Grid-free Weighted Particle method applied to the Vlasov-Poisson equation

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Abstract

We study a grid-free particle method based on following the evolution of the characteristics of the Vlasov-Poisson system, and we show that it converges for smooth enough initial data. This method is built as a combination of well-studied building blocks – mainly time integration and integral quadratures –, hence allows to obtain arbitrarily high orders. By making use of the Non-Uniform Fast Fourier Transform (NUFFT), the overall computational complexity is $\mathcal{O}(P + K^d \log K^d)$, where P is the total number of particles and where we only keep the Fourier modes $k \in (\mathbb{Z}^d)^*$ such that $k_1^2 + \cdots + k_d^2 \leq K^2$. Some numerical results are given for the Vlasov-Poisson system in the one-dimensional case.

Introduction

Meaningful physical simulations involving the Vlasov equations usually need up to six dimensions for the phase-space: three in position and three in velocity. For this reason, the development of efficient numerical scheme has attracted much interest, both in the mathematical and physics communities. After nearly forty years, two main "families" have emerged: Eulerian and Particle-In-Cell methods.

Eulerian schemes rely on a grid-discretization of the phase space. Famous examples are: Finite Differences, Finite Elements, (Backward/Forward) Semi-Lagrangian, Finite Volumes... In particular, the Backward Semi-Lagrangian methods have proved their efficiency for high-dimensional problems because it is possible – through some splitting – to only solve a sequence of one-dimensional problems. However, they suffer from a CFL-like condition as it is shown in [BM08] (an improved version of their estimate is given in [CDM13]), where the timestep has to be sufficiently small compared to some power of the space discretization. The paper [Fil01] proposes a finite volume scheme, and shows its convergence under a CFL condition. On the other hand, the Particle-In-Cell methods are based on a representation of the initial condition by a sum of Dirac masses – the so-called "particles" or "meta-particles" – which move freely in the phase space. After a time step, the particle are deposited on a grid in order to compute an approximation to the electrical energy. Because it relies on Monte-Carlo estimation one usually needs many particles in order to obtain meaningful results.

Many variants of both methods have been proposed in the literature, and in this work we will prove the convergence of a scheme proposed in [BOY11], where the authors introduce a grid-free Weighted Particle method to study the magnetization of the Hamiltonian Mean-Field model. Unfortunately, they only give a brief description of the algorithm with no convergence proof, even though the scheme is promising. Our goal is to detail thoroughly the method, and to prove its convergence. The approach presented is different from the Particle-In-Fourier method (see [MMBP19]), mainly in the way the charge density is computed, and how the approximate solution is represented. The main advantages of our approach is that it allows to obtain high-order estimates by combining well-studied high-order methods, such as integral quadratures and time integration schemes. The convergence estimate shows that with smooth enough initial data, the Fourier truncation error becomes neglictible, so that we don't need many Fourier modes in practice.

A by-product of our approach is that all the error terms are decoupled, yielding a relatively easy proof of convergence. We name our method "Weighted Particle method".

We start Section 1 by recalling the Vlasov-Poisson equation and an existence result in Sobolev regularity from [CCFM17]. Then we recall the main ideas of the Particle-In-Cell method and its variants, and review different ways of computing the electric field in the Vlasov-Poisson system. We end this section with a presentation of the Fourier approach to solve the Poisson equation, which will be at the core of the method presented and will allow the definition of a truncated Fourier kernel to the Vlasov-Poisson equation. This truncated Fourier kernel can be seen as an approximation to the exact Fourier kernel which involves an infinity of modes. In Section 2 we explain how the Weighted Particle method we propose is obtained naturally from the truncated Fourier kernel. The building blocks of this scheme are integral quadratures and time integration schemes, allowing a high-order method. Starting from the quadratures, we deduce the particle representation of the approximate solution in a natural way. Moreover, the method presented is totally grid-free since the particles don't require to be deposited onto some grid as it is done, for example, in the Particle-In-Cell method. Section 3 is dedicated to the Weighted Particle method. We start by discussing how this method differs from others in the literature, and then present our main result: the convergence of the approximate characteristics obtained through the Weighted Particle method towards the true characteristics of the Vlasov-Poisson system. One-dimensional numerical results are presented in Section 4 to illustrate the accuracy one can obtain with relatively few particles. Finally, Section 5 is dedicated to proving Theorem 3.1.

Contents

1 Preliminaries

Let $2d$ be the dimensions of the phase-space variable $z = (x, v)$. Usually, $d = 1, 2$, or 3. We consider periodic boundary conditions for the space variable x, i.e. $x \in \mathbb{T}^d_L$ where we denote $\mathbb{T}^d_L := \mathbb{R}/(L_1\mathbb{Z}) \times \cdots \times \mathbb{R}/(L_d\mathbb{Z})$ for $L_i > 0, i = 1, \ldots, d$. We let $v \in \mathbb{R}^d$. The Vlasov-Poisson system writes

Vlasov-Poisson

$$\left|\begin{array}{l} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + \Lambda \nabla_x \Phi[f](t,x) \cdot \nabla_v f(t,x,v) = 0, \\ \Phi[f](t,x) = \Delta^{-1}\rho(t,x), \\ f(0,x,v) = f_0(x,v), \end{array}\right. \tag{VP} \quad (1.1a)$$

where $t \geq 0$ and

$$\rho(t,x) := \int_{\mathbb{R}^d} \left(f(t,x,v) - \frac{1}{|\mathbb{T}^d_L|} \int_{\mathbb{T}^d_L} f(t,y,v)dy \right) dv.$$

The constant $\Lambda \in \{\pm 1\}$ distinguishes the attractive case $\Lambda = -1$, which models the dynamics of galaxies, from the repulsive one $\Lambda = +1$, which models the dynamics of plasmas. In this work we are only interested in the repulsive case.

1.1 Existence result

For a given multi-index $p = (p_1, \ldots, p_d) \in \mathbb{N}^d$, we denote by ∂^p_x the multi-derivative $\partial^{p_1}_{x_1} \ldots \partial^{p_d}_{x_d}$. Similarly, we set $v^m = v_1^{m_1} \ldots v_d^{m_d}$ for $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$ and $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$. We let $|\cdot|$ the usual euclidian norm on $\mathbb{R}^d$. As the functional framework, we will consider the spaces $\mathcal{H}^r_\nu$ equipped with the norms

$$||f||^2_{\mathcal{H}^r_\nu} = \sum_{\substack{(m,p,q)\in(\mathbb{N}^d)^3 \\ |p|+|q|\leq r \\ |m|\leq\nu}} \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} \left| v^m \partial^p_x \partial^q_v f(x,v)\right|^2 dxdv. \tag{1.2}$$

We have the following existence result from [CCFM17]:

Theorem 1.1

Let $\nu > d/2$, $r \geq 3\nu$. There exist constants $C_{r,\nu}$ and $L_{r,\nu}$ such that for all given $B > 0$ and $f_0 \in \mathcal{H}^{r+2\nu+1}_\nu$ such that $||f_0||_{\mathcal{H}^{r+2\nu+1}_\nu} \leq B$, then for all $\alpha, \beta \in [0,1]$, there exists a solution $f(t,x,v)$ of the Vlasov-Poisson equation

$$\partial_t f + \alpha v \cdot \nabla_x f + \beta \nabla_x \Phi[f] \cdot \nabla_v f = 0,$$

with initial value $f(0,x,v) = f_0(x,v)$ on the interval $[0,T]$ with

$$T := \frac{C_{r,\nu}}{1+B},$$

and we have the estimate

$$\forall t \in [0,T], \quad ||f(t)||_{\mathcal{H}^{r+2\nu+1}_\nu} \leq \min\left(2B, e^{L_{r,\nu}(1+B)t}\right) ||f_0||_{\mathcal{H}^{r+2\nu+1}_\nu}.$$

Moreover, for two initial conditions f_0 and g_0 satisfying the previous hypothesis, we have

$$\forall t \in [0,T], \quad ||f(t) - g(t)||_{\mathcal{H}^r_\nu} \leq e^{L_{r,\nu}(1+B)t} ||f_0 - g_0||_{\mathcal{H}^r_\nu}.$$

This result holds in the functional space $\mathcal{H}^{r+2\nu+1}_\nu$ which is a subspace of the usual Sobolev space $H^{r+2\nu+1}(\mathbb{T}^d_L \times \mathbb{R}^d) = W^{r+2\nu+1,2}(\mathbb{T}^d_L \times \mathbb{R}^d)$.

1.2 Particle methods

Before describing the ideas leading to the Particle-In-Cell method and its variants, we first have to discuss the characteristics of the Vlasov-Poisson equation. They are described for instance in [Gla96, Sect. 4.2], but for the sake of clarity we recall below their main properties. The characteristics of the Vlasov-Poisson system (VP) are the solutions to

$$\begin{cases} \dfrac{dX(t;s,x,v)}{dt} = V(t;s,x,v), & X(s;s,x,v) = x, \\ \dfrac{dV(t;s,x,v)}{dt} = E(t, X(t;s,x,v)), & V(s;s,x,v) = v. \end{cases} \tag{1.3}$$

The notation $X(t;s,x,v)$ (resp. $V(t;s,x,v)$) stands for the position (resp. velocity) component of the flow, starting from (x,v) at time s and evaluated at time t. Since

$$\frac{d}{dt}\left(f(t, X(t;s,x,v), V(t;s,x,v))\right) = \partial_t f + V(t;s,x,v)\cdot\nabla_x f + E(t, X(t;s,x,v))\cdot\nabla_v f = 0,$$

the solution f to the Vlasov-Poisson equation is constant along the characteristics, so that

$$f(t,x,v) = f(0, X(0;t,x,v), V(0;t,x,v)) = f_0(X(0;t,x,v), V(0;t,x,v)). \tag{1.4}$$

The last useful property about the Vlasov-Poisson characteristics is that they are measure-preserving. In other words, the mapping

$$\begin{cases} y = X(s;t,x,v) \\ w = V(s;t,x,v) \end{cases} \tag{1.5}$$

has unit Jacobian. Furthermore, we have

$$\begin{cases} x = X(t;s,y,w) \\ v = V(t;s,y,w) \end{cases} \tag{1.6}$$

and this mapping also has unit Jacobian.

We can now turn to the description of the Particle-In-Cell approach. It consists in following the evolution of some point particles. More precisely, given an initial distribution f_0, we approximate it by a sum of P Dirac masses, $P \in \mathbb{N}^*$:

$$f_0(x,v) \approx \sum_{p=1}^{P} \beta_p \delta(x - x_p)\delta(v - v_p) =: \tilde{f}_0(x,v), \tag{1.7}$$

where (x_p, v_p) are the initial coordinates of the particle p in the phase-space, $p = 1, \dots, P$. Here $\delta(\cdot)$ denotes the usual Dirac mass, and the number P is the total number of particles for which we want to follow the evolution. The quantity β_p is the weight of the particle numbered p, and the weights are usually chosen uniform in Particle-In-Cell methods.

In (1.7), each Dirac mass represents a collection of particles who are defined only by a point in the phase-space. Moreover we implicitely assume that the distribution of particles within a collection remains "close" for all times to the Dirac mass representing said collection. These Dirac masses are usually called *meta-particles*, because each one of them is treated numerically as one particle but may represent physically many particles. Some schemes, such as those presented in [Hew03, CPSF+14, CP15, FGG+17], allow the meta-particles to be deformed, and even to split or recombine. We will not study such possibilities in this work.

From (1.4), the approximate solution to the Vlasov-Poisson system with initial condition $\tilde{f}_0$ can be reconstructed at time t if we know the characteristics at time t. The approximate solution at time t writes

$$f(t,x,v) \approx \tilde{f}_0(X(0;t,x,v),V(0;t,x,v)) = \sum_{p=1}^{P} \beta_p \delta(X(0;t,x,v)-x_p)\delta(V(0;t,x,v)-v_p).$$

Each term in the sum is non zero if and only if

$$\begin{cases} X(0;t,x,v) = x_p \\ V(0;t,x,v) = v_p \end{cases}$$

From (1.5) and (1.6), this is equivalent to

$$\begin{cases} x = X(t;0,x_p,v_p) \\ v = V(t;0,x_p,v_p) \end{cases}$$

Therefore, the approximate solution to the Vlasov-Poisson system with initial condition $\tilde{f}_0$ can be written as

$$f(t,x,v) \approx \sum_{p=1}^{P} \beta_p \delta(x - X(t;0,x_p,v_p))\delta(v - V(t;0,x_p,v_p)).$$

Hence, it is sufficient to follow the characteristics forward in time in order to be able to reconstruct the approximate solution for all times. The main problem with this approach is that, after a time t, the particles are completely disorganized in the phase-space, and hence one needs a "pre-processing" step before being able to compute the electric field $E(t,\cdot)$ which is obtained as $E(t,\cdot) = \nabla_x \Phi[f](t,\cdot)$, where $\Phi[f](t,\cdot)$ is the solution to (1.1a).

1.3 Electric field

The simplest and most direct way to solve the Poisson equation (1.1a) is to use finite differences. However this would require solving a linear system. This is for instance the approach chosen in [HE88, BL91].

One issue in the particle method is that we do not know ρ at equally spaced points. We can however project the particles onto such a grid, and the way of doing so is not unique; the simplest one is the *Nearest-Grid-Point (NGP)*, but more elaborate ideas can be found for instance in [BL91, Sect. 2.6]. We can cite the *Cloud-In-Cell (CIC)* method introduced in [BF68], where the main idea is to introduce "shape functions" to replace the Dirac masses, in which case we talk about "finite-size particles".

Several authors have tried to find the best way of depositing the particles onto the grid, giving rise to many variants of the original PIC method, but they all suffer from the same problem: the deposition step is a very rough approximation when only a few particles are used.

Reviews of particle methods and their deposition steps are given in [Ver05] and [Oku72]. In [Lan70], the effect of spatial grid and its influence on plasma behavior are studied.

We can also cite the FLIP scheme [BKR88] which is in essence a PIC method where the authors try to only update the particle properties instead of completely resetting them from the grid estimations at each time step.

Another issue with the PIC methods that has been pointed out in [Bra16] is that they cannot preserve both energy and momentum, which are conserved at the continuous level.

1.4 Kernel-based computation

In order to solve the Poisson equation (1.1a), one may want to use a Green kernel G to compute E exactly:

$$E(t,x) = \int_{\mathbb{T}^d} \mathcal{K}(x,y)\cdot\left(\rho(t,y) - \frac{1}{|\mathbb{T}_L^d|}\int_{\mathbb{T}^d}\rho(t,\tilde{x})d\tilde{x}\right)dy, \tag{1.8}$$

where

$$\mathcal{K}(x,y) = -\nabla_x G(x,y), \quad -\Delta_x G(x,y) = \delta_0(x-y).$$

This approach can be found in [WO96, RS11, Bes04], and because it introduces a discontinuity in the electric field along the line $\{x=y\}$, there have been some attempts at smoothing it, see e.g. [Wol00].

However the way the electric field is smoothed depends on the authors, and it may seem arbitrary to choose one way or another. In the case of initial particles nonuniformly spaced, the authors of [WO96] write that a mollified version of the kernel G, depending on some mollification parameter, may be preferable to the unmollified version.

This Green kernel-based approach has also been used for numerical computations of fluid dynamics (e.g. Euler equations) in the so-called Vortex and Vortex Blob methods [AG85, Per85, ADL20]. These methods face the same issues, but the convergence of the former methods seems to be have treated more thoroughly (see [Hal79, BM82, Bea86, Cot87, GHL90, CGH91]). In particular, the authors of these papers have also faced the question of whether or not to mollify the Green kernel, and the overwhelming opinion is that the kernel has to be mollified in order to obtain realistic physical results. Because of the similarities between particle and vortex methods, we can assume this conclusion also holds for particle methods. We can also cite [HRCW13], where the authors obtain a smooth, high-order kernel approximating the Green kernel G.

The mollification of the Green kernel involved in plasma or fluid dynamic simulations depends on some mollification parameter which is chosen arbitrarily in the referenced papers. Hence it may not be satisfactory to rely on mollifying the Green kernel, even though its regularized version yields more physical results.

1.5 Fourier approach

We now present another way of approximating the Green kernel by some smooth function, and we hope that our method will prove to be more natural than manually mollifying the kernel.

Let $L := (L_1, \cdots, L_d)$, and for $z \in \mathbb{R}^d$ define

$$\frac{z}{L} := \left(\frac{z_1}{L_1}, \dots, \frac{z_d}{L_d}\right)$$

We use usual notations: $|z|$ for the ℓ^2 norm of a vector $z \in \mathbb{R}^d$, $z \cdot w$ for the ℓ^2 inner-product of two vectors $z, w \in \mathbb{R}^d$, and $|[0,L_1]\times[0,L_d]| = \prod_{i=1}^d L_i$. Moreover, we recall that for a multi-index $p \in \mathbb{N}^d$, we let

$$z^p = (z_1, \cdots, z_d)^{(p_1,\cdots,p_d)} = z_1^{p_1}\cdots z_d^{p_d}.$$

The convention we use for the Fourier transform $\hat{g}$ of a periodic function $g \in \mathbb{L}^2(\mathbb{T}_L^d)$ is the following:

$$\hat{g}(k) = \frac{1}{|\mathbb{T}_L^d|}\int_{\mathbb{T}_L^d} g(x)e^{-2i\pi k\cdot\frac{x}{L}}dx, \quad k \in \mathbb{Z}^d.$$

The solution $\Phi[f]$ of the Poisson equation (1.1a) can be obtained via straightforward computations:

$$\Phi[f](t,x) = \frac{-1}{|\mathbb{T}_L^d|}\sum_{k\in(\mathbb{Z}^d)^*}\frac{1}{4\pi^2\left|\frac{k}{L}\right|^2}\int_{\mathbb{T}_L^d\times\mathbb{R}^d} e^{2i\pi k\cdot\frac{x-y}{L}}f(t,y,v)dydv.$$

Moreover, since $\Phi[f]$ is a real quantity, the imaginary part of the right-hand side is equal to zero, so that

$$\Phi[f](t,x) = \frac{-1}{\left|\mathbb{T}_L^d\right|} \sum_{k\in(\mathbb{Z}^d)^*} \frac{1}{4\pi^2\left|\frac{k}{L}\right|^2}\left[\cos\left(2\pi k\cdot\frac{x}{L}\right)C_k(t) + \sin\left(2\pi k\cdot\frac{x}{L}\right)S_k(t)\right],$$

where

$$C_k(t) := \int_{\mathbb{T}_L^d\times\mathbb{R}^d} \cos\left(2\pi k\cdot\frac{y}{L}\right) f(t,y,v)dydv,$$
$$S_k(t) := \int_{\mathbb{T}_L^d\times\mathbb{R}^d} \sin\left(2\pi k\cdot\frac{y}{L}\right) f(t,y,v)dydv.$$

We easily obtain the electrical field E:

$$\begin{aligned} E(t,x) &= \nabla_x\Phi[f](t,x) \\ &= \frac{1}{\left|\mathbb{T}_L^d\right|} \sum_{k\in(\mathbb{Z}^d)^*} \frac{1}{2\pi\left|\frac{k}{L}\right|^2}\frac{k}{L}\left[\sin\left(2\pi k\cdot\frac{x}{L}\right)C_k(t) - \cos\left(2\pi k\cdot\frac{x}{L}\right)S_k(t)\right] \end{aligned} \tag{1.9}$$

The formula here, with a series over $k\in(\mathbb{Z}^d)^*$, corresponds to the Poisson framework. However, any truncation in the sum over k can be done in order to approximate E. It is intuitive to consider only a finite number of Fourier modes, and we choose to keep only the modes $\left\{k\in(\mathbb{Z}^d)^* : |k|\le K\right\}$ where $K\in\mathbb{N}^*$ is some parameter (think of it as *user-input*).

