What is functional thinking? Theoretical considerations and first results of an international interview study

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In this paper, we present the first results from the Erasmus+ project FunThink which focuses on enhancing functional thinking from primary to upper secondary school. In an international interview study (in Cyprus, Germany, Netherlands, Poland, and Slovakia) we investigated 35 educational experts' views on what they consider functional thinking to be. From each country between six and nine experts were interviewed. We analyze these semi-structured interviews using qualitative content analysis, with both deductive and inductive categories, related to different conceptualizations of functions, mathematization, activities supporting functional thinking, and cognitive aspects related to functions. These analyses are currently underway; therefore, we present our theoretical background, our coding scheme which is under construction, and excerpts of three interviews in this proposal.

Keywords: Functional thinking, expert interviews, empirical study.

Introduction

Functional thinking is required when relating two or more quantities, e.g., when understanding scientific laws such as the dependency between speed, distance, and time or when modelling something we read about in every newspaper such as the spread of a virus. Hence, it is not only a key element of (school) mathematics but also relevant for other disciplines and everyday situations (e.g. Selden & Selden, 1992; Vollrath, 1989). However, there is no consensus in the international literature on what exactly encompasses functional thinking and, hence, educators might also understand this notion differently, with different implications for teaching practice. This paper presents first findings of the Erasmus+ project *FunThink- Enhancing functional thinking from primary to upper secondary school*. The overarching goal of this project is to improve the teaching and learning of functional thinking across all school grades. As a basis for further steps in the project, the project members, inter alia, conducted a corresponding literature review, charted national curricular situations, and interviewed mathematics education experts[1] in order to portray their individual perspectives on functional thinking. Altogether, the interview study was conducted in five countries, yet, in this paper only interview excerpts from Germany and the Netherlands are presented regarding the question *what educational experts consider functional thinking to be*. To relate these empirical insights to relevant theoretical considerations on functional thinking, we present in the following section the corresponding theoretical background.

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[1] Further partners in the interview study are Martina Geisen, Veronika Hubeňáková, Monika Krišáková, Edyta Nowińska, Marios Pittalis, and Miroslawa Sajka.

Theoretical background

Based on the concept of function which reaches back to Bernoulli (1667 – 1748, Büchter & Henn, 2010), the notion of functional thinking was introduced over 100 years ago during the reforms of Meran in 1905. At that time, functional thinking was understood as conceptual interpretation of the mathematical object of function and was considered a "guiding category for teaching mathematics in order to concentrate, unify and structure different areas of mathematics taught in schools" (Krüger, 2019, p. 35). Since then, it has developed in different ways in the international context which led to a variation in definitions. In the following, we present three main strands in the understanding of functional thinking.

First of all, functional thinking can be seen as a major component of algebraic thinking (Warren & Cooper, 2005). More precisely, Pittalis et al. (2020) describe functional thinking "as the process of building, describing, and reasoning with and about functions" (p. 632) and relate this rather broad definition to Blanton and Kaput (2011), Stephens et al. (2017), and others.

The definitions by Markworth (2012) and Smith (2008) rather focus on the aspects of representation and generalization of functional thinking. They see functional thinking as a type of

> […] representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances. (Smith, 2008, p. 143)

Besides those two strands, Cañadas et al. (2016) describe functional thinking in a general sense composed of topics, methods, and relationships concerning functions. Moreover, these authors show examples that fit into the two previously outlined strands: Functional thinking includes functional relationships between quantities, the generalization, and representation, which all support the understanding of function behavior (Blanton & Kaput, 2011). Moreover, it is linked to the ideas of change, more explicitly to qualitative and quantitative change, the relationship between changes and the ability to use these relationships for solving problems (Warren & Cooper, 2005).

These three definitions illustrate that there is no clear consensus about what functional thinking entails. Although they appear disparate, they do share the idea that functional thinking involves reasoning about the relationship between quantities. Considering that, one could ask how functional thinking can be developed by learners and how teachers can support this process. Functional thinking cannot be learned as an independent topic but has to be considered in close connection to the concept of function (cf. Vollrath, 1989). With this regard, the literature describes four perspectives on functions that play an important role when dealing with concrete function tasks or preliminary activities. These so-called *function aspects* include characteristics of functions and can form a basis for the design and implementation of tasks in mathematics education. In the international context, usually four main aspects of functions are distinguished: input-output, covariation, correspondence, and mathematical object (e.g. Doorman et al., 2012; Pittalis et al., 2020).

Function as an input-output assignment stresses the operational and computational character of the function concept; in this sense, it is not necessary to be aware of the causal relation between the in- and output (Pittalis et al., 2020). It is for example relevant when dealing with patterns and structures:

within a sequence of values, *recursive patterning* describes the existing variation and indicates how a next element can be determined if the previous element or a number of elements is provided (Stephens et al., 2017).

