedness for S_1 follows from a bound for its integral kernel and Young's convolution inequality. The kernel bounds are established in Lemmas 4.2 and 4.3. Lemma 4.2 is in some sense the central ingredient of this paper, as it is the only result that uses the full mixed regularity in time and space. There, we use the spatial regularity of our coefficients to have $\mathrm{W}^{\alpha,p}_x$-multipliers at our disposal, which eventually leads to estimates against $\mathrm{W}^{1+\alpha,p}_x$.

The missing piece is the higher spatial regularity of weak solutions, the subject of Section 5. Recall for this that the $\mathrm{W}^{1+\alpha,p}_x$-norm can be given by $\|\cdot\|_{\mathrm{L}^p_x} + \|\partial^\alpha_x \cdot\|_{\mathrm{W}^{1,p}_x}$, where ∂^α_x is the fractional derivative of order α in x. Our plan is to control the latter term by showing that $\partial^\alpha_x u(t,x)$ is a weak (p,q)-solution for some admissible right-hand side. Formally, one has

$$\partial_t(\partial^\alpha_x u) - \operatorname{div}_x B(t,x)\nabla_x(\partial^\alpha_x u) + \kappa(\partial^\alpha_x u) = \partial^\alpha_x f - \operatorname{div}_x[B(t,\cdot), \partial^\alpha_x]\nabla_x u. \tag{3}$$

Then, the right-hand side is in $\mathrm{L}^q_t(\mathrm{W}^{-1,p}_x)$ if the commutator $[B(t,\cdot), \partial^\alpha_x]$ is $\mathrm{L}^q_t(\mathrm{L}^p_x)$-bounded. Owing to the spatial regularity of the coefficients, the latter fact is true according to Lemma 5.2. Nevertheless, there remains a technical difficulty. In the first place, u is only in $\mathrm{L}^q_t(\mathrm{W}^{1,p}_x)$, so neither can we plug $\partial^\alpha_x u$ into the equation, nor can we justify the necessary calculations to show (3). The way out is an approximation argument in which we use regularized coefficients and the difference quotient method (see Steps 1 and 2 in the proof of Proposition 5.3). Afterwards, when we want to take the limit in order to get back to our original equation, it is crucial to have control over the implied constants in the weak (p, q)-solution theory from Theorem 3.1 in terms of the coefficients.

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2. Uniform estimates for elliptic operators

In Section 1.1, we have introduced the elliptic operators $\{\mathcal{L}_t\}_{0<t<T}$. We will associate parts in L^2_x with these operators, and show *uniform* bounds for their associated semigroups and square roots. We will also transfer semigroup bounds to the space $\mathrm{W}^{-1,p}_x$. The cornerstone for the results in this section is a well-posedness result for parabolic systems in divergence form due to Dong and Kim [18].

2.1. Elliptic coefficients.

We stay slightly more general here, which will become handy for technical reasons later on in Section 5, for instance. That being said, we introduce the following class of regular elliptic coefficients, which includes the coefficients of the non-autonomous problems studied in this article.

Definition 2.1. Let $\gamma > 0$ and $N \geq 0$. Denote by $\mathcal{E}(\Lambda, \lambda, \gamma, N)$ the class of *elliptic coefficients* with coefficient bounds Λ and λ that are C^γ with norm at most N. More precisely, this class consists of all functions $B\colon \mathbb{R}^d \to \mathbb{C}^{dm\times dm}$ which satisfy

$$|B(x)| \leq \Lambda \quad \& \quad \sum_{k,\ell=1}^{m} \mathrm{Re}(B(x)^{k\ell}\xi^k \mid \xi^\ell) \geq \lambda|\xi|^2 \qquad (\xi \in \mathbb{C}^{dm}),$$

and the regularity condition

$$\frac{|B(x+h) - B(x)|}{|h|^\gamma} \leq N \qquad (h \in \mathbb{R}^d \setminus \{0\}).$$

Remark 2.2. Note that $A(t^*, \cdot) \in \mathcal{E}(\Lambda, \lambda, \alpha + \varepsilon, M)$.

2.2. Elliptic systems and weak (p, q)-solutions.

As in Section 1.1, we associate with a coefficient function B a form and an operator $\mathrm{W}^{1,2}_x \to \mathrm{W}^{-1,2}_x$.

Definition 2.3. Let $B \in \mathcal{E}(\Lambda, \lambda, \gamma, N)$. Define the form

$$b\colon \mathrm{W}^{1,2}_x \times \mathrm{W}^{1,2}_x \to \mathbb{C}, \qquad b(u, v) = \int_{\mathbb{R}^d} B(x)\nabla_x u(x) \cdot \overline{\nabla_x v(x)}\, \mathrm{d}x,$$

and associate with it the operator

$$\mathcal{B}\colon \mathrm{W}^{1,2}_x \to \mathrm{W}^{-1,2}_x \quad \text{via} \quad \langle \mathcal{B}u, v\rangle_{\mathrm{W}^{-1,2}_x, \mathrm{W}^{1,2}_x} = b(u, v) \qquad (u, v \in \mathrm{W}^{1,2}_x).$$

Given a family $\{\mathcal{B}_t\}_{0<t<T}$ induced by coefficients $B(t,\cdot) \in \mathcal{E}(\Lambda, \lambda, \gamma, N)$ and a parameter $\kappa \in \mathbb{R}$, associate with them the non-autonomous evolution problem

$$\text{(P')} \qquad \partial_t u(t) + \mathcal{B}_t u(t) + \kappa u(t) = f(t), \qquad u(0) = 0.$$

The following definition makes precise what we understand under a solution to (P'). With the choices $\mathcal{B}_t = \mathcal{L}_t$ and $\kappa = 0$, this clarifies in particular the solution concept for the problem (P) from the introduction.

Definition 2.4. Given $f \in \mathrm{L}^q_t(\mathrm{W}^{-1,p}_x)$, $p, q \in (1,\infty)$, and $\kappa \in \mathbb{R}$, call a function $u \in \mathrm{L}^q_t(\mathrm{W}^{1,p}_x)$ a *weak (p,q)-solution* of (P'), if $u(0) = 0$, and if the integral equation

$$\text{(IE)} \qquad \begin{aligned} \int_0^T -\varphi'(s)(u(s) \,|\, g) + \varphi(s) b_s(u(s), g) + \kappa \varphi(s)(u(s) \,|\, g) \, \mathrm{d}s \\ = \int_0^T \varphi(s) \langle f(s), g \rangle_{\mathrm{W}^{-1,p}_x, \mathrm{W}^{1,p'}_x} \, \mathrm{d}s \end{aligned}$$

holds for all $\varphi \in \mathrm{C}^\infty_0(0,T;\mathbb{C})$ and $g \in \mathrm{C}^\infty_0(\mathbb{R}^d;\mathbb{C})$.

Remark 2.5. (i) It follows from duality that a weak (p,q)-solution u of (P') has a weak derivative $\partial_t u(t)$ in $\mathrm{L}^q_t(\mathrm{W}^{-1,p}_x)$ that coincides with $f(t) - \mathcal{B}(t)u(t) - \kappa u(t)$ for almost all t.

(ii) A weak (p,q)-solution is continuous at 0 with values in $\mathrm{W}^{-1,p}_x$ according to [13, p. 483, Prop. 9], which renders the initial condition meaningful.

(iii) Existence and uniqueness of weak (p,q)-solutions are independent of the parameter κ. Indeed, if u is a weak (p,q)-solution to the parameter κ, then $v(t) = \mathrm{e}^{\kappa t} u(t)$ is a weak (p,q)-solution to the right-hand side $\mathrm{e}^{\kappa s} f$ with $\kappa = 0$, and vice versa.

(iv) The integral equation (IE) extends to $g \in \mathrm{W}^{1,p'}_x$ by continuity.

The parameter κ is supposed to be taken sufficiently large. This is quantified by the results in [18]. In particular, we can ensure ellipticity in this way. We emphasize that the choice of κ can be made uniform in the quantities mentioned in Agreement 1.

Let us agree for the rest of this section that B denotes any fixed coefficient function from the class $\mathcal{E}(\Lambda, \lambda, \varepsilon, M) \subseteq \mathcal{E}(\Lambda, \lambda, \alpha + \varepsilon, M)$. Implicit constants are allowed to depend on p, Λ, λ, ε, M, and dimensions.

As a consequence of ellipticity, there is some $\omega \in [0, \pi/2)$ depending on Λ, λ, and κ such that the numerical range of $b + \kappa(\cdot \,|\, \cdot)_2$ is contained in the closed sector $\overline{\mathrm{S}}_\omega$ of opening angle 2ω. Furthermore, using Definition 2.1 and the Lax–Milgram lemma, $\mathcal{B} + \kappa + \rho$ is invertible for all $\rho \geq 0$. In particular, $\mathcal{B} + \kappa$ is itself invertible as an operator $\mathrm{W}^{1,2}_x \to \mathrm{W}^{-1,2}_x$.

As a consequence of the Hölder regularity of the coefficients, $\mathcal{B} + \kappa$ extrapolates moreover to an isomorphism $\mathrm{W}^{1,p}_x \to \mathrm{W}^{-1,p}_x$ for *all* $p \in (1,\infty)$. The argument divides into two steps. First, the autonomous problem associated with $\mathcal{B} + \kappa$ is well-posed according to [18]. We will give further information on that result and its applicability in our context in Section 3, see in particular Lemma 3.2. Second, the well-posedness of the original elliptic problem together with an estimate for its solutions follow by applying a cutoff argument to a stationary solution [17, Proof of Thm. 2.2]. The result can then be summarized as follows.

Proposition 2.6. *Let $p \in (1, \infty)$. The operator $\mathcal{B} + \kappa$ extrapolates to an invertible operator $\mathrm{W}^{1,p}_x \to \mathrm{W}^{-1,p}_x$. Given $f \in \mathrm{W}^{-1,p}_x$, write $u \in \mathrm{W}^{1,p}_x$ for the unique solution to the equation $(\mathcal{B} + \kappa)u = f$. Then, one has the estimate $\|u\|_{\mathrm{W}^{1,p}_x} \lesssim \|f\|_{\mathrm{W}^{-1,p}_x}$.*

Remark 2.7. The solutions provided by Proposition 2.6 are *compatible* to Lax–Milgram solutions in the following sense. Given $f \in \mathrm{W}^{-1,p}_x \cap \mathrm{W}^{-1,2}_x$, let u be the solution in $\mathrm{W}^{1,p}_x$ provided by Proposition 2.6, and v be the solution in $\mathrm{W}^{1,2}_x$ provided by the Lax–Milgram lemma. Then u and v coincide. Indeed, this is a consequence of local compatibility in complex interpolation scales [30, Thm. 8.1] and the fact that Proposition 2.6 provides a solution for *all* $p \in (1, \infty)$.

Remark 2.8. The result in [18] only requires that κ is larger than a certain threshold quantified by the parameters fixed in Agreement 1. Hence, to ensure that all results in Section 2 remain true when κ is replaced by $\kappa/2$, we pick κ a bit larger for good measure. We will exploit this observation in Section 4.

2.3. **The elliptic operator on $\mathbf{L}^2_x$ and mapping properties.** In virtue of the embedding $\mathrm{L}^2_x \subseteq \mathrm{W}^{-1,2}_x$, define the part of $\mathcal{B}$ in L^2_x and denote it as an abuse of notation also by the symbol B (it will be clear from the context if B denotes the coefficient function or the part in L^2_x). Of course, the part of $\mathcal{B} + \kappa$ in L^2_x coincides with $B + \kappa$. One has that $B + \kappa$ is a densely defined, invertible, and m-ω-sectorial operator in L^2_x with domain $\mathrm{D}(B + \kappa) = \mathrm{D}(B)$. In particular, $-(B + \kappa)$ generates a holomorphic semigroup of contractions $\{\mathrm{e}^{-z(B+\kappa)}\}_{z \in \mathrm{S}_{\pi/2-\omega}}$ on L^2_x. We will tacitly employ some properties of the sectorial functional calculus of $B + \kappa$. The reader can consult [26, Chap. 7] for further background.

Owing to [10, Lem. 7.3], we deduce L^p_x-bounds for the semigroup generated by $-(B + \kappa)$ as a consequence of Proposition 2.6 and Remark 2.7.

Theorem 2.9. *Let $p \in (1, \infty)$ and $\varphi \in (0, \pi/2 - \omega)$. One has the estimate*

$$\|\mathrm{e}^{-z(B+\kappa)} f\|_{\mathrm{L}^p_x} \lesssim \|f\|_{\mathrm{L}^p_x} \qquad (z \in \mathrm{S}_\varphi, f \in \mathrm{L}^p_x \cap \mathrm{L}^2_x).$$

Remark 2.10. In [10, Lem. 7.3], only the case $p \geq 2$ is presented. The case $p \leq 2$ either follows by a duality argument with $\mathcal{B}^* + \kappa$, or by repeating the calculation in [10], but changing the order in which H^∞-calculus and $(\mathcal{B} + \kappa)^{-1}$ are applied.

2.4. **Square roots and bounds on $\mathbf{W}^{-1,p}_x$.** As an m-ω-sectorial operator, $B + \kappa$ possesses a square root $(B + \kappa)^{\frac{1}{2}}$. It acts as an isomorphism $\mathrm{W}^{1,2}_x \to \mathrm{L}^2_x$ according to the solution of the Kato square root problem [8]. As a consequence of coefficient regularity, $(B+\kappa)^{\frac{1}{2}}$ extrapolates to an isomorphism $\mathrm{W}^{1,p}_x \to \mathrm{L}^p_x$ for *all* $p \in (1, \infty)$. Similar ideas were already employed in [22], but relying on the Gaussian property, which ties these results to the scalar case and is notably more technical. Instead, we use recent results established by the first-named author in [10, Thm. 1.1]. Indeed, in the case $p \leq 2$, its application is justified by Theorem 2.9, whereas in the case $p \geq 2$, we appeal to Proposition 2.6 in conjunction with Remark 2.7.