The approximation to the field E for a given K then writes:

$$E^K(t,x) = \nabla_x\Phi^K[f^K](t,x) \tag{1.10}$$
$$= \frac{1}{\left|\mathbb{T}_L^d\right|} \sum_{\substack{k\in(\mathbb{Z}^d)^*\\ |k|\le K}} \frac{1}{2\pi\left|\frac{k}{L}\right|^2}\frac{k}{L}\left[\sin\left(2\pi k\cdot\frac{x}{L}\right)C_k^K(t) - \cos\left(2\pi k\cdot\frac{x}{L}\right)S_k^K(t)\right], \tag{1.11}$$

where

$$C_k^K(t) = \int_{\mathbb{T}^d\times\mathbb{R}^d} \cos\left(2\pi k\cdot\frac{y}{L}\right) f^K(t,y,v)dydv$$
$$S_k^K(t) = \int_{\mathbb{T}^d\times\mathbb{R}^d} \sin\left(2\pi k\cdot\frac{y}{L}\right) f^K(t,y,v)dydv$$

and where the function f^K is solution to the Vlasov-Poisson equation with a truncated kernel:

$$\left|\begin{array}{l} \partial_t f^K(t,x,v) + v\cdot\nabla_x f^K(t,x,v) + E^K(t,x)\cdot\nabla_v f^K(t,x,v) = 0 \\ f^K(0,x,v) = f_0(x,v) \end{array}\right. \tag{VP^K}$$

Similarly to Section 1.2, we can define for a given $K\in\mathbb{N}^*$ the characteristics of (VP^K) in the following way:

$$\begin{cases} \dfrac{dX^K(t;s,x,v)}{dt} = V^K(t;s,x,v), & X^K(s;s,x,v) = x \\ \dfrac{dV^K(t;s,x,v)}{dt} = E^K(t,X^K(t;s,x,v)), & V^K(s;s,x,v) = v \end{cases} \tag{1.13}$$

These characteristics exhibit the same properties as those given in Section 1.2, in particular the measure-preserving property. Thus, for all $k\in(\mathbb{Z}^d)^*$ such that $|k|\le K$, we have

$$\begin{aligned} C_k^K(t) &= \int_{\mathbb{T}^d\times\mathbb{R}^d} \cos\left(2\pi k\cdot\frac{X^K(t;0,y,v)}{L}\right) f_0(y,v)dydv, \\ S_k^K(t) &= \int_{\mathbb{T}^d\times\mathbb{R}^d} \sin\left(2\pi k\cdot\frac{X^K(t;0,y,v)}{L}\right) f_0(y,v)dydv. \end{aligned} \tag{1.14}$$

Remark 1.1

Our electric field E^K is presented here as an approximation to the exact E, however one could also understand (VP^K) as an intermediate system "between" Vlasov-HMF (in which case $K = 1$) and Vlasov-Poisson (in which case $K \to \infty$).

2 Building blocks of the Weighted Particle method

The difficulty in the computations of (1.14) resides in the fact that we cannot know in practice the characteristics $X^K(t; 0, y, v)$ and $V^K(t; 0, y, v)$ for all starting points $(y, v) \in \mathbb{T}_L^d \times \mathbb{R}^d$. Hence, it is natural to look at quadrature approximations, which would only involve the characteristics for a finite number of starting points.

2.1 Quadratures

Denote by $z = (x_1, \cdots, x_d, v_1, \cdots, v_d) \in \mathbb{R}^{2d}$ a variable of the phase-space, and suppose along the dimension i of the phase space we have a quadrature rule of order q_i over a closed interval I_i. The quadrature is defined by some nodes $\left\{z_i^j\right\}_j$, $z_i^j \in I_i$, and nonnegative weights $\left\{w_i^j\right\}_j$. We suppose the nodes are equispaced with step Δz_i, i.e. $z_i^{j_i} = z_i^0 + j_i \Delta z_i$ for some $\Delta z_i > 0$ and $z_i^0 \in I_i$. Under these conditions, the variable j_i belongs to some finite set $J_i := \{0, 1, \cdots, N_i\}$, where $N_i \in \mathbb{N}^*$ and $N_i \leq \left\lfloor \frac{|I_i|}{\Delta z_i} \right\rfloor$.

The error of the quadrature along dimension i is characterized as follows: there exists a constant $C > 0$ such that for all $g \in \mathcal{C}^{q_i+1}(I_i)$ we have

$$\left| \int_{I_i} g(\zeta_i) d\zeta_i - \sum_{j_i \in J_i} w_i^{j_i} g(z_i^{j_i}) \right| \leq C \left\| \partial_{\zeta_i}^{q_i+1} g(\zeta_i) \right\|_{\mathbb{L}^\infty(I_i)}.$$

Examples of quadratures satisfying these conditions are the rectangle rule and Newton-Cotes formulae of low order (high orders may involve negative weights).

Remark 2.1

We consider uniform quadratures nodes with nonnegative weights for simplicity, in order to obtain a convergence result. However it is also possible to consider in practice non-uniform quadratures (e.g. Gauss-Legendre or Gauss-Hermite quadratures) or negative weights (e.g. high-order Newton-Cotes formulae).

Our notations for the one-dimensional case have been set so that a generalization to the multi-dimensional case is straightforward. Let $j \in J := J_1 \times \cdots \times J_{2d}$ the label of the node $z^j = (z_1^{j_1}, \ldots, z_{2d}^{j_{2d}})$ in the multi-dimensional quadrature over $I_1 \times \cdots \times I_{2d}$. The weight of the node z^j is $w^j = w_1^{j_1} \ldots w_{2d}^{j_{2d}}$. The multi-dimensional quadrature over $I_1 \times \cdots \times I_{2d}$ is simply a cartesian product of one-dimensional quadratures over $I_1, \ldots, I_{2d}$.

In order to understand how (1.14) is approximated using this multi-dimensional integral, suppose for now that the initial condition f_0 has a compact support in velocity: this is only for the sake of understanding, and we will not use this hypothesis later. Under this assumption, let $I_v = I_d \times \cdots \times I_{2d}$ a cartesian product of finite intervals $I_d, \ldots, I_{2d}$, such that supp $f_0 \subset \mathbb{T}_L^d \times I_v$. Then, the integrals of (1.14) are integrals over $\mathbb{T}_L^d \times I_v$ and we are able to apply quadrature rules as described above to each dimension of the phase-space. We obtain, for all $k \in (\mathbb{Z}^d)^*$ such that

$|k| \le K$,

$$C_k^{K,h}(t) = \sum_{j=(j_1,\dots,j_{2d})\in J} \cos\left(2\pi k \cdot \frac{X^K(t;0,z^j)}{L}\right) f_0(z^j) w^j,$$
$$S_k^{K,h}(t) = \sum_{j=(j_1,\dots,j_{2d})\in J} \sin\left(2\pi k \cdot \frac{X^K(t;0,z^j)}{L}\right) f_0(z^j) w^j. \tag{2.1}$$

We give later in Proposition 5.5 an estimate on the approximation errors $\left|C_k^{K,h}(t) - C_k^K(t)\right|$ and $\left|S_k^{K,h}(t) - S_k^K(t)\right|$, depending on the order q_i of the quadratures and the quadrature steps Δz_i.

From the coefficients $C_k^{K,h}$ and $S_k^{K,h}$, one gets the following approximation to the electric field E^K:

$$E^{K,h}(t,x) := \frac{1}{\left|\mathbb{T}_L^d\right|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{1}{2\pi\left|\frac{k}{L}\right|^2} \frac{k}{L} \left[\sin\left(2\pi k \cdot \frac{x}{L}\right) C_k^{K,h}(t) - \cos\left(2\pi k \cdot \frac{x}{L}\right) S_k^{K,h}(t)\right] \tag{2.2}$$

In our notations, the exponent h denotes a phase-space discretization. With this electric field $E^{K,h}$, one can define an approximation to the equation (VP^K), which reads

$$\left|\begin{array}{l} \partial_t f^{K,h}(t,x,v) + v\cdot\nabla_x f^{K,h}(t,x,v) + E^{K,h}(t,x)\cdot\nabla_v f^{K,h}(t,x,v) = 0, \\ f^{K,h}(0,x,v) = f_0(x,v). \end{array}\right. \tag{VP^{K,h}}$$

Bearing in mind that we are trying to obtain a particle method, the sums in (2.1) suggest to have a particle corresponding to each j. We then have $P = |J| = |J_1| \times \dots \times |J_{2d}|$ particles in total. For each $p = 1,\dots,P$, we can find a unique index $j \in J$ such that $(x_p, v_p) := z^j$. The name “Weighted Particle method” stems from the fact that we can understand $f_0(z^j)w^j$ in (2.1) as the weight β_p of the particle numbered j (or equivalently, the particle labelled p). Finally, we can define the characteristics of equation (VP^{K,h}) as:

$$\begin{cases} \dfrac{dX_p^K(t)}{dt} = V_p^K(t), & X_p^K(0) = x_p \\ \dfrac{dV_p^K(t)}{dt} = E^{K,h}(t, X_p^K(t)), & V_p^K(0) = v_p \end{cases} \qquad p = 1,\dots,P. \tag{2.4}$$

The notations for these characteristics are deliberately distinct from those defined in (1.13) in order to distinguish them easily.

2.2 Time integration

We now have only a finite number of particles to follow, and their time evolution is defined by (2.4) which is an Ordinary Differential Equation (ODE). Therefore, integrating the ODE over $[0,t]$ gives the characteristics at time t. The problem of integrating numerically an ODE has been thoroughly studied and many numerical schemes exist.

Let $N_t \in \mathbb{N}$, we consider a uniform time-discretization $t^n = n\Delta t$, $0 \le n \le N_t$, of stepsize $\Delta t > 0$. We let $T := N_t\Delta t$. The ODE (2.4) is written as a first-order ODE, but it can be easily rewritten as a second-order ODE. Therefore, in order to integrate numerically (2.4), one can choose a time integration scheme to solve either first-order or second-order ODEs. We suppose the time integration scheme is globally of order γ. As an example, we could take the explicit Euler

Algorithm 1 Weighted Particle Method

Require:
- f_0: initial distribution
- The compact intervals $I_{d+1}, \dots, I_{2d}$.
- time integration scheme (specifying the timestep Δt and the number of timesteps N_t)
- Quadrature rule for each dimension (specifying, for each dimension $i = 1, \dots, 2d$, the number of nodes $N_i + 1$, their locations $\{z_i^j\}_{j=0,\dots,N_i}$, and their weights $\{w_i^j\}_{j=0,\dots,N_i}$)
- K: the truncation parameter

$P = (N_1 + 1) \times \cdots \times (N_{2d} + 1)$. *(Total number of particles)*
`x[p]`, `v[p]`, β`[p]` $\leftarrow (x_p, v_p, \beta_p)$, $p = 1, \dots, P$. *(Initial positions, velocities, and weights)*
for $n = 0, \dots, N_t$ **do**
  $t^n = n\Delta t$
  for all stages of the time integration over a timestep **do**
    Use NUFFT to compute approximate Fourier coefficients $C_k^{K,h}, S_k^{K,h}$ for $|k| \leq K$
    Update `x`, `v` with (2.4) by using (2.2).
    if Last stage of timestep **then**
      Compute Observables (e.g. electrical energy, momentum, total energy).
    end if
  end for
end for

method which is of order 1, or Runge-Kutta methods whose order depend on the coefficents. It would also be possible to use splitting methods in order to integrate (2.4).

Note that (2.4) exhibits a Hamiltonian structure since $E^{K,h} = \nabla_x \Phi^{K,h}[f^{K,h}]$ where $f^{K,h}$ is the solution to (VP^{K,h}) and where

$$\Phi^{K,h}[f^{K,h}](t,x) := \frac{-1}{|\mathbb{T}_L^d|} \sum_{\substack{k \in (\mathbb{Z}^d)^* \\ |k| \leq K}} \frac{1}{4\pi^2 \left|\frac{k}{L}\right|^2} \left[\cos\left(2\pi k \cdot \frac{x}{L}\right) C_k^{K,h}(t) + \sin\left(2\pi k \cdot \frac{x}{L}\right) S_k^{K,h}(t)\right].$$

Therefore, we may benefit from using a symplectic time integrator. Such time integration schemes have also been studied thoroughly, we can cite for instance [Qin96, HW75, FQ10].

For the numerical results that we will present in Section 4, we have chosen a symplectic, 3-stage, explicit, Runge-Kutta-Nyström scheme of order 4. Its Butcher tableau is given in [FQ10, p. 327]. For higher-order symplectic integrators, we refer to [Yos90] or more recently to [CCFM17].

Once we have applied our favorite time integration scheme to the particle numbered $p \in \{1, \dots, P\}$, we obtain an approximation to the solution $(X_p^K(t^n), V_p^K(t^n))_{p=1,\dots,P}$ of (2.4) . We will denote this approximation by

$$X_p^{K,n}, V_p^{K,n}.$$

These are the approximate characteristics that we will compute in practice. Finally, our method can be summed up via Algorithm 1.

3 Weighted Particle method

The Weighted Particle method simply consists in applying the ideas discussed above in Section 2. That is, for a given k, we have to compute the approximate coefficients $C_k^{K,h}$ and $S_k^{K,h}$ via quadratures as written in (2.1). This has a complexity $\mathcal{O}(P)$ where P is the total number of particles. Then we have to do this for all $k \in (\mathbb{Z}^d)^*$ such that $|k| \leq K$, in order to compute the approximate electric field $E^{K,h}$ as defined by (2.2). This amounts to computing $\mathcal{O}(K^d)$ times the coefficients $C_k^{K,h}, S^{K,h}$. Therefore, the overall complexity for one computation of the electric field is $\mathcal{O}(PK^d)$. Then, we can compute the approximate characteristics via a time integration scheme. We recall that a naive computation of the electric field via a quadrature approximation of (1.8) would require $\mathcal{O}(P^2)$ operations. Hence, the Weighted Particle method is an improvement for $K \lesssim P^{1/d}$.

Moreover, the complexity of order $\mathcal{O}(PK^d)$ may not be satisfying with many dimensions, even with K small. To reduce this, we can use the Non-Uniform Fast Fourier Transform (NUFFT). Thanks to this, the computational cost goes from $\mathcal{O}(PK^d)$ to $\mathcal{O}(P+K^d \log K^d)$, but the NUFFT introduces an additional approximation error.

The basic idea of this scheme has already been given in [BOY11]. However the algorithm proposed in the referenced paper, named "Weighted Particle code", imposes a regular lattice, does not consider Fourier modes other than $k = \pm 1$, imposes a normalization condition on the particle weights, and is only used to study the magnetization of the N-body simulation in the Hamiltonian Mean-Field framework. Finally, no proof of convergence of the algorithm is given, and the time integration scheme is not discussed. We do not have such restrictions here. Our proposed algorithm thus appears to be an extension of the "Weighted Particle code" from [BOY11], and it is guaranteed to converge by Theorem 3.1.

It can also be seen as an improvement of the grid-free method presented in [WO96]: in that work the authors use a smoothed Green kernel, and the rectangle rule to approximate integrals. We allow other types of quadratures here.

Finally, it can be seen as an application to the Vlasov-Poisson system of the method presented in [DM89], where the authors use the Weighted Particle method to approximate the solution to convection-diffusion equations. Our method could also be understood as a Vortex method with a Fourier regularization of the Green kernel.

We can find such approach to the Vlasov equations via the Fourier kernel mentioned in papers related to the Vlasov-HMF models – such as [dB10, AR95] – but no link to the general Poisson framework is discussed. A similar idea has been proposed in [PR21] to approximate the collision operator of the Boltzmann equation, called the Fourier-Galerkin spectral method.

The approach presented here is closely related to the Particle-In-Fourier method (PIF), see [MMBP19]. In the PIF method the charge density ρ is approximated as a sum of shape functions, which is similar to what is done in the Cloud-In-Cell method. The authors proposed Gaussian shapes as a natural choice, but one could argue that this is pretty arbitrary. Our Weighted Particle method does not require shape functions, and can compute ρ exactly up to the quadrature error. The PIF method also makes use of the Non-Uniform Fast Fourier Transform, so our method is not computationally worse than PIC or PIF. Finally, some ideas leading to the Weighted Particle method are very different from the PIC or PIF approach. In particular we do not seek an approximate solution as a sum of Dirac masses or shape functions, which is a simplifying assumption in PIC and PIF methods: in WPM this representation of the solution is simply a consequence of the quadrature rules. In our numerical examples, we use the library `FINUFFT.jl`, described in [BMaK19, Bar20].

To be coherent with the paper [BOY11] which first proposed the basic ideas presented here, we name our method "Weighted Particle method" (abbreviated WPM).

3.1 Convergence of the Weighted Particle method

The following result gives an estimate on how the numerical approximations of the characteristics of (VP^K) – with our notations, $X_p^{K,n}$ and $V_p^{K,n}$ – approach the true characteristics of the Vlasov-Poisson equation (VP) – with our notations, $X(t^n;0,x_p,v_p)$ and $V(t^n;0,x_p,v_p)$.

Theorem 3.1 (*Convergence of the Weighted Particle method*)

Let $j \in \mathbb{N}$ such that $j \geq 1 + \max_i q_i$, and $\nu, r, \alpha \in \mathbb{N}$ such that $\nu + j > d/2$, $r \geq \max(3(\nu + j), (j-1)(d/2+1))$, $\alpha > 1 + 2(r+d)$. Let $K \in \mathbb{N}$, and assume $f_0 \in \mathcal{H}_{\nu+j}^{r+\alpha}$.