The aspect of covariation emphasizes the simultaneous variation of two quantities, often a dependent and an independent variable, and relates to Thompson and Carlson (2017). In their work, they offer a definition of a function with a focus on covariational reasoning:

> A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (p. 436)

This definition highlights the connection of two variables and their interdependency without using the terms dependent and independent variable. Similarly, Confrey and Smith (1995) describe the covariational approach as comprehending, analyzing, and manipulating the relation between two changing quantities. The change in one quantity appears if a change in the related quantity occurs.

The view on a function as a correspondence relation focuses on the relation of the independent and dependent variable and on how this relation can be represented (Smith, 2008). In more formal definitions of functions this view is expressed as ordered pairs:

> […] a *function* from a set S to a set T is a rule that assigns to each element x of set S a unique element of set T. The set S is called the *domain* of the function. If f is the name of the function, then the unique element in T corresponding to an element x in S is denoted $f(x)$ […] and is called *image* of x. The set $\{f(x) \mid x \in S\}$ is called the *range* of the function. (Yandl, 1991, p. 72)

To conclude, the fourth aspect focuses on a function as a mathematical object with its own specific representations and properties which can be dealt with. This perspective is needed to compare a function with another function or with another mathematical object. Higher-order processes like differentiation or concatenation require this view of a function (Lichti & Roth, 2019).

Different to the international context, in Germany, only three aspects are commonly discussed. The aspect of input-output assignment is omitted as a separate aspect. It is rather included in the other aspects. For example, using a function machine where something is put in, which then results in an output relates to the aspect of correspondence due to the direct assignment. Moreover, considering the covariation between inputs and outputs can help finding the underlying rule. At the same time, the input-output assignment can refer to the object aspect if the calculation does not happen within a function but with the whole function (e.g. addition of two functions) which then results in a new output. This difference in the distinction of the aspects of functions might be due to country-related particularities or the historical development as in Germany the notion of functional thinking is clearly associated to Vollrath (1989) who only distinguishes these three aspects of functions.

The set of four aspects can be considered to show an increasing level of sophistication. Studies report a gradual development from a process view which is similar to the input-output-assignment aspect to a more structural view which can be compared to the function as a mathematical object aspect (Sfard, 1991). Activities with a focus on input-output assignment are often already included in primary school (e.g., Leinhardt et al., 1990; Lichti & Roth, 2019; Pittalis et al., 2020; Stephens et al., 2017).

Studies show that young students are able to reach sophisticated ways of reasoning with functions, or algebraically, if rich tasks are provided accompanied by fitting instruction (e.g., Blanton et al., 2015; Stephens et al., 2016; Stephens et al., 2017). The implementation of the three other aspects often follows later in the curricula, whereas the object aspect appears to be the most abstract one.

As stated above, functional thinking is closely intertwined with the concept of function and cannot be considered on its own. The topic of function has been found to cause difficulties for many secondary school students (Sproesser et al., 2020). Reasons for these difficulties might be found in the abstract character of functions which makes the concept only accessible through modelling in representations and focusing on the changes between such representations (cf. Duval, 2006). Tables, algebraic expressions, graphs, and verbal descriptions are the most common representations used in school. Each of these types of representation has advantages and disadvantages depending on the specific situation and task at hand. A flexible use of representation and changes between representations, can support students' learning and understanding of functions and therefore of functional thinking (e.g. Adu-Gyamfi, 2007).

Returning to what was stated at the beginning, functional thinking is considered a key aspect in mathematics and relates to many other disciplines, and everyday life. It is present in many situations even if we are not aware of it. The second part of this paper, which describes excerpts of an international interview study, provides insight into how international educational experts see functional thinking. This is particularly important in how they frame the development of students' functional thinking. Similarities and differences to the above-mentioned definitions of functional thinking will become visible from our analysis of the interviews.

Research question and methodology

The interview study was carried out in order to collect views and experiences of educational experts on functional thinking and to get insight of which elements described in the literature are particularly relevant for them. The research question for the main study is: *what do educational experts in Cyprus, Germany, the Netherlands, Poland, and Slovakia consider functional thinking to be?* In this paper, only exemplary results from Germany and the Netherlands are presented.

Sample

Experts of mathematics education in all five partner countries (Netherlands, Poland, Cyprus, Slovakia, and Germany) were informally approached by project members to participate in this study. The interviewees ranged from professionals for mathematics education from primary to tertiary education working at universities to experienced mathematics teachers for primary and secondary schools and curriculum developers. They were chosen in order to gather views from different professional perspectives but all were considered as experts referring to functional thinking in their embeddings. Between six and nine interviews were conducted in each partner country which led to a total of 35 interviews. In this paper, only excerpts from two interviews in Germany and one interview in the Netherlands are presented. A more detailed description of these three interviewees can be found in the results and discussion section.