Theorem 2.11. *Let $p \in (1,\infty)$. Then $(B+\kappa)^{\frac{1}{2}}$ extrapolates to a (compatible) isomorphism $\mathrm{W}^{1,p}_x \to \mathrm{L}^p_x$.*

Theorem 2.11 allows us to translate the L^p_x-bounds for $\{\mathrm{e}^{-z(B+\kappa)}\}_{\mathrm{S}_\varphi}$ from Theorem 2.9 to $\mathrm{W}^{-1,p}_x$-bounds.

Proposition 2.12. *Let $p \in (1,\infty)$ and $\varphi \in [0, \pi/2 - \omega)$. One has the estimate*

$$\|\mathrm{e}^{-z(\mathcal{B}+\kappa)} f\|_{\mathrm{W}^{-1,p}_x} \lesssim \|f\|_{\mathrm{W}^{-1,p}_x} \qquad (z \in \mathrm{S}_\varphi, f \in \mathrm{W}^{-1,p}_x \cap \mathrm{L}^2_x).$$

Proof. Let $z \in \mathrm{S}_\varphi$ and $f \in \mathrm{W}^{-1,p}_x \cap \mathrm{L}^2_x$. As a primer, let us show

$$\|(B+\kappa)^{-\frac{1}{2}} f\|_p \lesssim \|f\|_{\mathrm{W}^{-1,p}_x}. \tag{4}$$

We employ a duality argument. To this end, let $h \in \mathrm{L}^{p'}_x \cap \mathrm{L}^2_x$. Note that the coefficient class $\mathcal{E}(\Lambda, \lambda, \varepsilon, M)$ is invariant under taking adjoints. Calculate using Kato's square root property and Theorem 2.11 (applied with B^* and p' instead of B and p) that

$$\begin{aligned} |((B+\kappa)^{-\frac{1}{2}} f \,|\, h)| &= |(f \,|\, (B^*+\kappa)^{-\frac{1}{2}} h)| \\ &\leq \|f\|_{\mathrm{W}^{-1,p}_x} \|(B^*+\kappa)^{-\frac{1}{2}} h\|_{\mathrm{W}^{1,p'}_x} \\ &\lesssim \|f\|_{\mathrm{W}^{-1,p}_x} \|h\|_{p'}. \end{aligned}$$

Duality lets us conclude this first claim.

Next, since $\mathrm{e}^{-z(B+\kappa)}$ is the part of $\mathrm{e}^{-z(\mathcal{B}+\kappa)}$ in L^2_x, write

$$\mathrm{e}^{-z(\mathcal{B}+\kappa)} f = \mathrm{e}^{-z(B+\kappa)} (B+\kappa)^{\frac{1}{2}} (B+\kappa)^{-\frac{1}{2}} f = (B+\kappa)^{\frac{1}{2}} \mathrm{e}^{-z(B+\kappa)} (B+\kappa)^{-\frac{1}{2}} f.$$

Let $g \in \mathrm{W}^{1,p'}_x \cap \mathrm{L}^2_x$, and calculate similarly as above, but using furthermore Theorem 2.9, that

$$\begin{aligned} |\langle \mathrm{e}^{-z(\mathcal{B}+\kappa)} f, g\rangle| &= |(\mathrm{e}^{-z(B+\kappa)} (B+\kappa)^{-\frac{1}{2}} f \,|\, (B^*+\kappa)^{\frac{1}{2}} g)| \\ &\leq \|\mathrm{e}^{-z(B+\kappa)} (B+\kappa)^{-\frac{1}{2}} f\|_p \|(B^*+\kappa)^{\frac{1}{2}} g\|_{p'} \\ &\lesssim \|(B+\kappa)^{-\frac{1}{2}} f\|_p \|g\|_{\mathrm{W}^{1,p'}_x}. \end{aligned}$$

Duality and (4) lead to $\|\mathrm{e}^{-z(\mathcal{B}+\kappa)} f\|_{\mathrm{W}^{-1,p}_x} \lesssim \|(B+\kappa)^{-\frac{1}{2}} f\|_p \lesssim \|f\|_{\mathrm{W}^{-1,p}_x}$. □

2.5. **Uniform $\boldsymbol{R}$-sectoriality.** As a preparation for Section 4.4, we show R-sectoriality for the operator $B+\kappa$, with R-bound uniform in the quantified parameters from Agreement 1. For further background on R-boundedness and R-sectoriality, the reader can consult [31].

Proposition 2.13 (R-sectoriality of B)**.** *Let $p \in (1, \infty)$ and $\varphi \in [0, \pi/2 - \omega)$. Then, the semigroup $\{\mathrm{e}^{-z(B+\kappa)}\}_{z \in \mathrm{S}_\varphi}$ satisfies the square function estimate*

$$\Big\| \Big(\sum_{j=1}^k |\mathrm{e}^{-z_j(B+\kappa)} f_j|^2 \Big)^{\frac{1}{2}} \Big\|_p \lesssim \Big\| \Big(\sum_{j=1}^k |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_p \quad (k \geq 1, (z_j)_{j=1}^k \subseteq \mathrm{S}_\varphi, (f_j)_{j=1}^k \subseteq \mathrm{L}^p_x \cap \mathrm{L}^2_x).$$

In particular, for $z \in \mathrm{S}_\varphi$ fixed, the operator $\mathrm{e}^{-z(B+\kappa)}$ extends from $\mathrm{L}^p_x \cap \mathrm{L}^2_x$ to a bounded operator $S(z)$ on L^p_x, and the family $\{S(z)\}_{z \in \mathrm{S}_\varphi}$ is an R-bounded analytic semigroup with R-bound uniform in the parameters fixed in Agreement 1.

Remark 2.14. Proposition 2.13 shows in particular that the semigroup in L^p_x is R-sectorial of the same angle as the semigroup on L^2_x. Hence, we keep writing ω instead of, say, ω_R.

Before we come to the justification of Proposition 2.13, let us record an important consequence that we will need later on in Section 4.4.

Corollary 2.15. *Let $p \in (1, \infty)$ and $\psi \in [0, \pi - \omega)$. Denote by $-B^\kappa_p$ the generator of the semigroup $\{S(t)\}_{t>0}$ from Proposition 2.13. Then the family $\{z(z + B^\kappa_p)^{-1}\}_{z \in \mathrm{S}_\psi}$ of operators on L^p_x is R-bounded, and the R-bound is uniform in the quantities fixed in Agreement 1.*

Proof. Fix $z \in \mathrm{S}_\psi$. Split $\arg(z) = \varphi + \tilde{\varphi}$, where $|\varphi| \in [0, \pi/2 - \omega)$ and $|\tilde{\varphi}| \in [0, \pi/2)$. The operator $(z + B^\kappa_p)^{-1}$ can be represented using the Laplace transform [26, Prop. 3.4.1 d)] via

$$(z + B^\kappa_p)^{-1} = \mathrm{e}^{i\tilde{\varphi}} \int_0^\infty \mathrm{e}^{-t|z|\mathrm{e}^{i\varphi}} S(t\mathrm{e}^{-i\tilde{\varphi}}) \, \mathrm{d}t.$$

Then, the claim follows from [31, Ex. 2.15]. $\square$

Given $1 \leq r < 2 < s \leq \infty$ such that $p \in (r, s)$, Proposition 2.13 is a consequence of so-called $\mathrm{L}^r_x \to \mathrm{L}^s_x$ *off-diagonal estimates* for $\{\mathrm{e}^{-z(B+\kappa)}\}_{z \in \mathrm{S}_\varphi}$. The general approach in the context of homogeneous spaces was presented in [31], and for dependence of the implied constants, see [11, Sec. 5]. To be more precise, we suppose that, for some $c > 0$ and for all measurable sets $E, F \subseteq \mathbb{R}^d$ and $z \in \mathrm{S}_\varphi$, one has the bound

$$\|\mathbf{1}_F \mathrm{e}^{-z(B+\kappa)} \mathbf{1}_E f\|_s \lesssim |z|^{d/2s - d/2r} \mathrm{e}^{-c\frac{\mathrm{d}(E,F)^2}{|z|}} \|\mathbf{1}_E f\|_r \qquad (f \in \mathrm{L}^r_x \cap \mathrm{L}^2_x). \tag{5}$$

Inequality (5) for $r = s = 2$ is known under the name *Gaffney estimates* and is well-known in the literature. A proof of this result that carefully keeps track of the implicit constants can be found in [10, Prop. 3.3]. Likewise, (5) is known for $r = 2$, $s \in (2, \infty)$, and with $c = 0$, as a consequence of the L^p_x-bounds for the semigroup provided by Theorem 2.9 and [6, Prop. 3.2 (1)]. In this case, we speak of *hypercontractivity* of the semigroup. Finally, (5) is then a consequence of interpolation of Gaffney estimates with hypercontractivity, taking duality and composition into account.

3. Existence and uniqueness of weak (p,q)-solutions

In this section, we consider a family of operators $\{\mathcal{B}_t\}_{0<t<T}$ associated with coefficients $B(t,\cdot) \in \mathcal{E}(\Lambda,\lambda,\alpha,M)$ that *depend* C^β_x *on* t.[3] The prototype for such a family of operators is the family $\{\mathcal{L}_t\}_{0<t<T}$ from Section 1. We aim to prove the existence and uniqueness of solutions to the associated problem (P') in the sense of Definition 2.4. To do so, we recast our original problem in the framework originating from the works of Dong and Kim [15–18]. This includes the introduction of a global extension in time of our original problem on $\mathbb{R}$ as outlined in [15, Rem. 1]. Implicit constants in this section are allowed to depend on p, q, Λ, λ, α, β, Hölder regularity, and dimension.

We begin by extending our coefficient family $\{B_t\}_{0<t<T}$ to all of $\mathbb{R}$. We extend constantly at the endpoints, that is, we set $B_t \coloneqq B_0$ for all $t<0$ and $B_t \coloneqq B_T$ for all $t>T$. For such t, we associate of course also a form b_t with B_t. Note that this extension does not affect the assumed Hölder regularity of the coefficients. Furthermore, we isometrically extend the right-hand side $f \in \mathrm{L}^q_t(\mathrm{W}^{-1,p}_x)$ outside of $(0,T)$ by zero to arrive at a function in $\mathrm{L}^q(\mathbb{R};\mathrm{W}^{-1,p}_x)$, which we denote by F. Also in the sequel, we will systematically denote functions on $\mathbb{R}$ by capital letters to better distinguish them from their local analogues. Given the extensions of $\{B_t\}_{0<t<T}$ and F, we look for solutions $U \in \mathrm{L}^q(\mathbb{R};\mathrm{W}^{1,p}_x)$ fulfilling the extended integral equation

$$\tag{EIE} \begin{aligned} \int_{\mathbb{R}} -\Phi'(s)(U(s)\,|\,g) + \Phi(s) b_s(U(s),g) + \kappa\Phi(s)(U(s)\,|\,g)\,\mathrm{d}s \\ = \int_{\mathbb{R}} \Phi(s)\langle F(s), g\rangle_{\mathrm{W}^{-1,p}_x,\mathrm{W}^{1,p'}_x}\,\mathrm{d}s, \end{aligned}$$

where we use test functions $\Phi \in \mathrm{C}^\infty_0(\mathbb{R})$ and $g \in \mathrm{C}^\infty_0(\mathbb{R}^d)$. Dong and Kim solved a similar problem in [18]: They show that, for a given $F \in \mathbb{H}^{-1}_{p,q,1}(\mathbb{R}\times\mathbb{R}^d)$ with $F = F_0 + \sum_{i=1}^d \partial_i F_i$, $F_j \in \mathrm{L}^q(\mathbb{R};\mathrm{L}^p_x)$, there exists a solution $U \in \mathring{\mathcal{H}}^1_{p,q,1}(\mathbb{R}\times\mathbb{R}^d)$ satisfying the integral equation

$$\tag{DKIE} \int_{\mathbb{R}} -(U(s)\,|\,\Psi'(s)) + b_s(U(s),\Psi(s)) + \kappa(U(s)\,|\,\Psi(s))\,\mathrm{d}s = \int_{\mathbb{R}} \langle F(s),\Psi(s)\rangle\,\mathrm{d}s$$

for all test functions $\Psi \in \mathrm{C}^\infty_0(\mathbb{R}\times\mathbb{R}^d)$. We explain and compare the used function spaces in the sequel of this section. For the notion of weak solutions employed by Dong and Kim, see also [15, p. 896] and [16, p. 3286]. Furthermore, solutions to (DKIE) are subject to the *a priori* estimate

$$\tag{6} \kappa\|U\|_{\mathrm{L}^q(\mathbb{R};\mathrm{L}^p_x)} + \sum_{i=1}^d \kappa^{1/2}\|\partial_i U\|_{\mathrm{L}^q(\mathbb{R};\mathrm{L}^p_x)} \lesssim \|F_0\|_{\mathrm{L}^q(\mathbb{R};\mathrm{L}^p_x)} + \sum_{i=1}^d \kappa^{1/2}\|F_i\|_{\mathrm{L}^q(\mathbb{R};\mathrm{L}^p_x)}$$

according to [18, Thm. 7.2], where the implicit constant depends on p, q, Λ, λ, dimension, and the parameters γ and R_0 appearing in Lemma 3.2. In particular, choosing $F=0$ in (6) shows the uniqueness of solutions to (DKIE).

The rest of this section is divided into two steps: First, we will relate the solution concepts of (DKIE) and our extended integral equation (EIE) and show that the former implies

[3]Say that a family $\{B_t\}_{0<t<T} \subseteq \mathcal{E}(\Lambda,\lambda,\alpha,M)$ *depends* C^β_x *on* t if $B_t \in \mathcal{E}(\Lambda,\lambda,\alpha,M)$ and the mapping $t \mapsto B_t$ is β-Hölder continuous with values in C^α_x, that is, the scalar-valued function $t \mapsto \|B_t\|_{\mathrm{C}^\alpha_x}$ lies in the class C^β_t.

the latter and eventually leads to a solution for the original problem (P'). Second, we will check the validity of the regularity assumptions on $\{B_t\}$ from [18, Thm. 7.2] to harvest the results of the first step. Eventually, this will prove the following proposition.