Then there exists a constant $C > 0$ such that the following holds: for $\delta \geq 0$, define finite intervals $I_{d+1} := [a_1, b_1], \ldots, I_{2d} = [a_d, b_d]$ and $I_v := I_{d+1} \times \cdots \times I_{2d}$ such that

$$||f_0||_{\mathcal{H}_\nu^0(\mathbb{T}_L^d \times (\mathbb{R}^d \setminus I_v))} \leq \delta.$$

Then for all $K \in \mathbb{N}^*$, and $n = 1, \ldots, N_t$

$$\max_{p=1,\ldots,P} \left(\left|X_p^{K,n} - X(t^n;0,x_p,v_p)\right| + \left|V_p^{K,n} - V(t^n;0,x_p,v_p)\right|\right)$$
$$\leq C\left(K^d\left[\delta + K^{\gamma+1}\Delta t^\gamma + \sum_{i=1}^{2d} K^{q_i}\Delta z_i^{q_i}\right] + \frac{1}{(1+K)^{\frac{\alpha+1}{2}-d}}\right)$$

where C is independant of $n, \Delta t, \Delta z_i, K$.

The proof of this result relies on the following inequality:

$$\begin{aligned}
&|X_p^{K,n} - X^K(t^n;0,x_p,v_p)| + |V_p^{K,n} - V^K(t^n;0,x_p,v_p)| \\
&\quad \leq |X_p^{K,n} - X_p^K(t^n)| + |V_p^{K,n} - V_p^K(t^n)| \\
&\qquad + |X_p^K(t^n) - X^K(t^n;0,x_p,v_p)| + |V_p^K(t^n) - V^K(t^n;0,x_p,v_p)| \\
&\qquad\quad + |X^K(t^n;t^0,x_p,v_p) - X(t^n;t^0,x_p,v_p)| + |V^K(t^n;0,x_p,v_p) - V(t^n;0,x_p,v_p)|
\end{aligned} \tag{3.1}$$

We recall that $(X_p^K(t), V_p^K(t))_p$ are the solutions to (2.4), $(X^K(t;0,x_p,v_p), V^K(t;0,x_p,v_p))$ are the solutions to (1.13), and $(X(t;0,x_p,v_p), V(t;0,x_p,v_p))$ are the solutions to (1.3).

Each line corresponds to a different type of approximation: the first one is the time discretization error, the second one is the phase-space discretization error (i.e. the quadrature error), and the third one is the kernel truncature error.

Before proving our main result, which is achieved through several estimates and lengthy computations, we illustrate numerically the efficiency of our method.

4 Numerical Simulations

In this section we will give illustrations on how the Weighted Particle method performs on two standard one-dimensional benchmarks: Weak Landau damping and Two-Stream instability. The time integration scheme for all simulations is a symplectic, explicit, 3-stage Runge-Kutta-Nyström method of order 4, whose Butcher tableau was taken from [FQ10, p.327]. The Weighted Particle method is defined by some parameters:

- the truncation parameter K.
- the quadratures in x-space and v-space. We consider the rectangle rule in both dimensions, and let $N_1 + 1, N_2 + 1$ be the number of points for each quadrature. The total number of particles is defined as $P = (N_1+1)(N_2+1)$.

- the compact interval I_v for the v-quadrature. We consider an interval $I_v = [-v_{\max}, v_{\max}]$, where $v_{\max}$ is our parameter.
- the time step of the time integration scheme.

For each example, we display the time evolution of the electrical energy obtained with the WPM method. Moreover the total energy and momentum are conserved for the exact Vlasov-Poisson system, hence we can compare our WPM results with the exact quantities (computed analytically at time $t = 0$) and display the error. We also display the electrical energy as well as the errors obtained with a Backward semi-Lagrangian scheme which uses B-splines of order 5, and which also solves the problem (VP^K). For this Backward semi-Lagrangian scheme, labelled "BSL", we have used 256 points in the x-direction, and 512 points in the v-direction. Moreover, it uses the usual Strang splitting procedure for the time integration.

We do not give the evolution of the $\mathbb{L}^p$ norms from the WPM method because they are all conserved with respect to time by construction of the Weighted Particle method: the $\mathbb{L}^p$ norm of the approximate solution is $\left(\sum_{j\in J} f_0(z^j)^p w^j\right)^{1/p}$, and this does not depend on time. Hence the error between the true $\mathbb{L}^p$ norms and the numerical ones are simply the quadrature error at time $t = 0$.

4.1 Weak Landau damping

Description It is for example a test case in [Son16, p.54, Sect.4.4.2]. The initial condition reads:

$$f_0(x, v) = (1 + \alpha \cos(k_x x)) \exp(-v^2/2) \frac{1}{\sqrt{2\pi}}, \quad x \in [0, L], v \in [-v_{\max}, v_{\max}],$$

where $L := 2\pi/k_x$. This is one of the most famous examples. A numerical scheme has to recover accurately the damping rate and the period between oscillations in the electrical energy. There exists a theoretical formula giving the electrical energy for the dominating Fourier mode (see [Son16, p.56]). As other modes decay much faster, this formula is a good approximation of the exact electrical energy $E^{th}_{elec}(t)$ after a short time. For $k_x = 0.5$, the formula reads

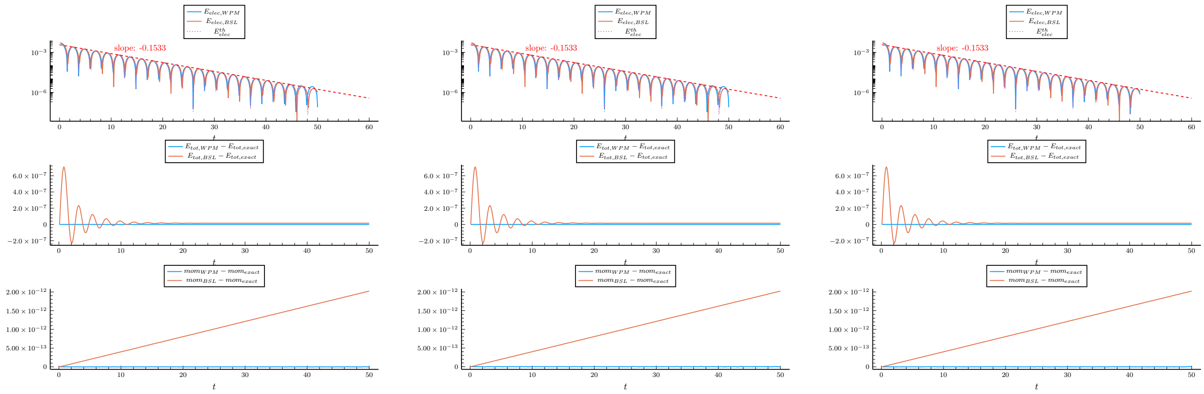
$$E^{th}_{elec}(t) \approx 0.004 \times 0.3677 e^{-0.1533t} \left|\cos(1.4156t - 0.536245)\right| \sqrt{L/2}.$$

WPM results The numerical parameters are $K = 1, v_{\max} = 12, k_x = 0.5, \alpha = 0.001, \Delta t = 0.1$. The results are given in Figure 1.

In the top row, we draw the results obtained with WPM (solid blue curve), the expected damping rate (red dashes), and the electrical energy of the dominating Fourier mode (green dots). For times up to $t = 45$, the Weighted Particle method can recover the electrical energy with a very good accuracy. In the second row of Figure 1, we draw the difference between the theoretical total energy and the total energy computed from WPM. We observe that the Weighted Particle method can recover the total energy with a very good accuracy (the difference is of order 10^{-10}), even better than the semi-Lagrangian scheme. In the third row, we compare the exact momentum with the momentum obtained from WPM. Here as well, the momentum is very well recovered (e.g. the difference is of order 10^{-14} for the example with the smallest number of particles), which is again better than BSL.

For this example we also observe an expected jump called the "Poincare recurrence", which is linked to the compact support in velocity (see [CPSF+14, Son16, EO14]). However, we are not able to explain the amplitude increase after the jump.

The relative $\mathbb{L}^2$ norm error is of order 10^{-14}.



(a) $N_x = 64, N_v = 128$ (b) $N_x = 128, N_v = 128$ (c) $N_x = 256, N_v = 256$

Figure 1: Top row (log-scale) : Electrical energy from WPM (resp. BSL), in blue (resp. red). Below: error between WPM (resp. BSL) results and exact quantities, in blue (resp. red) – middle row: total energy, bottom row: momentum.

4.2 Two-Stream Instability

Description This example can be found in [Son16, p.57] or [CRS09, p.1738]. Depending on the reference, the initial condition may be different. The idea of this example in both cases is to have two streams with opposite velocities. We will consider the formulation from [Son16]. The initial condition then reads:

$$f_0(x, v) = (1 + \alpha \cos(k_x x)) \frac{1}{2\sqrt{2\pi}} (\exp(-(v - v_0)^2/2) + \exp(-(v + v_0)^2/2)), ,$$

for $x \in [0, 2\pi/k_x], v \in [-v_{\max}, v_{\max}]$.

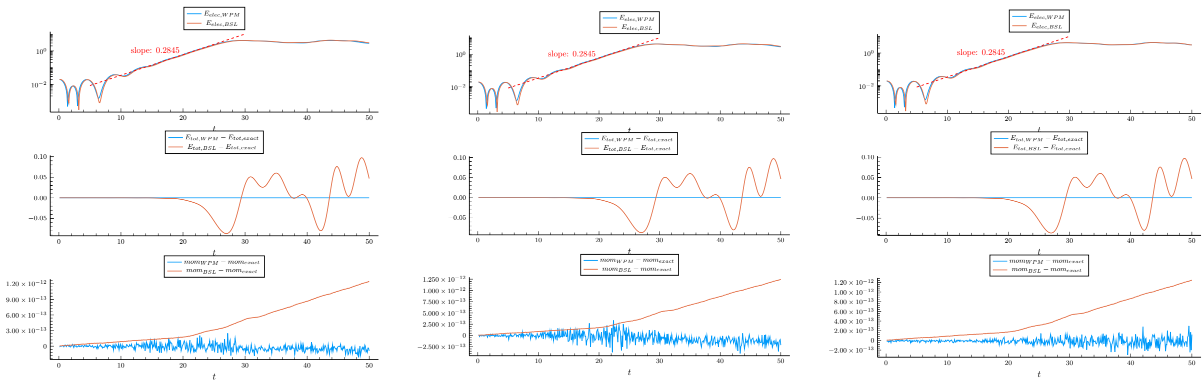
WPM results The numerical parameters are $K = 1, \alpha = 0.001, v_{\max} = 12, k_x = 0.2, v_0 = 3, \Delta t = 0.1$. The results are given in Figure 2.

For the Two-Stream instability, we know that there is first a short transitional state, followed by an instability, and then some periodic behavior. The instability rate is 0.2845.

As in the previous example, the first row in Figure 2 corresponds to the electrical energy obtained with the Weighted Particle method (solid blue curve), and we display the expected instability rate (red dashes). We can observe that the instability rate is recovered accurately with WPM. In the second row of Figure 2, we display the error between the theoretical total energy and the total energy obtained with WPM. The total energy is also recovered accurately (the difference is of order 10^{-6}), much more accurately than with the semi-Lagrangian scheme. In the third row, we compare the exact momentum with the momentum obtained from WPM. Here as well, the momentum is very well recovered (e.g. the difference is of order 10^{-13} for the example with the smallest number of particles). Again, the semi-Lagrangian yields by far the worst results.

The relative $\mathbb{L}^2$ norm error is of order 10^{-14}.

For all those examples we were able to recover very accurately the exact momentum, electrical energy and total energy. Relatively few particles were needed, compared to the usual PIC methods. As a comparison, we can cite for instance the paper [NSSF11] which uses a Particle-In-Wavelets scheme, where 2^{19} particles were necessary in order to obtain satisfying results with a tolerable statistical noise on the Landau Damping and Two-Stream instability examples. The authors of [CPSF+14] have done some Particle-In-Cell simulations and show that, on the



(a) $N_x = 64, N_v = 128$ (b) $N_x = 128, N_v = 128$ (c) $N_x = 256, N_v = 256$

Figure 2: Top row (log-scale) : Electrical energy from WPM (resp. BSL), in blue (resp. red). Below: error between WPM (resp. BSL) results and exact quantities, in blue (resp. red) – middle row: total energy, bottom row: momentum.

Strong and Weak Landau damping examples after a short time, the statistical noise with 256×256 particles prevents from drawing conclusions from the results. The method presented in [CPSF+14] does not have such a problem and can predict accurately the damping rates, but requires frequent remapping.

Moreover we have not displayed here the results of the comparison between a symplectic time integrator and a non-symplectic one, but experiments show that using a symplectic time integrator prevents from obtaining a drift in conservative quantities (e.g. total energy) which otherwise occurs. For this comparison, we have tested symplectic and non-symplectic versions of a 4th order Runge-Kutta-Nyström time integrator.

5 Proof of Theorem 3.1

The first thing to show is that the truncation of the kernel does not modify the existence result given by Theorem 1.1.

We recall that the spaces $\mathcal{H}^r_\nu$ are defined by (1.2). For functions in $\mathcal{H}^r_\nu$, we consider the Fourier transform along the space variable $x \in \mathbb{T}^d_L$ and denote it $\mathcal{F}_x$. Let P_K be the projection on the Fourier modes with frequency $|k| \leq K$.

We have the following lemma:

Lemma 5.1

Let $K \in \mathbb{N}^*$, define Φ, Φ^K as in (1.9) and (1.10). Then, for all $\nu, r \in \mathbb{N}$, we have

$$\forall g \in \mathcal{H}^r_\nu, \quad P_K \Phi[g] = \Phi^K[g] = \Phi[P_K g] \tag{5.1}$$

and

$$\forall g \in \mathcal{H}^r_\nu, \quad ||P_K g||_{\mathcal{H}^r_\nu} \leq ||g||_{\mathcal{H}^r_\nu}. \tag{5.2}$$

Proof: The first equality of (5.1) is just the definition of Φ^K. The second equality is straightforward by noting that $\Phi^K[g] = P_K \Phi[g]$, that the mapping $g \mapsto \Phi[g]$ is linear, and that the only dependance in the space variable x of $\Phi[g]$ is the dependance on x of g. It can also be shown by computing $P_K \Phi[g]$ and $\Phi[P_K g]$ explicitely and comparing the expressions.

For (5.2), we have by the Parseval equality

$$||P_K g||^2_{\mathcal{H}^r_\nu} = \sum_{\substack{(m,p,q)\in(\mathbb{N}^d)^3 \\ |p|+|q|\le r \\ |m|\le\nu}} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \int_{\mathbb{R}^d} |v^m \partial_v^q \mathcal{F}_x(g)(k,v) k^p|^2 \, dv \le ||g||^2_{\mathcal{H}^r_\nu},$$

and we recall that $k^p := k_1^{p_1}...k_d^{p_d}$. ■

It is now possible to follow step by step the proofs of [CCFM17, Thm. 5.1, Lemma 5.3], with the estimates holding thanks to Lemma 5.1, and we will obtain the following existence result:

Proposition 5.1

Let $K \in \mathbb{N}^*$, $\nu, r \in \mathbb{N}$, with $\nu > d/2$ and $r \ge 3\nu$. There exist constants $C_{r,\nu}$ and $L_{r,\nu}$ such that for all given $B > 0$ and $f_0 \in \mathcal{H}^{r+2\nu+1}_\nu$ with $||f_0||_{\mathcal{H}^{r+2\nu+1}_\nu} \le B$, for all $\alpha, \beta \in [0,1]$, there exists a solution $f^K(t,x,v)$ of the Vlasov-Poisson equation with truncated kernel (VP^K)

$$\begin{cases} \partial_t f^K + \alpha v \cdot \nabla_x f^K + \beta \nabla_x \Phi^K[f^K] \cdot \nabla_v f^K = 0 \\ f(0,x,v) = f_0(x,v) \end{cases}$$

on the interval $[0,T]$ with

$$T := \frac{C_{r,\nu}}{1+B},$$

and we have the estimate

$$\forall t \in [0,T], \quad \left|\left|f^K(t)\right|\right|_{\mathcal{H}^{r+2\nu+1}_\nu} \le \min\left(2B, e^{L_{r,\nu}(1+B)t}\right) ||f_0||_{\mathcal{H}^{r+2\nu+1}_\nu}.$$

Moreover, for two initial conditions f_0 and g_0 satisfying the previous hypotheses, we have

$$\forall t \in [0,T], \quad \left|\left|f^K(t) - g^K(t)\right|\right|_{\mathcal{H}^r_\nu} \le e^{L_{r,\nu}(1+B)t} ||f_0 - g_0||_{\mathcal{H}^r_\nu}.$$

We do not give the proof here as it would amount to copy *verbatim* the proof from [CCFM17], and we refer the reader to this paper for a self-contained proof. We have the following lemma:

Lemma 5.2

Let $\nu, r_1, r_2 \in \mathbb{N}$ such that $r_2 \ge r_1$. For all $f \in \mathcal{H}^{r_2}_\nu$, there exists a constant $C > 0$ such that for all $k \in \mathbb{Z}^d$, and all $q \in \mathbb{N}^d$ such that $|q| \le r_2$,

$$\forall m \in \mathbb{N}^d, \ |m| \le \nu, \quad \int_{\mathbb{R}^d} |v^m \partial_v^q \mathcal{F}_x(f)(k,v)|^2 \, dv \le \frac{C}{(1+|k|)^{2(r_2-|q|)+d}}$$

and, for all $K \in \mathbb{N}^*$,

$$||(I-P_K)f||^2_{\mathcal{H}^{r_1}_\nu} \le \frac{C||f||^2_{\mathcal{H}^{r_2}_\nu}}{(1+K)^{2(r_2-r_1)}}.$$

Proof: Recall the definition of the $\mathcal{H}^{r_2}_\nu$ norm:

$$||f||^2_{\mathcal{H}^{r_2}_\nu} = \sum_{\substack{(m,p,q)\in(\mathbb{N}^d)^3 \\ |p|+|q|\le r_2 \\ |m|\le\nu}} \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |v^m \partial_x^p \partial_v^q f(x,v)|^2 \, dx dv$$