Procedure and interview guideline

Prior to the interviews, a semi-structured interview guideline was created to answer, inter alia, the questions of what the experts understand by functional thinking and how it can be addressed in the classroom. Further questions included what students should learn to develop functional thinking in the interviewee's opinion and what exemplary tasks could look like. Moreover, some information was gathered about the interviewees' professional background. The interviews took place virtually or in person depending on the current situation (mostly related to COVID-19 restrictions) in each country. A recording, video tape (together with corresponding transcripts) or a detailed protocol of each interview was used for the analysis. The analysis is currently still in progress. The analysis methodology we use is qualitative content analysis according to Mayring (2014). This is used for building a coding scheme with inductive and deductive categories.

Coding scheme

As our coding scheme is currently under development, we only refer to the main categories we are working with. In a first step, we code educators' ideas in the perspective on functional thinking they referred to. Here, the four aspects of considering functions (input-output, covariation, correspondence, mathematical object) play the main role. Secondly, we code how functions are used for mathematization described by educators, which can take place inside (from informal to more formal mathematics, i.e., vertical mathematization) and outside (modelling a meaningful situation with mathematical tools, i.e., horizontal mathematization) of mathematics. In a third step, we code the activities educators described which they thought could support or require functional thinking. This especially addresses patterning and dealing with representations. Finally, we code semantic and syntactic elements and concepts related to functions and functional thinking, other related fields and counterexamples. As these codes are still under construction, in the following, we only show a first sketch of the analysis of interview excerpts.

Results and discussion

The first interviewee from Germany (G1) works at the transition from university to licensed teachers (a part-time seminary, where graduated college students gain their teaching license) with a focus in mathematics education. Interviewee G1 answered the question of what he considers functional thinking to be in the following way:

> Functional thinking [..] is everything that has to do with the dependence of two quantities, of two variables. […] It is so the upper goal, the upper principle, so on the one hand the one variable has a value, that affects the value of another variable that dependents on it. It would so rather be the static side, so the allocation, then also the change, if one variable changes, what consequences does it have for the other variable. Yes, and the third would be so basically the course that you can conclude, the overall picture of the dependency. […] It already goes in the direction of the idea of using mental representations of mathematical concepts (*Grundvorstellungen*), but above these basic ideas stands the consciousness of dependence and everything that is around it or what is subordinate, the calculating that must actually, that leans on this principle. […]

The description of functional thinking by interviewee G1 is rather broad and highlights the dependency of two variables. Concerning his perspective on functions, the aspects of covariation,

correspondence, and mathematical object are clearly mentioned and described as basic ideas. According to the interviewee, everything that follows, like calculations, can be derived from these principles. Due to the prominence of the aspects according to Vollrath (1989) in Germany, it is not surprising that the aspect of input-output assignment is not mentioned.

Another German interviewee (G2), a teacher from primary school (Grade 1-4), answered the same question. The interviewee is a longtime teacher who initially studied education for primary and lower secondary school with a focus in general studies, German, and mathematics. Besides the degree in education, the interviewee also has a postgraduate degree in pedagogy.

> […] what do they actually want with that in elementary school? […] it's about relationships for me in functional thinking, so not just functions according to the motto of a value is assigned to another value, but about relationships, about the discovery of relationships, and then again a bit of the science lesson plays into it for me, which then says laws of nature, you can make discoveries, you can observe them, you can explore them, you can measure something, math comes into it again […].

Interviewee G2 is rather general in her definition of functional thinking. G2 sees functional thinking in a wider sense than just related to functions. The focus is on relationships and connection to real life. The elements of discovery, observing, exploring, and measuring of relationships show this close connection to the real world. G2's description of functions indicates the aspect of correspondence. Later in the interview, as example, she mentions collections of tasks with continuous elements where students can recognize patterns (*starke Päckchen*) as an activity for addressing functional thinking which includes elements of the input-output assignment. In general, G2 seems less aware of the aspects of functions and functional thinking. In contrast to G1, G2 only mentions some aspects and does not address them explicitly.

An interviewee from the Netherlands (N1) has been a teacher for 16 years, mainly in the upper primary school grades (Grade 5 and 6). When asked about her definition of functional thinking, she mentioned "that must be about relating mathematics to a context and its utility." This is related to our code on horizontal mathematization (modelling extra-mathematical situations with mathematical tools), which is rather well established in the Netherlands, due to the implementation of realistic mathematics education. When prompted by the interviewer that functions could also be interpreted in a more mathematical sense, she referred to patterning tasks in the early grades of primary school, doing rows of calculations and observing what remains fixed and what changes, graphing activities, and summarized all these as "reasoning about relations." In this she clearly related to the covariational view of functional thinking while describing useful activities for eliciting it. Interestingly, she connected this reasoning about relations also to an attitude that students should develop in society, seeing relations, experimenting, encountering obstacles, and systematically try to deal with them.

These first excerpts indicate a clear difference in views between experts. Functional thinking is mostly understood in a way that is somewhat similar to one of or a mixture of the definitions mentioned in the theoretical background. Yet, the descriptions provided by the interviewees are less detailed and some lack a complete description of all aspects of functions and functional thinking. The detailed analysis which is to follow will provide more insights, from all the partner countries, into the extant conceptualizations of functional thinking in practice.

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