Theorem 3.1. *Given $f \in \mathrm{L}^q_t(\mathrm{W}^{-1,p}_x)$, there exists a unique weak (p,q)-solution u to (P'), and one has the estimate*

$$\|\partial_t u\|_{\mathrm{L}^q_t(\mathrm{W}^{-1,p}_x)} + \|\nabla_x u\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} + \kappa \|u\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{W}^{-1,p}_x)}.$$

Step 1: Compatibility with Dong and Kim. In order to solve (DKIE), Dong and Kim consider right-hand sides F in the spaces $\mathbb{H}^{-1}_{p,q,1}(\mathbb{R} \times \mathbb{R}^d)$. These spaces are isomorphic to the spaces $\mathrm{L}^q(\mathbb{R}; \mathrm{W}^{-1,p}_x)$ as can bee seen from a parabolic variant of [3, Thm. 3.9]. This means that the admissible right-hand sides for (DKIE) and (EIE) coincide. Now, [18, Sec. 8] gives the existence of a solution U to (DKIE) in the regularity class $\mathring{\mathcal{H}}^1_{p,q,1}(\mathbb{R} \times \mathbb{R}^d)$, which denotes the closure of $\mathrm{C}^\infty_0(\mathbb{R} \times \mathbb{R}^d)$ in the space $\mathcal{H}^1_{p,q,1}(\mathbb{R} \times \mathbb{R}^d)$. A function $U \in \mathcal{H}^1_{p,q,1}(\mathbb{R} \times \mathbb{R}^d)$ is by its very definition an element of $\mathrm{L}^q(\mathbb{R}; \mathrm{W}^{1,p}_x)$. Conversely a function in $\mathrm{L}^q(\mathbb{R}; \mathrm{W}^{1,p}_x)$ that satisfies (EIE) is a member of $\mathcal{H}^1_{p,q,1}(\mathbb{R} \times \mathbb{R}^d)$. For complete definitions of the above function spaces, the reader can consult [16, p. 3284] and [18, Sec. 4].

Comparing the classes of test functions employed in (EIE) and (DKIE), respectively, reveals that Dong and Kim use a larger class of test functions in their integral formulation. In particular, this shows that a solution to (DKIE) is also a solution to (EIE). On the other hand, recall that a function $U \in \mathrm{L}^q(\mathbb{R}; \mathrm{W}^{1,p}_x)$ solving (EIE) is also an admissible function for (DKIE). Using the fact that the tensors $\Phi(t)g(x)$ with $\Phi \in \mathrm{C}^\infty_0(\mathbb{R})$ and $g \in \mathrm{C}^\infty_0(\mathbb{R}^d)$ are dense in $\mathrm{L}^{q'}(\mathbb{R}; \mathrm{W}^{-1,p'}_x)$, we deduce by continuity (compare with Remark 2.5 (iv)) that (EIE) in particular remains to hold for test functions in $\mathrm{C}^\infty_0(\mathbb{R} \times \mathbb{R}^d)$. Hence, we get that U is also a solution for (DKIE), and is as such again unique.

Next, we focus on the *a priori* estimate (6) and its relation to the maximal regularity estimate in Theorem 3.1: Assuming $\kappa \geq 1$, we have

$$\|U\|_{\mathrm{L}^q(\mathbb{R}; \mathrm{W}^{1,p}_x)} \leq \kappa \|U\| + \sum_{i=1}^d \kappa^{1/2} \|\partial_i U\|_{\mathrm{L}^q(\mathbb{R}; \mathrm{W}^{1,p}_x)} \lesssim \kappa \|F\|_{\mathrm{L}^q(\mathbb{R}; \mathrm{W}^{-1,p}_x)}. \tag{7}$$

Note that the reasoning leading to the estimate (7) remains valid if $\kappa < 1$ with the consequence that $\leq$ has to replaced by $\lesssim$, and the implied constant then depends on κ.

Up to now, we have only worked out the existence and uniqueness of solutions to the extended integral equation (EIE). In the last part of this step, we will derive a full solution in the sense of Definition 2.4: Recall that $F = 0$ outside the interval $(0,T)$ by construction. Consequently, $U = 0$ on $(-\infty, 0)$ by uniqueness, hence $U(0) = 0$ by continuity (see Remark 2.5 (ii)). Additionally, the solution $U \in \mathrm{L}^q(\mathbb{R}, \mathrm{W}^{1,p}_x)$ that has been constructed via the method above gives rise to a restriction $u = U|_{(0,T)} \in \mathrm{L}^q_t(\mathrm{W}^{1,p}_x)$. Then u satisfies $u(0) = U(0) = 0$ by continuity and solves (IE). This shows that u is the unique (p,q)-solution of (P').

Step 2: Verification of the assumptions of Dong and Kim. The following lemma shows that the mean oscillation condition in Assumption [18, Asm. 7.1] is fulfilled. Hence, [18, Thm. 7.2 & Sec. 8] is applicable in our setting.

Lemma 3.2. *Let $\gamma \in (0, 1/4)$. Then there exists $R_0 \in (0,1]$ depending only on γ and the Hölder regularity of $\{B_t\}_{t\in\mathbb{R}}$ such that, for any $(t,x) \in \mathbb{R}^{d+1}$ and $r \in (0, R_0]$, we have*

$$\fint_{\mathrm{Q}_r(t,x)} \left| B_s^{k\ell}(y_1,\hat{y}) - \fint_{\mathrm{Q}'_r(t,\hat{x})} B_\tau^{k\ell}(y_1,\hat{z}) \,\mathrm{d}\hat{z}\,\mathrm{d}\tau \right| \mathrm{d}y\,\mathrm{d}s \leq \gamma \quad (k,\ell = 1,\dots,m),$$

where Q_r and Q'_r denote the parabolic cylinders *given by*

$$\mathrm{Q}_r(t,x) \coloneqq (t-r^2,t) \times \mathrm{B}_r(x) \quad \textit{and} \quad \mathrm{Q}'_r(t,\hat{x}) \coloneqq (t-r^2,t) \times \mathrm{B}'_r(\hat{x}),$$

respectively, and $x = (x_1,\hat{x})$ with $x_1 \in \mathbb{R}$ and $\hat{x} \in \mathbb{R}^{d-1}$.

Proof. Let $r > 0$ and $(t,x) \in \mathbb{R}^{d+1}$. Fix $(s,y) \in \mathbb{R}^{d+1}$. We decompose the integrand as

$$\begin{aligned} &\left| B_s^{k\ell}(y_1,\hat{y}) - \fint_{\mathrm{Q}'_r(t,\hat{x})} B_\tau^{k\ell}(y_1,\hat{z}) \,\mathrm{d}\hat{z}\,\mathrm{d}\tau \right| \\ &\qquad \leq \fint_{\mathrm{Q}'_r(t,\hat{x})} \left| B_s^{k\ell}(y) - B_\tau^{k\ell}(y) \right| + \left| B_\tau^{k\ell}(y_1,\hat{y}) - B_\tau^{k\ell}(y_1,\hat{z}) \right| \mathrm{d}\hat{z}\,\mathrm{d}\tau. \end{aligned}$$

Now, for the first term, we have using regularity of B that

$$\left| B_s^{k\ell}(y) - B_\tau^{k\ell}(y) \right| \leq \| B_s^{k\ell} - B_\tau^{k\ell} \|_\infty \lesssim |s-\tau|^\beta \lesssim |s-t|^\beta + |t-\tau|^\beta \lesssim r^{2\beta} \tag{8}$$

and, for the second term,

$$\left| B_\tau^{k\ell}(y_1,\hat{y}) - B_\tau^{k\ell}(y_1,\hat{z}) \right| \lesssim |\hat{y}-\hat{z}|^\alpha \| B_\tau^{k\ell} \|_{\mathrm{C}_x^\alpha} \lesssim (2r)^\alpha. \tag{9}$$

Observe that both estimates are uniform in s and y, to calculate the average over $\mathrm{Q}_r(t,x)$ as

$$\fint_{\mathrm{Q}_r(t,x)} \left| B_s^{k\ell}(y_1,\hat{y}) - \fint_{\mathrm{Q}'_r(t,\hat{x})} B_\tau^{k\ell}(y_1,\hat{z}) \,\mathrm{d}\hat{z}\,\mathrm{d}\tau \right| \mathrm{d}y\,\mathrm{d}s \lesssim r^{2\beta} + (2r)^\alpha,$$

where the implicit constant depends on the Hölder regularity of B and α, β. Now, given $\gamma \in (0, 1/4)$, choose $R_0 \in (0,1]$ small enough (depending on the implicit constant) to conclude. □

Remark 3.3. Note that the proof of Lemma 3.2 did not need the full mixed Hölder regularity of $\{B_t\}_{t\in\mathbb{R}}$. Indeed, the calculations in the proof show that estimates (8) and (9) both only rely on Hölder regularity in one of the two variables and uniformly for the other variable.

4. Estimates for the solution formula

In this section, we consider a family of operators $\{\mathcal{B}_t\}_{0<t<T}$ associated with coefficients $B(t,\cdot) \in \mathcal{E}(\Lambda, \lambda, \alpha+\varepsilon, M)$ that depend $\mathrm{C}_x^{\beta+\varepsilon}$ on t. The prototype for such a family of operators is the family $\{\mathcal{L}_t\}_{0<t<T}$ from Section 1. First, we derive a solution formula for weak (p,q)-solutions to the associated non-autonomous problem. Second, we derive

suitable estimates for it, which depend heavily on the regularity assumption for the coefficients. Implicit constants are throughout this section allowed to depend on p, q, Λ, λ, α, β, ε, Hölder regularity, and dimensions.

4.1. **Representation formula by Acquistapace and Terreni.** For a weak (p,q)-solution u of (P'), we rely on a well-known representation formula due to Acquistapace and Terreni in $\mathrm{W}^{-1,p}_x$ given pointwise by

$$(\heartsuit) \qquad u(t^*) = \int_0^{t^*} \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}(\mathcal{B}_{t^*}-\mathcal{B}_s)u(s)\,\mathrm{d}s + \int_0^{t^*} \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)} f(s)\,\mathrm{d}s.$$

The proof is well-known in the literature [2, 12, 22, 25], but we give a streamlined version that directly works with absolute continuity.

Proof of ($\heartsuit$). Consider on $[0,t^*]$ the function $v(s) = \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}u(s)$. Moreover, let $0 \leq \tau < t^*$. We claim the identity

$$(10) \qquad v(\tau) = v(0) + \int_0^{\tau} (\mathcal{B}_{t^*}+\kappa)\mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}u(s) + \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}u'(s)\,\mathrm{d}s.$$

Before we turn to the proof of (10), we show how it implies ($\heartsuit$). Note that the function $(\mathcal{B}_{t^*}+\kappa)\mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}u(s) + \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}u'(s)$ is in $\mathrm{L}^q_t(\mathrm{W}^{-1,p}_x)$, since u is a weak (p,q)-solution (keep Remark 2.5 (i) in mind) and the semigroup is bounded on $\mathrm{W}^{-1,p}_x$ owing to Proposition 2.12. Hence, by Lebesgue's theorem, we can take the limit $\tau \to t^*$ on the right-hand side of (10). Equally, we can take this limit on the left-hand side, owing to the facts that u is uniformly continuous over $[0,t^*]$ with values in $\mathrm{W}^{-1,p}_x$, and the family $\{\mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}\}_{0\leq s\leq t^*}$ is strongly continuous and bounded as a family of operators on $\mathrm{W}^{-1,p}_x$. Then, plugging in the actual definition of v yields ($\heartsuit$).

Let us come back to the proof of (10). On the interval $[0,\tau]$, $s \mapsto \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}$, considered as a family of operators on $\mathrm{W}^{-1,p}_x$, has a bounded derivative due to Proposition 2.12 and analyticity. As u is a weak (p,q)-solution, $u\colon [0,\tau] \to \mathrm{W}^{-1,p}_x$ is likewise absolutely continuous. Hence, observing $u(0) = 0$, deduce (10) from Lemma 4.1 below. $\square$

Lemma 4.1. *Let X, Y be Banach spaces, $\tau > 0$, $\{T(s)\}_{0\leq s\leq\tau}$ be a differentiable family of operators $X \to Y$ with bounded derivative, and $g\colon [0,\tau] \to X$ be absolutely continuous. Then $s \mapsto T(s)g(s) \in Y$ is an absolutely continuous function on $[0,\tau]$ with derivative $T'(s)g(s) + T(s)g'(s)$.*

Proof. The assumption on T implies in particular that $s \mapsto T(s)$ is absolutely continuous on $[0,\tau]$. Now, use absolute continuity of both T and g, and the Fubini–Tonelli theorem, to give

$$\begin{aligned}
&\int_0^{\tau} T'(s)g(s) + T(s)g'(s)\,\mathrm{d}s \\
&= \int_0^{\tau} T'(s)\Big(g(0) + \int_0^s g'(u)\,\mathrm{d}u\Big)\,\mathrm{d}s + \int_0^{\tau}\Big(T(0) + \int_0^s T'(u)\,\mathrm{d}u\Big)g'(s)\,\mathrm{d}s \\
&= \int_0^{\tau} T'(s)\,\mathrm{d}s\, g(0) + T(0)\int_0^{\tau} g'(s)\,\mathrm{d}s + \int_0^{\tau}\int_0^{\tau} T'(s)g'(u)\,\mathrm{d}u\,\mathrm{d}s.
\end{aligned}$$

All remaining integrals can now be evaluated using absolute continuity, and we only remain with $T(\tau)g(\tau)-T(0)g(0)$ after having canceled all superfluous terms. Rearranging terms gives the claim. □

Motivated by (♡), we are going to consider the operators

$$\begin{aligned} S_1(u)(t^*) &\mapsto \int_0^{t^*} (\mathcal{B}_{t^*}+\kappa)\mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}(\mathcal{B}_{t^*}-\mathcal{B}_s)u(s)\,\mathrm{d}s, \\ S_2(f)(t^*) &\mapsto (\mathcal{B}_{t^*}+\kappa)\int_0^{t^*} \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)} f(s)\,\mathrm{d}s. \end{aligned} \tag{11}$$

Up to some technicalities, boundedness of S_1 and S_2 will lead to the maximal regularity estimate for u later on in Section 6.