By the Parseval equality,

$$||f||^2_{\mathcal{H}^{r_2}_\nu} = \left|\mathbb{T}^d_L\right| \sum_{\substack{(m,\tilde p,q)\in(\mathbb{N}^d)^3\\ |\tilde p|+|q|\le r_2\\ |m|\le\nu}} \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} \left|\mathcal{F}_x\left(v^m \partial_x^{\tilde p}\partial_v^q f\right)(k,v)\right|^2 dv$$

$$= \left|\mathbb{T}^d_L\right| \sum_{\substack{(m,q)\in(\mathbb{N}^d)^2\\ |q|\le r_2\\ |m|\le\nu}} \sum_{k\in\mathbb{Z}^d} \sum_{\substack{\tilde p\in\mathbb{N}^d\\ |\tilde p|\le r_2-|q|}} (2\pi)^{2|\tilde p|}\left(\frac{k}{L}\right)^{2\tilde p} \int_{\mathbb{R}^d} \left|v^m\partial_v^q\mathcal{F}_x(f)(k,v)\right|^2 dv. \qquad (5.3)$$

We recall that with our convention, as $\tilde p\in\mathbb{N}^d$, $k, L\in\mathbb{R}^d$ we let

$$\left(\frac{k}{L}\right)^{2\tilde p} = \left(\frac{k_1}{L_1}\right)^{2\tilde p_1}\cdots\left(\frac{k_d}{L_d}\right)^{2\tilde p_d}.$$

A by-product of (5.3) is that, since the right-hand side is finite, the sum over k is also finite for every m, q. In the sum over $\tilde p\in\mathbb{N}^d$ with $|\tilde p|\le r_2-|q|$, we have in particular for each $i=1,\dots,d$, the term $\tilde p=(0,\cdots,0,r_2-|q|,0,\cdots,0)$ where only the $i-th$ coordinate is nonzero and its value is $r_2-|q|$. There is as well a $\tilde p$ such that $\tilde p=(0,\dots,0)$. Thus, for some constant C that does not depend on k,

$$\sum_{\substack{\tilde p\in\mathbb{N}^d\\ |\tilde p|\le r_2-|q|}} (2\pi)^{2|\tilde p|}\left(\frac{k}{L}\right)^{2\tilde p} \ge C\left(1+\sum_{i=1}^d k_i^{2(r_2-|q|)}\right).$$

The right-hand side of (5.3) being finite for every m, q, we then have

$$\left(1+\sum_{i=1}^d k_i^{2(r_2-|q|)}\right)\int_{\mathbb{R}^d}\left|v^m\partial_v^q\mathcal{F}_x(f)(k,v)\right|^2 dv \le \frac{C}{1+|k|^d},$$

for some C large enough. Finally, for all $|q|\le r_2$, $|m|\le\nu$, we have

$$\int_{\mathbb{R}^d}\left|v^m\partial_v^q\mathcal{F}_x(f)(k,v)\right|^2 dv \le \frac{C}{1+\sum_{i=1}^d k_i^{2(r_2-|q|)+d}} \le \frac{C}{(1+K)^{2(r_2-|q|)+d}},$$

where the last equality is a consequence of Jensen's inequality. This shows the first estimate we claim.

We now proceed to showing the second estimate. Coming back to (5.3), let $\tilde p = p+s$, where $p, s\in\mathbb{N}^d$ are such that $|s|\le r_2-r_1$ and $|p|\le r_1-|q|$. A given value of $\tilde p$ may be obtained by several combinations of s and p. However, the maximal number M of combinations yielding the same $\tilde p$ is finite and depends only on d, r_1, r_2. Therefore,

$$||f||^2_{\mathcal{H}^{r_2}_\nu} \ge \frac{\left|\mathbb{T}^d_L\right|}{M} \sum_{\substack{(m,q)\in(\mathbb{N}^d)^2\\ |q|\le r_2\\ |m|\le\nu}} \sum_{k\in\mathbb{Z}^d} \sum_{\substack{s\in\mathbb{N}^d\\ |s|\le r_2-r_1}} (2\pi)^{2|s|}\left(\frac{k}{L}\right)^{2s} \sum_{\substack{p\in\mathbb{N}^d\\ |p|\le r_1-|q|}} (2\pi)^{2|p|}\left(\frac{k}{L}\right)^{2p}\int_{\mathbb{R}^d}\left|v^m\partial_v^q\mathcal{F}_x(f)(k,v)\right|^2 dv$$

In the sum over $s\in\mathbb{N}^d$ with $|s|\le r_2-r_1$, we have in particular for each $i=1,\dots,d$ the term $s=(0,\cdots,0,r_2-r_1,0,\cdots,0)$ where only the $i-th$ coordinate is nonzero and its value is r_2-r_1. Thus,

$$||f||^2_{\mathcal{H}^{r_2}_\nu} \ge \frac{\left|\mathbb{T}^d_L\right|}{M} \sum_{\substack{(m,q)\in(\mathbb{N}^d)^2\\ |q|\le r_2\\ |m|\le\nu}} \sum_{k\in\mathbb{Z}^d} \left(\sum_{i=1}^d\left[2\pi\frac{k_i}{L_i}\right]^{2(r_2-r_1)}\right) \sum_{\substack{p\in\mathbb{N}^d\\ |p|\le r_1-|q|}} (2\pi)^{2|p|}\left(\frac{k}{L}\right)^{2p}\int_{\mathbb{R}^d}\left|v^m\partial_v^q\mathcal{F}_x(f)(k,v)\right|^2 dv$$

Again, by the Jensen inequality, there exists a constant $C_1 > 0$ such that

$$\sum_{i=1}^{d} \left[2\pi \frac{k_i}{L_i}\right]^{2(r_2-r_1)} \geq C_1 |k|^{2(r_2-r_1)}.$$

Hence

$$||f||^2_{\mathcal{H}^{r_2}_\nu} \geq C \sum_{\substack{(m,q)\in(\mathbb{N}^d)^2 \\ |q|\leq r_2 \\ |m|\leq \nu}} \sum_{k\in\mathbb{Z}^d} |k|^{2(r_2-r_1)} \sum_{\substack{p\in\mathbb{N}^d \\ |p|\leq r_1-|q|}} (2\pi)^{2|p|} \left(\frac{k}{L}\right)^{2p} \int_{\mathbb{R}^d} |v^m \partial_v^q \mathcal{F}_x(f)(k,v)|^2 \, dv,$$

where we let $C := C_1 \frac{|\mathbb{T}^d_L|}{M}$. In the sum over q, we can drop the terms corresponding to $|q| > r_1$ because it yields an empty set $\{p \in \mathbb{N}^d : |p| \leq r_1 - |q|\}$. Thus,

$$||f||^2_{\mathcal{H}^{r_2}_\nu} \geq C \sum_{\substack{(m,q)\in(\mathbb{N}^d)^2 \\ |q|\leq r_1 \\ |m|\leq \nu}} \sum_{k\in\mathbb{Z}^d} |k|^{2(r_2-r_1)} \sum_{\substack{p\in\mathbb{N}^d \\ |p|\leq r_1-|q|}} (2\pi)^{2|p|} \left(\frac{k}{L}\right)^{2p} \int_{\mathbb{R}^d} |v^m \partial_v^q \mathcal{F}_x(f)(k,v)|^2 \, dv. \quad (5.4)$$

Now, if $|k| > K$, then $\mathbb{N} \ni |k|^2 > K^2 \geq 1 + K^2$. As we want an estimate that depends on $(1+K)^{2(r_2-r_1)}$ and not on $(1+K^2)^{r_2-r_1}$, we use the following inequality that holds for $K \geq 1$:

$$(1+K)^2 \leq 2(1+K^2).$$

We truncate the sum over $k \in \mathbb{Z}^d$ to $|k| > K$ in (5.4), and we get

$$||f||^2_{\mathcal{H}^{r_2}_\nu} \geq C \left(\frac{(1+K)^2}{2}\right)^{r_2-r_1} \sum_{\substack{(m,q)\in(\mathbb{N}^d)^2 \\ |q|\leq r_1 \\ |m|\leq \nu}} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>K}} \sum_{\substack{p\in\mathbb{N}^d \\ |p|\leq r_1-|q|}} (2\pi)^{2|p|} \left(\frac{k}{L}\right)^{2p} \int_{\mathbb{R}^d} |v^m \partial_v^q \mathcal{F}_x(f)(k,v)|^2 \, dv.$$

Finally we can compare this expression to the one we had in (5.3), and obtain

$$||f||^2_{\mathcal{H}^{r_2}_\nu} \geq C(1+K)^{2(r_2-r_1)} \, ||(I-P_K)f||^2_{\mathcal{H}^{r_1}_\nu}.$$

■

We now have bounds for f and f^K, uniform in K, so we are able to obtain an estimate on their difference.

Proposition 5.2

Let $\nu, r \in \mathbb{N}$, with $\nu > d/2$, $r \geq 3\nu$, and $\alpha \geq 2\nu + 1$. Let f be the solution to the Vlasov-Poisson equation (VP), and f^K be the solution to the Vlasov-Poisson equation with truncated kernel (VP^K), both with the same initial condition $f_0 \in \mathcal{H}^{r+\alpha}_\nu$, such that $||f_0||_{\mathcal{H}^{r+\alpha}_\nu} \leq B$ for some $B > 0$. Then, there exists a constant $C > 0$ such that for all $K \in \mathbb{N}^*$ and all $t \in [0,T]$,

$$\left|\left|(f - f^K)(t)\right|\right|^2_{\mathcal{H}^r_\nu} \leq \frac{C}{(1+K)^\alpha}$$

Proof: We follow the end of the proof of Theorem 5.1 from [CCFM17].

By taking the difference (VP) - (VP^K), we obtain

$$\partial_t(f-f^K) + v\cdot\nabla_x(f-f^K) - \nabla_x\Phi[f]\cdot\nabla_v(f-f^K) = \nabla_x\Phi[P_K f^K - f]\cdot\nabla_v f^K.$$

We have by previous estimates

$$\forall t \in [0,T], \quad \begin{cases} ||f(t)||_{\mathcal{H}^{r+\alpha}_\nu} \leq C(t,r,\nu,B)||f_0||_{\mathcal{H}^{r+\alpha}_\nu} \\ ||f^K(t)||_{\mathcal{H}^{r+\alpha}_\nu} \leq C(t,r,\nu,B)||f_0||_{\mathcal{H}^{r+\alpha}_\nu} \end{cases}.$$

Since $\alpha \geq 2\nu+1$, Lemma 5.3 from [CCFM17], gives for all $t \in [0,T]$

$$\begin{aligned} ||(f-f^K)(t)||^2_{\mathcal{H}^r_\nu} \leq ||(f-f^K)(0)||^2_{\mathcal{H}^r_\nu} + C\int_{0,t} \left(1+||f(\sigma)||_{\mathcal{H}^r_\nu}\right) \left|\left|(f-f^K)(\sigma)\right|\right|^2_{\mathcal{H}^r_\nu} d\sigma \\ + 2\int_0^T \left|\left|\nabla_x \Phi[P_K f^K - f]\cdot \nabla_v f^K(\sigma)\right|\right|_{\mathcal{H}^r_\nu} \left|\left|(f-f^K)(\sigma)\right|\right|_{\mathcal{H}^r_\nu} d\sigma \end{aligned} \tag{5.5}$$

We have (we skip the details since they are given in [CCFM17])

$$\left|\left|v^m \partial_x^p \partial_v^q \left(\nabla_x \Phi[P_K f^K - f]\cdot \nabla_v f^K\right)\right|\right|_{\mathbb{L}^2} \leq C_{r,\nu} ||f||_{\mathcal{H}^{r+2\nu+1}_\nu} ||P_K f^K - f||_{\mathcal{H}^r_\nu}. \tag{5.6}$$

Moreover, from the decomposition $P_K f^K - f = P_K(f^K - f) + (P^K - I)f$ we have, using Lemma 5.2,

$$\begin{aligned} ||P_K f^K - f||_{\mathcal{H}^r_\nu} &\leq ||P_K(f^K - f)||_{\mathcal{H}^r_\nu} + ||(I-P_K)f||_{\mathcal{H}^r_\nu} \\ &\leq ||f^K - f||_{\mathcal{H}^r_\nu} + ||(I-P_K)f||_{\mathcal{H}^r_\nu} \\ &\leq ||f^K - f||_{\mathcal{H}^r_\nu} + \frac{C}{(1+K)^\alpha} \end{aligned}$$

Then (5.5) becomes, with the help of (5.6),

$$\begin{aligned} \forall t\in[0,T],\quad ||(f-f^K)(t)||^2_{\mathcal{H}^r_\nu} \leq ||(f-f^K)(0)||^2_{\mathcal{H}^r_\nu} + C(f_0)\int_0^t ||(f-f^K)(\sigma)||^2_{\mathcal{H}^r_\nu} d\sigma \\ + 2C_{r,\nu}(f_0)\int_0^t \left(||(f^K - f)(\sigma)||^2_{\mathcal{H}^r_\nu} + \frac{C}{(1+K)^\alpha}\right) d\sigma. \end{aligned}$$

Since (1.1a) and (VP^K) have the same initial condition, we the obtain by the Grönwall lemma the existence of a time-dependent function C, independant of K, that depends on r, ν, f_0, such that

$$\forall t\in[0,T],\quad \left|\left|(f-f^K)(t)\right|\right|^2_{\mathcal{H}^r_\nu} \leq \frac{C(t)}{(1+K)^\alpha}.$$

Since the function $C(t)$ depends continuously on $t\in[0,T]$, we get the result. ■

Proposition 5.3

Let $c\in\mathbb{N}^d$, $\nu\in\mathbb{N}, \alpha\in\mathbb{N}^*$, with $\nu > d/2$. Let $E[g] := \nabla_x\Phi[g]$ be the kernel to the Vlasov-Poisson equation (VP), computed with some function $g\in\mathcal{H}^\alpha_\nu$, and let $E^K[h] := \nabla_x\Phi^K[h]$ be the kernel to the Vlasov-Poisson equation with truncated kernel (VP^K), computed with $h\in\mathcal{H}^\alpha_\nu$. We do not require g and h to be respectively solutions of (VP) and (VP^K).

Assume there exists a constant $C>0$ such that for all $K\in\mathbb{N}^*$, $||(g-h)(t)||^2_{\mathcal{H}^0_\nu} \leq \frac{C}{(1+K)^\alpha}$. Then, for all $t\in[0,T]$ and all $x\in\mathbb{T}^d_L$,

$$\left|\partial_x^c \left(E[g](t,x) - E^K[h](t,x)\right)\right| \leq \frac{C}{(1+K)^{\frac{\alpha+1}{2}-d-\sum_i c_i}}.$$

Proof: For any $t \in [0,T]$ and $x \in \mathbb{T}_L^d$,

$$\begin{aligned}
&\partial_x^c \left(E^K[h](t,x) - E[g](t,x)\right) \\
&= \frac{1}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{\frac{k}{L}}{2\pi\left|\frac{k}{L}\right|^2} \left(2\pi\frac{k}{L}\right)^c \sin\left(2\pi\frac{k}{L}\cdot y\right)\left(C_k^K(t) - C_k(t)\right) \\
&\quad - \frac{1}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{\frac{k}{L}}{2\pi\left|\frac{k}{L}\right|^2} \left(2\pi\frac{k}{L}\right)^c \cos\left(2\pi\frac{k}{L}\cdot y\right)\left(S_k^K(t) - S_k(t)\right) \\
&\quad\quad + \frac{1}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|> K}} \frac{\frac{k}{L}}{2\pi\left|\frac{k}{L}\right|^2} \left(2\pi\frac{k}{L}\right)^c \left(\sin\left(2\pi\frac{k}{L}\cdot y\right)C_k(t) - \cos\left(2\pi\frac{k}{L}\cdot y\right)S_k(t)\right)
\end{aligned}$$

Let $\bar{c} := \sum_i c_i$. Notice that $|k^c| \le |k|^{\bar{c}}$, therefore

$$\begin{aligned}
\left|\partial_x^c \left(E^K[h](t,x) - E[g](t,x)\right)\right| &\le \frac{(2\pi)^{\bar{c}}}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{\left|\frac{k}{L}\right|^{\bar{c}+1}}{2\pi\left|\frac{k}{L}\right|^2} \left(\left|C_k^K(t) - C_k(t)\right| + \left|S_k^K(t) - S_k(t)\right|\right) \\
&\quad + \frac{(2\pi)^{\bar{c}}}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|> K}} \frac{\left|\frac{k}{L}\right|^{\bar{c}+1}}{2\pi\left|\frac{k}{L}\right|^2} \left(|C_k(t)| + |S_k(t)|\right) \\
&\le \frac{1}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{1}{2\pi\left|\frac{k}{L}\right|^{1-\bar{c}}} \left(\left|C_k^K(t) - C_k(t)\right| + \left|S_k^K(t) - S_k(t)\right|\right) \\
&\quad + \frac{1}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|> K}} \frac{1}{2\pi\left|\frac{k}{L}\right|^{1-\bar{c}}} \left(|C_k(t)| + |S_k(t)|\right).
\end{aligned} \tag{5.7}$$

We have, for $g, h \in \mathcal{H}_\nu^r$,

$$\begin{aligned}
\left|C_k^K(t) - C_k(t)\right| &\le \int_{\mathbb{T}_L^d\times\mathbb{R}^d} |g(t,y,v) - h(t,y,v)|\, dydv \\
&\le \int_{\mathbb{T}_L^d} \left(\int_{\mathbb{R}^d} \frac{dv}{(1+|v|^2)^\nu}\right)^{1/2} \left(\int_{\mathbb{R}^d} (1+|v|^2)^\nu |g(t,y,v) - h(t,y,v)|^2\, dv\right)^{1/2} dy \\
&\le C\left(\int_{\mathbb{T}_L^d\times\mathbb{R}^d} (1+|v|^2)^\nu |g(t,y,v) - h(t,y,v)|^2\, dydv\right)^{1/2} \\
&\le C\,||(g-h)(t)||_{\mathcal{H}_\nu^0},
\end{aligned}$$

for some constant C that does not depend on K or t, thanks to the assumption $\nu > d/2$. The same estimate holds naturally for $\left|S_k^K(t) - S_k(t)\right|$. Therefore, using our hypothesis $||(g-h)(t)||_{\mathcal{H}_\nu^0}^2 \le \frac{C}{(1+K)^\alpha}$, we get

$$\left|C_k(t) - C_k^K(t)\right|^2 \le \frac{C}{(1+K)^\alpha}.$$

The same estimate holds for $\left|S_k^K(t) - S_k(t)\right|^2$. Then, summing over k and applying a discrete

Cauchy-Schwarz inequality, we obtain for any $\gamma > d + 2(\bar{c} - 1)$, i.e. $2 - 2\bar{c} + \gamma > d$,

$$\sum_{\substack{k\in(\mathbb{Z}^d)^*\\|k|\le K}} \frac{1}{2\pi\left|\frac{k}{L}\right|^{1-\bar{c}+\gamma/2}} |k|^{\gamma/2}\left(\left|C_k^K(t) - C_k(t)\right| + \left|S_k^K(t) - S_k(t)\right|\right)$$

$$\le \left(\sum_{\substack{k\in(\mathbb{Z}^d)^*\\|k|\le K}} \frac{1}{4\pi^2\left|\frac{k}{L}\right|^{2-2\bar{c}+\gamma}}\right)^{1/2} \left(\sum_{\substack{k\in(\mathbb{Z}^d)^*\\|k|\le K}} |k|^{\gamma}\left[\left|C_k^K(t) - C_k(t)\right| + \left|S_k^K(t) - S_k(t)\right|\right]^2\right)^{1/2}$$

$$\le C\left(\frac{K^{d+\gamma}}{(1+K)^{\alpha}}\right)^{1/2}$$

$$\le \frac{C}{(1+K)^{(\alpha-d-\gamma)/2}} \tag{5.8}$$

where the constant C does not depend on K or t.

The second sum in (5.7) can be estimated with Lemma 5.2 by using the fact that $g \in \mathcal{H}_\nu^\alpha$. Indeed, we have

$$\begin{aligned}|C_k(t)| &= \left|\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{T}_L^d}\left(e^{2i\pi\frac{k}{L}\cdot y} + e^{-2i\pi\frac{k}{L}\cdot y}\right) g(t,y,v)dydv\right| \\ &= \left|\frac{|\mathbb{T}_L^d|}{2}\int_{\mathbb{R}^d}\left(\mathcal{F}_x(g)(k,v) + \mathcal{F}_x(g)(-k,v)\right)dv\right| \\ &\le C\left(\left[\int_{\mathbb{R}^d}(1+|v|^2)^\nu\left|\mathcal{F}_x(g)(k,v)\right|^2 dv\right]^{1/2} + \left[\int_{\mathbb{R}^d}(1+|v|^2)^\nu\left|\mathcal{F}_x(g)(-k,v)\right|^2 dv\right]^{1/2}\right).\end{aligned}$$

Now apply Lemma 5.2 to obtain, for all $|k| > K$,

$$|C_k(t)| \le \frac{C}{(1+K)^{\alpha+d/2}}, \tag{5.9}$$

the same estimate holding for $|S_k(t)|$. Hence,

$$\sum_{\substack{k\in(\mathbb{Z}^d)^*\\|k|> K}} \frac{1}{|k|^{1-\bar{c}}}\left(|C_k(t)| + |S_k(t)|\right) \le \frac{C}{(1+K)^{\alpha+d/2+1-\bar{c}-(d+1)}} = \frac{C}{(1+K)^{\alpha-d/2-\bar{c}}}. \tag{5.10}$$

It remains to compare the exponents in (5.8) and (5.10). Under the condition $\gamma > d + 2(\bar{c} - 1)$, we have

$$\alpha - d/2 - \bar{c} - \frac{\alpha - d - \gamma}{2} = \frac{\alpha}{2} - \bar{c} + \frac{\gamma}{2} > \frac{\alpha + d}{2} - 1 \ge 0,$$

since $\alpha \in \mathbb{N}^*$.

Finally, the error in (5.7) is dominated by the error of the first term. Taking for instance $\gamma = d + 2\bar{c} - 1$, we obtain

$$\left|\partial_x^c\left(E[g](t,x) - E^K[h](t,x)\right)\right| \le \frac{C}{(1+K)^{\frac{\alpha+1}{2}-d-\bar{c}}}$$

■

We will need at some point regularity in time for f^K, E^K, and this can be obtained at the expense of additional space regularity. The following lemma shows how to "exchange" space regularity with time regularity:

Proposition 5.4

Let $j \in \mathbb{N}^*$, $\nu, r, \alpha \in \mathbb{N}$ such that $\nu + j > d/2$, $r \geq \max\left(3(\nu+j), (j-1)(d/2+1)\right)$ and $\alpha > 1 + 2(r+d)$.

Let $K \in \mathbb{N}^*$. If $f_0 \in \mathcal{H}^{r+\alpha}_{\nu+j}$, then the solution f^K to (VP^K) as well as the solution f to (VP) are smooth with respect to time in $\mathcal{H}^r_\nu$. That is, for all $l \in \mathbb{N}$ with $l \leq j$,

$$\partial_t^l f^K \in \mathcal{H}^{r-(l-1)(d/2+1)}_{\nu+j-l}, \qquad \partial_t^l f \in \mathcal{H}^{r-(l-1)(d/2+1)}_{\nu+j-l},$$

and we have

$$E^K \in C^j([0,T] \times \mathbb{R}^d), \qquad E \in C^j([0,T] \times \mathbb{R}^d).$$

Proof: Because of the way the kernel E^K is defined, the joint regularity in (t,x) can be studied by studying the regularity in t and the regularity in x. Note that $\left(x \mapsto E^K(t,x)\right)$ is $C^\infty_{per}(\mathbb{T}^d_L)$, so it only remains to study the regularity with respect to time of the kernel, which boils down to studying the regularity with respect to time of the coefficients $C^K_k(t), S^K_k(t)$. Our proof will be done by induction on the derivative.

Base case With our assumptions we get $r + \alpha > 2(\nu+j)+1$, so that by Proposition 5.1 we have $f^K(t) \in \mathcal{H}^{r+\alpha}_{\nu+j}$ for short enough times. Thus,

$$\begin{aligned}
\partial_t C^K_k(t) &= \int_{\mathbb{T}^d \times \mathbb{R}^d} \cos\left(2\pi \frac{k}{L} \cdot y\right) \partial_t f^K(t,y,v) dy dv \\
&= -\int_{\mathbb{T}^d \times \mathbb{R}^d} \cos\left(2\pi \frac{k}{L} \cdot y\right) \left(v \cdot \nabla_x f^K(t,y,v) + E^K(t,y) \cdot \nabla_v f^K(t,y,v)\right) dy dv \\
&= -\int_{\mathbb{T}^d \times \mathbb{R}^d} v \cdot \frac{2\pi k}{L} \sin\left(2\pi \frac{k}{L} \cdot y\right) f^K(t,y,v) dy dv,
\end{aligned}$$

since

$$-\int_{\mathbb{T}^d \times \mathbb{R}^d} \cos\left(2\pi \frac{k}{L} \cdot y\right) E^K(t,y) \cdot \nabla_v f^K(t,y,v) dy dv = 0.$$

This can be rewritten

$$\begin{aligned}
\partial_t C^K_k(t) &= -\frac{1}{2i} \int_{\mathbb{R}^d} v \cdot \frac{2\pi k}{L} \left(\int_{\mathbb{T}^d_L} e^{2i\pi \frac{k}{L} \cdot y} f^K(t,y,v) dy - \int_{\mathbb{T}^d_L} e^{-2i\pi \frac{k}{L} \cdot y} f^K(t,y,v) dy \right) dv \\
&= -\frac{|\mathbb{T}^d_L|}{2i} \int_{\mathbb{R}^d} v \cdot \frac{2\pi k}{L} \left(\mathcal{F}_x(f^K)(-k,v) - \mathcal{F}_x(f^K)(k,v)\right) dv.
\end{aligned}$$

Therefore,

$$\left|\partial_t C^K_k(t)\right| \leq \frac{|\mathbb{T}^d_L|}{2} \left|\frac{2\pi k}{L}\right| \int_{\mathbb{R}^d} |v| \left(\left|\mathcal{F}_x(f^K)(-k,v)\right| + \left|\mathcal{F}_x(f^K)(k,v)\right|\right) dv.$$

The integral can be estimated using the first estimate of Lemma 5.2 after a Cauchy-Schwarz inequality with

$$|v| \left|\mathcal{F}_x(f^K)(k,v)\right| = \frac{(1+|v|)^\nu}{(1+|v|)^\nu} |v| \left|\mathcal{F}_x(f^K)(k,v)\right|.$$

We have, for some C which does not depend on K,

$$\begin{aligned}
\left|\partial_t C^K_k(t)\right| &\leq C|k| \left(\left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f^K)(k,v)\right|^2\right]^{1/2} + \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f^K)(-k,v)\right|^2\right]^{1/2} \right) \\
&\leq C|k| \left(\left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f-f^K)(k,v)\right|\right]^{1/2} + \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f)(k,v)\right|\right]^{1/2} \right. \\
&\qquad \left. + \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f-f^K)(-k,v)\right|^2\right]^{1/2} + \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f)(-k,v)\right|^2\right]^{1/2} \right)
\end{aligned}$$

The same estimate holds for $|\partial_t S_k^K(t)|$. The second and fourth terms are estimated by Lemma 5.2, using that $f \in \mathcal{H}^{r+\alpha}_{\nu+j}$:

$$\left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f)(-k,v)\right|^2\right]^{1/2} \leq \frac{1}{(1+K)^{r+\alpha+d/2}}.$$

Let $c \in \mathbb{N}^d$ and let $\bar{c} := \sum_i c_i$. We assume $\bar{c} \leq r+\alpha-2(\nu+j)-1$, so that

$$\begin{aligned}
\left|\partial_x^c \partial_t E^K(t,x)\right| &\leq \frac{(2\pi)^{\bar{c}}}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\leq K}} \frac{1}{2\pi\left|\frac{k}{L}\right|^{1-\bar{c}}} \left(\left|\partial_t C_k^K(t)\right| + \left|\partial_t S_k^K(t)\right|\right) \\
&\leq C \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\leq K}} |k|^{\bar{c}} \left(\begin{aligned} &\left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f-f^K)(k,v)\right|^2 dv\right]^{1/2} + \frac{1}{(1+K)^{r+\alpha+d/2}} \\ &+ \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f-f^K)(-k,v)\right|^2 dv\right]^{1/2} + \frac{1}{(1+K)^{r+\alpha+d/2}} \end{aligned} \right).
\end{aligned}$$

The sum

$$\sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\leq K}} |k|^{\bar{c}} \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f-f^K)(k,v)\right|^2 dv\right]^{1/2}$$

can be bounded by some quantity equivalent to $\left|\left|f-f^K\right|\right|_{\mathcal{H}^{\bar{c}}_{\nu+1}} \leq \left|\left|f-f^K\right|\right|_{\mathcal{H}^{r+\alpha-2(\nu+j)-1}_{\nu+j}}$. Since $f - f^K \in \mathcal{H}^{r+\alpha}_{\nu+j}$, by Proposition 5.2 we get

$$\sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\leq K}} |k|^{\bar{c}} \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(f-f^K)(k,v)\right|^2 dv\right]^{1/2} \leq \frac{C}{(1+K)^{\nu+j+1/2}}.$$

Hence,

$$\left|\partial_x^c \partial_t E^K(t,x)\right| \leq \frac{C}{(1+K)^{\nu+j+1/2}} + C \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\leq K}} \frac{1}{(1+K)^{d+1}}.$$

This can be bounded by a constant $C>0$ which does not depend on K. Hence, for $\beta_1 \in \mathbb{N}$,

$$\left|\left|\partial_t f^K\right|\right|_{\mathcal{H}^{\beta_1}_{\nu+j-1}} \leq \left|\left|v\cdot f^K\right|\right|_{\mathcal{H}^{\beta_1}_{\nu+j-1}} + \left|\left|E^K\cdot\nabla_v f^K\right|\right|_{\mathcal{H}^{\beta_1}_{\nu+j-1}} \leq C\left|\left|f^K\right|\right|_{\mathcal{H}^{\beta_1+1}_{\nu+j}},$$

where the last inequality holds if

$$\beta_1 \leq r+\alpha-2(\nu+j)-1. \tag{5.11}$$

For the right-hand side of the estimate to be finite, we need to have

$$\beta_1 + 1 \leq r+\alpha,$$

since we only have $f^K \in \mathcal{H}^{r+\alpha}_{\nu+j}$. However this is already satisfied by (5.11) since $\nu+j\geq 0$. From now on let $\beta_1 = r$, so that

$$\partial_t f^K \in \mathcal{H}^{\beta_1}_{\nu+j-1}.$$

This holds true for any $K\in\mathbb{N}^*$, let's now show this estimate also holds with the solution f to the non-truncated Vlasov-Poisson equation. Let $p\leq \beta_1 - 1$,

$$\begin{aligned}
&\left|\left|\partial_t(f-f^K)\right|\right|_{\mathcal{H}^p_{\nu+j-1}} \\
&\leq \left|\left|v\cdot\nabla_x(f-f^K)\right|\right|_{\mathcal{H}^p_{\nu+j-1}} + \left|\left|E^K\cdot\nabla_v f^K - E\cdot\nabla_v f\right|\right|_{\mathcal{H}^p_{\nu+j-1}} \\
&\leq \left|\left|v\cdot\nabla_x(f-f^K)\right|\right|_{\mathcal{H}^p_{\nu+j-1}} + \left|\left|E^K\cdot\nabla_v (f^K-f)\right|\right|_{\mathcal{H}^p_{\nu+j-1}} + \left|\left|(E-E^K)\cdot\nabla_v f\right|\right|_{\mathcal{H}^p_{\nu+j-1}} \\
&\leq \left|\left|f-f^K\right|\right|_{\mathcal{H}^{p+1}_{\nu+j}} + C\left|\left|f^K-f\right|\right|_{\mathcal{H}^{p+1}_{\nu+j}} + \max_{\substack{c\in\mathbb{N}^d \\ \bar{c}\leq p}} \left|\left|\partial_x^c\left(E^K-E\right)\right|\right|_{\mathbb{L}^\infty(\mathbb{T}_L^d)} \left|\left|f\right|\right|_{\mathcal{H}^{p+1}_{\nu+j}}.
\end{aligned} \tag{5.12}$$

Because $p+1 \leq \beta_1 \leq r$, we have

$$||f - f^K||_{\mathcal{H}^{p+1}_{\nu+j}} \leq ||f - f^K||_{\mathcal{H}^{r}_{\nu+j}} \leq \frac{C}{(1+K)^{\alpha/2}},$$

where the first inequality is clear and the second one comes from Proposition 5.2. For the third term of (5.12), we have

$$||f - f^K||_{\mathcal{H}^{0}_{\nu+j}} \leq ||f - f^K||_{\mathcal{H}^{r}_{\nu+j}} \leq \frac{C}{(1+K)^{\alpha/2}},$$

so that, by Proposition 5.3,

$$\max_{\substack{c\in\mathbb{N}^d\\ \bar{c}\leq p}} ||\partial_x^c \left(E^K - E\right)||_{\mathbb{L}^\infty(\mathbb{T}^d_L)} \leq \frac{C}{(1+K)^{(\alpha+1)/2-d-\bar{c}}} \leq \frac{C}{(1+K)^{(\alpha+1)/2-d-p}}.$$

Hence, (5.12) yields

$$\begin{aligned}
||\partial_t(f - f^K)||_{\mathcal{H}^{p}_{\nu+j-1}} &\leq (C+1)\,||f^K - f||_{\mathcal{H}^{p+1}_{\nu+j}} + \max_{\substack{c\in\mathbb{N}^d\\ \bar{c}\leq p}} ||\partial_x^c \left(E^K - E\right)||_{\mathbb{L}^\infty(\mathbb{T}^d_L)}\, ||f||_{\mathcal{H}^{p+1}_{\nu+j}} \\
&\leq \frac{C}{(1+K)^{\alpha/2}} + \frac{C}{(1+K)^{(\alpha+1)/2-d-p}}.
\end{aligned}$$

Thus,

$$||\partial_t(f - f^K)||_{\mathcal{H}^{p}_{\nu+j-1}} \leq \frac{C}{(1+K)^{(\alpha+1)/2-d-p}} = \frac{C}{(1+K)^{\gamma_1+\beta_1-p}},$$

where

$$\begin{aligned}
&\frac{\alpha+1}{2} - d - p =: \gamma_1 + \beta_1 - p \\
\iff\ &\gamma_1 := \frac{\alpha+1}{2} - \beta_1 - d.
\end{aligned}$$

Requiring $\gamma_1 > 0$ yields the condition

$$\alpha > 1 + 2(\beta_1 + d).$$

With our assumption $\alpha > 1 + 2(r+d)$, the above inequality is satisfied.