4.2. **Digression on interpolation spaces and fractional powers.** We recall some necessary facts from the theory of interpolation spaces and their relation with fractional powers of a sectorial operator. The reader can, for instance, consult the monographs [41] and [26] for further information on these topics.

Given Banach spaces $Y \subseteq X$, $\theta \in (0,1)$, and $r \in [1,\infty]$, write $[X,Y]_\theta$ for Calderón–Lion's *θ-complex*, and $(X,Y)_{\theta,r}$ for Petree's *(θ,r)-real interpolation space* between X and Y. The real and complex interpolation spaces do not coincide except when X and Y are Hilbert spaces and $r=2$. However, one always has the continuous inclusion $[X,Y]_\theta \subseteq (X,Y)_{\theta,\infty}$, see [41, Sec. 1.10.3, Thm. 1].

Suppose now that T is an invertible sectorial operator in X with $\mathrm{D}(T)=Y$. For $\alpha \in \mathbb{R}$, one can define *fractional powers* T^α of T inside its sectorial functional calculus, which are again invertible with $(T^\alpha)^{-1}=T^{-\alpha}$, and which satisfy the identity $T^\alpha T^\beta = T^{\alpha+\beta}$, where $\alpha,\beta \geq 0$. For $\alpha \in (0,1)$, the domain of T^α is given by the complex interpolation space $[X,Y]_\alpha$.

4.3. **Estimates for the kernel of S_1.** The following lemma is simple, but central in our argument, as it is the only result that uses the full *simultaneous regularity* in spacial and temporal variable.

Lemma 4.2. *Let $s \in (0,t^*)$. The operator $\mathcal{B}_{t^*}-\mathcal{B}_s$ acts as a bounded operator $\mathrm{W}^{1+\alpha,p}_x \to \mathrm{W}^{-1+\alpha,p}_x$ along with the estimate*

$$\|\mathcal{B}_{t^*}-\mathcal{B}_s\|_{\mathrm{W}^{1+\alpha,p}_x \to \mathrm{W}^{-1+\alpha,p}_x} \lesssim |t^*-s|^{\beta+\varepsilon}.$$

Proof. Let $s \in (0,t^*)$ and $f \in \mathrm{W}^{1+\alpha,p}_x \cap \mathrm{W}^{1,2}_x$. Recall that a $\mathrm{C}^{\alpha+\varepsilon}_x$-function is a multiplier on the space $\mathrm{W}^{\alpha,p}_x$, and the operator norm can be controlled by the respective Hölder norm. Hence, for $g \in \mathrm{W}^{1-\alpha,p'}_x \cap \mathrm{W}^{1,2}_x$, estimate

$$\begin{aligned} |\langle(\mathcal{B}_{t^*}-\mathcal{B}_s)f,g\rangle| &= \Big|\int_{\mathbb{R}^d}(B(t^*,x)-B(s,x))\nabla f(x)\cdot\overline{\nabla g(x)}\,\mathrm{d}x\Big| \\ &\leq \|(B(t^*,\cdot)-B(s,\cdot))\nabla f\|_{\mathrm{W}^{\alpha,p}_x}\|\nabla g\|_{\mathrm{W}^{-\alpha,p'}_x} \\ &\lesssim \|B(t^*,\cdot)-B(s,\cdot)\|_{\mathrm{C}^{\alpha+\varepsilon}}\|\nabla f\|_{\mathrm{W}^{\alpha,p}_x}\|g\|_{\mathrm{W}^{1-\alpha,p'}_x}. \end{aligned}$$

Using the regularity of A and duality, we deduce

$$\|(\mathcal{B}_{t^*} - \mathcal{B}_s)f\|_{\mathrm{W}_x^{-1+\alpha,p}} \lesssim |t^* - s|^{\beta+\varepsilon}\|\nabla f\|_{\mathrm{W}_x^{\alpha,p}} \leq |t^* - s|^{\beta+\varepsilon}\|f\|_{\mathrm{W}_x^{1+\alpha,p}}. \qquad \square$$

Lemma 4.3. *Let $s \in (0, t^*)$. The operator $(\mathcal{B}_{t^*} + \kappa)\mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}$ acts as a bounded operator $\mathrm{W}_x^{-1+\alpha,p} \to \mathrm{L}_x^p$, and satisfies the estimate*

$$\|(\mathcal{B}_{t^*} + \kappa)\mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}\|_{\mathrm{W}_x^{-1+\alpha,p} \to \mathrm{L}_x^p} \lesssim |t^* - s|^{-\beta-1}.$$

Proof. Let $s \in (0, t^*)$. Fix $\omega < \theta < \nu < \pi/2$. The operator

$$(\mathcal{B}_{t^*} + \kappa)^{\frac{3}{2}}\mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)} = (\mathcal{B}_{t^*} + \kappa)[\mathbf{z}^{\frac{1}{2}}\mathrm{e}^{-(t^*-s)\mathbf{z}}](\mathcal{B}_{t^*} + \kappa)$$

can be represented using the semigroup generated by $-(\mathcal{B}_{t^*} + \kappa)$ in the following way (see, for instance, [6, Sec. 2.2]): Write $\Gamma_\pm$ and $\gamma_\pm$ for the rays of angle $\pm(\pi/2 - \theta)$ and $\pm\nu$, respectively. Then, one has that $(\mathcal{B}_{t^*} + \kappa)[\mathbf{z}^{\frac{1}{2}}\mathrm{e}^{-(t^*-s)\mathbf{z}}](\mathcal{B}_{t^*} + \kappa)$ is a linear combination of the terms

$$\begin{aligned} &\int_{\Gamma_\pm} (\mathcal{B}_{t^*} + \kappa)\mathrm{e}^{-z(\mathcal{B}_{t^*}+\kappa)} \int_{\gamma_\pm} \mathrm{e}^{z\xi}\xi^{1/2}\mathrm{e}^{-(t^*-s)\xi}\,\mathrm{d}\xi\,\mathrm{d}z \\ = &\int_{\Gamma_\pm} z^{1-\alpha/2}(\mathcal{B}_{t^*} + \kappa)\mathrm{e}^{-z(\mathcal{B}_{t^*}+\kappa)} \int_{\gamma_\pm} z^{\alpha/2-1}\mathrm{e}^{z\xi}\xi^{1/2}\mathrm{e}^{-(t^*-s)\xi}\,\mathrm{d}\xi\,\mathrm{d}z. \end{aligned}$$

Let $f \in \mathrm{W}_x^{-1+\alpha,p} \cap \mathrm{L}_x^2$. Apply the L_x^p-norm to this representation, and use (4) and Proposition A.1 to give

$$\begin{aligned} &\|(\mathcal{B}_{t^*} + \kappa)\mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}f\|_p \\ \lesssim &\|(\mathcal{B}_{t^*} + \kappa)[\mathbf{z}^{\frac{1}{2}}\mathrm{e}^{-(t^*-s)\mathbf{z}}](\mathcal{B}_{t^*} + \kappa)\|_{\mathrm{W}_x^{-1,p}} \\ \lesssim &\int_0^\infty\int_0^\infty r^{\alpha/2-1}\mathrm{e}^{-cru}u^{1/2}\mathrm{e}^{-c(t^*-s)u}\|r^{1-\alpha/2}(\mathcal{B}_{t^*} + \kappa)\mathrm{e}^{-(r\mathrm{e}^{\pm i(\pi/2-\theta)})(\mathcal{B}_{t^*}+\kappa)}f\|_{\mathrm{W}_x^{-1,p}}\,\mathrm{d}u\,\mathrm{d}r \\ \lesssim &\int_0^\infty\int_0^\infty r^{\alpha/2-1}\mathrm{e}^{-cru}u^{1/2}\mathrm{e}^{-c(t^*-s)u}\,\mathrm{d}u\,\mathrm{d}r\|f\|_{\left(\mathrm{W}_x^{-1,p},\mathrm{D}(\mathcal{B}_{t^*}+\kappa)\right)_{\alpha/2,\infty}}. \end{aligned}$$

Using the transformations $\tilde{u} = (t^* - s)u$ followed by $\tilde{r} = r\tilde{u}(t^* - s)^{-1}$ in conjunction with Fubini's theorem, the factor in front of the norm of the real interpolation space can be controlled by $(t^*-s)^{(\alpha-3)/2}$. By the relation $2\beta+\alpha = 1$, we find $(t^*-s)^{(\alpha-3)/2} = (t^*-s)^{-\beta-1}$. Moreover, using that the (θ, ∞)-real interpolation norm is always controlled by the θ-complex interpolation norm [41, Sec. 1.10.3. Thm. 1], and the fact that the complex interpolation space $[\mathrm{W}_x^{-1,p}, \mathrm{W}_x^{1,p}]_{\alpha/2}$ coincides with $\mathrm{W}_x^{-1+\alpha,p}$, deduce

$$\|(\mathcal{B}_{t^*} + \kappa)\mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)}f\|_p \lesssim (t^* - s)^{-\beta-1}\|f\|_{\mathrm{W}_x^{-1+\alpha,p}}. \qquad \square$$

4.4. **Boundedness of S_2.** Recall the operator S_2 from (11). The aim of this subsection is to show the following.

Proposition 4.4. *Let $p, q \in (1, \infty)$, then one has the estimate*

$$\|S_2 f\|_{\mathrm{L}_t^q(\mathrm{L}_x^p)} \lesssim \|f\|_{\mathrm{L}_t^q(\mathrm{L}_x^p)} \qquad (f \in \mathrm{C}_0^\infty(\mathrm{L}_x^p \cap \mathrm{L}_x^2)).$$

It is well-known in the literature [22, 27, 35] that such bounds for S_2 follow from the boundedness of some vector-valued pseudo-differential operator. For the reader's convenience, we include a proof. For further background on vector-valued pseudo-differential operators, the reader may consult [29] and the references therein.

For technical reasons, we extend f by 0 outside $(0, T)$, and we extend the operator family $\{\mathcal{B}_t\}_{0<t<T}$ to $\mathbb{R}$ constantly at the endpoints, that is to say, $\mathcal{B}_t = \mathcal{B}_0$ for $t \leq 0$ and $\mathcal{B}_t = \mathcal{B}_T$ for $t \geq T$ (we performed the same extension already in Section 3). Using the vector-valued Fourier transform $\mathcal{F}$ (see [28, Sec. 2.4.c] for further information) and the Fubini–Tonelli theorem (its application is justified by integrability of $\mathcal{F}f$ and exponential decay of the semigroup), calculate

$$\begin{aligned}
\int_0^{t^*} \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)} f(s)\,\mathrm{d}s &= \int_{-\infty}^{\infty} \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)} \mathbf{1}_{[0,\infty)}(t^*-s) f(s)\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)} \mathbf{1}_{[0,\infty)}(t^*-s) \int_{-\infty}^{\infty} \mathcal{F}f(\tau)\mathrm{e}^{2\pi i s\tau}\,\mathrm{d}\tau\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-(t^*-s)(\mathcal{B}_{t^*}+\kappa)} \mathbf{1}_{[0,\infty)}(t^*-s)\mathrm{e}^{2\pi i s\tau}\,\mathrm{d}s\,\mathcal{F}f(\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} \mathrm{I}(\tau, t^*)\mathcal{F}f(\tau)\,\mathrm{d}\tau,
\end{aligned} \tag{12}$$

where $\mathrm{I}(\tau, t^*)$ is implicitly defined by the latest identity. Using the transformation $u = t^* - s$ and the relationship between a semigroup and its generator in virtue of the Laplace transform (apply for instance [26, Prop. 3.4.1 d)] to $\mathcal{B}_{t^*} + \kappa/2$), deduce

$$\mathrm{I}(\tau, t^*) = \mathrm{e}^{2\pi i\tau t^*} \int_0^{\infty} \mathrm{e}^{-u(\mathcal{B}_{t^*}+\kappa)}\mathrm{e}^{-2\pi i u\tau}\,\mathrm{d}u = \mathrm{e}^{2\pi i\tau t^*}(2\pi i\tau + (\mathcal{B}_{t^*}+\kappa))^{-1}.$$

Plug this back into (12) to conclude with the definition of S_2 that

$$S_2(f)(t^*) = (\mathcal{B}_{t^*}+\kappa)\int_{-\infty}^{\infty} (2\pi i\tau + (\mathcal{B}_{t^*}+\kappa))^{-1}\mathcal{F}f(\tau)\mathrm{e}^{2\pi i\tau t^*}\,\mathrm{d}\tau. \tag{13}$$

The integral $\int_{-\infty}^{\infty} \|(\mathcal{B}_{t^*}+\kappa)(2\pi i\tau + (\mathcal{B}_{t^*}+\kappa))^{-1}\mathcal{F}f(\tau)\mathrm{e}^{2\pi i\tau t^*}\|_{\mathrm{L}^2_x}\,\mathrm{d}\tau$ is finite, so we can commute $(\mathcal{B}_{t^*}+\kappa)$ with the integral in (13) owing to Hille's theorem. This means that $S_2(f)$ can be represented as the pseudo-differential operator with symbol

$$(\tau, s) \mapsto (\mathcal{B}_s+\kappa)(2\pi i\tau + (\mathcal{B}_s+\kappa))^{-1}.$$

Of course, by expansion, we can equally study boundedness of the pseudo-differential operator associated with the symbol $(\tau, s) \mapsto 2\pi i\tau(2\pi i\tau + (\mathcal{B}_s+\kappa))^{-1}$. In the following lemma, we study this symbol thoroughly.