Induction Let's now turn to the higher derivatives. Let $l \in \mathbb{N}$ with $l \leq j$, suppose that for any $m \leq l-1$, $\partial_t^m f^K, \partial_t^m f \in \mathcal{H}^{\beta_m}_{\nu+j-m}$ for some $r = \beta_1 \geq \dots \geq \beta_{l-1} > 0$, and assume there exists $C, \gamma_1 \geq \dots \geq \gamma_{l-1} > 0$ such that for all $m \leq l-1$, $p \leq \beta_m$,

$$||\partial_t^m(f - f^K)||_{\mathcal{H}^{p}_{\nu+j-m}} \leq \frac{C}{(1+K)^{\gamma_m+\beta_m-p}}. \tag{5.13}$$

Let $m \leq l$, we have

$$\begin{aligned}
\partial_t^m C_k^K(t) &= \int_{\mathbb{T}^d\times\mathbb{R}^d} \cos\left(2\pi\frac{k}{L}\cdot y\right) \partial_t^m f^K(t,y,v)dydv \\
&= -\int_{\mathbb{T}^d\times\mathbb{R}^d} \cos\left(2\pi\frac{k}{L}\cdot y\right) \begin{pmatrix} v\cdot\nabla_x\partial_t^{m-1} f^K(t,y,v) \\ \quad + \partial_t^{m-1}\left[E^K(t,y)\cdot\nabla_v f^K(t,y,v)\right] \end{pmatrix} dydv \\
&= -\int_{\mathbb{T}^d\times\mathbb{R}^d} \cos\left(2\pi\frac{k}{L}\cdot y\right) v\cdot\nabla_x\partial_t^{m-1} f^K(t,y,v)dydv,
\end{aligned}$$

since

$$\int_{\mathbb{T}^d} \cos\left(2\pi\frac{k}{L}\cdot y\right) E^K(t,y)\cdot\left(\int_{\mathbb{R}^d} \nabla_v f^K(t,y,v)dv\right)dy = 0.$$

As in the case $l = 1$, we have

$$\left|\partial_t^m C_k^K(t)\right| \le C|k| \left(\begin{aligned} &\left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1} f^K)(k,v)\right|^2 dv\right]^{1/2} \\ &\qquad + \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1} f^K)(-k,v)\right|^2 dv\right]^{1/2} \end{aligned} \right)$$

$$\le C|k| \left(\begin{aligned} &\left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1}(f-f^K))(k,v)\right| dv\right]^{1/2} + \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1} f)(k,v)\right| dv\right]^{1/2} \\ &+ \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1}(f-f^K))(-k,v)\right|^2 dv\right]^{1/2} + \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1} f)(-k,v)\right|^2 dv\right]^{1/2} \end{aligned} \right).$$

The same estimate holds for $|\partial_t^m S_k^K(t)|$. The second and fourth terms are estimated by Lemma 5.2, using that $\partial_t^{m-1} f \in \mathcal{H}^{\beta_{m-1}}_{\nu+j-(m-1)}$:

$$\left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1} f)(-k,v) dv\right|^2\right]^{1/2} \le \frac{1}{(1+K)^{\beta_{m-1}+d/2}}.$$

For $c \in \mathbb{N}^d$, $\bar{c} \le \beta_{m-1} - d/2 - 1$, we have

$$\left|\partial_x^c \partial_t^m E^K(t,x)\right| \le \frac{(2\pi)^{\bar{c}}}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{1}{2\pi \left|\frac{k}{L}\right|^{1-\bar{c}}} \left(\left|\partial_t^{m-1} C_k^K(t)\right| + \left|\partial_t^{m-1} S_k^K(t)\right|\right)$$

$$\le C \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} |k|^{\bar{c}} \left(\begin{aligned} &\left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1}(f-f^K))(k,v)\right|^2 dv\right]^{1/2} + \frac{1}{(1+K)^{\beta_{m-1}+d/2}} \\ &+ \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1}(f-f^K))(-k,v)\right|^2 dv\right]^{1/2} + \frac{1}{(1+K)^{\beta_{m-1}+d/2}} \end{aligned} \right).$$

The sum

$$\sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} |k|^{\bar{c}} \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1}(f-f^K))(k,v)\right|^2 dv\right]^{1/2}$$

can be bounded by some quantity equivalent to

$$\left|\left|\partial_t^{m-1}(f-f^K)\right|\right|_{\mathcal{H}^{\bar{c}}_{\nu+1}} \le \left|\left|\partial_t^{m-1}(f-f^K)\right|\right|_{\mathcal{H}^{\beta_{m-1}-d/2-1}_{\nu+j-(m-1)}} \le \frac{C}{(1+K)^{\gamma_{m-1}+d/2+1}},$$

where the last inequality is given by our induction hypothesis (5.13). That is,

$$\sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} |k|^{\bar{c}} \left[\int_{\mathbb{R}^d} |v|^{2(1+\nu)} \left|\mathcal{F}_x(\partial_t^{m-1}(f-f^K))(k,v)\right|^2 dv\right]^{1/2} \le \frac{C}{(1+K)^{\gamma_{m-1}+d/2+1}}.$$

Hence, for all $m \le l$, and $\bar{c} \le \beta_{m-1} - d/2 - 1$

$$\left|\partial_x^c \partial_t^{m-1} E^K(t,x)\right| \le \frac{C}{(1+K)^{\gamma_{m-1}+d/2+1}} + C \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{1}{(1+K)^{d+1}}.$$

This can be bounded by a constant $C > 0$ which does not depend on K. Thus, there exists a constant $C > 0$ such that for all $m \le l$, all $c \in \mathbb{N}^d$ with $\bar{c} \le \beta_{m-1} - d/2 - 1$ and all $K \in \mathbb{N}$,

$$\left|\partial_x^c \partial_t^{m-1} E^K(t,x)\right| \le C$$

for some constant $C>0$ which is independant of t, K, x. Hence, for $\beta_l \in \mathbb{N}$,

$$\begin{aligned}
||\partial_t^l f^K||_{\mathcal{H}^{\beta_l}_{\nu+j-l}} &\leq ||v\cdot\partial_t^{l-1}\nabla_x f^K||_{\mathcal{H}^{\beta_l}_{\nu+j-l}} + \sum_{m=0}^{l-1}\binom{l-1}{m}||\partial_t^m E^K\cdot\nabla_v\partial_t^{l-1-m}f^K||_{\mathcal{H}^{\beta_l}_{\nu+j-l}} \\
&\leq C\sum_{m=0}^{l-1}\binom{l-1}{m}||\partial_t^{l-1-m}f^K||_{\mathcal{H}^{\beta_l+1}_{\nu+j-(l-1)}}
\end{aligned}$$

where the last inequality holds when

$$\begin{aligned}
\beta_l &\leq \beta_{l-1} - d/2 - 1 \\
&\vdots \\
\beta_l &\leq \beta_1 - d/2 - 1,
\end{aligned}$$

which reduces to

$$\beta_l \leq \beta_{l-1} - d/2 - 1 \tag{5.14}$$

thanks to our assumption $\beta_1 \geq \cdots \geq \beta_{l-1}$. By induction on $l = j, j-1, \ldots, 1$, we get conditions on the β_l (recall we let $\beta_1 = r$):

$$\beta_l \leq \beta_1 - (l-1)(d/2+1) = r - (l-1)(d/2+1).$$

In order to have $\beta_j \geq 0$, we need to have $r \geq (j-1)(d/2+1)$ which is one of our assumptions on r. Let $p \leq \beta_l$,

$$\begin{aligned}
&||\partial_t^l(f-f^K)||_{\mathcal{H}^p_{\nu+j-l}} \leq ||v\cdot\nabla_x\partial_t^{l-1}(f-f^K)||_{\mathcal{H}^p_{\nu+j-l}} + ||\partial_t^{l-1}\left(E\cdot\nabla_v f\right) - \partial_t^{l-1}\left(E^K\cdot\nabla_v f^K\right)||_{\mathcal{H}^p_{\nu+j-l}} \\
&\leq ||\partial_t^{l-1}(f-f^K)||_{\mathcal{H}^{p+1}_{\nu+j-(l-1)}} + \sum_{m=0}^{l-1}\binom{l-1}{m}||\partial_t^m E\cdot\nabla_v\partial_t^{l-1-m}f - \partial_t^m E^K\cdot\nabla_v\partial_t^{l-1-m}f^K||_{\mathcal{H}^p_{\nu+j-l}} \\
&\leq ||\partial_t^{l-1}(f-f^K)||_{\mathcal{H}^{p+1}_{\nu+j-(l-1)}} + \sum_{m=0}^{l-1}\binom{l-1}{m}||\partial_t^m E\cdot\nabla_v\partial_t^{l-1-m}f - \partial_t^m E^K\cdot\nabla_v\partial_t^{l-1-m}f^K||_{\mathcal{H}^p_{\nu+j-l}} \\
&\leq ||\partial_t^{l-1}(f-f^K)||_{\mathcal{H}^{p+1}_{\nu+j-(l-1)}} + \sum_{m=0}^{l-1}\binom{l-1}{m}\begin{pmatrix} ||\left(\partial_t^m E - \partial_t^m E^K\right)\cdot\nabla_v\partial_t^{l-1-m}f||_{\mathcal{H}^p_{\nu+j-l}} \\ + ||\partial_t^m E^K\cdot\nabla_v\partial_t^{l-1-m}(f-f^K)||_{\mathcal{H}^p_{\nu+j-l}}\end{pmatrix} \\
&\leq ||\partial_t^{l-1}(f-f^K)||_{\mathcal{H}^{p+1}_{\nu+j-(l-1)}} + \sum_{m=0}^{l-1}\binom{l-1}{m}\begin{pmatrix} ||\left(E[\partial_t^m f] - E^K[\partial_t^m f^K]\right)\cdot\nabla_v\partial_t^{l-1-m}f||_{\mathcal{H}^p_{\nu+j-l}} \\ + ||\partial_t^m E^K\cdot\nabla_v\partial_t^{l-1-m}(f-f^K)||_{\mathcal{H}^p_{\nu+j-l}}\end{pmatrix} \\
&\leq ||\partial_t^{l-1}(f-f^K)||_{\mathcal{H}^{p+1}_{\nu+j-(l-1)}} \\
&\quad + \sum_{m=0}^{l-1}\binom{l-1}{m}\begin{pmatrix} \max_{\substack{c\in\mathbb{N}^d \\ \bar{c}\leq p}}||\partial_x^c\left(E[\partial_t^m f] - E^K[\partial_t^m f^K]\right)||_{\mathbb{L}^\infty(\mathbb{T}^d_L)}||\partial_t^{l-1-m}f||_{\mathcal{H}^{p+1}_{\nu+j-l}} \\ + \max_{\substack{c\in\mathbb{N}^d \\ \bar{c}\leq p}}||\partial_x^c\partial_t^m E^K||_{\mathbb{L}^\infty(\mathbb{T}^d_L)}||\partial_t^{l-1-m}(f-f^K)||_{\mathcal{H}^{p+1}_{\nu+j-l}}\end{pmatrix}.
\end{aligned} \tag{5.15}$$

Recall our assumption (5.13):

$$\forall m \leq l-1,\, p \leq \beta_m,\, ||\partial_t^m(f-f^K)||_{\mathcal{H}^p_{\nu+j-m}} \leq \frac{C}{(1+K)^{\gamma_m+\beta_m-p}},$$

hence

$$\begin{aligned}
&||\partial_t^{l-1}(f-f^K)||_{\mathcal{H}^{p+1}_{\nu+j-(l-1)}} + \sum_{m=0}^{l-1}\binom{l-1}{m}\max_{\substack{c\in\mathbb{N}^d \\ \bar{c}\leq p}}||\partial_x^c\partial_t^m E^K||_{\mathbb{L}^\infty(\mathbb{T}^d_L)}||\partial_t^{l-1-m}(f-f^K)||_{\mathcal{H}^{p+1}_{\nu+j-l}} \\
&\leq \frac{C}{(1+K)^{\gamma_{l-1}+\beta_{l-1}-p}} + \sum_{m=0}^{l-1}\binom{l-1}{m}\max_{\substack{c\in\mathbb{N}^d \\ \bar{c}\leq p}}||\partial_x^c\partial_t^m E^K||_{\mathbb{L}^\infty(\mathbb{T}^d_L)}\frac{C}{(1+K)^{\gamma_{l-1-m}+\beta_{l-1-m}-p}}.
\end{aligned}$$

By our previous estimates, $||\partial_x^c \partial_t^{l-1} E^K||_{\mathbb{L}^\infty(\mathbb{T}_L^d)} \leq C$ for any $c \in \mathbb{N}^d$ with $\bar{c} \leq \beta_{l-1} - d/2 - 1$. Thus,

$$||\partial_t^{l-1}(f - f^K)||_{\mathcal{H}^{p+1}_{\nu+j-(l-1)}} + \sum_{m=0}^{l-1} \binom{l-1}{m} \max_{\substack{c\in\mathbb{N}^d \\ \bar{c}\leq p}} ||\partial_x^c \partial_t^m E^K||_{\mathbb{L}^\infty(\mathbb{T}_L^d)} ||\partial_t^{l-1-m}(f - f^K)||_{\mathcal{H}^{p+1}_{\nu+j-l}}$$
$$\leq \frac{C}{(1+K)^{\gamma_{l-1}+\beta_{l-1}-p}} + \sum_{m=0}^{l-1} \binom{l-1}{m} \frac{C}{(1+K)^{\gamma_{l-1-m}+\beta_{l-1-m}-p}}.$$

Moreover, for any $m \leq l-1$, $\gamma_{l-1} + \beta_{l-1} - p \leq \gamma_m + \beta_m - p$, so that by (5.13),

$$||\partial_t^{l-1}(f - f^K)||_{\mathcal{H}^{p+1}_{\nu+j-(l-1)}} + \sum_{m=0}^{l-1} \binom{l-1}{m} \max_{\substack{c\in\mathbb{N}^d \\ \bar{c}\leq p}} ||\partial_x^c \partial_t^m E^K||_{\mathbb{L}^\infty(\mathbb{T}_L^d)} ||\partial_t^{l-1-m}(f - f^K)||_{\mathcal{H}^{p+1}_{\nu+j-l}}$$
$$\leq \frac{C}{(1+K)^{\gamma_{l-1}+\beta_{l-1}-p-1}}.$$

We need to estimate the first term in the sum of (5.15). Again, by (5.13) we have

$$||\partial_t^m(f - f^K)||_{\mathcal{H}^0_{\nu+j-m}} \leq \frac{C}{(1+K)^{\gamma_m+\beta_m}},$$

therefore Proposition 5.3 gives

$$\max_{\substack{c\in\mathbb{N}^d \\ \bar{c}\leq p}} ||\partial_x^c \left(E[\partial_t^m f] - E^K[\partial_t^m f^K]\right)||_{\mathbb{L}^\infty(\mathbb{T}_L^d)} \leq \frac{C}{(1+K)^{\gamma_m+\beta_m+\frac{1}{2}-d-p}}$$
$$\leq \frac{C}{(1+K)^{\gamma_{l-1}+\beta_{l-1}+\frac{1}{2}-d-p}}.$$

Finally, using the condition (5.14),

$$\begin{aligned}
||\partial_t^l(f - f^K)||_{\mathcal{H}^p_{\nu+j-l}} &\leq \frac{C}{(1+K)^{\gamma_{l-1}+\beta_{l-1}+\min(1/2-d,-1)-p}} \\
&\leq \frac{C}{(1+K)^{\gamma_{l-1}+\beta_l+d/2+1+\min(1/2-d,-1)-p}} \\
&\leq \frac{C}{(1+K)^{\gamma_{l-1}+\beta_l+\min(3/2-d/2,d/2)-p}}
\end{aligned}$$

Thus we deduce that we must have $\gamma_l = \gamma_{l-1} + \frac{\min(3-d,d)}{2} = \cdots = \gamma_1 + (l-1)\frac{\min(3-d,d)}{2}$. Moreover, $\gamma_1 = \frac{\alpha+1}{2} - \beta_1 - d$, hence

$$\gamma_l = \frac{\alpha+1}{2} - \beta_1 - d + (l-1)\frac{\min(3-d,d)}{2}.$$

The condition $\gamma_l > 0, l = 2, \ldots, j$, gives

$$\alpha > 2(\beta_1 + d) - (l-1)\min(3-d,d) - 1.$$

It is clear that $\min(3-d,d) \leq d$, thus $-\min(3-d,d) \geq -d$, so that

$$\alpha > 2(\beta_1 + d) - (j-1)d - 1 = 2(r+d) - (j-1)d - 1,$$

which is guaranteed to hold for every $j \in \mathbb{N}^*$ since we assume $\alpha > 1 + 2(r+d)$. ∎

We recall from Section 2.1 that q_i is the order of the quadrature along the i-th dimension and that Δz_i is the quadrature step along the i-th dimension, $1 \leq i \leq 2d$. We recall as well that the coefficients C_k^K, S_k^K are defined by (1.14), and the coefficients $C_k^{K,h}, S_k^{K,h}$ by (2.1). We have the following estimates on the quadrature error:

Proposition 5.5

Let $j \in \mathbb{N}$ such that $j \geq 1 + \max_i q_i$, and $\nu, r, \alpha \in \mathbb{N}$ such that $\nu + j > d/2$, $r \geq \max(3(\nu + j), (j-1)(d/2+1))$, and $\alpha > 1 + 2(r+d)$. Let $K \in \mathbb{N}$, and assume $f_0 \in \mathcal{H}^{r+\alpha}_{\nu+j}$.

Then there exists a constant $C > 0$ such that the following holds: for $\delta \geq 0$, define finite intervals $I_{d+1} := [a_1, b_1], \ldots, I_{2d} = [a_d, b_d]$ and $I_v := I_{d+1} \times \cdots \times I_{2d}$ such that

$$||f_0||_{\mathcal{H}^0_\nu(\mathbb{T}^d_L \times (\mathbb{R}^d \setminus I_v))} \leq \delta.$$

Then for all $k \in (\mathbb{Z}^d)^*$ and $K \in \mathbb{N}^*$, we have

$$\left|C_k^K(t) - C_k^{K,h}(t)\right| \leq C\delta + C \sum_{i=1}^{2d} \left(1 + C\frac{2\pi(q_i+1)}{\ln(q_i+2)}\left|\frac{k}{L}\right|\right)^{q_i+1} \Delta z_i^{q_i} \tag{5.16}$$

and

$$\left|S_k^K(t) - S_k^{K,h}(t)\right| \leq C\delta + C \sum_{i=1}^{2d} \left(1 + C\frac{2\pi(q_i+1)}{\ln(q_i+2)}\left|\frac{k}{L}\right|\right)^{q_i+1} \Delta z_i^{q_i} \tag{5.17}$$

where the estimates are uniform in time, and C does not depend on Δz_i.

As a consequence, for C large enough, we have the following estimates:

$$\left|C_k^K(t) - C_k^{K,h}(t)\right| \leq C\delta + C \sum_{i=1}^{2d} (1 + C|k|)^{q_i+1} \Delta z_i^{q_i} \tag{5.18}$$

and

$$\left|S_k^K(t) - S_k^{K,h}(t)\right| \leq C\delta + C \sum_{i=1}^{2d} (1 + C|k|)^{q_i+1} \Delta z_i^{q_i}. \tag{5.19}$$

Proof: We prove only the error estimate for $\left|C_k^K(t) - C_k^{K,h}(t)\right|$, since the treatment is exactly the same for $\left|S_k^K(t) - S_k^{K,h}(t)\right|$.

First of all, by the regularity assumption on f_0 we know from Proposition 5.4 that $E^K \in \mathcal{C}^j([0,T] \times \mathbb{R}^d)$. Therefore, the characteristics $(X^K, V^K) \in C^j([0,T] \times \mathbb{T}^d \times \mathbb{R}^d)$, so that the j-th space derivative is continuous in time.

The quadratures in velocity will be performed on the intervals $I_{d+i} = [a_i, b_i], i = 1, \ldots, d$, and the quadratures in space will be performed on $\mathbb{T}^d$. To make notations clearer and more general, define $I_i := [0, L_i]$ for $i = 1, \ldots, d$.

For $n = 1, \cdots, 2d$, we define

$$\tilde{z}_n := (z_n, \ldots, z_{2d}) \in I_n \times \cdots \times I_{2d},$$

$$g_t(\tilde{z}_n) := \int_{I_1 \times \cdots \times I_{n-1}} \cos\left(2\pi \frac{k}{L} \cdot X^K(t; 0, z)\right) f_0(z) dz_1 \cdots dz_{n-1},$$

$$h_t(\tilde{z}_n) = \sum_{j_1, \cdots, j_{n-1}} w_1^{j_1} \cdots w_{n-1}^{j_{n-1}} \cos\left(2\pi \frac{k}{L} \cdot X^K(t; 0, z_1^{j_1}, \cdots, z_{n-1}^{j_{n-1}}, \tilde{z}_n)\right) f_0(z_1^{j_1}, \cdots, z_{n-1}^{j_{n-1}}, \tilde{z}_n).$$

We will prove the estimates (5.16) and (5.17) by induction on the number of dimensions.

Base case For a fixed $\tilde z_2 \in I_2 \times \cdots \times I_{2d}$, the quadrature along the first dimension gives

$$\left| \int_{I_1} \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z_1,\tilde z_2)\right) f(0,z_1,\tilde z_2) dz_1 - \sum_{j_1} w_1^{j_1} \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z_1^{j_1},\tilde z_2)\right) f_0(z_1^{j_1},\tilde z_2)\right| \\ \leq C\Delta {z_1}^{q_1} \left|\left| \partial_{z_1}^{q_1+1} \left[\cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,\cdot,\tilde z_2)\right) f_0(\cdot,\tilde z_2)\right]\right|\right|_{\mathbb{L}^\infty(I_1)}. \tag{5.20}$$

We proceed to estimate the right-hand side, and consider a derivative along the n-th dimension instead of only along the first dimension:

$$\partial_{z_n}^{q_n+1} \left[\cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right) f_0(z)\right] = \sum_{l=0}^{q_n+1} \binom{q_n+1}{l} \partial_{z_n}^{q_n+1-l} f_0(z) \partial_{z_n}^l \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right). \tag{5.21}$$

By the Faà di Bruno formula (see [AS64, Sect. 24.1.2]), we have

$$\partial_{z_n}^l \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right) \\ = \sum_{m=0}^{l} \cos^{(m)}\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right) \sum (l;a_1,\ldots,a_l)' \prod_{c=1}^{l} \left(2\pi \frac{k}{L} \cdot \partial_{z_n}^c X^K(t;0,z)\right)^{a_c},$$

where the unindexed sum is performed over all l-tuples $(a_1,\ldots,a_l)$ such that

$$a_1 + 2a_2 + \cdots + la_l = l \quad \text{and} \quad a_1 + a_2 + \cdots + a_l = m.$$

The sum $\sum(l;a_1,\ldots,a_l)'$ is also called a Stirling number of the Second kind, of parameters (n,m). It counts the number of ways of partitioning a set of l elements into m non-empty subsets. We have

$$\begin{aligned} \left|\partial_{z_n}^l \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right)\right| &\leq \sum_{m=0}^{l} \sum (l;a_1,\ldots,a_l)' \prod_{c=1}^{l} \left|2\pi \frac{k}{L} \cdot \partial_{z_n}^c X^K(t;0,z)\right|^{a_c} \\ &\leq \sum_{m=0}^{l} \sum (l;a_1,\ldots,a_l)' \prod_{c=1}^{l} \left|2\pi \frac{k}{L}\right|_2^{a_c} \left|\partial_{z_n}^c X^K(t;0,z)\right|^{a_c} \\ &\leq (2\pi)^l \left|\frac{k}{L}\right|_2^l \sum_{m=0}^{l} \sum (l;a_1,\ldots,a_l)' \prod_{c=1}^{l} \left|\partial_{z_n}^c X^K(t;0,y,z)\right|^{a_c}, \end{aligned}$$

where the second inequality has been obtained by the discrete Cauchy-Schwarz inequality.

Since the characteristics X^K is of class $\mathcal{C}^j([0,T] \times \mathbb{T}_L^d \times \mathbb{R}^d)$, with $j \geq \max_i q_i + 1$, we know there exists a constant $C_{f_0}(K)$ that depends possibly on K such that for all $\mathbb{N} \ni c \leq q_n + 1$,

$$||\partial_{z_n}^c X^K||_{\mathbb{L}^\infty([0,T]\times\mathbb{T}_L^d\times I_v)} \leq C_{f_0}(K).$$

However, we want this constant C_{f_0} to be independant of K, and to be able to choose such a constant, we notice that as $K \to \infty$, we recover the non-truncated Vlasov-Poisson system's characteristics. For these characteristics, thanks to the regularity assumption, we know that there exists a constant denoted $C_{f_0}(\infty)$ such that

$$||\partial_{z_n}^c X||_{\mathbb{L}^\infty([0,T]\times\mathbb{T}_L^d\times I_v)} \leq C_{f_0}(\infty) < \infty.$$

So we can build a sequence of constants $\{C_{f_0}(K)\}_{K\geq 1}$ which is bounded. Then define $C_{f_0} := \max_{K\geq 1} C_{f_0}(K)$, and we have

$$||\partial_{z_n}^c X^K||_{\mathbb{L}^\infty([0,T]\times\mathbb{T}_L^d\times I_v)} \leq C_{f_0}.$$

Hence, for all $t \in [0,T]$,

$$\left|\partial_{z_n}^l \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right)\right| \le C_{f_0}^l \, (2\pi)^l \left|\frac{k}{L}\right|^l \sum_{m=0}^{l} \sum (l; a_1, \ldots, a_l)'.$$

The remaining sums correspond to the Bell number B_l, and it counts the number of ways to partition a set that has exactly l elements. We have the following bound (see [BT10]):

$$B_l \le \left(\frac{0.792 l}{\ln(l+1)}\right)^l \le \frac{l^l}{(\ln(l+1))^l}.$$

Therefore,

$$\left|\partial_{z_n}^l \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right)\right| \le C_{f_0}^l \left(\frac{2\pi l}{\ln(l+1)}\right)^l \left|\frac{k}{L}\right|^l \le C_{f_0}^l \left(\frac{2\pi (q_n+1)}{\ln(q_n+2)}\right)^l \left|\frac{k}{L}\right|^l.$$

By regularity of the initial condition f_0, we can choose the constant C_{f_0} large enough so that for all $n = 1, \cdots, 2d$,

$$||\partial_{z_n}^l f_0||_{\mathbb{L}^\infty(I_1\times\cdots\times I_{2d})} \le C_{f_0}, \quad l = 0, \ldots, q_n + 1.$$

We then get from (5.21)

$$\begin{aligned}
&\left|\partial_{z_n}^{q_n+1}\left[\cos\left(2\pi\frac{k}{L}\cdot X^K(t;0,z)\right) f_0(z)\right]\right| \\
&\le C_{f_0} \sum_{l=0}^{q_n+1} \binom{q_n+1}{l} \left|\left|\partial_{z_n}^l \cos\left(2\pi\frac{k}{L}\cdot X(t;0,z)\right)\right|\right|_{\mathbb{L}^\infty(I_1\times\cdots\times I_{2d})} \\
&\le C_{f_0}\left(1 + C_{f_0}\frac{2\pi(q_n+1)}{\ln(q_n+2)}\left|\frac{k}{L}\right|\right)^{q_n+1}
\end{aligned}$$

Note that the right-hand side does not depend on y or v, hence

$$\left|\left|\partial_{z_n}^{q_n+1}\left[\cos\left(2\pi\frac{k}{L}\cdot X^K(t;0,\cdot)\right) f_0(\cdot)\right]\right|\right|_{\mathbb{L}^\infty(I_1\times\cdots\times I_{2d})} \le C_{f_0}\left(1 + C_{f_0}\frac{2\pi(q_n+1)}{\ln(q_n+2)}\left|\frac{k}{L}\right|\right)^{q_n+1}. \tag{5.22}$$

Plugging this estimate with $n = 1$ back into (5.20), we obtain

$$\begin{aligned}
&\left|\int_{I_1} \cos\left(2\pi\frac{k}{L}\cdot X^K(t;0,z_1,\tilde{z}_2)\right) f_0(z_1,\tilde{z}_2)dz_1 - \sum_{j_1} w_1^{j_1} \cos\left(2\pi\frac{k}{L}\cdot X^K(t;0,z_1^{j_1},\tilde{z}_2)\right) f_0(z_1^{j_1},\tilde{z}_2)\right| \\
&= |g_t(\tilde{z}_2) - h_t(\tilde{z}_2)| \le C\left(1 + \frac{2\pi(q_1+1)}{\ln(q_1+2)}\left|\frac{k}{L}\right|\right)^{q_1+1} \Delta z_1^{q_1},
\end{aligned}$$

where the constant C does not depend on $k, \Delta z_1, q_1, \tilde{z}_2$.

Induction step We have

$$\begin{aligned}
&|g_t(\tilde{z}_{n+1}) - h_t(\tilde{z}_{n+1})| = \left|\int_{I_n} g_t(z_n,\tilde{z}_{n+1})dz_n - \sum_{j_n} w_n^{j_n} h_t(z_n^{j_n},\tilde{z}_{n+1})\right| \\
&\le \int_{I_n} |g_t(z_n,\tilde{z}_{n+1}) - h_t(z_n,\tilde{z}_{n+1})|\, dz_n + \left|\int_{I_n} h_t(z_n,\tilde{z}_{n+1})dz_n - \sum_{j_n} w_n^{j_n} h_t(z_n^{j_n},\tilde{z}_{n+1})\right|.
\end{aligned} \tag{5.23}$$

The first term on the right-hand side can be bounded using the previous step in the induction, which is assumed to give the following estimate:

$$|g_t(\tilde{z}_n) - h_t(\tilde{z}_n)| \le C\sum_{i=1}^{n-1}\left(1 + C\frac{2\pi(q_i+1)}{\ln(q_i+2)}\left|\frac{k}{L}\right|\right)^{q_i+1} \Delta z_i^{q_i}.$$

Since the right-hand side does not depend on $\tilde{z}_n$, we get

$$\int_{I_n} |g_t(z_n,\tilde{z}_{n+1}) - h_t(z_n,\tilde{z}_{n+1})|\, dz_n \leq |I_n|\, ||g_t(\tilde{z}_n) - h_t(\tilde{z}_n)||_{\mathbb{L}^\infty(I_n\times\cdots\times I_{2d})}$$
$$\leq C\sum_{i=1}^{n-1}\left(1 + C\frac{2\pi(q_i+1)}{\ln(q_i+2)}\left|\frac{k}{L}\right|\right)^{q_i+1}\Delta z_i^{q_i},$$

where the constant C does not depend on $k, \Delta z_i, q_i, \tilde{z}_{n+1}$.

It remains only to estimate the second term on the right-hand side of (5.23). We notice that it correspond to the quadrature error of the function $z_n \mapsto h_t(z_n,\tilde{z}_{n+1})$ over I_n. Thus,

$$\left|\int_{I_n} h_t(z_n,\tilde{z}_{n+1})dz_n - \sum_{j_n} w_n^{j_n} h_t(z_n^{j_n},\tilde{z}_{n+1})\right| \leq C\left|\left|\partial_{z_n}^{q_n+1} h_t(\cdot,\tilde{z}_{n+1})\right|\right|_{\mathbb{L}^\infty(I_n)}\Delta z_n^{q_n}. \tag{5.24}$$

We have

$$\partial_{z_n}^{q_n+1} h_t(z_n,\tilde{z}_{n+1}) = \partial_{z_n}^{q_n+1} h_t(\tilde{z}_n)$$
$$= \sum_{j_1,\cdots,j_{n-1}} w_1^{j_1}\cdots w_{n-1}^{j_{n-1}}\partial_{z_n}^{q_n+1}\left[\cos\left(2\pi\frac{k}{L}\cdot X^K(t;0,z_1^{j_1},\cdots,z_{n-1}^{j_{n-1}},\tilde{z}_n)\right) f_0(z_1^{j_1},\cdots,z_{n-1}^{j_{n-1}},\tilde{z}_n)\right],$$

and hence

$$\left|\left|\partial_{z_n}^{q_n+1} h_t(\cdot,\tilde{z}_{n+1})\right|\right|_{\mathbb{L}^\infty(I_n)}$$
$$\leq \sum_{j_1,\cdots,j_{n-1}}\left|w_1^{j_1}\cdots w_{n_1}^{j_{n-1}}\right|$$
$$\left|\left|\partial_{z_n}^{q_n+1}\left[\cos\left(2\pi\frac{k}{L}\cdot X^K(t;0,z_1^{j_1},\cdots,z_{n-1}^{j_{n-1}},\tilde{z}_n)\right) f_0(z_1^{j_1},\cdots,z_{n-1}^{j_{n-1}},\tilde{z}_n)\right]\right|\right|_{\mathbb{L}^\infty_{z_n}(I_n)}$$
$$\leq \sum_{j_1,\cdots,j_{n-1}}\left|w_1^{j_1}\cdots w_{n-1}^{j_{n-1}}\right|\left|\left|\partial_{z_n}^{q_n+1}\left[\cos\left(2\pi\frac{k}{L}\cdot X^K(t;0,\cdot)\right) f_0(\cdot)\right]\right|\right|_{\mathbb{L}^\infty(I_1\times\cdots\times I_{2d})}.$$

By (5.22), we get

$$\left|\left|\partial_{z_n}^{q_n+1} h_t(\cdot,\tilde{z}_{n+1})\right|\right|_{\mathbb{L}^\infty(I_n)} \leq C\left(1 + C\frac{2\pi(q_n+1)}{\ln(q_n+2)}\left|\frac{k}{L}\right|\right)^{q_n+1}\sum_{j_1,\cdots,j_{n-1}}\left|w_1^{j_1}\cdots w_{n-1}^{j_{n-1}}\right|.$$

Moreover, since the weights are nonnegative,

$$\sum_{j_1,\cdots,j_{n-1}}\left|w_1^{j_1}\cdots w_{n-1}^{j_{n-1}}\right| = \sum_{j_1,\cdots,j_{n-1}} w_1^{j_1}\cdots w_{n-1}^{j_{n-1}}.$$

The right-hand side corresponds to an approximation of the constant function equal to one on the hyperrectangle $I_1\times\cdots\times I_{n-1}$, hence the quadrature is exact and the value of the sum corresponds to the volume of the hyperrectangle. Therefore,

$$\left|\left|\partial_{z_n}^{q_n+1} h_t(\cdot,\tilde{z}_{n+1})\right|\right|_{\mathbb{L}^\infty(I_n)} \leq C\left(1 + C\frac{2\pi(q_n+1)}{\ln(q_n+2)}\left|\frac{k}{L}\right|\right)^{q_n+1}.$$

We can plug this into (5.24) to get

$$\left|\int_{I_n} h_t(z_n,\tilde{z}_{n+1})dz_n - \sum_{j_n} w_n^{j_n} h_t(z^{j_n},\tilde{z}_{n+1})\right| \leq C\left(1 + C\frac{2\pi(q_n+1)}{\ln(q_n+2)}\left|\frac{k}{L}\right|\right)^{q_n+1}\Delta z_n^{q_n}.$$

Finally, we obtain from (5.23)

$$|g_t(\tilde{z}_{n+1}) - h_t(\tilde{z}_{n+1})| \leq C\sum_{i=1}^{n}\left(1 + C\frac{2\pi(q_i+1)}{\ln(q_i+2)}\left|\frac{k}{L}\right|\right)^{q_i+1}\Delta z_i^{q_i}.$$

This achieves the induction step, so that this inequality holds for all $n = 1, \dots, 2d$.

When $n = 2d$,

$$\left| \int_{\mathbb{T}_L^d \times I_v} \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right) f_0(z) dz - C_k^{K,h}(t) \right| \leq C \sum_{i=1}^{2d} \left(1 + C \frac{2\pi(q_i+1)}{\ln(q_i+2)} \left|\frac{k}{L}\right|\right)^{q_i+1} \Delta z_i^{q_i},$$

where the constant C does not depend on $k, \Delta x, \Delta v, q_x, q_v$. Finally, by definition of the intervals I_i, we have

$$C_k(t) = \int_{\mathbb{T}^d \times I_v} \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right) f_0(z) dz + \int_{\mathbb{T}^d \times (\mathbb{R}^d \setminus I_v)} \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right) f_0(z) dz.$$

The second term on the left-hand side can be handled by using the fact that $f_0 \in \mathcal{H}_\nu^{r+2\nu+1}$, so that

$$\begin{aligned}
&\left| \int_{\mathbb{T}^d \times (\mathbb{R}^d \setminus I_v)} \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z)\right) f_0(z) dz \right| \\
&\leq \int_{\mathbb{T}^d \times (\mathbb{R}^d \setminus I_v)} |f_0(z)| dz \\
&\leq \left(\int_{\mathbb{T}^d \times (\mathbb{R}^d \setminus I_v)} \frac{1}{(1+|v|^2)^\nu} dx dv \right) \left(\int_{\mathbb{T}^d \times (\mathbb{R}^d \setminus I_v)} (1+|v|^2)^\nu |f_0(x,v)|^2 dx dv \right) \\
&\leq C \, ||f_0||_{\mathcal{H}_\nu^0(\mathbb{T}_L^d \times (\mathbb{R}^d \setminus I_v))} \leq C\delta
\end{aligned}$$

This achieves to show our claimed estimates. ∎

Finally, we are able to prove the convergence result.

Proof of Theorem 3.1 : We first show that any r-order time integration scheme for second order ODEs can be applied, then proceed to the claimed estimate. Throughout this proof we denote by C a quantity which is independant from $t, n, \Delta t, \Delta z_i, K$, and whose value may change from line to line.