Lemma 4.5. *The symbol $a(\tau, s) = 2\pi i\tau(2\pi i\tau + (\mathcal{B}_s+\kappa))^{-1}$ is an R-*Yamazaki symbol*, that is to say, there are some $\varepsilon > 0$ and $C > 0$ such that, for $k = 0, 1, 2,$ and $s, h \in \mathbb{R}$, one has the R-bound*

$$\mathcal{R}\Big\{(1+|\tau|)\partial_\tau^k\big[a(\tau, s) - a(\tau, s+h)\big]\ ;\ \tau \in \mathbb{R}\Big\} \leq C|h|^{\varepsilon}.$$

Proof. For brevity, we rescale $2\pi\tau$ to τ in the definition of the symbol a. Fix $\psi \in (\pi/2, \pi - \omega)$ and $s, h \in \mathbb{R}$. Define on S_ψ the function $A(\lambda) = (1+\lambda)\big[(\lambda + (\mathcal{B}_s+\kappa))^{-1} - (\lambda + (\mathcal{B}_{s+h}+\kappa))^{-1}\big]$.

Step 1: The case $k = 0$. We show that the function A is R-bounded with control against $|h|^\varepsilon$. Then, in particular, the case $k = 0$ is verified (keep the contraction principle in mind). For $\lambda \in \mathrm{S}_\psi$, expand $A(\lambda)$ using the functional calculus as

$$\text{(F1)} \qquad (1+\lambda)^{\frac{1}{2}} \left[\frac{\mathbf{z}^{\frac{1}{2}}}{\lambda + \mathbf{z} + \kappa/2} \right] (\mathcal{B}_s + \kappa/2)$$

$$\text{(F2)} \qquad \times (\mathcal{B}_s + \kappa/2)^{-\frac{1}{2}} \big[\mathcal{B}_{s+h} - \mathcal{B}_s \big] (\mathcal{B}_{s+h} + \kappa/2)^{-\frac{1}{2}}$$

$$\text{(F3)} \qquad \times (1+\lambda)^{\frac{1}{2}} \left[\frac{\mathbf{z}^{\frac{1}{2}}}{\lambda + \mathbf{z} + \kappa/2} \right] (\mathcal{B}_{s+h} + \kappa/2).$$

Using composition of R-bounds, we can treat all three factors separately. The decay in $|h|$ comes (F2), where as the other two are merely bounded. Moreover, (F1) and (F3) have the same structure, so we only present the estimate for (F1). Recall that, according to Remark 2.8, all results from Section 2 can be applied to $\mathcal{B}_s + \kappa/2$.

Put $g_\lambda = \mathbf{z}^{\frac{1}{2}}(\lambda + \mathbf{z} + \kappa/2)^{-1}$. The operator $g_\lambda(\mathcal{B}_s + \kappa/2)$ in (F1) is given as a linear combination of Cauchy integrals of the form

$$\int_0^\infty g_\lambda(t\mathrm{e}^{\pm i\nu}) t (t\mathrm{e}^{\pm i(\nu+\pi)} + (\mathcal{B}_s + \kappa/2))^{-1} \, \frac{\mathrm{d}t}{t},$$

where $\nu \in (\omega, \pi - \psi)$. According to [31, Cor. 2.14] and owing to Proposition 2.15,

$$\begin{aligned} \mathcal{R}\{(1+\lambda)^{\frac{1}{2}} g_\lambda(\mathcal{B}_s + \kappa/2) \, ; \, \lambda \in \mathrm{S}_\psi\} &\lesssim \sup_{\lambda \in \mathrm{S}_\psi} (1+|\lambda|)^{\frac{1}{2}} \int_0^\infty |g_\lambda(t\mathrm{e}^{\pm i\nu})| \, \frac{\mathrm{d}t}{t} \\ &=: \sup_{\lambda \in \mathrm{S}_\psi} (1+|\lambda|)^{\frac{1}{2}} \mathrm{I}(\lambda). \end{aligned}$$

Hence, it only remains to control $\mathrm{I}(\lambda)$ against $(1+|\lambda|)^{-\frac{1}{2}}$ to complete the treatment of (F1). To this end, calculate with the reverse triangle inequality for sectors

$$\text{(14)} \qquad \mathrm{I}(\lambda) = \int_0^\infty t^{\frac{1}{2}} |\lambda + \kappa/2 + t\mathrm{e}^{\pm i\nu}|^{-1} \, \frac{\mathrm{d}t}{t} \lesssim \int_0^\infty t^{\frac{1}{2}} (|\lambda| + \kappa/2 + t)^{-1} \, \frac{\mathrm{d}t}{t}.$$

Split the last integral in (14) as $\int_0^\infty \frac{\mathrm{d}t}{t} = \int_0^{|\lambda|+\kappa/2} \frac{\mathrm{d}t}{t} + \int_{|\lambda|+\kappa/2}^\infty \frac{\mathrm{d}t}{t}$ to deduce a bound (up to a constant) against $(\kappa/2 + |\lambda|)^{-\frac{1}{2}} \approx (1+|\lambda|)^{-\frac{1}{2}}$.

It remains to treat (F2). Here, a crucial observation is that (F2) is independent of λ, hence the R-bound can in fact be controlled by the operator norm [31, Rem. 2.6 c)]. Another important ingredient is the estimate

$$\text{(15)} \qquad \|\mathcal{B}_s - \mathcal{B}_{s+h}\|_{\mathrm{W}^{1,p}_x \to \mathrm{W}^{-1,p}_x} \lesssim |h|^\varepsilon,$$

whose proof follows the lines of Lemma 4.2, but it suffices to have coefficients in $\mathrm{C}^\varepsilon(\mathbb{R}; \mathrm{L}^\infty_x)$. Recall from Theorem 2.11 the estimate $\|(\mathcal{B}_s + \kappa/2)^{-\frac{1}{2}} f\|_{\mathrm{W}^{1,p}_x} \lesssim \|f\|_{\mathrm{L}^p_x}$ for $f \in \mathrm{L}^p_x \cap \mathrm{L}^2_x$. The same estimate holds of course if $\mathcal{B}_s$ is replaced by $(\mathcal{B}_s)^*$. Hence, we can estimate by duality and using (15) that, for $g \in \mathrm{L}^{p'}_x$,

$$\begin{aligned} &|((\mathcal{B}_s + \kappa/2)^{-\frac{1}{2}}(\mathcal{B}_{s+h} - \mathcal{B}_s)(\mathcal{B}_{s+h} + \kappa/2)^{-\frac{1}{2}} f \mid g)| \\ &= |((\mathcal{B}_{s+h} - \mathcal{B}_s)(\mathcal{B}_{s+h} + \kappa/2)^{-\frac{1}{2}} f \mid ((\mathcal{B}_s)^* + \kappa/2)^{-\frac{1}{2}} g)| \\ &\leq \|(\mathcal{B}_{s+h} - \mathcal{B}_s)(\mathcal{B}_{s+h} + \kappa/2)^{-\frac{1}{2}} f\|_{\mathrm{W}^{-1,p}_x} \|((\mathcal{B}_s)^* + \kappa/2)^{-\frac{1}{2}} g\|_{\mathrm{W}^{1,p'}_x} \\ &\lesssim |h|^\varepsilon \|(\mathcal{B}_{s+h} + \kappa/2)^{-\frac{1}{2}} f\|_{\mathrm{W}^{1,p}_x} \|g\|_{\mathrm{L}^{p'}_x} \\ &\lesssim |h|^\varepsilon \|f\|_{\mathrm{L}^p_x} \|g\|_{\mathrm{L}^{p'}_x}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\mathcal{R}\{(\mathcal{B}_s + \kappa/2)^{-\frac{1}{2}}(\mathcal{B}_{s+h} - \mathcal{B}_s)(\mathcal{B}_{s+h} + \kappa/2)^{-\frac{1}{2}}\} \\ &\lesssim \|(\mathcal{B}_s + \kappa/2)^{-\frac{1}{2}}(\mathcal{B}_{s+h} - \mathcal{B}_s)(\mathcal{B}_{s+h} + \kappa/2)^{-\frac{1}{2}}\|_{\mathrm{L}^p_x \to \mathrm{L}^p_x} \\ &\lesssim |h|^\varepsilon. \end{aligned}$$

Step 2: The case $k \geq 1$. Since the function A is holomorphic in λ and defined on a sector that strictly includes the half-plane, this is a simple consequence of Cauchy's formula, see [31, Ex. 2.16]. □

Proof of Proposition 4.4. We have already seen that the bound for S_2 follows from the bound for the pseudo-differential operator associated with the symbol $a(\tau, t) = 2\pi i\tau(2\pi i\tau + (\mathcal{B}_t + \kappa))^{-1}$. It was shown in [29, Thm. 17 & Rem. 20] that boundedness for such a pseudo-differential operator follows if the symbol a is an R-Yamazaki symbol. However, this was just verified in Lemma 4.5. □

5. Higher regularity of weak solutions

In this section, we consider a family of operators $\{\mathcal{B}_t\}_{0<t<T}$ associated with coefficients $B(t,\cdot) \in \mathcal{E}(\Lambda, \lambda, \alpha + \varepsilon, M)$. Note that we do not require any regularity in time in this section. Provided that the associated problem (P') admits a solution, we show higher spatial regularity for this solution in Proposition 5.3. This is based on a commutator argument that already appeared in [7]. Implicit constants are allowed to depend on p, q, Λ, λ, α, ε, κ, Hölder constants, and dimensions.

Definition 5.1. The operator ∂^α_x is defined as the (unbounded) Fourier multiplication operator on L^2_x with symbol $|\xi|^\alpha$. It extrapolates to a bounded operator $\mathrm{W}^{\alpha,p}_x \to \mathrm{L}^p_x$ and we keep writing ∂^α_x.

Recall that $f \in \mathrm{W}^{s+\alpha,p}_x$ if, and only if, $f \in \mathrm{W}^{s,p}_x$ and $\partial^\alpha_x f \in \mathrm{W}^{s,p}_x$, see [40, p. 133].

We use the representation of ∂^α_x as a hypersingular integral to show the following commutator estimate.

Lemma 5.2. *Let $p \in (1,\infty)$. Assume that b is a smooth and bounded scalar function on $\mathbb{R}^d$. Then the commutator $[\partial^\alpha_x, b]$, initially defined on $\mathrm{W}^{\alpha,p}_x$, extends to a bounded operator* *on L^p_x, and satisfies the estimate*

$$\|[\partial^\alpha_x, b]f\|_{\mathrm{L}^p_x} \lesssim \|b\|_{\mathrm{C}^{\alpha+\varepsilon}_x}\|f\|_{\mathrm{L}^p_x} \qquad (f \in \mathrm{W}^{\alpha,p}_x). \tag{16}$$

Proof. Observe that, since $[\partial^\alpha_x, b]\colon \mathrm{W}^{\alpha,p}_x \to \mathrm{L}^p_x$ is bounded, it suffices, in virtue of density and Fatou's lemma, to establish (16) for f smooth and bounded.

According to [38, Sec. 25.4], the fractional derivative ∂^α_x acts on bounded and smooth functions g as the *hypersingular integral* given for $x \in \mathbb{R}^d$ by

$$\partial^\alpha_x g(x) = c\int_{\mathbb{R}^d} \frac{g(y)-g(x)}{|y-x|^{d+\alpha}}\,\mathrm{d}y = c\int_{\mathbb{R}^d} \frac{g(x-y)-g(x)}{|y|^{d+\alpha}}\,\mathrm{d}y.$$

We can apply this identity to f and bf in virtue of the assumption on b and the reduction at the beginning of this proof. Consequently, the commutator can be written as

$$\begin{aligned}[\partial^\alpha_x, b]f(x) &= c\Big(\int_{\mathbb{R}^d} \frac{b(x-y)f(x-y)-b(x)f(x)}{|y|^{d+\alpha}}\,\mathrm{d}y - b(x)\int_{\mathbb{R}^d} \frac{f(x-y)-f(x)}{|y|^{d+\alpha}}\,\mathrm{d}y\Big)\\ &= c\int_{\mathbb{R}^d} \frac{(b(x-y)-b(x))f(x-y)}{|y|^{d+\alpha}}\,\mathrm{d}y.\end{aligned}$$

Split the integral into the regions $|y| \le 1$ and $|y| \ge 1$ and use Hölder-regularity of b in the first, and boundedness of b in the second case, to bound the absolute value of the expression from above by

$$[b]_{\mathrm{C}^{\alpha+\varepsilon}_x}\int_{|y|\le 1} |y|^{-d+\varepsilon}|f(x-y)|\,\mathrm{d}y + \|b\|_\infty \int_{|y|\ge 1} |y|^{-d-\alpha}|f(x-y)|\,\mathrm{d}y.$$

In summary, we have shown

$$|[\partial^\alpha_x, b]f(x)| \lesssim \|b\|_{\mathrm{C}^{\alpha+\varepsilon}_x}\int_{\mathbb{R}^d} \big[\mathbf{1}_{|y|\le 1}|y|^{-d+\varepsilon} + \mathbf{1}_{|y|\ge 1}|y|^{-d-\alpha}\big]|f(x-y)|\,\mathrm{d}y.$$

Since $\mathbf{1}_{|y|\le 1}|y|^{-d+\varepsilon} + \mathbf{1}_{|y|\ge 1}|y|^{-d-\alpha}$ is integrable over $\mathbb{R}^d$, the claim follows from Young's convolution inequality. □

Proposition 5.3. *Given a weak (p,q)-solution u of (P') for some right-hand side $f \in \mathrm{L}^q_t(\mathrm{L}^p_x)$, one has higher spatial regularity in the sense $u \in \mathrm{L}^q_t(\mathrm{W}^{1+\alpha,p}_x)$ together with the estimate*

$$\|u\|_{\mathrm{L}^q_t(\mathrm{W}^{1+\alpha,p}_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}.$$

Proof. The proof divides into four steps.

Step 1: Regularization of the equation. Let $\rho \in \mathrm{C}^\infty_0(\mathbb{R}^d)$ be positive with integral one and define the usual mollifier sequence $\rho_n(x) \coloneqq n^d\rho(nx)$. Put $B_n \coloneqq \rho_n *_x B$, where $*_x$ denotes convolution in the x-variable. One has

$$B_n(t,x)\xi\cdot\eta = \int_{\mathbb{R}^d} \rho_n(y)B(t,x-y)\xi\cdot\eta\,\mathrm{d}y \qquad (\xi,\eta \in \mathbb{C}^{dm}),$$

hence B_n is elliptic with the same bounds as B. In conjunction with the calculation

$$\begin{aligned}\|B_n(t,x) - B_n(t,y)\| &\le \int_{\mathbb{R}^d} \rho_n(z)\|B(t,x-z)-B(t,y-z)\|\,\mathrm{d}z\\ &\le M|x-y|^{\alpha+\varepsilon},\end{aligned}$$

this shows that B_n is again in the class $\mathcal{E}(\Lambda, \lambda, \alpha + \varepsilon, M)$. Similarly, we derive for fixed n using smoothness of ρ that B_n is Lipschitz in the x variable uniformly in t. Now, according to Theorem 3.1, there exist unique weak (p,q)-solutions u_n to equation (P') with B replaced by B_n in the definition of $\mathcal{B}_t$.

Step 2: Qualitative higher regularity for solutions of the regularized equations. Using the method of difference quotients, we show that the solutions u_n from Step 1 belong to the class $\mathrm{L}^q_t(\mathrm{W}^{2,p}_x)$. This is a non-quantitative technical necessity to justify certain calculations in Step 3. To keep the notation concise, we will omit the subscript n and simply write u instead of u_n for the solution and B instead of B_n for the coefficients. We emphasize that, in this step, the only quantitative property of the regularized coefficients that we are going to use is the Lipschitz property in x uniform in t.

For $y \in \mathbb{R}^d$, define the translation operator S_y in the x-variable by $f \mapsto f(\cdot + y)$. We extend S_y by pointwise application in t to parabolic spaces like $\mathrm{L}^q_t(\mathrm{L}^p_x)$ (for simplicity, we keep writing the symbol S_y for this extension). Then, set for $j = 1, \dots, d$ and $h \in \mathbb{R}$ the difference quotient operator $D^j_h u \coloneqq \frac{1}{h}(S_{h\mathrm{e}_j} u - u)$, where e_j is the jth unit vector in $\mathbb{R}^d$. Observe that the operator D^j_h leaves the space of test functions invariant.

Using the chain rule and translation in the x-variable, one gets for t^* fixed, $y \in \mathbb{R}^d$, and $g \in \mathrm{W}^{1,p'}_x$ the identity

$$
\begin{aligned}
&b_{t^*}(S_y u(t^*), g)\\
&= \int_{\mathbb{R}^d} B(t^*, x)\nabla u(t^*, x+y) \cdot \overline{\nabla g(x)} \,\mathrm{d}x\\
&= \int_{\mathbb{R}^d} [B(t^*, x) - B(t^*, x+y)]\nabla u(t^*, x+y) \cdot \overline{\nabla g(x)} \,\mathrm{d}x + b_{t^*}(u(t^*), S_{-y} g).
\end{aligned}
\tag{17}
$$

Note that S_{-y} is the adjoint of S_y, and, consequently, $-D^j_{-h}$ is the adjoint of D^j_h. Hence, if we plug $D^j_h u$ in (IE), and use the adjoint of D^j_h for the first and third, and (17) for the second term, we obtain

$$
\begin{aligned}
&\int_0^T -\varphi'(s)(D^j_h u(s) \,|\, g) + \varphi(s) b_s(D^j_h u(s), g) + \varphi(s)\kappa(D^j_h u(s) \,|\, g) \,\mathrm{d}s\\
&= \int_0^T \varphi(s) \int_{\mathbb{R}^d} \Big(\frac{B(s,x) - B(s, x+h\mathrm{e}_j)}{h}\Big) \nabla u(s, x+h\mathrm{e}_j) \cdot \overline{\nabla g(x)} \,\mathrm{d}x \,\mathrm{d}s\\
&\quad - \int_0^T -\varphi'(s)(u(s) \,|\, D^j_{-h} g) + \varphi(s) b_s(u(s), D^j_{-h} g) + \varphi(s)\kappa(u(s) \,|\, D^j_{-h} g) \,\mathrm{d}s\\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

To bound term II, we use first that u is a solution for the right-hand side f, followed by the fact that we can estimate the difference quotients of g by ∇g, see for instance [19, Sec. 5.8.2. Thm. 3]. So, write

$$
\begin{aligned}
&\int_0^T -\varphi'(s)(u(s) \,|\, D^j_{-h} g) + \varphi(s) b_s(u(s), D^j_{-h} g) + \varphi(s)\kappa(u(s) \,|\, D^j_{-h} g) \,\mathrm{d}s\\
&\qquad = \int_0^T \varphi(s)(f(s) \,|\, D^j_{-h} g) \,\mathrm{d}s,
\end{aligned}
$$

and for $s \in (0,T)$ fixed and all $h \in \mathbb{R}$, estimate the pairing in its integrand by

$$|(f(s) \,|\, D^j_{-h} g)| \leq \|f(s)\|_{\mathrm{L}^p_x} \|D^j_{-h} g\|_{\mathrm{L}^{p'}_x} \lesssim \|f(s)\|_{\mathrm{L}^p_x} \|\nabla g\|_{\mathrm{L}^{p'}_x}.$$

Using Hölder's inequality in the t-variable reveals that term II is induced by a function in $\mathrm{L}^q_t(\mathrm{W}^{-1,p}_x)$, with bound independent of h. For term I, use that B is Lipschitz in the x-variable uniformly in $s \in (0,T)$, along with Hölder's inequality and translation invariance of the L^p_x-norm.

Eventually, we see that $D^j_h u$ is a weak (p,q)-solution to some right-hand side in $\mathrm{L}^q_t(\mathrm{W}^{-1,p}_x)$, where the norm of the right-hand side can be controlled independently of h. Consequently, the estimate from Theorem 3.1 gives

$$\|D^j_h u\|_{\mathrm{L}^q_t(\mathrm{W}^{1,p}_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \qquad (j = 1, \dots, d, h \in \mathbb{R}). \tag{18}$$

In particular, we deduce from (18) that there is a sequence $(h_n)_n$ of positive numbers such that h_n converges to 0, and such that $D^j_{h_n} u$ converges to a weak limit point $v \in \mathrm{L}^q_t(\mathrm{W}^{1,p}_x)$. We claim that, for almost every $s \in (0,T)$, the function $v(s)$ is the jth weak derivative in the x-variable of $u(s)$. Indeed, it follows from the "integration by parts"-identity

$$\int_{\mathbb{R}^d} (D^j_h f) g \, \mathrm{d}x = - \int_{\mathbb{R}^d} f (D^j_{-h} g) \, \mathrm{d}x,$$

which is a simple consequence of translation in the integral, that one has, for $\varphi \in \mathrm{C}^\infty_0(\mathbb{R}^d)$ and $\psi \in \mathrm{C}^\infty_0(0,T)$, the identity

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}^d} \partial_j \varphi(x) u(s,x) \, \mathrm{d}x \; \psi(s) \, \mathrm{d}s &= - \lim_n \int_0^T \int_{\mathbb{R}^d} D^j_{-h_n} \varphi(x) u(s,x) \, \mathrm{d}x \; \psi(s) \, \mathrm{d}s \\ &= \lim_n \int_0^T \int_{\mathbb{R}^d} \varphi(x) D^j_{h_n} u(s,x) \, \mathrm{d}x \; \psi(s) \, \mathrm{d}s. \end{aligned}$$

Integration against $\varphi(x)\psi(s)$ gives rise to a functional on $\mathrm{L}^q_t(\mathrm{L}^p_x)$, hence weak convergence of $D^j_{h_n} u$ identifies the latest limit with

$$\int_0^T \int_{\mathbb{R}^d} \varphi(x) v(s,x) \, \mathrm{d}x \; \psi(s) \, \mathrm{d}s.$$

Finally, the fundamental lemma of the calculus of variations shows

$$- \int_{\mathbb{R}^d} \partial_j \varphi(x) u(s,x) \, \mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x) v(s,x) \, \mathrm{d}x \qquad \text{for almost every } s \in (0,T),$$

which reveals $\partial_j u(s,x) = v(s,x)$ for almost every $s \in (0,T)$ and $j = 1, \dots, d$. But as $v \in \mathrm{L}^q_t(\mathrm{W}^{1,p}_x)$, the lifting property for Sobolev spaces shows $u \in \mathrm{L}^q_t(\mathrm{W}^{2,p}_x)$.

Step 3: Uniform bounds using a commutator argument. The current step is the essence of this proof, filling in the details of the heuristic given in the roadmap in Section 1.2. As in Step 2, we continue to work with the regularized coefficients, but still omitting the subscript n in the notation. However, now we will also rely on the properties established in Step 1 that are uniform in n.

Recall that ∂^α_x is the fractional derivative of order α in x from Definition 5.1. Note that ∂^α_x commutes with ∇_x and ∂_t, which is a consequence of its definition as a Fourier multiplier.

Our goal is to show that $\partial_x^\alpha u$ is a weak (p,q)-solution to some admissible right-hand side. Note that $\partial_x^\alpha u \in \mathrm{L}_t^q(\mathrm{W}_x^{1,p})$, owing to the higher spatial regularity of u established in Step 2, which allows us to plug this term into the equation. That being said, calculate

$$\begin{aligned}\int_0^T &-\varphi'(s)(\partial_x^\alpha u(s) \mid g) + \varphi(s) b_s(\partial_x^\alpha u(s), g) + \varphi(s)\kappa(\partial_x^\alpha u(s) \mid g)\,\mathrm{d}s \\ &= \int_0^T -\varphi'(s)(u(s) \mid \partial_x^\alpha g) + \varphi(s) b_s(u(s), \partial_x^\alpha g) + \varphi(s)\kappa(u(s) \mid \partial_x^\alpha g)\,\mathrm{d}s \\ &\qquad + \int_0^T \varphi(s)\big[b_s(\partial_x^\alpha u(s), g) - b_s(u(s), \partial_x^\alpha g)\big]\,\mathrm{d}s.\end{aligned}$$

Note that $\partial_x^\alpha g \in \mathrm{W}_x^{1,p'}$ since g is smooth. Hence, in the light of Remark 2.5 (iv), use the equation for u, and expand the definition of b_s, to rewrite the last expression as

$$\begin{aligned}\int_0^T \varphi(s)(f(s) \mid \partial_x^\alpha g)\,\mathrm{d}s + \int_0^T \varphi(s) \int_{\mathbb{R}^d} B(s,x)\nabla\partial_x^\alpha u(s) \cdot \overline{\nabla g} - B(s,x)\nabla u(s) \cdot \overline{\nabla \partial_x^\alpha g}\,\mathrm{d}x\,\mathrm{d}s \\ =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

We have to check that the terms I and II are induced by right-hand sides in $\mathrm{L}_t^q(\mathrm{W}_x^{-1,p})$. For term I, this is a direct consequence of the mapping properties of ∂_x^α, and the $\mathrm{L}_t^q(\mathrm{W}_x^{-1,p})$-norm can be controlled by $\|f\|_{\mathrm{L}_t^q(\mathrm{L}_x^p)}$.

Let us proceed with term II. Keep in mind that $B(s,x)$ is Lipschitz in x, and thus is a multiplier on $\mathrm{W}_x^{1,p}$. We use this fact and higher regularity of u from Step 2 to commute ∂_x^α with ∇_x to rewrite the integral over $\mathbb{R}^d$ in II as

$$\int_{\mathbb{R}^d} B(s,x)\nabla\partial_x^\alpha u(s) \cdot \overline{\nabla g} - B(s,x)\nabla u(s) \cdot \overline{\nabla\partial_x^\alpha g}\,\mathrm{d}x = \int_{\mathbb{R}^d} \Big[B(s,x), \partial_x^\alpha\Big]\nabla u(s) \cdot \overline{\nabla g}\,\mathrm{d}x.$$

Then, we apply the commutator estimate from Lemma 5.2 for all times s (keep in mind that $B(s,\cdot)$ is smooth and bounded by the regularization in Step 1) along with Hölder's inequality to deduce that term II belongs to class $\mathrm{L}_t^q(\mathrm{W}_x^{-1,p})$ as well. In this case, the $\mathrm{L}_t^q(\mathrm{W}_x^{-1,p})$-norm is controlled by $\|u\|_{\mathrm{L}_t^q(\mathrm{W}_x^{1,p})}$, where implicit constants depend on the Hölder regularity of the coefficients, which are also under control by Step 1.

In summary, Theorem 3.1 gives $\partial_x^\alpha u \in \mathrm{L}_t^q(\mathrm{W}_x^{1,p})$ with estimate

$$\|u\|_{\mathrm{L}_t^q(\mathrm{W}_x^{1+\alpha,p})} \lesssim \|u\|_{\mathrm{L}_t^q(\mathrm{L}_x^p)} + \|\partial_x^\alpha u\|_{\mathrm{L}_t^q(\mathrm{W}_x^{1,p})} \lesssim \|f\|_{\mathrm{L}_t^q(\mathrm{L}_x^p)} + \|u\|_{\mathrm{L}_t^q(\mathrm{W}_x^{1,p})} \lesssim \|f\|_{\mathrm{L}_t^q(\mathrm{L}_x^p)},$$

where we have applied Theorem 3.1 once more, but this time for u instead of $\partial_x^\alpha u$, to get the final estimate.