Recall the the characteristics of the Vlasov equation with a truncated Fourier kernel:

$$\begin{cases} \dfrac{d}{dt} X^K(t;0,x,v) = V^K(t;0,x,v) \\ \dfrac{d}{dt} V^K(t;0,x,v) = E^K(t, X^K(t;0,x,v)) \end{cases}$$

where E^K is defined by (1.11). Therefore,

$$\frac{d^2}{dt^2} X^K(t;0,x,v) = E^K(t, X^K(t;0,x,v)).$$

However this function E^K is not a function we can compute in practice in the Weighted Particle method, since it requires a knowledge of the mapping $(x,v) \to (X^K, V^K)(t; t^0, x, v)$ for all $(x,v) \in \mathbb{T}_L^d \times \mathbb{R}^d$ in order to compute $C_k^K(t)$ and $S_k^K(t)$. We instead use the approximations $C_k^{K,h}, S_k^{K,h}$ of C_k^K, S_k^K, given in (2.1):

$$C_k^{K,h}(t) = \sum_{j \in J} \cos\left(2\pi \frac{k}{L} \cdot X^K(t;0,z^j)\right) f(0,z^j) w^j,$$

$$S_k^{K,h}(t) = \sum_{j \in J} \sin\left(2\pi \frac{k}{L} \cdot X^K(t;0,z^j)\right) f(0,z^j) w^j.$$

We recall that from these approximates coefficients, we defined in (2.2) an approximate kernel $E^{K,h}$:

$$E^{K,h}(t,x) = \frac{1}{|\mathbb{T}_L^d|} \sum_{\substack{k \in (\mathbb{Z}^d)^* \\ |k| \leq K}} \frac{1}{2\pi \left|\frac{k}{L}\right|^2} \frac{k}{L} \left[\sin\left(2\pi k \cdot \frac{x}{L}\right) C_k^{K,h}(t) - \cos\left(2\pi k \cdot \frac{x}{L}\right) S_k^{K,h}(t) \right].$$

Let $p = 1, \dots, P$, the quantity $X_p^K(t^n)$, defined in (2.4), is the solution to the second-order ODE:

$$\frac{d^2}{dt^2} X_p^K(t) = E^{K,h}(t, X_p^K(t)), \qquad X_p^K(t^0) = x_p.$$

Moreover we have

$$E^K(t,x) = E^{K,h}(t,x) + (\delta E)^K(t,x)$$

where

$$\begin{aligned}
&(\delta E)^K(t,y) := E^K(t,y) - E^{K,h}(t,y) \\
&= \frac{1}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{1}{2\pi \left|\frac{k}{L}\right|^2} \frac{k}{L} \left[\sin\left(2\pi k \cdot \frac{y}{L}\right)(C_k^{K,h} - C_k^K)(t) - \cos\left(2\pi k \cdot \frac{y}{L}\right)(S_k^{K,h} - S_k^K)(t)\right]
\end{aligned}$$

Thus we deduce

$$\begin{aligned}
|(\delta E)^K(t,y)| &\le \frac{1}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{1}{2\pi \left|\frac{k}{L}\right|^2} \left|\frac{k}{L}\right| \left(\left|C_k^{K,h} - C_k^K\right|(t) + \left|S_k^{K,h} - S_k^K\right|(t)\right) \\
&\le C \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{1}{|k|} \left(\left|(C_k^{K,h} - C_k^K)\right|(t) + \left|(S_k^{K,h} - S_k^K)\right|(t)\right) \\
&\le CK^d\delta + C\sum_{i=1}^{2d} \Delta z_i^{q_i} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{(1+C|k|)^{q_i+1}}{|k|} \\
&\le CK^d \left(\delta + \sum_{i=1}^{2d} \Delta z_i^{q_i} K^{q_i}\right) =: \mathcal{E}(K, \Delta x, \Delta v)
\end{aligned} \tag{5.25}$$

where the third inequality is from (5.18) and (5.19). The exact characteristics $X^K(t; t^0, x_p, v_p)$, defined in (1.13), then satisfy

$$\frac{d^2}{dt^2} X^K(t;t^0,x_p,v_p) = E^{K,h}(X^K(t;t^0,x_p,v_p)) + (\delta E)^K(t, X^K(t;t^0,x_p,v_p)).$$

We recall inequality (3.1), so that we can prove the claimed result in three steps, each one corresponding to a line of this inequality. Each line corresponds to a different type of approximation: the first one is the time discretization error, the second one the phase-space discretization error, and the third one the kernel truncature error.

Because $E^{K,h}$ is more appropriately dealt with by vector variables, we use the following notations: $\mathbb{V}^K(t) := \frac{d}{dt}\mathbb{X}^K(t)$, $\mathcal{X}^K(t) := (X^K(t;t^0,x_1,v_1), \dots, X^K(t;t^0,x_P,v_P))$, and $\mathcal{V}^K(t) := \frac{d}{dt}\mathcal{X}^K(t)$.

Step 1: time discretization error Notice that the time dependence of the function $E^{K,h}$ is only due to the time dependence of the finite-dimensional vector $\mathbb{X}^K(t) \in \mathbb{R}^{dP}$. Therefore we may write $C_k^{K,h}(t) \equiv C_k^{K,h}(\mathbb{X}^K(t))$ by abuse of notations, in which case $C_k^{K,h}(\mathbb{x})$ is a $\mathcal{C}^\infty(\mathbb{R}^{dP}, \mathbb{R})$ function of $\mathbb{x}$. It is possible to write $\frac{d^2}{dt^2}\mathbb{X}^K(t) = \mathbf{E}^{K,h}(\mathbb{X}^K(t))$ for some function

$$\begin{gathered}
\mathbb{R}^{dP} \to \mathbb{R}^{dP} \\
\mathbb{x} = (x_1, \dots, x_P) \mapsto \mathbf{E}^{K,h}(\mathbb{x}) = \left(\mathbf{E}_1^{K,h}(\mathbb{x}), \dots, \mathbf{E}_P^{K,h}(\mathbb{x})\right)
\end{gathered}$$

where, for $i = 1, \dots, P$ we let $x_i \in \mathbb{R}^d$ and

$$\mathbb{R}^d \ni \mathbf{E}_i^{K,h}(\mathbb{x}) = \frac{1}{|\mathbb{T}_L^d|} \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\le K}} \frac{1}{2\pi \left|\frac{k}{L}\right|^2} \frac{k}{L} \left[\sin\left(2\pi k \cdot \frac{x_i}{L}\right) C_k^{K,h}(\mathbb{x}) - \cos\left(2\pi k \cdot \frac{x_i}{L}\right) S_k^{K,h}(\mathbb{x})\right].$$

The coefficients $C_k^{K,h}(\mathbb{x})$ and $S_k^{K,h}(\mathbb{x})$ are defined by

$$C_k^{K,h}(\mathbb{x}) = \sum_{p=1}^{P} \cos\left(2\pi k \cdot \frac{\mathbb{x}_p}{L}\right) \beta_p,$$

$$S_k^{K,h}(\mathbb{x}) = \sum_{p=1}^{P} \sin\left(2\pi k \cdot \frac{\mathbb{x}_p}{L}\right) \beta_p.$$

Therefore, the mapping $\left(\mathbb{x} \mapsto \mathbf{E}^K(\mathbb{x})\right) \in \mathcal{C}^\infty(\mathbb{R}^{dP}, \mathbb{R}^{dP})$. Moreover, from the definition of the characteristics $(\mathbb{X}^K, \mathbb{V}^K)$, we have

$$\begin{cases} \dfrac{d}{dt}\mathbb{X}^K(t) = \mathbb{V}^K(t) \\ \dfrac{d}{dt}\mathbb{V}^K(t) = \mathbf{E}^{K,h}(\mathbb{X}^K(t)) \end{cases}$$

The right-hand side is a $\mathcal{C}^\infty(\mathbb{R}^{2dP}, \mathbb{R}^{2dP})$ function of $(\mathbb{X}^K(t), \mathbb{V}^K(t))$, therefore we know that the characteristics $t \mapsto (\mathbb{X}^K(t), \mathbb{V}^K(t))$ are $\mathcal{C}^\infty([0,T])$.

In order to apply the error estimate for the time integration scheme to solve second-order ODE, we recall that the error depends on the $(\gamma+1)$-th derivative of the function $x \mapsto \mathbf{E}^{K,h}(x)$. If the time integration scheme solves first-order ODEs, the error would depend on the $(\gamma+1)-th$ derivative of the function $(x,v) \mapsto (v, \mathbf{E}^{K,h}(x))$.

It can be shown with the Faà di Bruno formula that for any $l \in \mathbb{N}^{dP}$, $|l| \leq \gamma$,

$$\left|\left|\partial_{\mathbb{x}}^l \left[\sin\left(2\pi k \cdot \frac{x_i}{L}\right) C_k^{K,h}(\mathbb{x}) - \cos\left(2\pi k \cdot \frac{x_i}{L}\right) S_k^{K,h}(\mathbb{x})\right]\right|\right|_{\mathbb{L}^\infty(\mathbb{R}^{dP})} \leq CK^{\gamma+1},$$

where the constant C does not depend on K.

Therefore, no matter if the time integration scheme approximates first-order or second-order ODEs, we obtain for any $n = 1, \ldots, N_t$

$$\max_{p=1,\ldots,P} \left(\left|X_p^{K,n} - X_p^K(t^n)\right| + \left|V_p^{K,n} - V_p^K(t^n)\right|\right) \leq CK^{d+\gamma+1}\Delta t^\gamma$$

where the constant C does not depend on K or Δt.

Step 2: phase-space discretization The assumptions that characteristics and their approximations have the same initial conditions can be rewritten as $\mathbb{X}^K(t^0) = \mathcal{X}^K(t^0)$ and $\mathbb{V}^K(t^0) = \mathcal{V}^K(t^0)$. We have, for $s \in [t^0, t^0 + T]$,

$$\begin{aligned} \begin{pmatrix} \mathbb{X}^K(s) \\ \mathbb{V}^K(s) \end{pmatrix} &= \begin{pmatrix} \mathbb{X}^K(t^0) \\ \mathbb{V}^K(t^0) \end{pmatrix} + \int_{t^0}^s \begin{pmatrix} \mathbb{V}^K(\tau) \\ \mathbf{E}^{K,h}(\mathbb{X}^K(\tau)) \end{pmatrix} d\tau \\ &= \begin{pmatrix} \mathbb{X}^K(t^0) \\ \mathbb{V}^K(t^0) \end{pmatrix} + \int_{t^0}^s \begin{pmatrix} \mathbb{V}^K(\tau) \\ E^K(\tau, \mathbb{X}^K(\tau)) \end{pmatrix} d\tau + \int_{t^0}^s \begin{pmatrix} 0 \\ (\delta E)^K(\tau, \mathbb{X}^K(\tau)) \end{pmatrix} d\tau. \end{aligned}$$

Note that we also have

$$\begin{pmatrix} \mathcal{X}^K(s) \\ \mathcal{V}^K(s) \end{pmatrix} = \begin{pmatrix} \mathbb{X}^K(t^0) \\ \mathbb{V}^K(t^0) \end{pmatrix} + \int_{t^0}^s \begin{pmatrix} \mathcal{V}^K(\tau) \\ E^K(\tau, \mathcal{X}^K(\tau)) \end{pmatrix} d\tau,$$

so that

$$\begin{pmatrix} \mathbb{X}^K(s) \\ \mathbb{V}^K(s) \end{pmatrix} = \begin{pmatrix} \mathcal{X}^K(s) \\ \mathcal{V}^K(s) \end{pmatrix} + \int_{t^0}^s \begin{pmatrix} \mathbb{V}^K(\tau) - \mathcal{V}^K(\tau) \\ E^K(\mathbb{X}^K(\tau)) - E^K(\mathcal{X}^K(\tau)) \end{pmatrix} d\tau + \int_{t^0}^s \begin{pmatrix} 0 \\ (\delta E)^K(\tau, \mathbb{X}^K(\tau)) \end{pmatrix} d\tau.$$

From the mean value theorem we get:

$$\left|E^K(\tau, \mathbb{X}^K(\tau)) - E^K(\tau, \mathcal{X}^K(\tau))\right| \leq C \left(\sum_{\substack{k \in (\mathbb{Z}^d)^* \\ |k| \leq K}} |C_k^K(t)| + |S_k^K(t)| \right) \left|\mathbb{X}^K(\tau) - \mathcal{X}^K(\tau)\right|.$$

Using the fact that the function $f_0 \in \mathcal{H}^{r+\alpha}_{\nu+j}$, we can apply the same ideas as those leading to (5.9), in order to obtain

$$|C_k(t)| \leq \frac{C}{(1+|k|)^{r+\alpha+d/2}}$$

for some $C>0$ which does not depend on k. The same estimate holds for $\left|S^K_k(t)\right|$. Hence

$$\begin{aligned}\left|E^K(\tau,\mathbb{X}^K(\tau)) - E^K(\tau,\mathcal{X}^K(\tau))\right| &\leq C \sum_{\substack{k\in(\mathbb{Z}^d)^* \\ |k|\leq K}} \frac{1}{(1+|k|)^{r+\alpha+d/2}} \left|\mathbb{X}^K(\tau) - \mathcal{X}^K(\tau)\right| \\ &\leq C\left|\mathbb{X}^K(\tau) - \mathcal{X}^K(\tau)\right| \end{aligned} \tag{5.26}$$

where the constant C can be taken independant of K because $r+\alpha > d/2$. Thus, using (5.25),

$$\begin{aligned}\left|\begin{pmatrix}\mathbb{X}^K(s) - \mathcal{X}^K(s) \\ \mathbb{V}^K(s) - \mathcal{V}^K(s)\end{pmatrix}\right| &\leq \int_{t^0}^s \left|\begin{pmatrix}\mathbb{V}^K(\tau) - \mathcal{V}^K(\tau) \\ E^K(\mathbb{X}^K(\tau)) - E^K(\mathcal{X}^K(\tau))\end{pmatrix}\right| d\tau + T\left|\mathcal{E}(K,\Delta x,\Delta v)\right| \\ &\leq C\int_{t^0}^s \left|\begin{pmatrix}\mathbb{V}^K(\tau) - \mathcal{V}^K(\tau) \\ \mathbb{X}^K(\tau) - \mathcal{X}^K(\tau)\end{pmatrix}\right| d\tau + T\left|\mathcal{E}(K,\Delta x,\Delta v)\right|,\end{aligned}$$

and we conclude by using the Grönwall lemma:

$$\left|\begin{pmatrix}\mathbb{X}^K(s) - \mathcal{X}^K(s) \\ \mathbb{V}^K(s) - \mathcal{V}^K(s)\end{pmatrix}\right| \leq CTe^{CT}K^d\left(\delta + \sum_{i=1}^{2d} \Delta z_i^{q_i} K^{q_i}\right),$$

where C is independant of $K, \Delta z_i, s$.

Step 3: kernel truncature error We estimate the approximation in the characteristics that is due to the truncation error in the Fourier Kernel. For $p=1,\dots,P$,

$$X^K(t;t^0,x_p,v_p) = x_p + \int_{t^0}^t V^K(\tau;t^0,x_p,v_p)d\tau, \quad X(t;t^0,x_p,v_p) = x_p + \int_{t^0}^t V(\tau;t^0,x_p,v_p)d\tau,$$

$$V^K(t;t^0,x_p,v_p) = v_p + \int_{t^0}^t E^K(\tau, X^K(t^0,x_p,v_p))d\tau, \quad V(t;t^0,x_p,v_p) = v_p + \int_{t^0}^t E(\tau; X(\tau;t^0,x_p,v_p))d\tau,$$

so that we have

$$\begin{aligned}\begin{pmatrix}\mathcal{X}^K(s) \\ \mathcal{V}^K(s)\end{pmatrix} &= \begin{pmatrix}\mathcal{X}^K(t^0) \\ \mathcal{V}^K(t^0)\end{pmatrix} + \int_{t^0}^s \begin{pmatrix}\mathcal{V}^K(s) \\ E^K(\tau,\mathcal{X}^K(\tau))\end{pmatrix} d\tau \\ &= \begin{pmatrix}\mathcal{X}(s) \\ \mathcal{V}(s)\end{pmatrix} + \int_{t^0}^s \begin{pmatrix}\mathcal{V}^K(s) - \mathcal{V}(s) \\ E^K(\tau,\mathcal{X}^K(\tau)) - E(\tau,\mathcal{X}(\tau))\end{pmatrix} d\tau \\ &= \begin{pmatrix}\mathcal{X}(s) \\ \mathcal{V}(s)\end{pmatrix} + \int_{t^0}^s \begin{pmatrix}\mathcal{V}^K(s) - \mathcal{V}(s) \\ E^K(\tau,\mathcal{X}^K(\tau)) - E(\tau,\mathcal{X}^K(\tau))\end{pmatrix} + \begin{pmatrix}0 \\ E(\tau,\mathcal{X}^K(\tau)) - E(\tau,\mathcal{X}(\tau))\end{pmatrix} d\tau.\end{aligned}$$

Thus,

$$\left|\begin{pmatrix}\mathcal{X}^K(s) - \mathcal{X}(s) \\ \mathcal{V}^K(s) - \mathcal{V}(s)\end{pmatrix}\right| \leq \int_{t^0}^s \left|\begin{pmatrix}\mathcal{V}^K(s) - \mathcal{V}(s) \\ E^K(\tau,\mathcal{X}^K(\tau)) - E^K(\tau,\mathcal{X}(\tau))\end{pmatrix}\right| + \left|\begin{pmatrix}0 \\ E^K(\tau,\mathcal{X}(\tau)) - E(\tau,\mathcal{X}(\tau))\end{pmatrix}\right| d\tau.$$

Since $f_0 \in \mathcal{H}^{r+\alpha}_{\nu+j}$, by Proposition 5.2 we have $\left|\left|(f-f^K)(t)\right|\right|^2_{\mathcal{H}^r_\nu} \leq \frac{C}{(1+K)^\alpha}$, hence by Proposition 5.3 we obtain

$$\left|\begin{pmatrix}0 \\ E^K(\tau,\mathcal{X}(\tau)) - E(\tau,\mathcal{X}(\tau))\end{pmatrix}\right| \leq \frac{C}{(1+K)^{\frac{\alpha+1}{2}-d}},$$

where C does not depend on K. We get

$$\left|\begin{pmatrix}\mathcal{X}^K(s) - \mathcal{X}(s) \\ \mathcal{V}^K(s) - \mathcal{V}(s)\end{pmatrix}\right| \leq \int_{t^0}^s \left|\begin{pmatrix}\mathcal{V}^K(s) - \mathcal{V}(s) \\ E^K(\tau,\mathcal{X}^K(\tau)) - E^K(\tau,\mathcal{X}(\tau))\end{pmatrix}\right| d\tau + \frac{C}{(1+K)^{\frac{\alpha+1}{2}-d}}$$

For the same reasons as those leading to (5.26), we obtain

$$\left|\begin{pmatrix}\mathcal{X}^K(s)-\mathcal{X}(s)\\ \mathcal{V}^K(s)-\mathcal{V}(s)\end{pmatrix}\right| \leq C\int_{t^0}^{s}\left|\begin{pmatrix}\mathcal{V}^K(s)-\mathcal{V}(s)\\ \mathcal{X}^K(\tau)-\mathcal{X}(\tau)\end{pmatrix}\right| d\tau + \frac{CT}{(1+K)^{\frac{\alpha+1}{2}-d}}.$$

Finally, the Grönwall lemma yields

$$\left|\begin{pmatrix}\mathcal{X}^K(s)-\mathcal{X}(s)\\ \mathcal{V}^K(s)-\mathcal{V}(s)\end{pmatrix}\right| \leq \frac{CTe^{CT}}{(1+K)^{\frac{\alpha+1}{2}-d}}$$

which completes the proof. ■

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