Step 4: Taking the limit in Step 1. The solutions u_n to the regularized equations from Step 1 satisfy the identity

$$\begin{aligned}&\int_0^T \varphi'(s)(u_n(s)\,|\,g) + \varphi(s)(f(s)\,|\,g) - \varphi(s)\kappa(u_n(s)\,|\,g)\,\mathrm{d}s \\ &= \int_0^T \varphi(s)\int_{\mathbb{R}^d} B_n(s,x)\nabla u_n(s)\cdot\overline{\nabla g}\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{19}$$

Moreover, we have seen in Step 3 that $\|u_n\|_{\mathrm{L}^q_t(\mathrm{W}^{1+\alpha,p}_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}$ holds uniformly in n. Since p and q are in the reflexive range, we find a subsequence (which we still denote by u_n) for which u_n and ∇u_n converge weakly in $\mathrm{L}^q_t(\mathrm{L}^p_x)$ to some limit $v \in \mathrm{L}^q_t(\mathrm{W}^{1+\alpha,p}_x)$. Moreover, $\|v\|_{\mathrm{L}^q_t(\mathrm{W}^{1+\alpha,p}_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}$. The former fact directly enables us to pass to the limit

$$\int_0^T \varphi'(s)(v(s)\,|\,g) + \varphi(s)(f(s)\,|\,g) + \varphi(s)\kappa(v(s)\,|\,g)\,\mathrm{d}s$$

on the left-hand side of (19). For the right-hand side, write

$$\int_0^T \varphi(s)\int_{\mathbb{R}^d} B_n(s,x)\nabla u_n(s,x)\cdot\overline{\nabla g(x)}\,\mathrm{d}x\,\mathrm{d}s = \int_0^T \varphi(s)\int_{\mathbb{R}^d} \nabla u_n(s,x)\cdot\overline{B_n(s,x)^*\nabla g(x)}\,\mathrm{d}x\,\mathrm{d}s.$$

Clearly, $B_n(s,x)^*$ is uniformly bounded, and, by regularity in the x-variable of B, one has $B_n(s,x)^* \to B(s,x)^*$ pointwise. Hence, the dominated convergence theorem gives $\varphi(s)B_n(s,x)^*\nabla g(x) \to \varphi(s)B(s,x)^*\nabla g(x)$ strongly in $\mathrm{L}^{q'}_t(\mathrm{L}^{p'}_x)$. Hence, the right-hand side of (19) converges to

$$\int_0^T \varphi(s)\int_{\mathbb{R}^d} B(s,x)\nabla v(s,x)\cdot\overline{\nabla g(x)}\,\mathrm{d}x\,\mathrm{d}s.$$

In summary, taking the limit in (19) results in

$$\begin{aligned}&\int_0^T \varphi'(s)(v(s)\,|\,g) + \varphi(s)(f(s)\,|\,g) - \varphi(s)\kappa(v(s)\,|\,g)\,\mathrm{d}s \\ &= \int_0^T \varphi(s)\int_{\mathbb{R}^d} B(s,x)\nabla v(s,x)\cdot\overline{\nabla g(x)}\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

This shows that u and v solve the same equation. Uniqueness of solutions leads to $u = v \in \mathrm{L}^q_t(\mathrm{W}^{1+\alpha,p}_x)$ as desired. The corresponding estimate was already mentioned above. □

Remark 5.4. In Step 3, we have used that the fractional derivative can be written as a Fourier multiplier, and hence commutes with ∇. This is the central reason that ties us to whole-space in the x variable. Moreover, the limiting argument in Step 4 relies on the control of the implied constants from Theorem 3.1.

6. Proof of Theorem 1.1

Following the plan outlined in the roadmap in Section 1.2 we assemble the results from the previous sections to prove Theorem 1.1.

Proof of Theorem 1.1. Let $f \in \mathrm{L}^q_t(\mathrm{L}^p_x)$. In virtue of Remark 2.5 (iii), we consider the shifted problem (P') with $\mathcal{B}_t = \mathcal{L}_t$ instead of (P). Let u be its unique (p,q)-solution from Theorem 3.1. We want to show $\mathcal{L}_t u(t) \in \mathrm{L}^q_t(\mathrm{L}^p_x)$ with estimate against $\|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}$. This happens in three steps.

Step 1: Reduction to right-hand sides in $\mathrm{C}^\infty_0(\mathrm{L}^p_x \cap \mathrm{L}^2_x)$. Let $(f_n)_n$ be a sequence in $\mathrm{C}^\infty_0(\mathrm{L}^p_x \cap \mathrm{L}^2_x)$ that converges to f in $\mathrm{L}^q_t(\mathrm{L}^p_x)$. Let u_n be the weak (p,q)-solution of (P') with $\mathcal{B}_t = \mathcal{L}_t$ and right-hand side f_n provided by Theorem 3.1. Suppose the maximal regularity estimate

$$\|\mathcal{L}_t u_n(t)\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \lesssim \|f_n\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \tag{20}$$

with implicit constant independent of n. Since the u_n are weak (p,q)-solutions, arguing as in Step 4 of the proof of Proposition 5.3, we see that u_n converges weakly to u in $\mathrm{L}^q_t(\mathrm{L}^p_x)$ and that ∇u_n converges weakly to ∇u in $\mathrm{L}^q_t(\mathrm{L}^p_x)$. Let $v \in \mathrm{L}^{q'}_t(\mathrm{L}^{p'}_x) \cap \mathrm{L}^2_t(\mathrm{L}^2_x)$. Then, in particular, $(\mathcal{L}_{t^*} u_n(t^*) \,|\, v(t^*)) \to (\mathcal{L}_{t^*} u(t^*) \,|\, v(t^*))$ for almost every t^*. Consequently, we find by Fatou's lemma and (20) that

$$\begin{aligned}\Big|\int_0^T \langle \mathcal{L}_t u(t), v(t)\rangle \,\mathrm{d}t\Big| &\leq \liminf_n \Big|\int_0^T \langle \mathcal{L}_t u_n(t), v(t)\rangle \,\mathrm{d}t\Big| \\ &\lesssim \liminf_n \|f_n\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \|v\|_{\mathrm{L}^{q'}_t(\mathrm{L}^{p'}_x)} \\ &= \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \|v\|_{\mathrm{L}^{q'}_t(\mathrm{L}^{p'}_x)}.\end{aligned}$$

Hence, duality yields $\|\mathcal{L}_t u(t)\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}$, provided we can show (20).

Step 2: Treating the first term in (♡). We write u instead of u_n for this part to emphasize that this step does not rely on the regularization of the right-hand side. Let $v \in \mathrm{L}^{q'}_t(\mathrm{L}^{p'}_x) \cap \mathrm{L}^2_t(\mathrm{L}^2_x)$. We aim to estimate S_1 by duality. To this end, write

$$\begin{aligned}&\Big|\int_0^T \int_0^t ((\mathcal{L}_t + \kappa)\mathrm{e}^{-(t-s)(\mathcal{L}_t+\kappa)}(\mathcal{L}_t - \mathcal{L}_s)u(s) \,|\, v(t)) \,\mathrm{d}s \,\mathrm{d}t\Big| \\ = &\Big|\int_0^T \int_0^t (u(s) \,|\, ((\mathcal{L}_t + \kappa)\mathrm{e}^{-(t-s)(\mathcal{L}_t+\kappa)}(\mathcal{L}_t - \mathcal{L}_s))^* v(t)) \,\mathrm{d}s \,\mathrm{d}t\Big|.\end{aligned} \tag{21}$$

For t and s fixed, the operator $(\mathcal{L}_t + \kappa)\mathrm{e}^{-(t-s)(\mathcal{L}_t+\kappa)}(\mathcal{L}_t - \mathcal{L}_s)$ maps $\mathrm{W}^{1+\alpha,p}_x \to \mathrm{L}^p_x$ with norm controlled by $|t-s|^{-1+\varepsilon}$ as combining Lemmas 4.2 and 4.3 shows. Consequently, its adjoint maps $\mathrm{L}^{p'}_x \to \mathrm{W}^{-1-\alpha,p'}_x$ with the same bound. Use this together with the

$\mathrm{W}^{1+\alpha,p}_x$–$\mathrm{W}^{-1-\alpha,p'}_x$ duality pairing in (21) to bound its right-hand side by

$$\begin{aligned}&\int_0^T\int_0^t \|u(s)\|_{\mathrm{W}^{1+\alpha,p}_x} \|((\mathcal{L}_t+\kappa)\mathrm{e}^{-(t-s)(\mathcal{L}_t+\kappa)}(\mathcal{L}_t-\mathcal{L}_s))^* v(t)\|_{\mathrm{W}^{-1-\alpha,p'}_x}\,\mathrm{d}s\,\mathrm{d}t \\ \lesssim &\int_0^T\int_0^t \|u(s)\|_{\mathrm{W}^{1+\alpha,p}_x}|t-s|^{-1+\varepsilon}\|v(t)\|_{\mathrm{L}^{p'}_x}\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

By Hölder's inequality, this can be bounded by

$$\Big\|\int_0^t |t-s|^{-1+\varepsilon}\|u(s)\|_{\mathrm{W}^{1+\alpha,p}_x}\,\mathrm{d}s\Big\|_{\mathrm{L}^q_t}\|v(t)\|_{\mathrm{L}^{q'}_t(\mathrm{L}^{p'}_x)}. \tag{22}$$

By Young's convolution inequality (the convolution kernel $s\mapsto |s|^{-1+\varepsilon}$ is integrable over $(0,T)$) and Proposition 5.3, control (22) by $\|u\|_{\mathrm{L}^q_t(\mathrm{W}^{1+\alpha,p}_x)}\|v\|_{\mathrm{L}^{q'}_t(\mathrm{L}^{p'}_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}\|v\|_{\mathrm{L}^{q'}_t(\mathrm{L}^{p'}_x)}$. Hence, duality shows in summary

$$\|\int_0^t (\mathcal{L}_t+\kappa)\mathrm{e}^{-(t-s)(\mathcal{L}_t+\kappa)}(\mathcal{L}_t-\mathcal{L}_s)u(s)\,\mathrm{d}s\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}. \tag{23}$$

In particular, the above calculation (applied with v constant) shows that

$$\int_0^t \|(\mathcal{L}_t+\kappa)\mathrm{e}^{-(t-s)(\mathcal{L}_t+\kappa)}(\mathcal{L}_t-\mathcal{L}_s)u(s)\|\,\mathrm{d}s<\infty \qquad \text{for almost every } t\in(0,T).$$

Whence, Hille's theorem shows

$$\begin{aligned}&(\mathcal{L}_t+\kappa)\int_0^t \mathrm{e}^{-(t-s)(\mathcal{L}_t+\kappa)}(\mathcal{L}_t-\mathcal{L}_s)u(s)\,\mathrm{d}s\\ &=\int_0^t (\mathcal{L}_t+\kappa)\mathrm{e}^{-(t-s)(\mathcal{L}_t+\kappa)}(\mathcal{L}_t-\mathcal{L}_s)u(s)\,\mathrm{d}s\\ &=S_1(u)(t),\end{aligned}$$

so that (23) translates to

$$\|S_1(u)\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}.$$

Step 3: Treating the second term in (♡). Thanks to the reduction to more regular right-hand sides in Step 1, Proposition 4.4 directly yields $\|S_2(f_n)\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \lesssim \|f_n\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}$.

In summary, Steps 2 and 3 in conjunction with Theorem 3.1 give

$$\|\mathcal{L}_t u_n(t)\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \lesssim \|S_1(u_n)\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} + \|S_2(f_n)\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} + \kappa\|u_n\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)},$$

which is (20). Hence, $\|\mathcal{L}_t u(t)\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)} \lesssim \|f\|_{\mathrm{L}^q_t(\mathrm{L}^p_x)}$ as was demonstrated in Step 1. This completes the proof. □

Appendix A. Interpolation

We derive a representation of the real interpolation spaces used in Section 4 using analytic semigroups. While such descriptions are of course part of the literature, the following exposition aims to give explicit control for the implicit constants and is tailored to our desired application.

Let $\psi \in [0, \pi/2)$. Assume that we are given a reflexive Banach space X and an analytic semigroup $\{\mathrm{e}^{-zT}\}_{z \in \mathrm{S}_\psi}$ on X that satisfies for some $M > 0$ the bound

$$\|\mathrm{e}^{-zT}\|_{X \to X} + \|zT\mathrm{e}^{-zT}\|_{X \to X} \leq M \qquad (z \in \mathrm{S}_\psi). \tag{24}$$

Proposition A.1. *Let $\varphi \in (-\psi, \psi)$, and put $\gamma(t) \coloneqq t\mathrm{e}^{\mathrm{i}\varphi}$. For $\theta \in (0,1)$, the space*

$$\mathrm{D}_T(\theta, \varphi) \coloneqq \Big\{ x \in X \; ; \; t \mapsto \|t^{1-\theta} T \mathrm{e}^{-\gamma(t)T} x\|_X \in \mathrm{L}^\infty(0,\infty) \Big\},$$

equipped with the norm

$$\|x\|_{\mathrm{D}_T(\theta,\varphi)} \coloneqq \|x\|_X + \|t^{1-\theta} T \mathrm{e}^{-\gamma(t)T} x\|_{\mathrm{L}^\infty(0,\infty;X)},$$

coincides with the real interpolation space $(X, \mathrm{D}(T))_{\theta,\infty}$ up to equivalent norms, and the constants in the norm equivalence depend only on θ and M.

The (θ,∞)-real interpolation space in Proposition A.1 can be defined using the trace method as follows. Suppose that we are given two Banach spaces $Y \subseteq X$ and $\theta \in (0,1)$. Then, define the space

$$\begin{aligned} \mathrm{V}(\theta, X, Y) \coloneqq \Big\{ u \colon (0,\infty) \to X \; ; \; & t \mapsto u_\theta(t) \coloneqq t^{1-\theta} u(t) \in \mathrm{L}^\infty(0,\infty;Y) \\ & \& \quad t \mapsto v_\theta(t) \coloneqq t^{1-\theta} u'(t) \in \mathrm{L}^\infty(0,\infty;X) \Big\}, \end{aligned}$$

and equip it with the norm $\|u\|_{\mathrm{V}(\theta,X,Y)} \coloneqq \|u_\theta\|_{\mathrm{L}^\infty(0,\infty;Y)} + \|v_\theta\|_{\mathrm{L}^\infty(0,\infty;X)}$. As usual, u' denotes the weak derivative of u. It turns out that functions in $\mathrm{V}(\theta,X,Y)$ are absolutely continuous with values in X. This enables us to put

$$(X,Y)_{\theta,\infty} \coloneqq \{u(0) \; ; \; u \in \mathrm{V}(\theta,X,Y)\}, \qquad \|x\|_{(X,Y)_{\theta,\infty}} \coloneqq \inf_u \|u\|_{\mathrm{V}(\theta,X,Y)},$$

where the infimum is taken over all $u \in \mathrm{V}(\theta,X,Y)$ with $u(0) = x$, confer [33, Prop. 1.2.10].

The proof of Proposition A.1 follows the lines of [33, Prop. 2.2.2]. However, we include three marginal adaptations. First, we explicitly trace the dependence of the implicit constants. Second, our space $\mathrm{D}_T(\theta,\varphi)$ is defined using complex rays in the sector of analyticity of T. Third, we drop the restriction $t \in (0,1)$ from the definition of the space $\mathrm{D}_T(\theta,\infty)$ in [33].

Proof of Proposition A.1. To ease the notation, all norms without a subscript shall denote the norm on X.

Step 1: $\|x\|_{\mathrm{D}_T(\theta,\varphi)} \lesssim \|x\|_{(X,\mathrm{D}(T))_{\theta,\infty}}$. Let $u \in \mathrm{V}(\theta, X, \mathrm{D}(T))$ with $u(0) = x$. Write using absolute continuity $x = u(t) - \int_0^t u'(s)\,\mathrm{d}s$. First, use the decomposition of x and the triangle inequality to give

$$\|t^{1-\theta} T\mathrm{e}^{-\gamma(t)T} x\| \leq \|t^{1-\theta} T\mathrm{e}^{-\gamma(t)T} u(t)\| + \|t^{1-\theta} T\mathrm{e}^{-\gamma(t)T} \int_0^t u'(s)\,\mathrm{d}s\|.$$

For the first term, use (24) and the definition of the graph norm to deduce

$$\|t^{1-\theta} T\mathrm{e}^{-\gamma(t)T} u(t)\| \leq M \|t^{1-\theta} Tu(t)\| \leq M \|t^{1-\theta} u(t)\|_{\mathrm{D}(T)}.$$

For the second term, continue likewise using (24) and $|\gamma(t)| = t$ to get

$$\|t^{1-\theta}T\mathrm{e}^{-\gamma(t)T}\int_0^t u'(s)\,\mathrm{d}s\| = \|t^{1-\theta}\gamma(t)T\mathrm{e}^{-\gamma(t)T}\frac{1}{t}\int_0^t u'(s)\,\mathrm{d}s\| \leq M\|t^{1-\theta}\frac{1}{t}\int_0^t u'(s)\,\mathrm{d}s\|.$$

Using [33, Cor. 1.2.9], proceed by

$$M\|t^{1-\theta}\frac{1}{t}\int_0^t u'(s)\,\mathrm{d}s\| \leq \frac{M}{1-\theta}\|t^{1-\theta}u'(t)\|.$$

So far, we have shown that

$$\|t^{1-\theta}T\mathrm{e}^{-\gamma(t)T}x\| \lesssim \|t^{1-\theta}u(t)\|_{\mathrm{D}(T)} + \|t^{1-\theta}u'(t)\| \leq \|u\|_{\mathrm{V}(\theta,X,\mathrm{D}(T))}, \tag{25}$$

with implicit constant only depending on θ and M.

Second, use again the decomposition of x together with averaging and Fubini's theorem to calculate

$$\begin{aligned}\|x\| &\leq \int_0^1 \|u(t)\|\,\mathrm{d}t + \int_0^1\int_0^t \|u'(s)\|\,\mathrm{d}s\,\mathrm{d}t \\ &\leq \|t^{1-\theta}u(t)\|_{\mathrm{L}^\infty(0,1;X)}\int_0^1 t^\theta\,\frac{\mathrm{d}t}{t} + \int_0^1\int_s^1 \|u'(s)\|\,\mathrm{d}t\,\mathrm{d}s \\ &\lesssim \|t^{1-\theta}u(t)\|_{\mathrm{L}^\infty(0,\infty;X)} + \|s^{1-\theta}u'(s)\|_{\mathrm{L}^\infty(0,\infty;X)}\int_0^1 s^\theta\,\frac{\mathrm{d}s}{s} \\ &\lesssim \|u\|_{\mathrm{V}(\theta,X,\mathrm{D}(T))}.\end{aligned}$$

Implicit constants only depend on θ. Apply the $\mathrm{L}^\infty(0,\infty)$-norm to estimate (25) and add it to this second inequality. Then, take the infimum over all admissible u to conclude this step.

Step 2: $\|x\|_{(X,\mathrm{D}(T))_{\theta,\infty}} \lesssim \|x\|_{\mathrm{D}_T(\theta,\varphi)}$. Let $x \in \mathrm{D}_T(\theta,\varphi)$, and put $u(t) \coloneqq \eta(t)\mathrm{e}^{-\gamma(t)T}x$, where $\eta\colon (0,\infty) \to [0,1]$ is a smooth cutoff function that is 1 around 0, supported in $[0,1]$, and which satisfies $\|\eta'\|_\infty \leq 2$. Observe that u and u' are again supported in $[0,1]$. Hence, for the first term in $\|u\|_{\mathrm{V}(\theta,X,\mathrm{D}(T))}$, we calculate for $t \in [0,1]$ using (24) that

$$\begin{aligned}\|t^{1-\theta}u(t)\|_{\mathrm{D}(T)} &\leq \|t^{1-\theta}\mathrm{e}^{-\gamma(t)T}x\| + \|t^{1-\theta}T\mathrm{e}^{-\gamma(t)T}x\| \\ &\leq M\|x\| + \|t^{1-\theta}T\mathrm{e}^{-\gamma(t)T}x\|_{\mathrm{L}^\infty(0,\infty;X)}.\end{aligned}$$

Similarly, the second term of $\|u\|_{\mathrm{V}(\theta,X,\mathrm{D}(T))}$ can be controlled in virtue of

$$\begin{aligned}\|t^{1-\theta}u'(t)\| &\leq \|t^{1-\theta}T\mathrm{e}^{-\gamma(t)T}x\| + \|\eta'\|_\infty\|t^{1-\theta}\mathrm{e}^{-\gamma(t)T}x\| \\ &\leq \|t^{1-\theta}T\mathrm{e}^{-\gamma(t)T}x\|_{\mathrm{L}^\infty(0,\infty;X)} + 2M\|x\|.\end{aligned}$$

In summary, $u \in \mathrm{V}(\theta,X,\mathrm{D}(T))$ with $\|u\|_{\mathrm{V}(\theta,X,\mathrm{D}(T))} \lesssim \|x\|_{\mathrm{D}_T(\theta,\varphi)}$, with implicit constant depending only on M. In particular, $\|x\|_{(X,\mathrm{D}(T))_{\theta,\infty}} \lesssim \|x\|_{\mathrm{D}_T(\theta,\varphi)}$. □

References

[1] M. Achache and E.M. Ouhabaz. *Lions' maximal regularity problem with $H^{\frac{1}{2}}$-regularity in time*. J. Differential Equations **266** (2019), no. 6, 3654–3678. doi:10.1016/j.jde.2018.09.015

[2] P. Acquistapace and B. Terreni. *A unified approach to abstract linear nonautonomous parabolic equations*. Rend. Sem. Mat. Univ. Padova **78** (1987), 47–107.

[3] R.A. Adams, J.J.F Fournier. *Sobolev Spaces*. Elsevier/Academic Press, Amsterdam, 2003.

[4] H. Amann. *Quasilinear parabolic problems via maximal regularity*. Adv. Differential Equations **10** (2005), no. 10, 1081–1110.

[5] W. Arendt, D. Dier, and S. Fackler. *J. L. Lions' problem on maximal regularity*. Arch. Math. **109** (2017), no. 1, 59–72. doi:10.1007/s00013-017-1031-6

[6] P. Auscher. *On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates*. Mem. Amer. Math. Soc. **186** (2007), no. 871.

[7] P. Auscher and M. Egert. *On non-autonomous maximal regularity for elliptic operators in divergence form*. Arch. Math. (Basel) **107** (2016), no. 3, 271–284. doi:10.1007/s00013-016-0934-y

[8] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian. *The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$*. Ann. of Math. (2) **156** (2002), no. 2, 633–654. doi:10.2307/3597201

[9] P. Auscher and P. Tchamitchian. Square root problem for divergence operators and related topics. Astérisque, vol. 249. Société mathématique de France, 1988.

[10] S. Bechtel. arXiv preprint, available at `https://arxiv.org/abs/2201.09561`.

[11] S. Bechtel and E.M. Ouhabaz. arXiv preprint, available at `https://arxiv.org/abs/2207.09115`.

[12] G.D. Blasio. *Maximal L^p regularity for nonautonomous parabolic equations in extrapolation spaces*. J. Evol. Equ. **6** (2006), no. 2, 229–245. doi:10.1007/s00028-006-0241-3

[13] R. Dautray and J.L. Lions. Mathematical Analysis and Numerical Methods for Science and Technology, vol. 5. Springer, Berlin, 1992.

[14] D. Dier and R. Zacher. *Non-autonomous maximal regularity in Hilbert spaces*. J. Evol. Equ. **17** (2017), no. 3, 883–907. doi:10.1007/s00028-016-0343-5

[15] H. Dong and D. Kim. *On the L_p-solvability of higher order parabolic and elliptic systems with BMO coefficients*. Arch. Ration. Mech. Anal **109** (2011), no. 3, 889–889. doi:10.1007/s00205-010-0345-3

[16] H. Dong and D. Kim. *Higher order elliptic and parabolic systems with variably partially BMO coefficients in regular and irregular domains*. J. Funct. Anal. **261** (2011), no. 11, 3279–3327. doi:10.1016/j.jfa.2011.08.001

[17] H. Dong and D. Kim. *L_p solvability of divergence type parabolic and elliptic systems with partially BMO coefficients*. Calc. Var. Partial Differential Equations **40** (2011), no. 3-4, 357–389. doi:10.1007/s00526-010-0344-0

[18] H. Dong and D. Kim. On L_p-estimates for elliptic and parabolic equations with A_p weights. Trans. Amer. Math. Soc. **370** (2018), no. 7, 5081–5130. doi:10.1090/tran/7161

[19] L.C. Evans. Partial Differential Equations. Second edition. Graduate Studies in Mathematics, vol. 19. Amer. Math. Soc., Providence, RI, 2010. doi:10.1090/gsm/019

[20] S. Fackler. arXiv preprint, available at `https://arxiv.org/abs/1511.06207`.

[21] S. Fackler. *J.-L. Lions' problem concerning maximal regularity of equations governed by non-autonomous forms*. Ann. Inst. H. Poincaré C Anal. Non Linéaire **34** (2017), no. 3, 699–709. doi:10.1016/j.anihpc.2016.05.001

[22] S. Fackler. *Nonautonomous maximal L^p-regularity under fractional Sobolev regularity in time*. Anal. PDE **11** (2018), no. 5, 1143–1169. doi:10.2140/apde.2018.11.1143

[23] C. Gallarati and M.C. Veraar. *Maximal regularity for non-autonomous equations with measurable dependence on time*. Potential Anal. **46** (2017), no. 3, 527–567. doi:10.1007/s11118-016-9593-7

[24] C. Gallarati and M.C. Veraar. *Evolution families and maximal regularity for systems of parabolic equations*. Adv. Differential Equations **22** (2017), no. 3-4, 169–190.

[25] B. Haak and E.M. Ouhabaz. *Maximal regularity for non-autonomous evolution equations*. Math. Ann. **363** (2015), no. 3-4, 1117–1145. doi:10.1007/s00208-015-1199-7

[26] M. Haase. The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications, vol. 169. Birkhäuser, Basel, 2006. doi:10.1007/3-7643-7698-8

[27] M. Hieber and S. Monniaux. *Pseudo-differential operators and maximal regularity results for non-autonomous parabolic equations*. Proc. Amer. Math. Soc. **128** (2000), 1047–1053. doi:10.1090/S0002-9939-99-05145-X

[28] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory*. Springer, Cham, 2016. doi:10.1007/978-3-319-48520-1

[29] T. Hytönen and P. Portal. *Vector-valued multiparameter singular integrals and pseudodifferential operators*. Adv. Math. **217** (2008), no. 2, 519–536. doi:10.1016/j.aim.2007.08.002

[30] N. Kalton, S. Mayboroda, and M. Mitrea. Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin spaces and applications to problems in partial differential equations. In *Interpolation theory and applications*, volume 445 of Contemp. Math., 121–177. Amer. Math. Soc., Providence, RI, 2007. doi:10.1090/conm/445/08598

[31] P.C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*, Lecture Notes in Math. 1855, 65–311. Springer, Berlin, 2004. doi:10.1007/978-3-540-44653-8_2

[32] J.L. Lions. Équations différentielles opérationnelles et problèmes aux limites. Die Grundlehren der mathematischen Wissenschaften, Bd. 111. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961.

[33] A. Lunardi. Analytic Semigroups and Optimal Regularity in Parabolic Problems. Progress in Nonlinear Differential Equations and Their Applications, vol. 16. Birkhäuser, Basel, 1995.

[34] E.M. Ouhabaz and C. Spina. *Maximal regularity for non-autonomous Schrödinger type equations*. J. Differential Equations **248** (2010), no. 7, 1668–1683. doi:10.1016/j.jde.2009.10.004

[35] P. Portal and Š. Željko. *Pseudodifferential operators on Bochner spaces and an application*. Math. Z. **253** (2006), no. 4, 805–819. doi:10.1090/S0002-9939-99-05145-X

[36] J. Prüss. *Maximal regularity for evolution equations in L^p-spaces*. Conf. Semin. Mat. Univ. Bari **285** (2002), 1–39.

[37] J. Prüss and R. Schnaubelt. *Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time*. J. Math. Anal. Appl. **256** (2001), no. 2, 405–430. doi:10.1006rjmaa.2000.7247

[38] S.G. Samko, A.A. Kilbas, and O.I. Marichev. Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach Science Publishers, Yverdon, 1993.

[39] L. de Simon. *Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine*. Rend. Sem. Mat. Univ. Padova **34** (1964), 205–223.

[40] E.M. Stein. Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, 1970. doi:10.1515/9781400883882

[41] H. Triebel. Interpolation Theory, Function Spaces, Differential Operators. North-Holland Mathematical Library, vol. 18. North-Holland Publishing, Amsterdam, 1978.

[42] L. Weis. *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*. Math. Ann. **319** (2001), no. 4, 735–758. doi:10.1007/PL00004457.

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